The attractor of an equation of Tricomi's type.

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Abstract: Consider the attractor $\mathcal{A}$ of an equation of pendulum type with friction driven by a constant torque. The results of M. Levi and also obtained independently by Q. Min, S. Xian, and Z. Jinyan show that if the friction coefficient is larger than a certain bound then $\mathcal{A}$ is homeomorphic to the circle. We shall study the bifurcation diagram of a particular class of equations of pendulum type and show that the bounds on the friction coefficient obtained before are optimal.

1 Introduction

Motivated by the applications to the synchronous electrical motors, Tricomi [7], [8], studied in detail the equation

$$x'' + cx' + \sin x = \beta, \quad c, \beta > 0, \tag{1}$$

that is also the well known model for a pendulum with friction driven by a constant torque. When $\beta > 1$ there exist no equilibria and Tricomi proved the existence of a running periodic solution, i.e. a solution x so that $x(t+\tau) = x(t) + 2\pi$ for some $\tau > 0$, that attracts all the other solutions. On the other hand, when $\beta < 1$ there exists equilibria and he proves the existence of a constant $c_0(\beta)$ so that if $0 < c < c_0(\beta)$ there exist one running periodic solution but when the parameter c is moved to values above $c_0(\beta)$ the running periodic solution is destroyed in a homoclinic bifurcation and the equilibria become globally attracting. Both situations are sketched in Figure 1. We drew the situation where the stable equilibrium is a stable spiral, actually it is a stable node for some values of c. We have thus the bifurcation diagram given in Figure 2. For an exhaustive discussion and rigorous proofs of the

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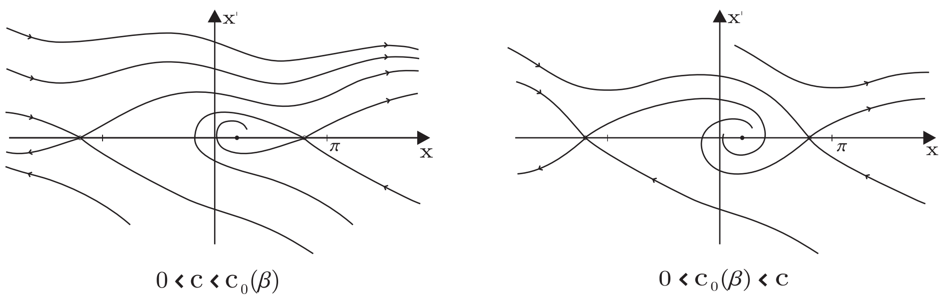

Figure 1:

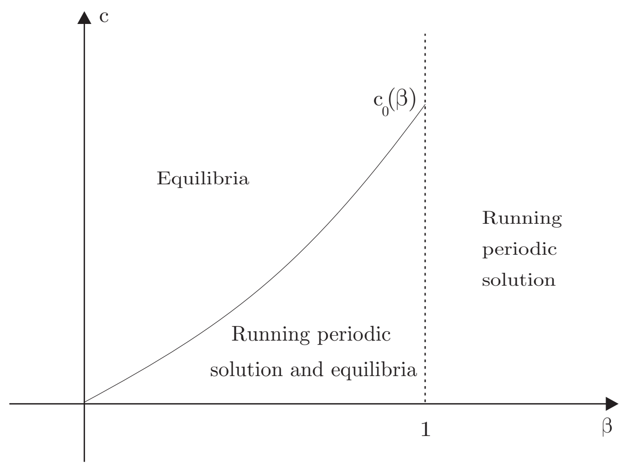

Figure 2:

remaining cases see [1]. More generally we can study the flux associated to the pendulum like equation

$$x'' + cx' + g(t,x) = 0, \quad c > 0,$$

where $g \in C(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z})$ for some $T > 0$. Like in the pendulum case, the phase space of the associated equation

