On the Distribution of the Product and Ratio of Independent Central and Doubly Non-central Generalized Gamma Ratio random variables

Carlos A. Coelho and João T. Mexia [1]

Abstract

Using a decomposition of the characteristic function of the logarithm of the product of independent Generalized Gamma Ratio random variables we obtain explicit expressions for both the probability density and cumulative distribution functions of the product of independent central or non-central random variables with generalized F or Generalized Gamma Ratio distributions under the form of particular mixtures of Pareto and inverted Pareto distributions. The expressions obtained do not involve any unsolved integrals and are much adequate for computer implementation and the development of asymptotic and near-exact distributions. By considering not necessarily positive power parameters we were able to obtain as particular cases not only the product of Beta prime, folded T, folded Cauchy and F random variables but also the densities and distributions for the ratio of two independent Generalized Gamma Ratio random variables or two independent products of such variables. Products of Generalized Gamma Ratio distributions may be applied in the study of multivariate linear functional models. As a by-product we also obtain closed form representations for the distribution of the difference of two independent sums of a finite number of Gamma random variables with different rate parameters and integer shape parameters, under the form of finite mixtures of Gamma distributions, as well as the distributions for the product and ratio of generalized Pareto distributions, under the form of finite mixtures of Pareto and inverted Pareto distributions.

Key words: particular mixtures, Pareto and inverted Pareto distributions, GIG distribution, sum of Exponentials, difference of Exponentials, folded T, folded Cauchy, Beta prime, Beta second kind

1 Introduction

The problem of obtaining an explicit expression, without involving any unsolved integrals, for both the probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of the product of independent Generalized Gamma Ratio random variables (r.v.'s) or Generalized F r.v.'s, as they are also called, is a challenging one, moreover since the characteristic function is not readily available for such r.v.'s. In this paper we present the distribution for the product of independent central and doubly non-central Generalized Gamma Ratio (GGR) random variables under the form of particular mixtures of Pareto and inverted Pareto distributions. Expressions for the p.d.f. (and c.d.f. ?) of the central case of such a product were obtained by Shah and Rathie (1974) in terms of Fox's H function. Yet, only for the central case, also Pham-Gia and Turkkan (2002) obtained expressions for the p.d.f. of the product of only two independent generalized F r.v.'s in terms of the Lauricella hypergeometric D-function of two variables. However, even nowadays when good softwares for symbolic and numeric computation are available the computation of Fox's H function and the Lauricella function is not readily available, being usually computed in terms of the integrals that define them. Although Pham-Gia and Turkkan (2002) developed an efficient computer code to compute the Lauricella hypergeometric function, their approach is not extensible to the product of more than two r.v.'s and as these authors strengthen, when we consider the product of more than two GGR r.v.'s it seems that "frequently, however, no closed form solution for these operations can be obtained and one has to resort to approximate approaches, including simulation". The same authors, when referring to the results in Shah and Rathie (1974)

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say that "these results, although very convenient notationwise, are difficult to be programmed on a computer and hence are difficult to be used in applications". Yet the same authors say that "they are, however, essential when the number of variables in the product, or quotient, is larger than 2". In this paper our aim is to obtain explicit simpler expressions for the p.d.f. and c.d.f. of the product of independent Generalized Gamma Ratio r.v.'s which may be readily implemented computationally and that, given its structure, may also give us ready access to asymptotic and near-exact distributions. Given the approach followed, not only the distributions for the non-central case are readily at hand but also the distribution for the product of any non-null power of GGR r.v.'s.

As particular immediate cases we have the product of central and non-central independent generalized second kind Beta or Beta prime, folded T and folded Cauchy r.v.'s and yet, of course, F r.v.'s.

Given the fact that the characteristic functions for the GGR random variables are not readily available and given the fact that we are dealing with a product of random variables, it is handier to carry our work through the decomposition of the characteristic function of the logarithm of the product of the GGR random variables. Thus, this was our choice.

Another novelty is that, although usually only positive power parameters are considered for the GGR distributions, actually nothing forces those parameters to be positive, given that the correct approach is taken, being the case that actually a negative power parameter in the GGR distribution only takes us to consider the reciprocal of that given random variable with the symmetrical positive power parameter. Given the way he problem is approached, even negative power parameters may be easily considered in the GGR distributions and also the distribution of the ratio of two GGR random variables or of the ratio of two products of GGR random variables are particular cases of the results obtained in the paper.

Products of several independent GGR random variables are related to a test statistic used in the multivariate linear functional model (Provost, 1986).

2 Some preliminary results

2.1 The Generalized Gamma Ratio (GGR) distribution

In order to establish some of the notation, nomenclature and a result used ahead we will start to define what we intend by a Generalized Gamma Ratio (GGR) distribution. Let

$$X_1 \sim \Gamma(r_1,\lambda_1) \quad \text{and} \quad X_2 \sim \Gamma(r_2,\lambda_2)$$

be to independent r.v.'s with Gamma distributions with shape parameters r_1 and r_2 and rate parameters λ_1 and λ_2, that is, for example, X_1 has p.d.f. (probability density function)

$$f_{X_1}(x) = \frac{\lambda_1^{r_1}}{\Gamma(r_1)}\, e^{-\lambda_1 x}\, x^{r_1-1}\,, \qquad r_1,\lambda_1>0;\, x_1>0\,.$$

Let then

$$Y_1 = X_1^{1/\beta}\,, \quad Y_2 = X_2^{1/\beta}\,, \quad \beta\in I\!R\backslash\{0\}$$

and

$$Z = Y_1/Y_2\,.$$

We will say that Y_1 and Y_2 have Generalized Gamma distributions and that Z has a GGR distribution. Using standard methods we have the p.d.f.^{s} of Y_i $(i=1,2)$ and Z given by

$$f_{Y_i}(y_i) = \frac{|\beta|\,\lambda_i^{r_i}}{\Gamma(r_i)}\, e^{-\lambda_i y_i^\beta}\, y_i^{\beta r_i-1}\,, \quad y_i>0\,, \quad (i=1,2)$$

and

$$f_Z(z) = \frac{|\beta|\,k^{r_1}}{B(r_1,r_2)}\,\left(1+kz^\beta\right)^{-r_1-r_2}\, z^{\beta r_1-1}\,, \quad z>0$$

where $k=\lambda_1/\lambda_2$ and $B(\cdot,\cdot)$ is the Beta function.

We will denote the fact that Z has the GGR distribution with parameters k, r_1, r_2 and β by

$$Z \sim GGR(k,r_1,r_2,\beta)\,.$$

The non-central moments of Z are easily derived as

$$E\left(Z^h\right) = k^{-h}\,\frac{\Gamma(r_1+h)}{\Gamma(r_1)}\,\frac{\Gamma(r_2-h)}{\Gamma(r_2)}\,, \qquad (-r_1<h<r_2)\,.$$

If $\beta=1$, $r_1=m/2$ and $r_2=n/2$, with $m,n\in I\!N$, then Z has an F distribution with m and n degrees of freedom. This is the reason why the distribution of Z is also called a Generalized F distribution (Shah and Rathie, 1974).

2.2 The Generalized Integer Gamma (GIG) distribution

In this subsection and in the two following ones we will establish some distributions that will be used in the next section. Let

$$X_j \sim \Gamma(r_j,\lambda_j) \qquad j=1,\ldots,p$$

be p independent r.v.'s with Gamma distributions with shape parameters $r_j\in I\!N$ and rate parameters $\lambda_j>0$ $(j=1,\ldots,p)$. We will say that then the r.v.

