On some generalizations of the Cox-Markov process

Carlos A. Coelho and João T. Mexia [1]

Abstract

In this paper we first show how a mixture of particular Generalized Integer Gamma (GIG) distributions is the distribution of time till absorption into state 0 for a Cox-Markov process. Based on this approach, several generalizations of the Cox-Markov model are then proposed. For all the cases the distribution of the time till absorption into state 0 is derived, using the GIG distribution as a basis. The moments of these waiting times are derived for all cases and modules for the computation of their exact distributions are provided.

Key words: Cox distribution, mixtures, Generalized Integer Gamma distribution, moments.

1 Introduction

1.1 The usual Cox-Markov process

A Cox-Markov process is a Markov process with $n+1$ states as the one depicted in Figure 1 (Augustin and Büscher, 1982). The process remains in state k $(1 \leq k \leq n)$ an amount of time X_k, exponentially distributed with parameter λ_k, and upon departure from state k the process moves to state 0 with probability α_k and to state $k-1$ with probability $\overline{\alpha}_k = 1 - \alpha_k$. We will suppose that the amount of time the process remains in a given state is independent of the amount of time the system remains in any other state. To avoid trivial situations we will assume that $\alpha_k < 1$ for all $k > 1$. Clearly, we always have $\alpha_1 = 1$.

A Cox distribution is then the distribution of the random variable Y_k that represents the time till absorption into state 0, starting from state k, for the Markov process above.

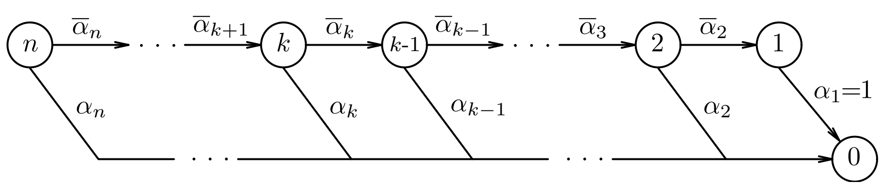

Figure 1 – A Cox-Markov process with $n+1$ states, and corresponding transition probabilities.

The probability that, starting from state k, the system reaches state 0 in $m \leq k$ steps is

$$p_{mk} \;=\; \alpha_{k-m+1} \prod_{j=1}^{m-1} (1 - \alpha_{k-j+1})\,, \qquad m = 1, \ldots, k\,, \tag{1}$$

obviously with $p_{1k} = \alpha_k$ and $\sum_{m=1}^{k} p_{mk} \;=\; 1$.

Let

$$X_k \;\sim\; Exp(\lambda_k)$$

be the time that the Markov process remains in state k $(1 \leq k \leq n)$.

[1]corresponding author: Carlos A. Coelho (cmac@fct.unl.pt) is Associate Professor and João T. Mexia is Professor of Statistics at the Department of Mathematics of the Faculty of Sciences and Technology of the New University of Lisbon, 2829-516 Caparica, Portugal

Then

$$P(Y_k < y) \;=\; \sum_{m=1}^{k} p_{mk}\, P\Bigg(\underbrace{\sum_{j=1}^{m} X_{k-j+1}}_{X^*_{mk}} < y\Bigg) \;=\; \sum_{m=1}^{k} p_{mk}\, P\left(X^*_{mk} < y\right) \qquad (2)$$

where $X^*_{mk} = \sum_{j=1}^{m} X_{k-j+1}$ is the sum of m independent random variables with Exponential distributions, having what we will call a simple Generalized Integer Gamma (GIG) distribution with shape parameters all equal to one and rate parameters $\lambda_k, \ldots,\ \lambda_{k-m+1}$ (see the next section for a complete definition of the GIG distribution as well as its probability density and cumulative distribution functions).

Although it is not hard, after some algebraic manipulation, to show that the mixture representation in (2) is equivalent to the representation used by Koole (2004), we will stick with this representation since it is not only more convenient for the generalizations of the Cox-Markov process presented in the next section but also for the derivation of the moments as well as of the probability density function (p.d.f.) and the cumulative distribution function (c.d.f.), since these will be immediately available from the p.d.f. and c.d.f. of the GIG distribution shown in the next section, since once obtained the p.d.f. and c.d.f. for X^*_{mk}, the p.d.f. and c.d.f. of Y_k is readily at hand.

From (2) we may then see Y_k as a mixture of k simple GIG distributions, the m-th $(m = 1, \ldots, k)$ of which with depth at most m, since it is the sum of m independent Exponentially distributed random variables. If X_j $(j = 1, \ldots, p)$ are p independent random variables, using the Multinomial Theorem we may express, for $h \in I\!N_0$ the h-th moment of $Z = \sum_{j=1}^{p} X_j$, as

$$E\left(Z^h\right) \;=\; h! \sum_{\substack{\forall_{\oplus} n_j = h \\ j=1:p}} \prod_{j=1}^{p} \frac{E\left(X_j^{n_j}\right)}{n_j!}\,, \qquad (3)$$

where $\displaystyle\sum_{\substack{\forall_{\oplus} n_j = h \\ j=1:p}}$ stands for the summation over all sets of p non-negative integers n_j such that they add up to h.

