DISTRIBUTION, LIMIT DISTRIBUTION AND PREDICTION BANDS FOR THE MAXIMUM CLAIM

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SYNOPTIC ABSTRACT

Exact and limit distributions are obtained for the maximum claim, using the collective model. Prediction bands are derived, falling either into a controlled and or a non-controlled case. These cases are separated by a crucial value of an estimable parameter. An application to automobile insurance data is given.

Key words and phrases: exact distribution, limit distribution, maximum claim.

1. INTRODUCTION

Let $Y(t)$ be the maximum claim paid up to time $t \in \mathbb{R}^+$. Let us assume that the claims $\{X_i > 0 : i \in \mathbb{N}\}$ are random variables *i.i.d.*, with common distribution function $F_X(\cdot)$. We also take the number of claims up to time t, $N(t)$ to be generated by a Poisson process with rate $\lambda(t)$ $(t \geq 0)$, independent of the claims. We obtain the exact distribution of the random variable $Y(t)$, $F_{Y(t)}(\cdot)$, and conditions for the limit distribution of the random variable $Z(t) = \lambda(t)g(Y(t))$, with $g(\cdot)$ a known function, to be exponential. The expression of $Z(t)$ will be obtained for some special (and theoretically important) cases.

Our results refer to the Collective Risk model, for instance see Bowers (1986), but instead of the sum of claims up to instant t we are interested in the maximum of those claims. Thus we will be interested in the tails of the claims distribution, namely their representation, using the Pareto generalized distribution, see McNeil (1997).

Moreover, we will obtain prediction bounds for $Z(t)$, showing that these fall into two situations corresponding to controlled and non-controlled maximum claim. The two situations are separated by a critical value for an estimable parameter. An example with real data on automobile insurance is shown.

2. EXACT DISTRIBUTION

Let $Y(t) = \max\{X_1, ..., X_n\}$ be the most severe claim up to time $t \in \mathbb{R}^+$ with n the total number of claims occurred up to that time, and let $\phi_{N(t)}(u)$ be the moment generating function of $N(t)$. We have for the distribution function of $Y(t)$, $F_{Y(t)}(y)$, with $y > 0$:

$$\begin{aligned} F_{Y(t)}(y) &= P[Y(t) \leq y] = \sum_{n=0}^{\infty} P[N(t) = n] P[\bigcap_{i=1}^{n} X_i \leq n] \\ &= \sum_{n=0}^{\infty} P[N(t) = n] F_X^n(y) = \phi_{N(t)}(\ln(F_X(y))) - P[N(t) = 0] \end{aligned}$$

By definition $Y(t) = 0$ whenever $N(t) = 0$.

Since we suppose that $N(t)$ is Poisson distributed with parameter $\lambda^*(t) = \int_0^t \lambda(u)\, du > 0$, we immediately know that

$$F_{Y(t)}(y) = \exp(-\lambda^*(t)(1 - F_X(y))),\ y \geq 0. \tag{1}$$

3. LIMIT LAW

3.1. Discrete Case

If $g(\cdot)$ is a continuous strictly decreasing function, verifying $\lim_{v\to\infty} g(v) = 0$ and $\lim_{v\to 0} g(v) = \infty$, the inverse function $g^{-1}(\cdot)$ will have the same monotonicity properties as the direct function. Let $W(t) = g(Y(t))$, then, according to (1), the distribution function of $W(t)$ will be

$$\begin{aligned} F_{W(t)}(w) &= P[g(Y(t)) \leq w] = P[Y(t) \geq g^{-1}(w)] = 1 - F_{Y(t)}(g^{-1}(w)) \\ &= 1 - \exp[\lambda t(1 - F_X(g^{-1}(w)))],\ w \geq 0. \end{aligned} \tag{2}$$

Defining the random variable $Z(t) = \lambda^*(t) g(Y(t))$, we have

PROPOSITION 3: *If* $L(w) = 1 - F_X(g^{-1}(w))$ *has an expansion in MacLaurin series then, the limit distribution function of* $Z(t)$ *is*

$$\lim_{t\to\infty} F_{Z(t)}(z) \quad = \quad 1 - \exp\left\{L'(0)\, z\right\},\ z \geq 0$$

with $L'(0) \geq 0$.