$$Y=\sum_{j=1}^{p} X_j$$

has a GIG distribution of depth p, with shape parameters r_j and rate parameters λ_j, $(j=1,\ldots,p)$, and we will denote this fact by

$$Y \sim GIG(r_j,\lambda_j;p) \qquad j=1,\ldots,p\,.$$

The p.d.f. and c.d.f. (cumulative distribution function) of Y are, see Coelho (1998), respectively given by

$$f_Y(y) = K\sum_{j=1}^{p} P_j(y)\,e^{-\lambda_j\,y} \tag{1}$$

and

$$F_Y(y) = 1-K\sum_{j=1}^{p} P_j^*(y)\,e^{-\lambda_j\,y}$$

where

$$K=\prod_{j=1}^{p}\lambda_j^{r_j}\,, \qquad P_j(y)=\sum_{k=1}^{r_j} c_{jk}\,y^{k-1} \tag{2}$$

and

$$P_j^*(y)=\sum_{k=1}^{r_j} c_{jk}\,(k-1)!\sum_{i=0}^{k-1}\frac{y^i}{i!\,\lambda_i^{k-i}}$$

with

$$c_{j,r_j}(p,\underline{r})=\frac{1}{(r_j-1)!}\prod_{\substack{i=1\\ i\neq j}}^{p}(\lambda_i-\lambda_j)^{-r_i}\,, \qquad j=1,\ldots,p\,, \tag{3}$$

and

$$c_{j,r_j-k}=\frac{1}{k}\sum_{i=1}^{k}\frac{(r_j-k+i-1)!}{(r_j-k-1)!}\,R(i,j,p,\underline{r},\underline{\lambda})\,c_{j,r_j-(k-i)}\,, \qquad \begin{array}{l}(k=1,\ldots,r_j-1)\\ (j=1,\ldots,p)\end{array} \tag{4}$$

where

$$\underline{r}=[r_1,r_2,\ldots,r_p]'\,, \quad \text{and} \quad R(i,j,p,\underline{r},\underline{\lambda})=\sum_{\substack{k=1\\ k\neq j}}^{p} r_k\,(\lambda_j-\lambda_k)^{-i} \quad (i=1,\ldots,r_j-1)\,. \tag{5}$$

2.3 The distribution of the sum of random variables with Exponential distribution and the distribution of the difference of two of these random variables

Let

$$X_i \sim Exp(\lambda_i) \qquad i = 1, \ldots, p$$

be p independent Exponential r.v.'s with rate parameters λ_i ($i = 1, \ldots, p$), and let

$$Y = \sum_{i=1}^{p} X_i \,.$$

The distribution of the r.v. Y is a particular case of the GIG distribution (Coelho, 1998) of depth p, with all the shape parameters equal to 1, whose p.d.f. may be written as

$$f_Y(y) = K \sum_{j=1}^{p} c_j \, e^{-\lambda_j y}$$

where

$$K = \prod_{j=1}^{p} \lambda_j \qquad \text{and} \qquad c_j = \prod_{\substack{k=1 \\ k \neq j}}^{p} \frac{1}{\lambda_j - \lambda_k} \quad (j = 1, \ldots, p).$$

We will denote the fact that Y has this distribution by

$$Y \sim SE(\lambda_j, \, j \in \{1, \ldots, p\}) \,.$$

Let then

$$Y_1 \sim SE(\lambda_j, \, j \in \{1, \ldots, p\}) \quad \text{and} \quad Y_2 \sim SE(\nu_j, \, j \in \{1, \ldots, p\})$$

be two independent r.v.'s and let

$$Z = Y_1 - Y_2 \,.$$

The p.d.f.^{s} of Y_1 and Y_2 may then be respectively written as

$$f_{Y_1}(y_1) = K_1 \sum_{j=1}^{p} c_j \, e^{-\lambda_j y_1} \quad \text{and} \quad f_{Y_2}(y_2) = K_2 \sum_{j=1}^{p} d_j \, e^{-\nu_j y_2}$$

where

$$K_1 = \prod_{j=1}^{p} \lambda_j \; , \quad K_2 = \prod_{j=1}^{p} \nu_j \tag{6}$$

and, for $j = 1, \ldots, p$,

$$c_j = \prod_{\substack{k=1 \\ k \neq j}}^{p} \frac{1}{\lambda_j - \lambda_k} \; , \quad d_j = \prod_{\substack{k=1 \\ k \neq j}}^{p} \frac{1}{\nu_j - \nu_k} \tag{7}$$

so that the p.d.f. of Z will be given by

$$\begin{aligned} f_Z(z) &= \int_{\max(z,0)}^{+\infty} K_1 K_2 \left(\sum_{j=1}^{p} c_j \, e^{-\lambda_j \, y_1} \right) \left(\sum_{j=1}^{p} d_j \, e^{-\nu_j (y_1 - z)} \right) dy_1 \\ &= K_1 K_2 \sum_{j=1}^{p} \sum_{k=1}^{p} e^{\nu_k z} \, c_j \, d_k \int_{\max(z,0)}^{+\infty} e^{-(\lambda_j + \nu_k) y_1} \, dy_1 \end{aligned}$$

or,

$$f_Z(z) \;=\; \begin{cases} K_1K_2 \displaystyle\sum_{j=1}^{p} H_{1j}\, c_j\, e^{-\lambda_j\, z} & z \geq 0 \\[2ex] K_1K_2 \displaystyle\sum_{j=1}^{p} H_{2j}\, d_j\, e^{\nu_j\, z} & z \leq 0 \end{cases} \tag{8}$$

where

$$H_{1j} = \sum_{h=1}^{p} \frac{d_h}{\lambda_j + \nu_h} \qquad \text{and} \qquad H_{2j} = \sum_{h=1}^{p} \frac{c_h}{\lambda_h + \nu_j}\,.$$

Then if we take

$$W \;=\; k^{-1} e^{Z}$$

we will have

$$f_W(w) \;=\; \begin{cases} K_1K_2 \displaystyle\sum_{j=1}^{p} H_{1j}\, c_j\, (kw)^{-\lambda_j} \frac{1}{w} & w \geq k^{-1} \\[2ex] K_1K_2 \displaystyle\sum_{j=1}^{p} H_{2j}\, d_j\, (kw)^{\nu_j} \frac{1}{w} & 0 < w \leq k^{-1}\,. \end{cases} \tag{9}$$

The c.d.f.^{s} of Z and W are, respectively,

$$F_Z(z) \;=\; \begin{cases} K_1K_2 \displaystyle\sum_{j=1}^{p} \left(H_{2j} \frac{d_j}{\nu_j} + H_{1j} \frac{c_j}{\lambda_j} \left(1 - e^{-\lambda_j\, z}\right)\right) & z \geq 0 \\[2ex] K_1K_2 \displaystyle\sum_{j=1}^{p} H_{2j} \frac{d_j}{\nu_j}\, e^{\nu_j\, z} & z \leq 0 \end{cases} \tag{10}$$

and

$$F_W(w) \;=\; \begin{cases} K_1K_2 \displaystyle\sum_{j=1}^{p} \left(H_{2j} \frac{d_j}{\nu_j} + H_{1j} \frac{c_j}{\lambda_j} \left(1 - (kw)^{-\lambda_j}\right)\right) & w \geq k^{-1} \\[2ex] K_1K_2 \displaystyle\sum_{j=1}^{p} H_{2j} \frac{d_j}{\nu_j}\, (kw)^{\nu_j} & 0 < w \leq k^{-1}\,. \end{cases} \tag{11}$$

The distribution of Z is also the distribution of the sum of p independent r.v.'s with the distribution of the difference of two independent r.v.'s with exponential distribution (either with similar or different parameters).

We should note that although namely in (6) it may seem that it would not be reasonable to take $p \to \infty$, as a matter of fact in both (8) and (9) taking $p \to \infty$ will yield proper legitimate distributions (p.d.f.^{s}).

If we take into account that if the r.v. X has an Exponential distribution with rate parameter λ, with p.d.f.

$$f_X(x) \;=\; \lambda\, e^{-\lambda x}\,, \qquad \lambda > 0; x > 0\,,$$

the r.v. $Y = k * e^X$ has a Pareto distribution with rate parameter λ and lower bound parameter k, with p.d.f.

$$f_Y(y) \;=\; \lambda \left(\frac{y}{k}\right)^{-\lambda} \frac{1}{y}\,, \qquad \lambda > 0; y\, k\,.$$

We will also say that the r.v. $X_1 = -X$ with p.d.f.

$$f_{X_1}(x) \;=\; \lambda\, e^{\lambda x}\,, \qquad \lambda > 0; x < 0$$

has a symmetrical Exponential distribution with rate parameter λ and that the r.v. $Y_1 = 1/Y$ with p.d.f.

$$f_{Y_1}(y) \;=\; \lambda \left(\frac{y}{k^{-1}}\right)^{\lambda} \frac{1}{y}$$

has an inverted Pareto distribution with rate parameter λ and lower bound parameter k^{-1}.