Thus, we have the h-th moment $(h \in I\!N_0)$ of Y_k given by

$$\begin{aligned} E\left(Y_k^h\right) \;&=\; \sum_{m=1}^{k} p_{mk}\, E\left[\left(\sum_{i=1}^{m} X_{k-i+1}\right)^h\right] \;=\; \sum_{m=1}^{k} p_{mk}\, h! \sum_{\substack{\forall_{\oplus} n_i = h \\ i=1:m}} \prod_{i=1}^{m} \frac{\Gamma(1+n_i)}{\Gamma(1)\, n_i!}\, \lambda_{k-i+1}^{-n_i} \\ &=\; h! \sum_{m=1}^{k} p_{mk} \sum_{\substack{\forall_{\oplus} n_i = h \\ i=1:m}} \prod_{i=1}^{m} \lambda_{k-i+1}^{-n_i}\,. \end{aligned} \qquad (4)$$

1.2 The Generalized Integer Gamma distribution

Let

$$X_j \;\sim\; \Gamma(r_j, \lambda_j)\,, \qquad j = 1, \ldots, p$$

b p independent random variables with a Gamma distribution with integer shape parameters r_j and all different rate parameters $\lambda_j > 0$, that is, the probability density function (p.d.f.) of X_j is written as

$$f_{X_j}(x) \;=\; \frac{\lambda_j^{r_j}}{\Gamma(r_j)}\, e^{-\lambda_j x}\, x^{r_j - 1} \qquad x > 0\,.$$

Then the distribution of

$$Z \;=\; \sum_{j=1}^{p} X_j$$

is what Coelho (1998) called a GIG distribution of depth p, with p.d.f. and c.d.f.

$$f_Z(z) \;=\; K \sum_{j=1}^{p} P_j(z)\, e^{-\lambda_j\, z}\,, \qquad\quad F_Z(z) \;=\; 1 - K \sum_{j=1}^{p} P_j^*(z)\, e^{-\lambda_j\, z} \tag{5}$$

where

$$K = \prod_{j=1}^{p} \lambda_j^{r_j}\;, \qquad P_j(z) = \sum_{k=1}^{r_j} c_{jk}\, z^{k-1} \tag{6}$$

and

$$P_j^*(z) = \sum_{k=1}^{r_j} c_{jk}\,(k-1)! \sum_{i=0}^{k-1} \frac{z^i}{i!\,\lambda_i^{k-i}} \tag{7}$$

with

$$c_{j,r_j} = \frac{1}{(r_j-1)!} \prod_{\substack{i=1\\ i\neq j}}^{p} (\lambda_i - \lambda_j)^{-r_i}\;, \qquad j = 1,\ldots,p\,, \tag{8}$$

and

$$c_{j,r_j-k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(r_j-k+i-1)!}{(r_j-k-1)!}\, R(i,j,p,\underline{r},\underline{\lambda})\, c_{j,r_j-(k-i)}\,, \quad \begin{matrix}(k=1,\ldots,r_j-1)\\ (j=1,\ldots,p)\end{matrix} \tag{9}$$

where

$$\underline{r} = [r_1, r_2, \ldots, r_p]'\,, \quad \text{and} \quad R(i,j,p,\underline{r},\underline{\lambda}) = \sum_{\substack{k=1\\ k\neq j}}^{p} r_k\,(\lambda_j - \lambda_k)^{-i} \quad (i = 1,\ldots,r_j-1)\,. \tag{10}$$

Of course, for $p = 1$ the GIG distribution yields the integer Gamma distribution.

Using (3) and the Multinomial Theorem, for $h \in I\!N_0$, the h-th non-central moment of Z is given by

$$E\left(Z^h\right) \;=\; h! \sum_{\substack{\forall_{\oplus} n_j = h\\ j=1:p}} \prod_{j=1}^{p} \frac{E\left(X_j^{n_j}\right)}{n_j!} \;=\; h! \sum_{\substack{\forall_{\oplus} n_j = h\\ j=1:p}} \prod_{j=1}^{p} \frac{\Gamma(r_j+n_j)}{\Gamma(r_j)\, n_j!}\, \lambda_j^{-n_j}\,.$$

We will use the notation

$$Z \sim GIG\big(\underbrace{r_1,\ldots,r_p}_{p}\,;\,\underbrace{\lambda_1,\ldots,\lambda_p}_{p}\big)$$

to denote the fact that the random variable Z has a GIG distribution of depth p with shape parameters $r_1,\ldots,r_p$ and rate parameters $\lambda_1,\ldots,\lambda_p$, avoiding the use of the underbrace notation whenever the depth is clear from the list of parameters.

We may note that the distribution of $\sum_{i=1}^{m} X_{k-i+1}$ $(m = 1,\ldots,k)$ in (2) is a particular case of the above GIG distribution. In case all λ_{k-i+1} are different (for $i = 1,\ldots,m$), all r_i's will be equal to 1. For short, we will call this distribution a 'simple' GIG distribution.

Anyway, whenever there are say $p - m$ $(m < p)$ groups of X_i's with the same rate parameter, all we have to do is to add the corresponding shape parameters and consider then a GIG distribution with depth $m (< p)$. More precisely, if among the p random variables X_i there are only $m (< p)$ different rate parameters, we may consider, without any loss of generality, that there are p_1 of them with rate parameter λ_1, p_2 with rate parameter λ_2, ..., and p_m of them with rate parameter λ_m, with $\sum_{j=1}^{m} p_j = p$. The distribution of Z will be in this case a GIG distribution with depth m, with shape parameters $s_1,\ldots,s_m$ and rate parameters $\lambda_1,\ldots,\lambda_m$, where s_j $(j = 1,\ldots,m)$ is the sum of the shape parameters of the p_j random variables X_i with rate parameter λ_j. Ahead in this paper, for short, we will call this process just the 'grouping' of the shape and rate parameters. In section 4 a Mathematica module is provided to do this job.

In the next section, in order to be able to better precise the parameters involved in the GIG distributions, we will use

$$F_Y\Big(y;\, GIG(\underbrace{r_1,\ldots,r_p}_{p}\,;\,\underbrace{\lambda_1,\ldots,\lambda_p}_{p})\Big)\,. \tag{11}$$

to denote the c.d.f. of a random variable Y with a GIG distribution of depth p with shape parameters $r_1,\ldots,r_p$ and rate parameters $\lambda_1,\ldots,\lambda_p$. We will be using the underbraced indication of the depth mainly in some cases where the indexation may be not so simple. In any case the expressions for either the p.d.f. or the c.d.f. may be obtained directly from (5)-(10) by adequately replacing the depth and the shape and rate parameters.