Proof: Since $\lim_{w\to 0} L(w) = 0$, we will have

$$L(w) = \sum_{j=0}^{\infty} \frac{d^j\, L(w)}{dw^j}\bigg|_{w=0} \frac{w^j}{j!},$$

so that, by (2),

$$F_{Z(t)}(z) = F_{W(t)}(z/(\lambda^*(t))) = 1 - \exp\left\{-\lambda^*(t) \sum_{j=1}^{\infty} \frac{d^j L(w)}{dw^j}\bigg|_{w=0} \frac{(z/(\lambda^*(t)))^j}{j!}\right\}$$

and so

$$\lim_{t\to\infty} F_{Z(t)}(z) = 1 - \exp\left\{L'(0)\, z\right\}, \qquad z \in \mathbb{R}^+ .$$

To finish the proof we just have to show that $L'(0) \geq 0$, but

$$\begin{aligned} L'(0) \quad &= \quad -\frac{dF_X(g^{-1}(w)))}{dg^{-1}(w)} \frac{dg^{-1}(w)}{dw}\bigg|_{w=0} \\ &= \quad -f_X(g^{-1}(0)) \frac{dg^{-1}(w)}{dw}\bigg|_{w=0} \geq 0 \end{aligned}$$

with $f_X(\cdot)$ the probability density function of X. □

So, when $L'(0) > 0$ the limit distribution is exponencial. Moreover, if $\lim_{x\to\infty} f_X(x) x^r = c > 0$ with $r > 1$, we will have

$$\lim_{w\to 0} f_X(g^{-1}(w))\, g^{-r}(w) = c > 0\,,$$

hence

$$-\lim_{w\to 0} L'(w)\, \frac{g^{-r}(w)}{\frac{dg^{-1}(w)}{dw}} = c.$$

For instance, if we take $g^{-1}(w) = w^{-s}$, then, $-g^{-r}(w)\frac{dg^{-1}(w)}{dw} = s w^{(r-1)s-1}$ and with $s = 1/(r-1)$, we obtain $-g^{-r}(w)\frac{dg^{-1}(w)}{dw} = (r-1)^{-1}$ and consequently,

$$\lim_{w\to 0} L'(w) \quad = \quad L'(0) = \frac{c}{r-1}.$$

Since, the inverse function of $g^{-1}(w) = w^{-1/(r-1)} = y$ is $w = y^{1-r}$, we have

PROPOSITION 4: *If* $L(w) = 1 - F_X(g^{-1}(w))$ *has an expansion in MacLaurin series, and* $\lim_{x\to\infty} f_X(x)x^r = c > 0$ *with* $r > 1$*, the limit distribution of* $Z(t) = \lambda^*(t)Y(t)^{1-r}$ *is* $1 - \exp\{-c/(r-1)z\}$, $z \in \mathbb{R}^+{}_0$.

For the Pareto generalized distribution

$$F(x|\delta,\gamma) \;=\; 1 - (1+\gamma\, x/\delta)^{-1/\gamma}\,; \qquad x>0;\;\; \delta>0;\;\; \gamma>0$$

we have

$$c = \frac{1}{\delta}\left(\frac{\gamma}{\delta}\right)^{-\frac{1}{\gamma}-1} \quad \text{and} \quad r = \frac{\gamma+1}{\gamma}\,. \tag{3}$$

3.2. Excedances and adjustment

According to the Pickands-Balkema-de Haan theorem (see McNeil, 1997) we assume that the distribution of the excedances over the high threshold v, whenever $F_X(v) < 1$,

$$F_X(x|v) = P(X - v \le x | X > v),$$

as the threshold tends to the right endpoint, has as limit distribution the generalized Pareto distribution $F(x - v|\delta, \gamma)$.

Let $N_v(t)$ be the number of claims up to time t that exceede v. This will be a Poisson process with rate $\lambda^*_v(t) = \lambda^*(t)\,(1 - F_X(v))$. So

$$Z_v(t) \;=\; \lambda^*_v(t)Y(t)^{1-r}$$

will have, according to Proposition 4, limit distribution $1-\exp\{-c/(r-1)z\}$, where r and c may be estimated following the procedure due to de Haan (1994).

Let $X_{n-k,n} \le \ldots \le X_{n,n}$ be the claims exceeding v. Then (see de Haan, 1994), with

$$M_n^{(r)} \;=\; \frac{1}{k}\sum_{l=1}^{k}\left[\log\left(\frac{x_{n-l+1,n}}{x_{n-k,n}}\right)\right]^r \qquad \text{(scale invariant)}$$

we have the estimators

$$\hat{\gamma}_n \;=\; 1 + M_n^{(1)} - \frac{1}{2}\left(1 - \frac{{M_n^{(1)}}^2}{M_n^{(2)}}\right)^{-1} \qquad \text{(scale invariant)}$$

$$\hat{\delta}_n = v \frac{M_n^{(1)}}{\rho} \qquad \begin{cases} \rho = 1 & \text{if } \hat{\gamma}_n \geq 0 \\ \rho = \frac{1}{1-\gamma} & \text{if } \hat{\gamma}_n < 0 \end{cases} .$$

Applying these estimators to a sample of 16 trimestral claim totals for an automobile insurance portfolio for 2001/2006, we obtained the estimates

$$\begin{aligned} \hat{\gamma} &= 0.05204 \\ \hat{\delta} &= 320.518 \,, \end{aligned}$$

so that

$$\hat{r} = 20.21599$$

and $\hat{c}$ may be derived from (3).