We may then also note that while the distribution of Z, for $z \geq 0$, may be seen as a particular mixture of Exponential distributions with rate parameters λ_j $(j = 1, \ldots, p)$, with weights

$$p_j \;=\; K_1 K_2 H_{1j} \frac{c_j}{\lambda_j}\,, \qquad j = 1, \ldots, p\,,$$

with

$$\sum_{j=1}^{p} p_j \;=\; P[Z \geq 0]$$

and for $z \leq 0$ as a particular mixture of symmetrical Exponential distributions with rate parameters ν_j, with weights

$$s_j \;=\; K_1 K_2 H_{2j} \frac{d_j}{\nu_j}\,, \qquad j = 1, \ldots, p\,,$$

with

$$\sum_{j=1}^{p} s_j \;=\; P[Z \leq 0]\,,$$

the distribution of W may, for $w \geq k^{-1}$, be seen as a particular mixture of Pareto distributions with rate parameters λ_j $(j = 1, \ldots, p)$ and lower bound parameters k^{-1}, with weights p_j $(j = 1, \ldots, p)$, with

$$\sum_{j=1}^{p} p_j \;=\; P[W \geq k^{-1}]\,,$$

for $w \leq k^{-1}$, it may be seen as a mixture of inverted Pareto distributions with rate parameters ν_j $(j = 1, \ldots, p)$ and lower bound parameters k, with weights s_j $(j = 1, \ldots, p)$, with

$$\sum_{j=1}^{p} s_j \;=\; P[W \leq k^{-1}]\,.$$

2.4 The distribution of the difference of two GIG distributions

Let, for $j = 1, \ldots, p_1$ and $l = 1, \ldots, p_2$,

$$Y_1 \sim GIG(r_{1j}, \lambda_j, p_1)\,, \qquad \text{and} \qquad Y_2 \sim GIG(r_{2l}, \nu_l, p_2)\,,$$

be two independent r.v.'s and let

$$Z \;=\; Y_1 - Y_2\,.$$

Then, considering (1) and taking K_1 and c_{jk} defined in a similar manner to K and c_{jk} in (2) and (3)-(5) respectively, and K_2 and d_{lh} defined in a corresponding manner, using p_2 instead of p_1, r_{2l} instead of r_{1j}, and ν_l instead of λ_j, for $l = 1, \ldots, p_2$ and $j = 1, \ldots, p_1$, the p.d.f. of Z is given by

$$f_Z(z) \;=\; K_1 K_2 \int_{\max(z,0)}^{+\infty} \left(\sum_{j=1}^{p_1} \left(\sum_{k=1}^{r_{ij}} c_{jk}\, y_1^{k-1} \right) e^{-\lambda_j y_1} \right) \left(\sum_{l=1}^{p_2} \left(\sum_{h=1}^{r_{2l}} d_{lh} (y_1 - z)^{h-1} \right) e^{-\nu_l (y_1 - z)} \right) dy_1$$

$$
\begin{aligned}
&= K_1 K_2 \int_{\max(z,0)}^{+\infty} \sum_{j=1}^{p_1} \sum_{l=1}^{p_2} \left(\sum_{k=1}^{r_{1j}} c_{jk}\, y_1^{k-1} \right) \left(\sum_{h=1}^{r_{2l}} d_{lh} (y_1 - z)^{h-1} \right) e^{-(\lambda_j + \nu_l) y_1}\, e^{\nu_l\, z}\, dy_1 \\
&= K_1 K_2 \sum_{j=1}^{p_1} \sum_{l=1}^{p_2} \int_{\max(z,0)}^{+\infty} \sum_{k=1}^{r_{1j}} \sum_{h=1}^{r_{2l}} c_{jk}\, d_{lh}\, y_1^{k-1} (y_1 - z)^{h-1}\, e^{-(\lambda_j + \nu_l) y_1}\, e^{\nu_l z}\, dy_1 \\
&= K_1 K_2 \sum_{j=1}^{p_1} \sum_{l=1}^{p_2} e^{\nu_l z} \sum_{k=1}^{r_{1j}} \sum_{h=1}^{r_{2l}} c_{jk}\, d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} (-z)^{h-1-i} \int_{\max(z,0)}^{+\infty} e^{-(\lambda_j + \nu_l) y_1}\, y_1^{k+i-1}\, dy_1
\end{aligned}
\tag{12}
$$

what, taking, for $m > 0$ and $k \in I\!N_0$,

$$
\int_{\max(z,0)}^{+\infty} e^{-my}\, y^k\, dy = \begin{cases} e^{-mz} \displaystyle\sum_{i=0}^{k} \frac{k!}{i!} \frac{z^i}{m^{k-i+1}} & z \geq 0 \\[2ex] \dfrac{k!}{m^{k+1}} & z \leq 0 \end{cases}
\tag{13}
$$

gives

$$
f_Z(z) = \begin{cases} K_1 K_2 \displaystyle\sum_{j=1}^{p_1} P_{1j}^{**}(z)\, e^{-\lambda_j\, z} & z \geq 0 \\[2ex] K_1 K_2 \displaystyle\sum_{j=1}^{p_1} P_{2j}^{**}(z)\, e^{\nu_j\, z} & z \leq 0 \end{cases}
\tag{14}
$$

where

$$
P_{1j}^{**}(z) = \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} (-z)^{h-1-i} \sum_{t=0}^{k+i-1} \frac{(k+i-1)!}{t!} \frac{z^t}{(\lambda_j + \nu_l)^{k+i-t}}
\tag{15}
$$

and

$$
P_{2j}^{**}(z) = \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} (-z)^{h-1-i} \frac{(k+i-1)!}{(\lambda_j + \nu_j)^{k+i}} \,.
\tag{16}
$$

It is however interesting and useful to observe that, given that the distribution of $Y_1 - Y_2$ and $Y_2 - Y_1$ are symmetrical, and that we may in (12) integrate in order to y_2 instead of y_1, we may obtain the p.d.f. of Z given by a similar expression to the one in (14), with

$$
P_{1j}^{**}(z) = \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{k-1} \binom{k-1}{i} z^{k-1-i} \frac{(h+i-1)!}{(\lambda_j + \nu_l)^{h+i}}
\tag{17}
$$

and

$$
P_{2j}^{**}(z) = \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{k-1} \binom{k-1}{i} z^{k-1-i} \sum_{t=0}^{h+i-1} \frac{(h+i-1)!}{t!} \frac{z^t}{(\lambda_j + \nu_l)^{h+i-t}} \,,
\tag{18}
$$

moreover since indeed

$$
\sum_{i=0}^{h-1} \binom{h-1}{i} (-z)^{h-1-i} \sum_{t=0}^{k+i-1} \frac{(k+i-1)!}{t!} \frac{z^t}{(\lambda_j + \nu_l)^{k+i-t}} = \sum_{i=0}^{k-1} \binom{k-1}{i} z^{k-1-i} \frac{(h+i-1)!}{(\lambda_j + \nu_l)^{h+i}} \,.
$$

This way, in order to obtain a simpler expression for the p.d.f. of Z we may consider the p.d.f. in (14) with $P_{1j}^{**}(z)$ given by (17) and $P_{2j}^{**}(z)$ given by (16).

Then, using (13), the c.d.f. of Z may be written as

$$F_Z(z) \;=\; \begin{cases} K_1K_2 \displaystyle\sum_{j=1}^{p_1} P_{1j}^{***}(z)\, e^{-\lambda_j\, z} & z \geq 0 \\ K_1K_2 \displaystyle\sum_{j=1}^{p_1} P_{2j}^{***}(z)\, e^{\nu_j\, z} & z \leq 0 \end{cases} \tag{19}$$

with

$$P_{1j}^{***}(z) \;=\; \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(h+i-1)!}{(\lambda_j+\nu_l)^{h+i}} \sum_{t=0}^{k-1-i} \frac{(k-1-i)!}{t!} \frac{z^t}{\lambda_j^{k-i-t}} \tag{20}$$

and

$$P_{2j}^{***}(z) \;=\; \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} \frac{(k+i-1)!}{(\lambda_j+\nu_j)^{k+i}} \sum_{t=0}^{h-1-i} \frac{(h-1-i)!}{t!} \frac{(-z)^t}{\nu_l^{h-i-t}}\,. \tag{21}$$

If we consider, for $k>0$, the r.v.