1.3 The usual Cox-Markov process revisited

As we saw above, in this case the distribution of Y_k is a mixture of GIG distributions. If all the k rate parameters λ_{k-j+1} are different for $j=1,\ldots,k$, from (2) we may take the distribution of Y_k as a mixture of k simple GIG distributions, the m-th ($m=1,\ldots,k$) of which has depth m,

$$X^*_{mk} \sim GIG(\underbrace{1,\ldots,1}_{m}\,;\,\underbrace{\lambda_k,\ldots,\lambda_{k-m+1}}_{m})$$

so that the c.d.f. of Y_k may in this case be written as

$$F_{Y_k}(y) \,=\, \sum_{m=1}^{k} p_{mk}\left(1 - K\sum_{j=1}^{m} c_j\, e^{-\lambda_{k-j+1}y}\right)$$

where

$$K \,=\, \prod_{j=1}^{m}\lambda_{k-j+1}\,, \qquad c_j \,=\, \prod_{\substack{i=1\\ i\neq j}}^{m}(\lambda_{k-i+1}-\lambda_{k-j+1})^{-1}\,.$$

If among the m rate parameters λ_{k-j+1} ($j=1,\ldots,m$) there are only $n_m < m$ different ones, then each X^*_{mk} has a GIG distribution of depth n_m. If after adequate 'grouping', we denote by $\lambda^*_1,\ldots,\lambda^*_{m^*}$ the $n_m < m$ different rate parameters and by $r^*_1,\ldots,r^*_{m^*}$ the corresponding shape parameters, using the notation in (11), we may write the c.d.f. of Y_k as

$$F_{Y_k}(y) \,=\, \sum_{m=1}^{k} p_{mk}\, F_{X^*_{mk}}\Big(y;\, GIG(r^*_1,\ldots,r^*_{n_m};\, \lambda^*_1,\ldots,\lambda^*_{n_m})\Big)\,. \tag{12}$$

We may note that the above way to compute the parameters c_{jk} is much simpler than the one usually used, see for example (Koole, 2004), including for the case when there are some rate parameters λ_{k-j+1} that are equal. Also, their computation is rendered precise when we use the computation capabilities of a software like Mathematica. Modules for the complete computation of the above c.d.f. are provided in section 4.

2 Generalizations of the Cox-Markov process

2.1 A first generalization of the Cox-Markov process

A first generalization of the Cox-Markov process may be thought of when we consider that the time the process remains in state k ($1 \le k \le n$) is the waiting time for the N_k-th event of a non-homogeneous Poisson process. In this case the distribution of X_k, the waiting time in state k, will be the sum of N_k independent $Exp(\lambda_{ik})$ ($i=1,\ldots,N_k$) random variables, that is a GIG distribution of depth at most N_k. This distribution will be a simple GIG distribution of depth N_k if all the N_k rates in the Poisson process are different and it will be a GIG distribution of depth $N^*_k < N_k$ if there are $N^*_k < N_k$ sets of different rate parameters. Since the X_k are considered independent and since the GIG distribution is closed for sums, that is, the distribution of the sum of independent GIG distributions is still a GIG distribution, as it is easy to see from the definition of the GIG distribution in the previous section, the distribution of Y_k will still be in this case a mixture of GIG distributions, since the distribution of each sum $\sum_{i=1}^{m} X_{k-i+1}$ $(m = 1, \ldots, k)$ will be a GIG distribution.

More precisely, we may say that in this case, from (2), Y_k is a mixture of k sums, the m-th $(m = 1, \ldots, k)$ of which of m independent GIG random variables, the j-th $(j = 1, \ldots, m)$ of these with depth at most N_{k-j+1}. We say of depth at most N_{k-j+1}, since although we assume that the waiting time X_{k-j+1} is the waiting time for the N_{k-j+1}-th event from a non-homogeneous Poisson process, in the more general case, some of the N_{k-j+1} rates may be equal, what would yield for X_{j-k+1} a GIG distribution of depth smaller than N_{k-j+1}, more precisely, of depth $N_{k-j+1} - n$, where n is the number of rates that are equal to other 'previous' ones. Since the random variables X_{k-j+1} are assumed independent for $j = 1, \ldots, k$, any sum of these random variables is also a GIG distributed random variable and as such Y_k will be a mixture of k GIG distributions, the m-th $(m = 1, \ldots, k)$ of which with depth at most $\sum_{j=1}^{m} N_{k-j+1}$.

However, in order to obtain the moments of Y_k it will be easier to stick with the first definition, given the difficulty in specifying the depth and the shape parameters for the GIG in the case where some of the rate parameters would be equal. We have

$$\begin{aligned} E\left(Y_k^h\right) &= \sum_{m=1}^{k} p_{mk}\, E\left[\left(\sum_{i=1}^{m} X_{k-i+1}\right)^h\right] = \sum_{m=1}^{k} p_{mk}\, h! \sum_{\substack{\forall_{\oplus} n_j = h \\ j=1:m}} \prod_{j=1}^{m} \frac{E\left(X_{k-i+1}^{n_j}\right)}{n_j!} \\ &= h! \sum_{m=1}^{k} p_{mk} \sum_{\substack{\forall_{\oplus} n_j = h \\ j=1:m}} \prod_{j=1}^{m} \sum_{\substack{\forall_{\oplus} k_i = n_j \\ i=1:N_{k-j+1}}} \prod_{i=1}^{N_{k-j+1}} \lambda_{i,k-j+1}^{-k_i} \end{aligned}$$

where it actually doesn't matter whether some of the rate parameters $\lambda_{i,k-j+1}$ $(i = 1, \ldots, N_{k-j+1};\ j = 1, \ldots, k)$ are equal or not.