4. PREDICTION BANDS

The limit distribution for $Z_v(t)$ has the quantiles $z_q = -\frac{r-1}{c}\log(1-q)$. Since

$$Y(t) = \left(\frac{Z_v}{\lambda_v^*(t)}\right)^{\frac{1}{1-r}}$$

increases with $Z_v = Z_v(\infty)$, we have the quantiles

$$y_q(t) = \left(\frac{z_q}{\lambda_v^*(t)}\right)^{\frac{1}{1-r}} = a(q,r)\left(\frac{\lambda_v^*(t)}{\frac{r-1}{c}}\right)^{\frac{1}{r-1}}$$

with

$$a(q,r) = \Big(-\log(1-q)\Big)^{\frac{1}{r-1}} > 0$$

when $0 << t$. We point out that the ratios

$$\frac{y_{q_2}(t)}{y_{q_1}(t)} = \frac{a(q_2,r)}{a(q_1,r)} = \left(\frac{\log(1-q_2)}{\log(1-q_1)}\right)^{\frac{1}{r-1}}$$

do not depend on time.

Using the new time scale $t^* = \frac{c}{r-1}\lambda^*(t)$ we get $y_q(t^*) = a(q,r)\, {t^*}^{\frac{1}{r-1}}$ with

$$y_q'(t^*) = \frac{a(q,r)}{r-1}(t^*)^{\frac{2-r}{r-1}} > 0$$

and

$$y_q''(t^*) = \frac{a(q,r)}{r-1}\frac{2-r}{r-1}(t^*)^{\frac{3-2r}{r-1}} \begin{cases} < 0, & r < 0 \\ = 0, & r = 2 \\ > 2, & r > 2 \,. \end{cases}$$

To illustrate the cases with $r < 2$ and $r > 2$ we present the graphs of the 80% prediction bands for the maximum claim, for $r = 1.9$ and $r = 7.5$, in Figure 1.

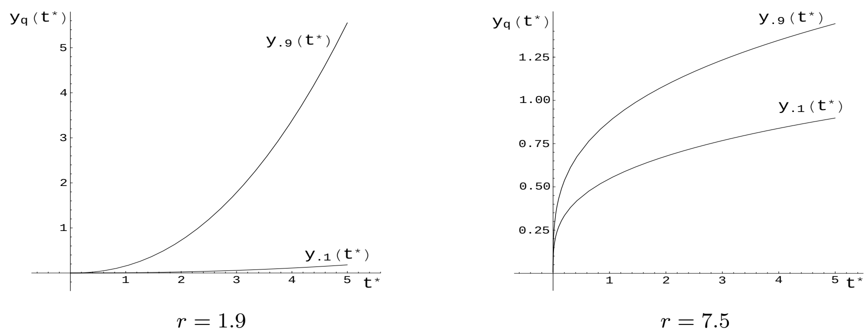

Figure 1. – 80% prediction bands for the maximum claim.

For the automobile insurance data we have the 90% prediction band in Figure 2, which displays a situation with good control of the increase of the maximum claims over time.

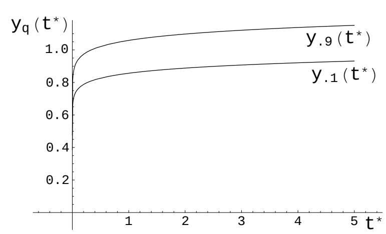

Figure 2. – 90% prediction band for the maximum claim for the automobile insurance.

5. TIME RANDOMIZATION

We now randomize time t^*. Since the actual value of this parameter at a given time will depend on a great number of factors we can avail ourselves of Fisher's metatheorem (Fisher, 1918) to assume that t^* is normally distributed with mean vector $\mu(t)$ and variance $b\mu(t)$. Thus, the q-th quantile of t^* is given by

$$t_q^*(t) = \mu(t) + z_q \sqrt{b\,\mu(t)}\,.$$

Accordingly, for the random variable

$$g(t^*) = c\,(t^*)^s\,, \quad s > 0,$$

we have the quantiles

$$g_q(t) \;=\; c\left(t^*_{1-q}(t)\right)^s \;=\; c\left(\mu(t) + z_{1-q}\sqrt{b\,\mu(t)}\right)^s$$

from which we can obtain tollerance intervals for the $y_q(t) = y_q(t^*(t))$

$$y_{q^*,q}(t) \;=\; c\,a(q^*, r)\left(\mu(t) + z_{1-q}\sqrt{b\,\mu(t)}\right)^s\,.$$

Taking

$$q = p/2\,, \quad q^* = p^*/2$$

we get

$$P\Big(Y(t) \,\in\, [y_{q^*,q}(t)\,;\; y_{1-q^*,1-q}(t)]\Big) \;\geq\; 1 - (p + p^*)\,.$$

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