$$\begin{cases} x' = v \\ v' = -cv - g(t,x) \end{cases} \tag{2}$$

is invariant for translations along the vector $\begin{pmatrix} 2\pi \\ 0 \end{pmatrix}$. This means that we can regard the phase space of (2) as a cylinder $\mathcal{C}$ formed by the identification of the points of the form $y + k\begin{pmatrix} 2\pi \\ 0 \end{pmatrix}$, $k \in \mathbb{Z}$. We shall denote by the same letter y the element of $\mathbb{R}^2$ or the class of $y \in \mathbb{R}^2$ in $\mathcal{C}$. The Poincaré map P is the function defined in $\mathcal{C}$ that to each initial condition y_0 corresponds $y(T; y_0)$, where $y(t; y_0)$ is the solution of (2) satisfying $y(0) = y_0$. Given the dissipative nature of (2), it is well known that we can define a maximal invariant set $\mathcal{A}$ for P(see [4]), that is also a global attractor for the iterates of the Poincaré map. If the motion is overdamped then $\mathcal{A}$ is homeomorphic to $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, in such case the restriction of P to $\mathcal{A}$ becomes an homeomorphism from the circle to itself. It is thus interesting to give criteria to decide if $\mathcal{A}$ is homeomorphic to $\mathbb{T}^1$ or not. The following theorem was proven in [4], motivated by previous results in [2] and [5].

Theorem 1.1 *If there exists a constant c_1 such that*

$$c_1 < \frac{g(t, x_1) - g(t, x_2)}{x_1 - x_2} < \frac{c^2}{4}$$

for each $(t, x_1, x_2) \in \mathbb{R}^3$, with $x_1 \neq x_2$, then $\mathcal{A}$ is homeomorphic to $\mathbb{T}^1$.

An equivalent characterization of $\mathcal{A}$ is (see [4])

$$\mathcal{A} = \{y(0) : \ y \text{ is a solution of (2) bounded in } \mathcal{C}\}. \tag{3}$$

So in the particular case of Tricomi and when $0 < c < c_0(\beta)$ the attractor $\mathcal{A}$ is formed by the equilibria, the running periodic solution and the heteroclinics joining them; clearly $\mathcal{A}$ is not homeomorphic to $\mathbb{T}^1$. Therefore, from the last theorem we conclude that $c_0(\beta) < 2$ for every $\beta \in]0, 1[$. Actually Tricomi also gave estimations to the curve $c_0(\beta)$ (see [6] for a survey of this kind of estimations) that show that $c_0(\beta) < \sqrt{2}$. The purpose of this paper is to show that the constant $c^2/4$ is optimal in the above theorem. To this end we shall consider a variation of Tricomi's equation

$$\begin{cases} x' = v \\ v' = -cv - g(x) + \beta \end{cases} \tag{4}$$

where the sine function has been replaced by the 2π-periodic function defined by

$$g(x) = \begin{cases} x \text{ if } x \in [-\pi/2, \pi/2] \\ -x + \pi \text{ if } x \in [\pi/2, 3\pi/2] \end{cases} .$$

For this nonlinearity equilibria exist if $\beta \in]0, \pi/2[$ and we prove that:

Theorem 1.2 *If $c < 2$, there exists $\beta_1 \in]0, \pi/2[$ so that if $\beta_1 < \beta < \pi/2$ then equation (4) has a running periodic solution.*

Once this result is proved it is easy to deduce:

Corollary 1.3 *If $c < 2$, there exists $\beta_1 \in]0, \pi/2[$ so that if $\beta_1 < \beta < \pi/2$ then the attractor $\mathcal{A}$ associated to equation* (4) *is not homeomorphic to $\mathbb{T}^1$.*

Therefore, in this modified case, the curve $c_0(\beta)$ attains values arbitrarily close to 2. Since the Lipschitz constant of f is 1 we conclude that the estimation obtained in the last theorem is optimal. We observe that in [3] there were already discussions about the optimality of $c^2/4$, using a non-autonomous equation and a different approach. However in the example in [3] the period T was an additional parameter. A problem that remains open is the following: given $c < 2$ there exists a periodic forcing p such that the attractor associated to the pendulum equation

$$x'' + cx' + \sin x = p(t)$$

is not homeomorphic to the circle?