Assuming that the rates $\lambda_{i,k-j+1}$ are all different for $i = 1, \ldots, N_{k-j+1}$, since X_{k-j+1} is the sum of N_{k-j+1} independent $Exp(\lambda_{i,k-j+1})$ random variables $(i = 1, \ldots, N_{k-j+1})$, we have

$$X_{k-j+1} \sim GIG(\underbrace{1, \ldots, 1}_{N_{k-j+1}};\ \underbrace{\lambda_{1,k-j+1}, \ldots, \lambda_{N_{k-j+1},k-j+1}}_{N_{k-j+1}})$$

that is a simple GIG distribution of depth N_{k-j+1}. If we further assume that the rates $\lambda_{i,k-j+1}$ are all different (for $j = 1, \ldots, k$), then X_{mk}^* has also a simple GIG distribution of depth $\sum_{j=1}^{m} N_{k-j+1}$. In this case the c.d.f. of Y_k may be written as

$$F_{Y_k}(y) = \sum_{m=1}^{k} p_{mk} \left(1 - K \sum_{j=1}^{m} \sum_{i=1}^{N_{k-j+1}} c_{ij}\, e^{-\lambda_{i,k-j+1} y}\right)$$

with

$$K = \prod_{j=1}^{m} \prod_{i=1}^{N_{k-j+1}} \lambda_{i,k-j+1}\,, \qquad c_{ij} = \prod_{\substack{n=1 \\ n \neq j}}^{m} \prod_{\substack{h=1 \\ \text{for } h=i}}^{N_{k-n+1}} \left(\lambda_{h,k-n+1} - \lambda_{i,k-j+1}\right)^{-1}.$$

If some of the $\lambda_{i,k-j+1}$ $(i = 1, \ldots, N_{k-j+1}; j = 1, \ldots, k)$ are equal we will denote the c.d.f. of Y_k and the GIG distribution of X_{mk}^* as in (12), now with

$$n_m < \sum_{j=1}^{m} N_{k-j+1}$$

and where now $\lambda_1^*, \ldots \lambda_{n_m}^*$ stand for the set of different rate parameters among the rate parameters in the set $\{\lambda_{1,k}, \ldots, \lambda_{N_k,k}; \ldots; \lambda_{1,k-m+1}, \ldots, \lambda_{N_{k-m+1},k-m+1}\}$.

2.2 A second generalization of the Cox-Markov process

A second generalization may be obtained if in each state k ($k = 1, \ldots, n$) we suppose that $t \in \{0, 1, \ldots, N_k\}$ trials are administered, where $N_k \in I\!N$ is a fixed number for each k. Let p_k be the probability of undergoing the N_k trials at state k. Then, if N represents the number of trials the process undergoes at state k, we have

$$P(N = N_k) = p_k$$

with

$$P(N = t) = \binom{N_k}{t} p_k^{t/N_k} \left(1 - p_k^{1/N_k}\right)^{N_k - t}, \qquad t = 0, 1, \ldots, N_k \,.$$

Further, let us suppose that the time spent in each trial, in state k, is

$$Z_{jk} \sim \Gamma(r_{jk}, \lambda_{jk}), \qquad r_{jk} \in I\!N; \quad \begin{array}{l} j = 0, 1, \ldots, N_k \\ k = 1, \ldots, n, \end{array}$$

that is, it has a Gamma distribution with shape parameter $r_{jk} \in I\!N$ and rate parameter λ_{jk}. Let us suppose that the Z_{jk} are independent in j and k. Let X_{tk} be the time spent till the completion of $t \in \{0, 1, \ldots, N_k\}$ trials. We have

$$X_{tk} = \sum_{j=0}^{t} Z_{jk}$$

with $X_{0k} = Z_{0k}$ being the 'residual', 'base' or minimum waiting time in state k, that is, the waiting time in state k when no trials are undergone.

The random variables X_{tk} ($t = 0, 1, \ldots, N_k; k = 1, \ldots, n$) will have integer Gamma distributions if all the λ_{jk} are equal for $j = 0, \ldots, N_k$ and will have a GIG distribution if some or all of them are different.

Then, being X_k the waiting time in state k, we have

$$\begin{aligned} P(X_k \le x) &= \sum_{t=0}^{N_k} P(N = t)\, P(X_k \le x | N = t) \\ &= \sum_{t=0}^{N_k} \underbrace{\binom{N_k}{t} p^{t/N_k} \left(1 - p^{1/N_k}\right)^{N_k - t}}_{\theta_{tk}} P(X_{tk} \le x) = \sum_{t=0}^{N_k} \theta_{tk}\, P(X_{tk} \le x)\,, \end{aligned}$$

clearly with $\sum_{t=0}^{N_k} \theta_{tk} = 1$, being thus the distribution of X_k, in the general case, a mixture of GIG distributions. Then, from (1) the distribution of Y_k, the waiting time till absorption in state 0 will be a mixture of sums of mixtures of GIG distributions, or, since the sum of mixtures of GIG distributions is itself a mixture of GIG distributions, so that we may just say that the distribution of Y_k is a mixture of mixtures of GIG distributions, or yet just a mixture of GIG distributions. However, reporting the distribution of the random variables X^*_{mk} under the form of just a mixture of GIG distributions, instead of under the form of the sum of mixtures of GIG distributions may indeed be more complicated and not much useful, since then it would be hard to report the depth of the GIG distributions involved and their parameters, mainly in the case where there are some rate parameters that are equal.

More precisely, Y_k is a mixture of k sums, the m-th ($m = 1, \ldots, k$) of which is a sum of m mixtures (the X_{k-j+1}, $j = 1, \ldots, m$), the j-th ($j = 1, \ldots, m$) of which is a mixture of $N_{k-j+1} + 1$ distributions, the t-th of which is a GIG with depth at most $t+1$ with shape parameters $r_{i,k-j+1}$ and rate parameters $\lambda_{i,k-j+1}$, $i = 0, \ldots, t$, that is, if all rate parameters $\lambda_{i,k-j+1}$ are different, for $i = 0, \ldots, t$,

$$X_{t,k-j+1} \sim GIG\Big(\underbrace{r_{0,k-j+1}, \ldots, r_{t,k-j+1}}_{t+1}; \underbrace{\lambda_{0,k-j+1}, \ldots, \lambda_{t,k-j+1}}_{t+1}\Big) \qquad \begin{array}{l} t = 0, \ldots, N_k \\ j = 1, \ldots, m\,. \end{array}$$