$$W \;=\; k^{-1} e^Z$$

we have

$$f_W(w) \;=\; \begin{cases} K_1K_2 \displaystyle\sum_{j=1}^{p_1} Q_{1j}^{**}(\log(kw))\, (kw)^{-\lambda_j}\, \frac{1}{w} & w \geq k^{-1} \\ K_1K_2 \displaystyle\sum_{j=1}^{p_1} Q_{2j}^{**}(\log(kw))\, (kw)^{\nu_j}\, \frac{1}{w} & 0 < w \leq k^{-1} \end{cases} \tag{22}$$

where

$$Q_{1j}^{**}(\log(kw)) \;=\; \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{k-1} \binom{k-1}{i} (\log(kw))^{k-1-i} \frac{(h+i-1)!}{(\lambda_j+\nu_l)^{h+i}} \tag{23}$$

and

$$Q_{2j}^{**}(\log(kw)) \;=\; \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} (-\log(kw))^{h-1-i} \frac{(k+i-1)!}{(\lambda_j+\nu_j)^{k+i}}\,. \tag{24}$$

$$F_W(w) \;=\; \begin{cases} K_1K_2 \displaystyle\sum_{j=1}^{p_1} Q_{1j}^{***}(\log(kw))\, (kw)^{-\lambda_j} & w \geq k^{-1} \\ K_1K_2 \displaystyle\sum_{j=1}^{p_1} Q_{2j}^{***}(\log(kw))\, (kw)^{\nu_j} & 0 < w \leq k^{-1} \end{cases} \tag{25}$$

with

$$Q_{1j}^{***}(\log(kw)) \;=\; \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(h+i-1)!}{(\lambda_j+\nu_l)^{h+i}} \sum_{t=0}^{k-1-i} \frac{(k-1-i)!}{t!} \frac{(\log(kw))^t}{\lambda_j^{k-i-t}} \tag{26}$$

and

$$Q_{2j}^{***}(\log(kw)) \;=\; \sum_{k=1}^{r_{1j}} c_{jk} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} \frac{(k+i-1)!}{(\lambda_j+\nu_j)^{k+i}} \sum_{t=0}^{h-1-i} \frac{(h-1-i)!}{t!} \frac{(-\log(kw))^t}{\nu_l^{h-i-t}}\,. \tag{27}$$

As we did with (8) and (9), also in (14) through (18) we may take both $p_1 \to \infty$ and $p_2 \to \infty$, still holding proper distributions.

3 The distribution of the product of m independent random variables with GGR distributions

3.1 The case with all distinct shape parameters for the numerator and denominator

Let

$$X_j \sim GGR(k_j, r_{2j}, r_{1j}, \beta_j) \qquad j = 1, \ldots, m$$

be m independent r.v.'s.

We want an explicit and concise expression for the p.d.f. and c.d.f. of the r.v.

$$W = \prod_{j=1}^{m} X_j \,. \tag{28}$$

Based on the results in subsection 1.1 we have

$$E\left(X_j^h\right) = k_j^{-h/\beta_j} \frac{\Gamma(r_{2j}+h/\beta_j)\,\Gamma(r_{1j}-h/\beta_j)}{\Gamma(r_{1j})\,\Gamma(r_{2j})}$$

and if we take

$$Y_j = k_j^{1/\beta_j} X_j$$

then we have

$$E\left(Y_j^h\right) = \frac{\Gamma(r_{2j}+h/\beta_j)\,\Gamma(r_{1j}-h/\beta_j)}{\Gamma(r_{1j})\,\Gamma(r_{2j})}\,,$$

so that if we take

$$W' = \prod_{j=1}^{m} Y_j$$

we have

$$W = \prod_{j=1}^{m} X_j = \underbrace{\left(\prod_{j=1}^{m} k_j^{-1/\beta_j}\right)}_{=K^*} \prod_{j=1}^{m} Y_j = K^*\, W'$$

and then if

$$Z = \log\, W' = \sum_{j=1}^{m} \log\, Y_j$$

we have, for $i = \sqrt{-1}$,

$$\begin{aligned} \Phi_Z(t) &= \prod_{j=1}^{m} \Phi_{\log Y_j}(t) = \prod_{j=1}^{m} E\left(Y_j^{it}\right) \\ &= \prod_{j=1}^{m} \frac{\Gamma(r_{2j}+it/\beta_j)\,\Gamma(r_{1j}-it/\beta_j)}{\Gamma(r_{1j})\,\Gamma(r_{2j})} \end{aligned} \tag{29}$$

what, using

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{i=1}^{\infty} \left(\frac{i}{i+z}\, e^{z/i}\right)$$

where γ is the Euler gamma constant, may, after some simplifications, be written as

$$\Phi_Z(t) = \prod_{j=1}^{m} \prod_{k=0}^{\infty} \beta_j(r_{2j}+k)\left(\beta_j(r_{2j}+k)+it\right)^{-1}\, \beta_j(r_{1j}+k)\left(\beta_j(r_{1j}+k)-it\right)^{-1}\,.$$

Since the β_j $(j=1,\dots,m)$ are not necessarily all positive we take

$$\beta_j^* = |\beta_j|\,, \qquad s_{1j} = \begin{cases} r_{1j} & \text{if} \quad \beta_j > 0 \\ r_{2j} & \text{if} \quad \beta_j < 0 \end{cases} \qquad \text{and} \qquad s_{2j} = \begin{cases} r_{2j} & \text{if} \quad \beta_j > 0 \\ r_{1j} & \text{if} \quad \beta_j < 0 \end{cases} \,,$$

so that we may write

$$\Phi_Z(t) \;=\; \prod_{j=1}^{m}\prod_{k=0}^{\infty} \beta_j^*(s_{2j}+k)\left(\beta_j^*(s_{2j}+k)+it\right)^{-1}\, \beta_j^*(s_{1j}+k)\left(\beta_j^*(s_{1j}+k)-it\right)^{-1}\,,$$

what shows that, if $\beta_j^* s_{1j} \neq \beta_k^* s_{1k}$ and $\beta_j^* s_{2j} \neq \beta_k^* s_{2k}$, $\forall j \neq k,\ j,k \in \{1,\dots,m\}$, the distribution of Z is the same as the distribution of a sum of infinitely many independent r.v.'s distributed as the difference of two independent Exponential distributions, with parameters $\beta_j^*(s_{2j}+k)$ and $\beta_j^*(s_{1j}+k)$ $(j=1,\dots,m; h=0,1,\dots)$. Alternatively, Z is distributed as the difference of two independent r.v.'s, each one with the distribution of the sum of infinitely many independent Exponential distributions.

Thus, since

$$W \;=\; K^*\, e^Z \;; \qquad K^* = \prod_{j=1}^{m} k_j^{-1/\beta_j}\,, \tag{30}$$

taking $s_{1jh}^* = \beta_j^*(s_{1j}+h)$ and $s_{2jh}^* = \beta_j^*(s_{2j}+h)$ and using (10) and (11) above as a basis, we get

$$F_W(w) \;=\; \begin{cases} \displaystyle\lim_{n\to\infty} K_1K_2 \sum_{j=1}^{m}\sum_{h=0}^{n} \left(H_{2jh}\,\frac{d_{hj}}{s_{2jh}^*} + H_{1jh}\,\frac{c_{hj}}{s_{1jh}^*}\left(1-(w/K^*)^{-s_{1jh}^*}\right)\right)\,, & w \geq K^* \\ \displaystyle\lim_{n\to\infty} K_1K_2 \sum_{j=1}^{m}\sum_{h=0}^{n} H_{2jh}\,\frac{d_{hj}}{s_{2jh}^*}\,(w/K^*)^{s_{2jh}^*}\,, & 0 < w \leq K^* \end{cases} \tag{31}$$

where K_1 and K_2 are defined in a similar manner as above, that is,

$$K_1 = \prod_{j=1}^{m}\prod_{h=0}^{n} \beta_j^*(s_{1j}+h)\,, \qquad K_2 = \prod_{j=1}^{m}\prod_{h=0}^{n} \beta_j^*(s_{2j}+h)\,,$$

and, for $j=1,\dots,m$ and $n=0,1,\dots$,

$$c_{hj} = \prod_{\substack{\eta=1\\ \eta\neq j}}^{m}\prod_{\substack{\nu=0\\ \nu\neq h}}^{n} \frac{1}{\beta_j^*(s_{1j}+h)-\beta_\eta^*(s_{1\eta}+\nu)}\,, \qquad d_{hj} = \prod_{\substack{\eta=1\\ \eta\neq j}}^{m}\prod_{\substack{\nu=0\\ \nu\neq h}}^{n} \frac{1}{\beta_j^*(s_{2j}+h)-\beta_\eta^*(s_{2\eta}+\nu)}\,,$$

and

$$H_{1jh} = \sum_{k=1}^{m}\sum_{l=0}^{n} \frac{d_{kl}}{\beta_k^*(s_{2k}+l)-\beta_j^*(s_{1j}+h)}\,, \qquad H_{2jh} = \sum_{k=1}^{m}\sum_{l=0}^{n} \frac{c_{kl}}{\beta_k^*(s_{1k}+l)-\beta_j^*(s_{2j}+h)}\,.$$

To lighten the writing we may replace the pair of indexes (k,j) by $h=km+j$, setting

$$s_{ih}^* = \beta_j^*(s_{ij}+k)\,, \quad \text{for} \;\; i=1,2; \;\; j=1,\dots,m; \;\; k=0,\dots,n\,. \tag{32}$$