The reader can find related results in the author's web page http://ptmat.lmc.fc.ul.pt/~rmartins

2 Proofs

Consider the equation (4) where $\beta \in]0, \pi/2[$. We shall show that for each $c < 2$ there exists $\beta_1 \in]0, \pi/2[$ so that if $\beta_1 \leq \beta < \pi/2$ then (4) has a running periodic solution and simultaneously equilibria, so $\mathcal{A}$ is not homeomorphic to $\mathbb{T}^1$.

The points of the form $\begin{pmatrix} \beta + 2k\pi \\ 0 \end{pmatrix}$, $k \in \mathbb{Z}$, are stable spirals and $\begin{pmatrix} \pi - \beta + 2k\pi \\ 0 \end{pmatrix}$, $k \in \mathbb{Z}$, are saddles in the phase space of (4). Notice that

$$\overline{x}_\beta(t) = (\pi/2 + \beta)e^{\frac{-c+\sqrt{c^2+4}}{2}t} - \pi - \beta$$

is the solution of $x'' + cx' - x - \pi = \beta$ so that

$$\lim_{t \to -\infty} \begin{pmatrix} \overline{x}_\beta(t) \\ \overline{x}'_\beta(t) \end{pmatrix} = \begin{pmatrix} -\pi - \beta \\ 0 \end{pmatrix},$$

$\overline{x}_\beta$ is increasing, and $\overline{x}_\beta(0) = -\pi/2$. We conclude that $(\overline{x}_\beta, \overline{x}'_\beta)$ is a solution of (4) in $] - \infty, 0[$ and is inside of the unstable manifold of $\begin{pmatrix} -\pi - \beta \\ 0 \end{pmatrix}$. Let us denote by $y_\beta = (x_\beta, v_\beta)$ the solution of (4) that coincides with $(\overline{x}_\beta, \overline{x}'_\beta)$ in

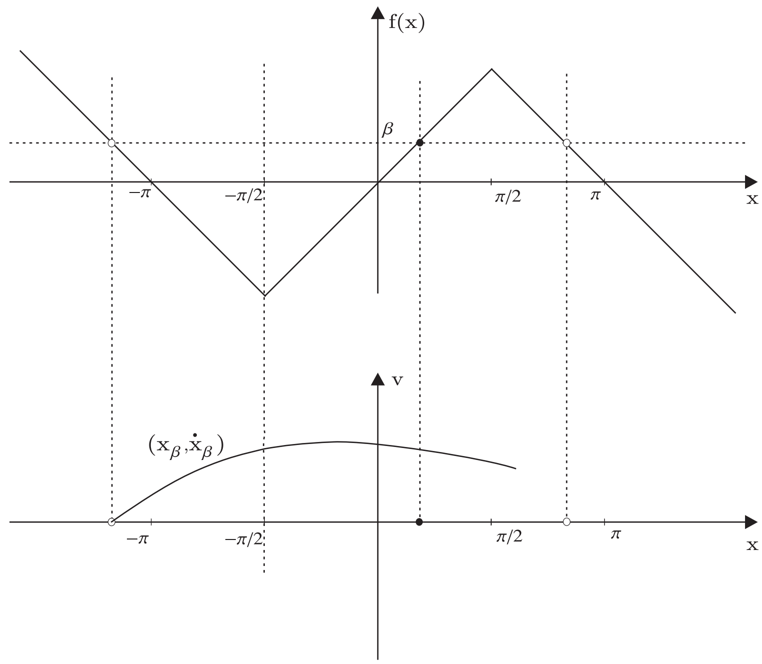

Figure 3:

the interval $]-\infty,0[$(see Figure 3). The function v_β is positive in the interval $]-\infty,0[$; let $t_\beta \geq 0$ be the first value where $v_\beta(t_\beta)=0$, if $v_\beta(t)>0$ for all $t>0$ we define $t_\beta=+\infty$.

We shall start to show that there exists $\beta_1 \in]0,\pi/2[$ so that if $\beta_1 \leq \beta < \pi/2$ then $t_\beta=+\infty$. This will be done in two lemmas.