We thus have the h-th ($h \in I\!N_0$) non-central moment of Y_k given by

$$\begin{aligned}
E\left(Y_k^h\right) &= \sum_{m=1}^{k} p_{mk}\, E\left[\left(\sum_{j=1}^{m} X_{k-j+1}\right)^h\right] \\
&= \sum_{m=1}^{k} p_{mk}\, h! \sum_{\substack{\forall_{\oplus} n_j = h \\ j=1:m}} \prod_{j=1}^{m} \frac{E\left(X_{k-j+1}^{n_j}\right)}{n_j!} \\
&= h! \sum_{m=1}^{k} p_{mk} \sum_{\substack{\forall_{\oplus} n_j = h \\ j=1:m}} \prod_{j=1}^{m} \frac{1}{n_j!} \sum_{t=0}^{N_{k-j+1}} \theta_{t,k-j+1}\, E\left(X_{t,k-j+1}^{n_j}\right) \\
&= h! \sum_{m=1}^{k} p_{mk} \sum_{\substack{\forall_{\oplus} n_j = h \\ j=1:m}} \prod_{j=1}^{m} \sum_{t=0}^{N_{k-j+1}} \theta_{t,k-j+1} \sum_{\substack{\forall_{\oplus} k_i = n_j \\ i=0:t}} \prod_{i=0}^{t} \frac{\Gamma(r_{i,k-j+1}+k_i)}{\Gamma(r_{i,k-j+1})\, k_i!}\, \lambda_{i,k-j+1}^{-k_i}.
\end{aligned}$$

Since the sum of independent GIG distributions is again a GIG distribution, the sum of mixtures of GIG distributions is itself a mixture of GIG distributions. However, although it may be not too easy to express the distribution of each X_{mk}^* under this form of a mixture of GIG distributions, such is necessary in order to be able to obtain an expression for the c.d.f. of Y_k in this case.

Summarizing, we have

$$P(Y_k \le y) = \sum_{m=1}^{k} p_{mk}\, P(X_{mk}^* \le y)$$

where

$$X_{mk}^* = \sum_{j=1}^{m} X_{k-j+1}$$

with

$$P(X_{k-j+1} \le y) = \sum_{t=0}^{N_{k-j+1}} \theta_{t,k-+1}\, P(X_{t,k-j+1} \le y)$$

where, for $t = 0, \ldots, N_{k-j+1}$,

$$X_{t,k-j+1} \sim GIG(r_{0,k-j+1}, \ldots, r_{t,k-j+1};\, \lambda_{0,k-j+1}, \ldots, \lambda_{t,k-j+1})$$

is a GIG of depth $t+1$ if all rate parameters involved are different and in general will be a GIG of depth at most $t+1$.

Thus, the distribution of X_{mk}^* will be a mixture of

$$K_m = \prod_{j=1}^{m} (N_{k-j+1} + 1)$$

GIG distributions of varying depth, which will be at most, that is, if all rate parameters involved are different,

$$N_{t_1,\ldots,t_m} = \sum_{j=1}^{m} t_j \qquad \text{with} \qquad t_j \in \{0, \ldots, N_{k-j+1}\}\,.$$

The K_m weights in this mixture are

$$\theta^*_{t_1,\ldots,t_m} = \prod_{j=1}^{m} \theta_{t_j,k-j+1} \qquad \text{for all} \qquad t_j \in \{0, \ldots, N_{k-j+1}\}$$

and the corresponding GIG distribution in the mixture is, assuming that all the rate parameters involved are different,

$$GIG(r_{0,k},\ldots,r_{t_1,k},r_{0,k-1},\ldots,r_{t_2,k-1},\ldots,r_{0,k-m+1},\ldots,r_{t_m,k-m+1};\\ \lambda_{0,k},\ldots,\lambda_{t_1,k},\lambda_{0,k-1},\ldots,\lambda_{t_2,k-1},\ldots,\lambda_{0,k-m+1},\ldots,\lambda_{t_m,k-m+1})$$

that is a GIG with depth $N_{t_1,\ldots,t_m}$. We may note that this GIG distribution is the distribution of

$$\sum_{j=1}^{m} X_{t_j,k-j+1} \qquad \Big(t_j \in \{0,\ldots,N_{k-j+1}\}\Big)$$

where the $X_{t_j,k-j+1}$ are assumed independent for $j=1,\ldots,m$.

Thus, for the re-indexation from 1 through K_m of all the K_m different sequences $t_1,\ldots,t_m$ (with $t_j \in \{0,\ldots,N_{k-j+1}\}$, $j=1,\ldots,m$), the c.d.f. of X^*_{mk}, using the notation in (11), may be written as

$$F_{X^*_{mk}}(y) = \sum_{j=1}^{K_m} \theta^*_j\, F\Big(y;\, GIG(r_{0,k},...,r_{t_1,k},r_{0,k-1},...,r_{t_2,k-1},...,r_{0,k-m+1},...,r_{t_m,k-m+1};\\ \lambda_{0,k},...,\lambda_{t_1,k},\lambda_{0,k-1},...,\lambda_{t_2,k-1},...,\lambda_{0,k-m+1},...,\lambda_{t_m,k-m+1})\Big), \tag{13}$$

where the re-indexation of the weights $\theta^*_{t_1,\ldots,t_m}$ and the corresponding GIG distributions follow the natural order of the weights (see Appendix A onto this account).

In the case where some of the rate parameters are equal, an adequate 'grouping' of the shape and rate parameters has to be done before writing and considering the c.d.f. for the GIG distributions involved, which in this case will have a depth smaller than $N_{t_1,\ldots,t_m}$.