We may now define K_1, K_2, c_j, d_j, H_{1j} and H_{2j} $(j=1,\dots,m(n+1))$, (with $n\to\infty$) in a way similar to the one used in subsection 1.2, that is,

$$K_1 = \prod_{j=1}^{m(n+1)} s_{1j}^*\,, \qquad K_2 = \prod_{j=1}^{m(n+1)} s_{2j}^*\,,$$

and, for $j = 1, \ldots,$

$$c_j = \prod_{\substack{k=1 \\ k\neq j}}^{m(n+1)} \frac{1}{s_{1j}^* - s_{1k}^*}\,, \qquad d_j = \prod_{\substack{k=1 \\ k\neq j}}^{m(n+1)} \frac{1}{s_{2j}^* - s_{2k}^*}$$

and

$$H_{1j} = \sum_{h=1}^{m(n+1)} \frac{d_h}{s_{1j}^* - s_{2h}^*}\,, \qquad H_{2j} = \sum_{h=1}^{m(n+1)} \frac{c_h}{s_{1h}^* - s_{2j}^*}$$

so that we may write the p.d.f. of the r.v. W as

$$F_W(w) = \begin{cases} \displaystyle\lim_{n\to\infty} K_1 K_2 \sum_{j=1}^{m(n+1)} \left(H_{2j}\, \frac{d_j}{s_{2j}^*} + H_{1j}\, \frac{c_j}{s_{1j}^*} \left(1 - (w/K^*)^{-s_{1j}^*}\right)\right)\,, & w \geq K^* \\ \displaystyle\lim_{n\to\infty} K_1 K_2 \sum_{j=1}^{m(n+1)} H_{2j}\, \frac{d_j}{s_{2j}^*}\, (w/K^*)^{s_{2j}^*}\,, & 0 < w \leq K^*\,. \end{cases} \tag{33}$$

3.2 The general case

In the general case we will admit that it is possible that some of the parameters $\tilde{s}_{1j} = \beta_j^* s_{1j}$, $j \in \{i, \ldots, m\}$, will be equal and that also some of the parameters $\tilde{s}_{2j} = \beta_j^* s_{2j}$, $j \in \{i, \ldots, m\}$, will be equal. More precisely, and without any loss of generality, let us suppose that τ_1 of the m parameters $\tilde{s}_{1j}$ are equal to $\tilde{s}_{11}$, τ_2 are equal to $\tilde{s}_{12}$, and so on, and that τ_{p_1} are equal to $\tilde{s}_{1p_1}$, with $p_1 < m$ and

$$\sum_{j=1}^{p_1} \tau_j = m\,,$$

and that, similarly, η_1 of the m parameters $\tilde{s}_{2j}$ are equal to $\tilde{s}_{21}$, η_2 are equal to $\tilde{s}_{22}$, and so on, and that η_{p_2} are equal to $\tilde{s}_{2p_2}$, with $p_2 < m$ and

$$\sum_{j=1}^{p_2} \eta_j = m\,.$$

Then the characteristic function in (29) may be written as

$$\begin{aligned} \Phi_Z(t) &= \prod_{j=1}^{p_1} \prod_{k=0}^{\infty} \left(\tilde{s}_{1j} + \beta_j^* k\right)^{\tau_j} \left(\tilde{s}_{1j} + \beta_j^* k - it\right)^{-\tau_j} \prod_{j=1}^{p_2} \prod_{k=0}^{\infty} \left(\tilde{s}_{2j} + \beta_j^* k\right)^{\eta_j} \left(\tilde{s}_{2j} + \beta_j^* k - it\right)^{-\eta_j} \\ &= \lim_{n\to\infty} \prod_{j=1}^{p_1} \prod_{k=0}^{n} \left(\tilde{s}_{1j} + \beta_j^* k\right)^{\tau_j} \left(\tilde{s}_{1j} + \beta_j^* k - it\right)^{-\tau_j} \prod_{j=1}^{p_2} \prod_{k=0}^{n} \left(\tilde{s}_{2j} + \beta_j^* k\right)^{\eta_j} \left(\tilde{s}_{2j} + \beta_j^* k - it\right)^{-\eta_j} \end{aligned}$$

that is the characteristic function of the difference of two independent r.v.'s with GIG distributions, the first one, that is, the one with positive sign, with depth $p_1{\times}(n+1)$ (with $n \to \infty$), with rate parameters and associated shape parameters

$$\begin{array}{ccccc} \underbrace{\tilde{s}_{11} + \beta_1^* k}_{k=0,\ldots,n}, & \ldots, & \underbrace{\tilde{s}_{1j} + \beta_j^* k}_{k=0,\ldots,n}, & \ldots, & \underbrace{\tilde{s}_{1p_1} + \beta_{p_1}^* k}_{k=0,\ldots,n} \\ \underbrace{\tau_1, \ldots, \tau_1}_{n+1}, & \ldots, & \underbrace{\tau_j, \ldots, \tau_j}_{n+1}, & \ldots, & \underbrace{\tau_{p_1}, \ldots, \tau_{p_1}}_{n+1} \end{array}$$

and the second one, that is, the one with negative sign, with depth $p_2{\times}(n+1)$ (with $n \to \infty$), with shape parameters and associated rate parameters

$$\begin{array}{ccccc} \underbrace{\tilde{s}_{21} + \beta_1^* k}_{k=0,\ldots,n}, & \ldots, & \underbrace{\tilde{s}_{2j} + \beta_j^* k}_{k=0,\ldots,n}, & \ldots, & \underbrace{\tilde{s}_{2p_1} + \beta_{p_2}^* k}_{k=0,\ldots,n} \\ \underbrace{\eta_1, \ldots, \eta_1}_{n+1}, & \ldots, & \underbrace{\eta_j, \ldots, \eta_j}_{n+1}, & \ldots, & \underbrace{\eta_{p_2}, \ldots, \eta_{p_2}}_{n+1} \end{array}.$$

Let us consider the vectors

$$\underline{\tau}^* = \Big[\underbrace{\tau_1,\ldots,\tau_{p_1},\tau_1,\ldots,\tau_{p_1},\ \ldots\ ,\tau_1,\ldots,\tau_{p_1}}_{n+1\ \text{times}}\Big]', \qquad \underline{\eta}^* = \Big[\underbrace{\eta_1,\ldots,\eta_{p_2},\eta_1,\ldots,\eta_{p_2},\ \ldots\ ,\eta_1,\ldots,\eta_{p_2}}_{n+1\ \text{times}}\Big]',$$

where, for $j = 1,\ldots,p_1(n+1)$ and $l = 1,\ldots,p_2(n+1)$, with $j = kp_1 + h$ and $l = kp_2 + i$, for $k = 0,\ldots,n$, $h = 1,\ldots,p_1$ and $i = 1,\ldots,p_2$,

$$\tau_j^* = \tau_h\,, \qquad \text{and} \qquad \eta_l^* = \eta_i\,,$$

and, similarly to the vectors $\underline{s}_1^*$ and $\underline{s}_2^*$ considered in the previous subsection, the vectors

$$\underline{\tilde{s}}_1^* = \Big[\tilde{s}_{11},\ldots,\tilde{s}_{1p_1},\tilde{s}_{11}+\beta_1^*,\ldots,\tilde{s}_{1p_1}+\beta_{p_1}^*,\ \ldots\ ,\tilde{s}_{11}+\beta_1^* n,\ldots,\tilde{s}_{1p_1}+\beta_{p_1}^* n\Big]'$$

and

$$\underline{\tilde{s}}_2^* = \Big[\tilde{s}_{21},\ldots,\tilde{s}_{2p_1},\tilde{s}_{21}+\beta_1^*,\ldots,\tilde{s}_{2p_1}+\beta_{p_1}^*,\ \ldots\ ,\tilde{s}_{21}+\beta_1^* n,\ldots,\tilde{s}_{2p_1}+\beta_{p_1}^* n\Big]'$$

where, once again, for $j = 1,\ldots,p_1(n+1)$ and $l = 1,\ldots,p_2(n+1)$, with j, l, k, h and i defined as above,

$$\tilde{s}_{1j}^* = \tilde{s}_{1h} + \beta_h^* k \qquad \text{and} \qquad \tilde{s}_{2l}^* = \tilde{s}_{2i} + \beta_i^* k\,.$$

The c.d.f. of $W = K^* e^Z$, for K^* defined as in (30), may then be derived from (25), taking into account the shape and rate parameters mentioned above, as

$$F_W(w) = \begin{cases} \displaystyle\lim_{n\to\infty} K_1 K_2 \sum_{j=1}^{p_1(n+1)} Q_{1j}^{***}(\log(w/K^*))\,(w/K^*)^{-\tilde{s}_{1j}^*} & w \geq K^* \\ \displaystyle\lim_{n\to\infty} K_1 K_2 \sum_{j=1}^{p_2(n+1)} Q_{2j}^{***}(\log(w/K^*))\,(w/K^*)^{\tilde{s}_{2j}^*} & 0 < w \leq K^* \end{cases} \tag{34}$$

where now K_1, K_2, $Q_{1j}^{***}(\cdot)$ and $Q_{2j}^{***}(\cdot)$ are defined as in subsection 2.4, with p_1 replaced by $p_1(n+1)$, p_2 replaced by $p_2(n+1)$, λ_j replaced by $\tilde{s}_{1j}^*$, ν_l replaced by $\tilde{s}_{2j}^*$, r_{1j} replaced by τ_j^* and r_{2l} replaced by η_l^*.