Lemma 2.1 *If $c<2$ there exists $\beta_0 \in]0,\pi/2[$ in such a way that for each $\beta_0 \leq \beta < \pi/2$ there exists $0<t_\beta^1<t_\beta$ so that $x_\beta(t_\beta^1)>\pi/2$ and $v_\beta(t)>0$ for every $t \in]-\infty,t_\beta^1[$.*

Proof: Notice that

$$\widetilde{x}_\beta(t)=(\pi/2+\beta)e^{-\frac{c}{2}t}\left[\frac{-2c+\sqrt{c^2+4}}{\sqrt{4-c^2}}\sin\left(\frac{\sqrt{4-c^2}}{2}t\right)-\cos\left(\frac{\sqrt{4-c^2}}{2}t\right)\right]+\beta$$

is the solution of the equation $x''+cx'+x=\beta$ so that

$$\widetilde{x}_\beta(0)=-\pi/2=\overline{x}_\beta(0);$$

$$\widetilde{x}'_\beta(0)=(\pi/2+\beta)\left(\frac{-c+\sqrt{c^2+4}}{2}\right)=\overline{x}'_\beta(0).$$

Defining

$$f(t) = e^{-\frac{c}{2}t}\left[\frac{-2c+\sqrt{c^2+4}}{\sqrt{4-c^2}}\sin\left(\frac{\sqrt{4-c^2}}{2}t\right) - \cos\left(\frac{\sqrt{4-c^2}}{2}t\right)\right],$$

we observe that the smallest value $t_2 > 0$ where $f'(t_2) = 0$ is so that $f(t_2) > 0$. We conclude that the function $\widetilde{x}_\beta(t) = (\pi/2+\beta)f(t)+\beta$ is increasing in $]0, t_2[$ and $\widetilde{x}_\beta(t_2) = (\pi/2+\beta)f(t_2)+\beta > \pi/2$ if

$$\beta > \beta_0 = \max\left\{0, \frac{\pi}{2}\frac{1-f(t_2)}{1+f(t_2)}\right\}.$$

If $t_3 > 0$ is the first value where $\widetilde{x}_\beta(t) \notin]-\pi/2, \pi/2[$ then $x_\beta(t) = \widetilde{x}_\beta(t)$ in $]0, t_3[$, we conclude that for each $\beta > \beta_0$ there exists t^1_β so that $x_\beta(t^1_\beta) > \pi/2$ and $v_\beta(t) > 0$ for every $t \in]-\infty, t^1_\beta[$.

Lemma 2.2 *If $c < 2$ then there exists $\beta_1 \in]0, \pi/2[$ so that if $\beta_1 < \beta < \pi/2$ then $t_\beta = +\infty$.*

Proof: Let β_0 be given by the last lemma. Take $\beta_1 \in]0, \pi/2[$ so that $\beta_0 \leq \beta_1$ and $x_{\beta_0}(t^1_{\beta_0}) > \pi - \beta_1$. Consider $\beta_1 < \beta < \pi/2$. By the last lemma v_β is well defined as a function of x and is positive in the interval $]-\pi-\beta, x_\beta(t^1_\beta)[$(with $x_\beta(t^1_\beta) > \pi/2$). On the other hand, $v_\beta(x)$ is solution of

$$\frac{dv}{dx} = -c + \frac{\beta - g(x)}{v} \tag{5}$$

in the same interval. Since

$$\frac{dv_{\beta_0}}{dx} = -c + \frac{\beta_0 - g(x)}{v} < -c + \frac{\beta - g(x)}{v},$$

v_{β_0} is a lower solution of (5) in the interval $]-\pi-\beta_0, \pi-\beta_1[$. Considering a sufficiently large constant L we obtain an upper solution of (5). This pair of lower and upper solutions form a funnel for equation (5) in the interval $]-\pi-\beta_0, \pi-\beta_1[$(see Figure 4). Since $v_\beta(-\pi-\beta_0) \in]0, L[$ we conclude that $v_\beta(x)$ is well defined and positive in the interval $]-\pi-\beta, \pi-\beta[$. Similarly we can conclude that v_β is well defined and positive in the interval $]\pi-\beta, 3\pi-\beta[$, using $v_\beta(x-2\pi)$ as a lower solution and L as an upper solution. By induction we conclude that $v_\beta(x)$ is well defined and positive in $]-\pi-\beta, +\infty[$. Hence $t_\beta = +\infty$.