Now we may re-write the expression for the moments of Y_k as

$$\begin{aligned} E\left(Y_k^h\right) &= \sum_{m=1}^{k} p_{mk}\, E\left[(X^*_{mk})^h\right] \\ &= h! \sum_{m=1}^{k} p_{mk} \sum_{j=1}^{K_m} \theta^*_j \sum_{\substack{\forall_{\oplus} n_l = h \\ l=1:N_{t_1,\ldots,t_m}}} \prod_{l=1}^{N_{t_1,\ldots,t_m}} \frac{\Gamma(r_l+n_l)}{\Gamma(r_l)\, n_l!}\, \lambda_l^{-n_l}\,, \end{aligned}$$

generally with

$$r_l \equiv r_{d_j,k-j+1} \qquad \text{for} \qquad l = 1 + d_j + \sum_{k=1}^{j-1} t_k \qquad (j=1,\ldots,m;\, 0 \le d_j \le t_j)\,.$$

The c.d.f. of Y_k may then, using again the notation in (11) and from (2) and (13), be written as

$$F_{Y_k}(y) = \sum_{m=1}^{k} p_{mk} \sum_{j=1}^{K_m} \theta^*_j\, F\Big(y;\, GIG(r_{0,k},...,r_{t_1,k},r_{0,k-1},...,r_{t_2,k-1},...,r_{0,k-m+1},...,\\ r_{t_m,k-m+1};\, \lambda_{0,k},...,\lambda_{t_1,k},\lambda_{0,k-1},...,\lambda_{t_2,k-1},...,\lambda_{0,k-m+1},...,\lambda_{t_m,k-m+1})\Big),$$

We may stress that once we have the adequate programmed modules to handle the possible repetitions of the rate parameters in the GIG distributions and the modules to handle the correct way the summations the computation of the above c.d.f. is not too complicated. See section 3 ahead.

2.3 A third generalization of the Cox-Markov process

A third and more elaborate generalization is originated if we consider that the staying time in state k is determined by the following stochastic process. In each state k $(1 \le k \le n)$ after a time with

Exponential duration $Exp(\lambda_i)$, there is a trial with two possible outcomes: i) remain in state k, or ii) pass on to state $k-1$, being p_k the probability of this latter event. The number N_k of proofs before passing to the next state will have a Geometric distribution with parameter p_k, that is,

$$P(N_k = N) \;=\; (1-p_k)^{N-1}\, p_k\,, \qquad N = 1, 2, \ldots\,.$$

Being $X_{k,1}, \ldots, X_{k,N_k}$ i.i.d. $Exp(\lambda_k)$, the distribution of the staying time in state k is given by

$$\begin{aligned} P(X_k \le x) \;&=\; \sum_{N=1}^{\infty} P(N_k = N)\, P(X_k \le x | N_k = N) \\ &=\; \sum_{N=1}^{\infty} (1-p_k)^{N-1} p_k\, P\left(\sum_{j=1}^{N} X_{k,j} \le x\right) = \sum_{N=1}^{\infty} (1-p_k)^{N-1} p_k\, P\left(X^*_{Nk} \le x\right) \end{aligned}$$

where

$$X^*_{Nk} \;=\; \sum_{j=1}^{N} X_{k,j} \;\sim\; \Gamma(N, \lambda_k)\,.$$

Thus, the distribution of X_k is an infinite mixture of integer Gamma distributions with weights $(1-p_k)^{N-1} p_k$.

Summarizing, in this case we have,

$$P(Y_k \le y) \;=\; \sum_{m=1}^{k} p_{mk}\, P(X^*_{mk} \le y)$$

where

$$X^*_{mk} \;=\; \sum_{j=1}^{m} X_{k-j+1}$$

where X_{k-j+1} is an infinite mixture of integer Gamma distributions, since

$$P(X_{k-j+1} \le y) \;=\; \sum_{N=1}^{\infty} \theta_{N,k-j+1}\, P(X^*_{N,k-j+1} \le y)$$

where

$$\theta_{N,k-j+1} = (1-p_{k-j+1})^{N-1} p_{k-j+1}\,, \qquad X^*_{N,k-j+1} = \sum_{l=1}^{N} X_{k-j+1,l} \;\sim\; \Gamma(N, \lambda_{k-j+1})\,.$$

Thus, directly from above, X^*_{mk} may be seen as a sum of m infinite mixtures of integer Gamma random variables, the j-th ($j = 1, \ldots, m$) of which with weights $\theta_{N,k-j+1}$ of $\Gamma(N, \lambda_{k-j+1}$ random variables ($N = 1, 2, \ldots$), what would lead us to write the h-th non-central moment of Y_k as

$$\begin{aligned} E\left(Y_k^h\right) \;&=\; \sum_{m=1}^{k} p_{mk}\, E\left[\left(\sum_{j=1}^{m} X_{k-j+1}\right)^h\right] = \sum_{m=1}^{k} p_{mk}\, h! \sum_{\substack{\forall_{\oplus} n_j = h \\ j=1:m}} \prod_{j=1}^{m} \frac{E\left(X^{n_j}_{k-j+1}\right)}{n_j!} \\ &=\; h! \sum_{m=1}^{k} p_{mk} \sum_{\substack{\forall_{\oplus} n_j = h \\ j=1:m}} \prod_{j=1}^{m} \frac{\lambda^{n_j}_{k-j+1}}{n_j!} \sum_{N=1}^{\infty} \theta_{N,k-j+1}\, \frac{\Gamma(N+n_j)}{\Gamma(N)}\,. \end{aligned}$$

However, since a sum of m infinite mixtures of integer Gamma distributions is an infinite mixture of GIG distributions of depth (at most) m, the distribution of X^*_{mk} may be equivalently seen as an infinite mixture with weights

$$p_{i_1,\ldots,i_m} \;=\; \prod_{j=1}^{m} \theta_{i_j,k-j+1}\,, \qquad \text{for } i_j = 1, 2, \ldots$$

being the associated GIG distribution, supposing the rate parameters λ_{k-j+1} all different for $j = 1, \ldots, m$,

$$GIG(\underbrace{i_1, \ldots, i_m}_{m}; \underbrace{\lambda_k, \ldots, \lambda_{k-m+1}}_{m})$$

and thus the distribution of X^*_{mk} given by, in this case,

$$P\left(X^*_{mk} \leq y\right) = \sum_{i_1,\ldots,i_m=1}^{\infty} p_{i_1,\ldots,i_m} F\Big(y; GIG(\underbrace{i_1, \ldots, i_m}_{m}; \underbrace{\lambda_k, \ldots, \lambda_{k-m+1}}_{m})\Big). \qquad (14)$$