3.3 The double non-central case

The following Lemma is a useful result for some of the work ahead ad its proof is straightforwardly obtained from the Theorem of global probability (Robbins, 1948; Robbins and Pitman, 1949).

LEMMA 1: Let Z_1 and Z_2 be two independent r.v.'s taking values on the non-negative integers, such that

$$P(Z_1 = i) = u_i \qquad \text{and} \qquad P(Z_2 = j) = v_j\,, \qquad i,j = 0,1,\ldots\,.$$

Let further X and Y be two r.v.'s and let $X_i = X|Z_1 = i$ and $Y_j = Y|Z_2 = j$, for $i,j = 0,1,\ldots$, and let yet $g(\cdot,\cdot)$ be a measurable function.

Then

$$\Phi_X(t) = \sum_{i=0}^{\infty} u_i\,\Phi_{X_i}(t)\,, \qquad \Phi_Y(t) = \sum_{j=0}^{\infty} v_j\,\Phi_{Y_j}(t)$$

and if

$$Z = g(X,Y)$$

then

$$\Phi_Z(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_i\,v_j\,\Phi_{g(X_i,Y_j)}(t)\,.$$

A particular case of the above Lemma is clearly the case where the r.v.'s X and Y are mixtures. We will use the above Lemma exactly in this case.

We will say that the r.v. Y has a non-central Generalized Gamma distribution with shape parameter r, rate parameter λ, power parameter β and non-centrality parameter δ if its p.d.f. may be written as a mixture with Poisson weights with rate $\delta/2$ of Generalized Gamma p.d.f.s with shape parameters $r+i$ $(i=0,1,\ldots)$, rate parameter λ and power parameter β, that is, if

$$f_Y(y) \,=\, \sum_{i=0}^{\infty} p_i \, \frac{|\beta|\,\lambda^{r+i}}{\Gamma(r+i)}\, e^{-\lambda y^\beta}\, y^{\beta(r+i)+1}\,,$$

where

$$p_i \,=\, \frac{(\delta/2)^i}{i!}\, e^{-\delta/2} \qquad i=0,1,\ldots \tag{35}$$

clearly with $\sum_{i=0}^{\infty} p_i \,=\, 1$. We will denote the fact that the r.v. Y has a non-central Generalized Gamma distribution with the above parameters by

$$Y \,\sim\, \Gamma(r,\lambda,\beta;\delta)\,, \tag{36}$$

If the r.v. Y has the non-central Generalized Gamma distribution in (36), then it is straightforward to show that the r.v.

$$X \,=\, Y^\beta\,, \qquad \beta \in I\!R\backslash\{0\}\,,$$

has a non-central Gamma distribution with shape parameter r, rate parameter λ and non-centrality parameter δ.

Let us suppose that

$$Y_1 \,\sim\, \Gamma(r_1,\lambda_1,\beta;\delta_1) \qquad \text{and} \qquad Y_2 \,\sim\, \Gamma(r_2,\lambda_2,\beta;\delta_2)$$

are two independent r.v.'s and let

$$Z \,=\, Y_1/Y_2\,.$$

Then, the r.v. Z will have what we call a double non-central Generalized Gamma Ratio or double non-central Generalized F distribution. Using Lemma 1 above, the p.d.f. of Z is, for $k=\lambda_1/\lambda_2$,

$$f_Z(z) \,=\, \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_i\,\nu_j\, \frac{|\beta|\,k^{r_1+i}}{B(r_1+i,r_2+j)}\, \left(1+k\,z^\beta\right)^{-r_1-r_2-i-j}\, z^{\beta(r_1+i)-1}$$

where $k=\lambda_1/\lambda_2$ and

$$p_i \,=\, \frac{(\delta_1/2)^i}{i!}\, e^{-\delta_1/2} \qquad \text{and} \qquad \nu_j \,=\, \frac{(\delta_2/2)^j}{j!}\, e^{-\delta_2/2}\,.$$

We will denote the fact that the r.v. Z has this distribution by

$$Z \,\sim\, GGR(r_1,r_2,k,\beta;\delta_1,\delta_2)\,.$$

In this section we will be interested in obtaining the distribution of

$$W \,=\, \prod_{j=1}^{n} Z_j \tag{37}$$

where

$$Z_j \,\sim\, GGR(r_{1j},r_{2j},k_j,\beta_j;\delta_{1j},\delta_{2j})\,.$$

Using Lemma 1 and expression (31), in subsection 3.1, for the c.d.f. of W in the central case, we obtain, for the case where all shape parameters in the numerator are different and all shape parameters in the denominator are also different, that is the case where,

$$\beta_j^* s_{1j} \neq \beta_k^* s_{1k} \quad \text{and} \quad \beta_j^* s_{2j} \neq \beta_k^* s_{2k} \quad \text{for all } j\neq k \text{ with } j,k\in\{1,\ldots,m\}\,,$$

defining s^*_{1k} and s^*_{2k} as in (31), we have for $s^*_{1khi}=\beta^*_k(s_{1k}+h+i)$ and $s^*_{2khj}=\beta^*_k(s_{2k}+h+j)$,

$$F_W(w) = \begin{cases} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lim_{n\to\infty} K_{1i}\,K_{2j}\sum_{h=0}^{n}\sum_{k=1}^{m} p_{ik}\,\nu_{jk}\left(H_{2khij}\,\frac{d_{hkj}}{s^*_{2khj}}\right. \\ \qquad\qquad \left. + H_{1khij}\,\frac{c_{hki}}{s^*_{1khi}}\left(1-(w/K^*)^{-s^*_{1khi}}\right)\right) & w\ge K^* \\ \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lim_{n\to\infty} K_{1i}\,K_{2j}\sum_{h=0}^{n}\sum_{k=1}^{m} p_{ik}\,\nu_{jk}\,H_{2khij}\,\frac{d_{hkj}}{s^*_{2khj}}\,(w/K^*)^{s^*_{2khj}} & 0<w\le K^* \end{cases}$$

where

$$K_{1i} = \prod_{\eta=1}^{m}\prod_{h=0}^{n}\beta^*_\eta(s_{1\eta}+i+h) \qquad \text{and} \qquad K_{2j} = \prod_{\eta=1}^{m}\prod_{h=0}^{n}\beta^*_\eta(s_{2\eta}+j+h)\,,$$

$$H_{1khij}=\sum_{\eta=1}^{m}\sum_{l=0}^{n}\frac{d_{\eta lj}}{\beta^*_\eta(s_{2\eta}+j+l)-\beta^*_k(s_{1k}+i+h)}\,,\quad H_{2khij}=\sum_{\eta=1}^{m}\sum_{l=0}^{n}\frac{c_{\eta li}}{\beta^*_\eta(s_{1\eta}+i+l)-\beta^*_k(s_{2k}+j+h)}\,,$$

with

$$c_{hki} = \prod_{\substack{\eta=1\\ \eta\neq k}}^{m}\prod_{\substack{\nu=0\\ \nu\neq h}}^{n}\frac{1}{\beta^*_k(s_{1k}+i+h)-\beta^*_\eta(s_{1\eta}+i+\nu)}\,,\quad d_{hkj} = \prod_{\substack{\eta=1\\ \eta\neq k}}^{m}\prod_{\substack{\nu=0\\ \nu\neq h}}^{n}\frac{1}{\beta^*_k(s_{2k}+j+h)-\beta^*_\eta(s_{2\eta}+j+\nu)}$$

and yet, for $i,j=0,1,\dots$ and $k=1,\dots,m$,

$$p_{ik} = \frac{(\delta_{1k}/2)^i}{i!}\,e^{-\delta_{1k}/2} \qquad \text{and} \qquad \nu_{jk} = \frac{(\delta_{2k}/2)^j}{j!}\,e^{-\delta_{2k}/2}\,. \tag{38}$$