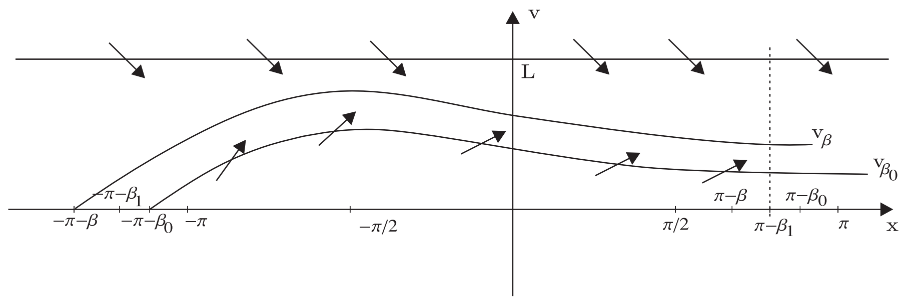

Figure 4:

In the proof of the last lemma we used the concept of lower and upper solutions to construct a funnel. In the next proof we shall use the same concept to prove the existence of a solution for a periodic boundary value problem. We recall the following classical theorem.

Theorem 2.3 *Let $f \in C(\mathbb{R}/T\mathbb{Z} \times \mathbb{R})$ and $\alpha, \beta \in C^1[0,T]$ be functions satisfying*

$$\alpha'(t) \le f(t, \alpha(t)), \quad \beta'(t) \ge f(t, \beta(t)), \quad \forall t \in [0,T],$$

and $\alpha(0) \le \alpha(T) < \beta(T) \le \beta(0)$. Then α, β are so-called an ordered pair of lower and upper solutions and there exists a T-periodic solution x of $x' = f(t,x)$ so that $\alpha(t) \le x(t) \le \beta(t)$, for all $t \in [0,T]$.

Proof of Theorem 1.2: Consider β_1 given by the last lemma and $\beta_1 < \beta < \pi/2$. Using the function $v_\beta(x)$ as a lower solution and L as an upper solution in the interval $[-\pi-\beta, \pi-\beta]$, we obtain an ordered pair of lower and upper solutions. We conclude that (5) has a positive periodic solution $v_0(x)$. To this solution corresponds a running periodic solution (x_0, v_0) of system (4).

Proof of Corollary 1.3: Consider the running periodic solution (x_0, v_0) given by the last theorem. Using (3) we conclude that

$$\{(x_0(t), v_0(t)) : t \in \mathbb{R}\} \cup \{(-\pi-\beta, 0)\} \cup \{(\beta, 0)\} \subset \mathcal{A}.$$

Notice that by uniqueness the points $(-\pi-\beta, 0)$, $(\beta, 0)$ do not belong to the orbit (x_0, v_0). Since $\{(x_0(t), v_0(t)) : t \in \mathbb{R}\}$ is itself homeomorphic to $\mathbb{T}^1$ and is a proper subset of $\mathcal{A}$, we conclude that $\mathcal{A}$ is not homeomorphic to $\mathbb{T}^1$.

We remark that the proof of Lemma 2.1 depends on the fact that the equilibrium $\begin{pmatrix} \beta \\ 0 \end{pmatrix}$ is a stable spiral for every $\beta \in]0, \pi/2[$, therefore on the fact that g is not differentiable at $x = \pi/2$. However, using the continuous dependence of solutions on the equation, it is not difficult to show that given β such that Corollary 1.3 holds it is possible to find $g_1 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ sufficiently close to g in the norm of the uniform convergence such that the atractor associated to

$$\begin{cases} x' = v \\ v' = -cv - g_1(x) + \beta \end{cases}$$

is similar to the atractor associated to (4).

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