From (14) we may get both a second expression for the moments and an expression for the c.d.f. of Y_k. Directly from (14) above we get

$$\begin{aligned} E\left(Y_k^h\right) &= \sum_{m=1}^{k} p_{mk} E\left[\left(X^*_{mk}\right)^h\right] \\ &= h! \sum_{m=1}^{k} p_{mk} \sum_{i_1,\ldots,i_m=1}^{\infty} p_{i_1,\ldots,i_m} \sum_{\substack{\forall_{\oplus} n_j = h \\ j=1:m}} \prod_{j=1}^{m} \frac{\Gamma(i_j + n_j)}{\Gamma(i_j)\, n_j!} \lambda_{k-j+1}^{-n_j} \end{aligned}$$

and

$$F_{Y_k}(y) = \sum_{m=1}^{k} p_{mk} \sum_{i_1,\ldots,i_m=1}^{\infty} p_{i_1,\ldots,i_m} F\Big(y; GIG(\underbrace{i_1, \ldots, i_m}_{m}; \underbrace{\lambda_k, \ldots, \lambda_{k-m+1}}_{m})\Big),$$

supposing, as above, the rate parameters λ_{k-j+1} all different for $j = 1, \ldots, m$. Otherwise, the adequate 'grouping' has to be done before defining the GIG distributions.

2.4 A fourth generalization of the Cox-Markov process

A fourth generalization of the Cox-Markov process may be thought of, if we consider a situation in all similar to the one in the previous subsection, now with the distribution of each X_{kj} being a $\Gamma(r_{kj}, \lambda_{kj})$ distribution instead of being an Exponential distribution, being the waiting time for the r_{kj}-th event of a Poisson process with rate λ_{kj}.

Now we have

$$X^*_{N,k-j+1} = \sum_{l=1}^{N} X_{k-j+1,l} \sim GIG(r_{k-j+1,1}, \ldots, r_{k-j+1,N}; \lambda_{k-j+1,1}, \ldots, \lambda_{k-j+1,N}),$$

so that now the distribution of X^*_{mk} is an infinite mixture of GIG distributions with the GIG distribution corresponding to the weight $p_{i_1,\ldots,i_m}$ having depth $\sum_{j=1}^{m} i_j$. The c.d.f. of Y_k is thus

$$\begin{aligned} F_{Y_k}(y) = \sum_{m=1}^{k} p_{mk} \sum_{i_1,\ldots,i_m=1}^{\infty} p_{i_1,\ldots,i_m} \\ F\Big(y; GIG(\underbrace{r_{k1}, ..., r_{ki_1}, r_{k-1,1}, ..., r_{k-1,i_2}, ..., r_{k-m+1,1}, ..., r_{k-m+1,i_m}}_{i_1+i_2+\ldots+i_m}; \\ \underbrace{\lambda_{k1}, ..., \lambda_{ki_1}, \lambda_{k-1,1}, ..., \lambda_{k-1,i_2}, ..., \lambda_{k-m+1,1}, ..., \lambda_{k-m+1,i_m}}_{i_1+i_2+\ldots+i_m})\Big), \end{aligned}$$

with

$$E\left(Y_k^h\right) = h! \sum_{m=1}^{k} p_{mk} \sum_{i_1,\ldots,i_m=1}^{\infty} p_{i_1,\ldots,i_m} \sum_{\substack{\oplus \mathbb{N}_0 \ni n_{jl} \\ j=1:m \\ l=1:i_j}} \prod_{j=1}^{m} \prod_{l=1}^{i_j} \frac{\Gamma(r_{k-j+1,l} + n_{jl})}{\Gamma(r_{k-j+1,l})\, n_{jl}!} \lambda_{k-j+1,l}^{-n_{jl}}.$$

3 Mathematica modules for the computation of the distribution of Y_k

We provide modules programmed using version 5.1 of the Mathematica software (from Wolfram Research), to compute or obtain the expression for either the p.d.f. or the c.d.f. of the distribution of Y_k. The base module is the module GIG, in Figure 3, used to compute either the p.d.f. or the c.d.f. of a GIG distribution. This module uses the module Makec, in Figure 2, to compute the parameters c_{jk} in the GIG distribution. We should note that the computation of these parameters is done using the integer computation capabilities of Mathematica by previously rationalizing all the rate parameters λ_j and receiving only for integer shape parameters r_j. This way the computation problems that might arise (Koole, 2004) are overcome. This module uses as arguments a first list with the shape parameters, a second list with the rate parameters and a third argument with the common length of these lists. Although this third argument could be avoided to be passed on, it seemed more functional to do it this way.

```
Makec[r_, l_, p_] := Module[{c},
  c = Table[Table[1, {j, 1, Max[r]}], {i,1,p}];
  Table[c = ReplacePart[c, (Product[(l[[j]]-l[[i]])^(-r[[j]]), {j,1,i-1}]*
                Product[(l[[j]]-l[[i]])^(-r[[j]]), {j,i+1,p}])/(r[[i]]-1)!, {i,r[[i]]}], {i,1,p}];
  Table[Table[c=ReplacePart[c,Sum[((r[[i]]-k+j-1)!*(Sum[r[[h]]/(l[[i]]-l[[h]])^j,{h,1,i-1}]
                +Sum[r[[h]]/(l[[i]]-l[[h]])^j, {h,i+1,p}])*c[[i]][[r[[i]]-(k-j)]]])/(r[[i]]-k-1)!,
                                 {j,1,k}]/k, {i,r[[i]]-k}], {k, 1, r[[i]]-1}], {i,1,p}];
  c ]
```

Figure 2 – Mathematica module for the computation of the parameters c_{jk} in the GIG distribution.