In case that all the non-centrality parameters in the numerator of W are the same, say $\delta_{1k}=\delta_1$, $\forall k\in\{1,\dots,m\}$, and all non-centrality parameters in the denominator are also the same, with say $\delta_{2k}=\delta_2$, $\forall k\in\{1,\dots,m\}$, the only difference in the distribution of W would be that the weights p_{ik} and ν_{jk} would be no more a function of k and the c.d.f. of W could be written as

$$F_W(w) = \begin{cases} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_i\,\nu_j\lim_{n\to\infty} K_{1i}\,K_{2j}\sum_{h=0}^{n}\sum_{k=1}^{m}\left(H_{2khij}\,\frac{d_{hkj}}{s^*_{2khj}}\right. \\ \qquad\qquad \left. + H_{1khij}\,\frac{c_{hki}}{s^*_{1khi}}\left(1-(w/K^*)^{-s^*_{1khi}}\right)\right) & w\ge K^* \\ \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_i\,\nu_j\lim_{n\to\infty} K_{1i}\,K_{2j}\sum_{h=0}^{n}\sum_{k=1}^{m} H_{2khij}\,\frac{d_{hkj}}{s^*_{2khj}}\,(w/K^*)^{s^*_{2khj}} & 0<w\le K^* \end{cases}$$

Let $k|m$ represent the remainder of the integer ratio of k by m and let $k_{|m}=1+k|m$. Then, an alternative representation for the c.d.f. of W in this case may be derived from the c.d.f. in (33) in subsection 3.1, obtained for the central case of W, we have, the c.d.f. of W in (37) is, for

$$s^{**}_{1ki} = s^*_{1k}+i\,\beta^*_{k_{|p_1}} \qquad \text{and} \qquad s^{**}_{2kj} = s^*_{2k}+j\,\beta^*_{k_{|p_2}}\,,$$

with s^*_{1k} and s^*_{2k} defined by (32),

$$F_W(w) = \begin{cases} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \lim_{n\to\infty} K_{1i}\, K_{2j} \sum_{k=1}^{m(n+1)} p_{i,k_{|m}}\, \nu_{j,k_{|m}} \left(H_{2kij}\, \frac{d_{kj}}{s^{**}_{2kj}} \right. \\ \displaystyle\qquad\qquad \left. + H_{1kij}\, \frac{c_{ki}}{s^{**}_{1ki}} \left(1-(w/K^*)^{-s^{**}_{1ki}}\right)\right) & w \geq K^* \\ \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \lim_{n\to\infty} K_{1i}\, K_{2j} \sum_{k=1}^{m(n+1)} p_{i,k_{|m}}\, \nu_{j,k_{|m}}\, H_{2kij}\, \frac{d_{kj}}{s^{**}_{2kj}}\, (w/K^*)^{s^{**}_{2kj}} & 0 < w \leq K^* \end{cases}$$

where

$$K_{1i} = \prod_{k=1}^{m(n+1)} s^{**}_{1ki}\,, \quad K_{2j} = \prod_{k=1}^{m(n+1)} s^{**}_{2kj}\,, \quad H_{1kij} = \sum_{h=1}^{m(n+1)} \frac{d_{hj}}{s^{**}_{1hi}-s^{**}_{2kj}}\,, \quad H_{2kij} = \sum_{h=1}^{m(n+1)} \frac{c_{hi}}{s^{**}_{2hi}-s^{**}_{1kj}}$$

with

$$c_{ki} = \prod_{\substack{h=1\\h\neq k}}^{m(n+1)} \frac{1}{s^{**}_{1hi}-s^{**}_{1ki}} \qquad \text{and} \qquad d_{kj} = \prod_{\substack{h=1\\h\neq k}}^{m(n+1)} \frac{1}{s^{**}_{2hj}-s^{**}_{2kj}}$$

and yet

$$p_{ik} = \frac{(\delta_{1k}/2)^i}{i!}\, e^{-\delta_{1k}/2} \qquad \text{and} \qquad \nu_{jk} = \frac{(\delta_{2k}/2)^j}{j!}\, e^{-\delta_{2k}/2}\,.$$

In case where all the non-centrality parameters in the numerator of W are the same, say $\delta_{1k} = \delta_1$, $\forall k \in \{1,\dots,m\}$, and all non-centrality parameters in the denominator are also the same, with say $\delta_{2k} = \delta_2$, $\forall k \in \{1,\dots,m\}$, the only difference in the distribution of W would be that the weights p_{ik} and ν_{jk} would be no more a function of k and the c.d.f. of W could be written as

$$F_W(w) = \begin{cases} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_i\, \nu_j \lim_{n\to\infty} K_{1i}\, K_{2j} \sum_{k=1}^{m(n+1)} \left(H_{2kij}\, \frac{d_{kj}}{s^{**}_{2kj}} \right. \\ \displaystyle\qquad\qquad \left. + H_{1kij}\, \frac{c_{ki}}{s^{**}_{1ki}} \left(1-(w/K^*)^{-s^{**}_{1ki}}\right)\right) & w \geq K^* \\ \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_i\, \nu_j \lim_{n\to\infty} K_{1i}\, K_{2j} \sum_{k=1}^{m(n+1)} H_{2kij}\, \frac{d_{kj}}{s^{**}_{2kj}}\, (w/K^*)^{s^{**}_{2kj}} & 0 < w \leq K^*\,. \end{cases}$$

For the double non-central case corresponding to the general case studied in subsection 3.2 above, where some of the parameters $\tilde{s}_{1j} = \beta^*_j s_{1j}$, $j \in \{i,\dots,m\}$, will be equal and also some of the parameters $\tilde{s}_{2j} = \beta^*_j s_{2j}$, $j \in \{i,\dots,m\}$, will be equal, we have, from (34),

$$F_W(w) = \begin{cases} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \lim_{n\to\infty} K_{1i} K_{2j} \sum_{k=1}^{p_1(n+1)} Q^{***}_{1kij}\left(\log(w/K^*)\right)\, (w/K^*)^{-\tilde{s}^{**}_{1ki}} & w \leq K^* \\ \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \lim_{n\to\infty} K_{1i} K_{2j} \sum_{k=1}^{p_1(n+1)} Q^{***}_{2kij}\left(\log(w/K^*)\right)\, (w/K^*)^{\tilde{s}^{**}_{2kj}} & w \leq K^* \end{cases}$$

with

$$\tilde{s}^{**}_{1ki} = \tilde{s}^*_{1k} + i\, \beta^*_{k_{|p_1}}\,, \quad \tilde{s}^{**}_{2kj} = \tilde{s}^*_{2k} + j\, \beta^*_{k_{|p_2}}\,, \quad K_{1i} = \prod_{k=1}^{p_1(n+1)} \left(\tilde{s}^{**}_{1ki}\right)^{\tau^*_k}\,, \quad K_{2j} = \prod_{k=1}^{p_2(n+1)} \left(\tilde{s}^{**}_{2kj}\right)^{\eta^*_k}\,,$$

$$Q^{***}_{1kij}\left(\log(w/K^*)\right)=\sum_{g=1}^{\tau_k^*}c_{kgi}\sum_{l=1}^{p_2(n+1)}\sum_{h=1}^{\eta_l^*}d_{lhj}\sum_{u=0}^{g-1}\binom{g-1}{u}\frac{(h-u-1)!}{\left(\tilde{s}^{**}_{1ki}+\tilde{s}^{**}_{2lj}\right)^{h+u}}\sum_{t=0}^{g-1-u}\frac{(g-1-u)!}{t!}\frac{\left(\log(w/K^*)\right)^t}{(\tilde{s}^{**}_{1ki})^{g-u-t}}$$

$$Q^{***}_{2kij}\left(\log(w/K^*)\right)=\sum_{g=1}^{\tau_k^*}c_{kgi}\sum_{l=1}^{p_2(n+1)}\sum_{h=1}^{\eta_l^*}d_{lhj}\sum_{u=0}^{h-1}\binom{h-1}{u}\frac{(g-u-1)!}{\left(\tilde{s}^{**}_{1ki}+\tilde{s}^{**}_{2lj}\right)^{g+u}}\sum_{t=0}^{g-1-u}\frac{(h-1-u)!}{t!}\frac{\left(\log(w/K^*)\right)^t}{\left(\tilde{s}^{**}_{2lj}\right)^{h-u-t}}$$

where, for $k=1,\ldots,p_1$ and for $l=1,\ldots,p_2$,

$$c_{k,\tau_k^*,i}\;=\;\frac{1}{(\tau_k^*-1)!}\prod_{\substack{l=1\\l\neq k}}^{p_1}\left(\tilde{s}^{**}_{1li}-\tilde{s}^{**}_{1ki}\right)^{-\tau_l^*}\;,\qquad d_{l,\eta_l^*,j}\;=\;\frac{1}{(\eta_l^*-1)!}\prod_{\substack{k=1\\k\neq l}}^{p_2}\left(\tilde{s}^{**}_{2kj}-\tilde{s}^{**}_{2lj}\right)^{-\eta_k^*}$$

with, for $l=1,\ldots,\tau_k^*-1$ and $k=1,\ldots,p_1$,

$$c_{k,\tau_k^*-l,i}\;=\;\frac{1}{l}\sum_{h=1}^{l}\frac{(\tau_k^*-l+h-1)!}{(\tau_k^*-l-1)!}\,R_1(h,k,p_1,\underline{\tau}^*,\underline{\tilde{s}}_1^{**})\,c_{k,\tau_k^*-(l-h),i}$$

and, for $m=1,\ldots,\eta_l^*-1$ and $l=1,\ldots,p_2$,

$$d_{l,\eta_l^*-m,j}\;=\;\frac{1}{m}\sum_{h=1}^{m}\frac{(\eta_l^*-m+h-1)!}{(\eta_l^*-m-1)!}\,R_2(h,l,p_2,\underline{\eta}^*,\underline{\tilde{s}}_2^{**})\,d_{l,\eta_l^*-(m-h),j}$$

where, for $h=0,\ldots,\tau_k^*-1$ and $m=0,\ldots,\eta_l^*-1$,

$$R_1(h,k,p_1,\underline{\tau}^*,\underline{\tilde{s}}_1^{**})\;=\;\sum_{\substack{l=1\\l\neq k}}^{p_1}\tau_k^*\left(\tilde{s}^{**}_{1ki}-\tilde{s}^{**}_{1li}\right)^{-h}\;,\qquad R_2(m,l,p_2,\underline{\eta}^*,\underline{\tilde{s}}_2^{**})\;=\;\sum_{\substack{n=1\\n\neq l}}^{p_2}\eta_l^*\left(\tilde{s}^{**}_{2lj}-\tilde{s}^{**}_{2nj}\right)^{-m}\;.$$

4 Conclusions and Final Remarks

We should strengthen that the results obtained may be easily and directly generalized to the case where we are interested in the distribution of the r.v.

$$Z\;=\;\prod_{j=1}^{n}\gamma_j\,Y_j^{\alpha_j}$$

where $\gamma_j\in I\!R^+$, $\alpha_j\in I\!R\backslash\{0\}$ and Y_j are independent r.v.'s with GGR distributions, since it is straightforward to show that if

$$Y_j\;\sim\;GGR(r_{1j},r_{2j},k_j,\beta_j;\delta_{1j},\delta_{2j}) \tag{39}$$

then

$$\gamma_j\,Y_j^{\alpha_j}\;\sim\;GGR\left(r_{1j},r_{2j},\frac{k_j}{\gamma_j^{\beta_j/\alpha_j}},\frac{\beta_j}{\alpha_j};\delta_{1j},\delta_{2j}\right)\,.$$

If the r.v. X has a standard Beta distribution, the distribution of either $(1-X)/X$ or $X/(1-X)$ is then usually called a standard Beta prime or Beta second kind distribution. However we should note that the distribution of either $(1-X)/X$ or $X/(1-X)$ is actually only a particular GGR distribution. Actually it is easy to show that if

$$Y_j\;\sim\;GGR(r_{1j},r_{2j},k_j,\beta_j;0,0)$$

with $k_j = 1$ and $\beta_j = 1$ then the r.v.'s $1/(1+Y_j)$ and $Y_j/(1+Y_j)$ have standard Beta distributions with parameters r_{2j} and r_{1j} or r_{1j} and r_{2j}, respectively, while for general $k_j > 0$ and general $\beta_j \in I\!R\backslash\{0\}$ the r.v. $X_j = Y_j/(1+Y_j)$ has what we call a generalized Beta distribution with p.d.f.

$$f_{X_j}(x) \;=\; \frac{|\beta_j|\, k_j^{r_{1j}}}{B(r_{1j}, r_{2j})} \left(1 + k_j\, \frac{x}{1-x}\right)^{-r_{1j}-r_{2j}} \; (1-z)^{-\beta_j r_{1j}-1}\, z^{\beta_j r_{1j}-1}$$

which clearly reduces to the standard Beta p.d.f. for $k_j = 1$ and $\beta_j = 1$, while the r.v. $1 - X_j = 1/(1+Y_j)$ has of course a similar p.d.f. with r_{1j} and r_{2j} swapped. For this reason the distribution of Y_j is also called, for $k_j = 1$ and $\beta_j = 1$ a Beta prime distribution and for general k_j and β_j a generalized Beta prime distribution. Thus, for the non-central distribution in (39) we may say that Y_j has also a non-central generalized Beta prime distribution and thus, the distributions obtained in section 3 are also the distributions of the product of independent central and non-central generalized Beta prime r.v.'s.

Clearly if in (39) we have $r_{1j} = m/2$, $r_{2j} = n/2$, with $m, n \in I\!N$ and $k_j = \beta_j = 1$ we have the r.v.'s Y_j with either central or non-central F distributions, according to the case that $\delta_j = 0$ or $\delta_j \neq 0$, with m and n degrees of freedom. The results in section 3 may then be readily applied to both the central and non-central cases.

Also, if in (39) we have $r_{1j} = 1/2$, $r_{2j} = n_j/2$, $k_j = 1/n_j$ and $\beta_j = 2$, with $n_j \in I\!N$, we have Y_j with the so-called folded T distribution, that is the distribution of a r.v. $Y_j = |T_j|$, where T_j is a r.v. with a Student T distribution with n_j degrees of freedom. If instead we take $r_{2j} = 1/2$ and $k_j = 1$ we will have Y_j with the so-called folded Cauchy distribution, that is the distribution of the absolute value of a r.v. with a standard Cauchy distribution, or a Student T distribution with only 1 degree of freedom. In both cases, once again, the results in section 3 may be readily applied to both the central and non-central cases.

Besides, given the way our approach was conducted, even the doubly non-central case, where each of the GGR r.v.'s is the ratio of two non-central Gamma r.v.'s, was then readily at hand. Also, since, from the beginning, in subsection 2.1, and opposite to what is commonly done, we considered the power parameters as real (only non-null), allowing them to be negative, the results obtained may be directly extended to the distribution of the ratio of two independent GGR random variables or the distribution of the ratio of two products of independent GGR random variables. In order to obtain the distribution of the ratio of two independent GGR random variables one simply has to consider $m = 2$ in (28), taking then for the random variable in the denominator the symmetric of its power parameter. The distribution for the ratio of two products of independent GGR random variables may then be obtained by taking the distribution of the product of the whole set of random variables, taking the symmetric of the power parameters for the random variables in the denominator.

As a by-product, in subsections 2.3 and 2.4 we also obtain closed form representations, not involving any infinite series or unsolved integrals, for the distribution of the difference of two independent sums of a finite number of Exponential random variables with all different rate parameters or Gamma random variables with all different rate parameters and integer shape parameters, under the form of particular mixtures of either Exponential or Gamma distributions, according to the case. Also, if we consider the exponential of a Gamma random variable with integer shape parameter as a generalized Pareto distribution, then the distribution of the random variable W in subsection 2.3 is the distribution of the ratio of two independent products of Pareto distributions, while the distribution of the same random variable in subsection 2.4 is the distribution of the ratio of two independent products of generalized Pareto distributions, expressed as a particular mixture of Pareto and inverted Pareto distributions.

Given the form of the exact distributions obtained for the product of either central and non-central GGR r.v.'s asymptotic and near-exact distributions are readily at hand, being not treated in this paper due to length limitations. They are intended to be published in a separate paper. We may think of asymptotic distributions by simple truncation of the series obtained what would indeed give mainly unsatisfactory results, mainly in terms of c.d.f. and quantiles, owing to the fact that then the weights would not add up to the right values, preventing this way the c.d.f.^{s} from reaching the value 1. Much better results may be obtained if we consider near-exact distributions, based on the concept of keeping a good part of the exact characteristic function unchanged and approaching the remaining, and desirably, much smaller part, by an asymptotic result (Coelho, 2003, 2004) what, given the form obtained for the exact distributions, would lead us to consider for example the truncation of the infinite series obtained, coupled with one or two more terms that would both make the weights add up to the right value and the first two, three or four moments to match the first exact ones.

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