The module GIG in Figure 3 uses four arguments. The first is an optional argument, with default value of 1, that indicates whether to compute the p.d.f. or the c.d.f.. If it is given the value 0, the p.d.f. is computed. The two following arguments are the same as the two first arguments of the module Makec, that is, two lists with the shape and rate parameters. The fourth argument is the running variable for the p.d.f. or c.d.f. or the value at which these are to be computed. If we give an explicit numerical value to this fourth argument we obtain the computed value of either the GIG p.d.f. or c.d.f. at that point and if we give this fourth argument as a letter we will obtain the expression for the p.d.f. or c.d.f. with that letter representing the running value.

```
GIG[pdfcdf_:1,r_, li_, z_] := Module[{p, l, c}, p=Length[r];
  If[r∈Integers && And@@Positive[r] && And@@Positive[li] && p==Length[li],
  l=Rationalize[li, 0]; c=Makec[r, l, p];
  If[pdfcdf==0, P[w_]:=Table[Sum[c[[j]][[k]]*w^(k - 1), {k, 1, r[[j]]}], {j, 1, p}];
                Product[l[[j]]^r[[j]], j, 1, p]*Sum[P[z][[j]]*Exp[-l[[j]]*z], {j, 1, p}],
        P[w_]:=Table[Sum[c[[j]][[k]]*(k-1)!*Sum[w^i/(i!*l[[j]]^(k-i)), {i,0,k-1}],
                                                              {k,1,r[[j]]}],{j,1,p}];
        1 - Product[l[[j]]^r[[j]], {j, 1, p}]*Sum[P[z][[j]]*Exp[-l[[j]]*z], {j, 1, p}] ] ]]
```

Figure 3 – Mathematica module for both the p.d.f. and the c.d.f. of the GIG distribution.

The module Group in Figure 4 receives as arguments a vector of shape parameters and a second vector of rate parameters which are supposed to have the same length and is used to group together all equal rate parameters and add the corresponding shape parameters. The output of this module is thus the set of a shape and a rate parameter vector after grouping. The module does not require the shape parameters to be integer and does not check for equal length of the two argument vectors since in the context of its use such is assured a priori.

We may illustrate the use of the module Group with the following call

```
Group[{1.5, 2, 2.5, 3, 3.5, 4, 4.5}, {1.23, 4.6, 3.4, 4.6, 1.23, 1.23, 3.4}]
```

that produces the output

```
{{9., 7., 5}, {1.23, 3.4, 4.6}} .
```

The module Weights also in Figure 4 is used to compute the weights p_{mk} in (2) for the mixture distribution of Y_k. It receives as argument a vector with the values of $\alpha_k, \ldots, \alpha_2$ and it adds by itself the value of $\alpha_1 = 1$ at the end of the list.

The module Params, yet in Figure 4, which is actually just a Mathematica function, taking as argument the set of rate parameters $\lambda_k, \ldots, \lambda_1$, computes the sets of shape and rate parameters for the k r.v.'s $X^*_{m,k}$ in (2), passing them through the Group module in order to group possible equal rate parameters. The parameters for X^*_{1k} are not passed through the Group module since this is one only Exponential r.v..

Finally the module Distwt in Figure 5 is used to compute either the p.d.f. or the c.d.f. of Y_k in section 1. This module receives four arguments, the first of which is similar to the first argument of the module GIG being used to choose between the p.d.f. and the c.d.f., the second one is the vector of values of $\alpha_k, \ldots, \alpha_2$, the third is the vector of rate parameters $\lambda_k, \ldots, \lambda_1$ and the fourth is the running variable for either the p.d.f. or the c.d.f., which works as the similar argument described for the module GIG.

The module checks if the values of α_j $(j = 2, \ldots, k)$ are between 0 and 1 and if the lengths of the vectors with the probabilities α_j and the rate parameters λ_j do match.

The module Momwt in Figure 6 is used to compute directly the moments of Y_k, using (4). This module has as arguments the vector of probabilities $\alpha_k, \ldots, \alpha_2$, the vector of rate parameters $\lambda_k, \ldots, \lambda_1$ and the order of the moment. This module needs the Mathematica package Combinatorica, in order to be able to use the Compositions module therein to compute the integer partitions of h.

```
Group[r_, l_] := Module[{ls, rs, ds, ln, rn, flag0, radd},
  ls=Sort[l]; rs=r[[Ordering[l]]]; ds=Drop[ls,-1]-Drop[ls,1]; ln=rn={}; flag0=radd=0;
  Do[If[ds[[i]]==0,
    If[flag0==0, {flag0=1; radd=radd+rs[[i]]+rs[[i+1]]; ln=Append[ln,ls[[i]]]},
                 radd=radd+rs[[i+1]]],
    If[flag0==1, {flag0=0; rn=Append[rn,radd]; radd=0},
                 {ln=Append[ln,ls[[i]]]; rn=Append[rn,rs[[i]]]}]],   {i,1,Length[r]-1}];
  If[ds[[Length[r]-1]]==0, rn=Append[rn,radd],
                    {rn=Append[rn,rs[[Length[r]]]]; ln=Append[ln,ls[[Length[r]]]]}];
  {rn, ln} ]
```

```
Weights[alpha_] := Module[{k, nalpha},
  k = 1+Length[alpha]; nalpha = Prepend[Reverse[alpha],1];
  Table[nalpha[[k-m+1]]*Product[(1-nalpha[[k-j+1]]), {j,1,m-1}], {m,1,k}] ]
```

```
Params[lambda_]:=Prepend[Table[Group[Table[1,{i,1,j}],Table[lambda[[i]],{i,1,j}]],
                 {j,2,Length[lambda]}],{{1},{lambda[[1]]}}]
```

Figure 4 – Mathematica modules for the grouping of equal rate parameters and the computation of the weights p_{mk} in (2) and Mathematica function for the computation of the rate parameters for the r.v.'s X^*_{mk} in (2).

```
Distwt[pdfcdf_:1,alpha_,lambda_,z_]:=Module[{k,w,par}, k=Length[lambda];
  If[And @@ Positive[alpha] && And @@ Negative[alpha-1],
    If[k==1+Length[alpha] && And @@ Positive[lambda],
      w=Weights[alpha]; par=Params[lambda];
      Sum[w[[j]]*GIG[pdfcdf,par[[j]][[1]],par[[j]][[2]],z],{j,1,k}]   ]]]
```

Figure 5 – Mathematica module for the computation of both the p.d.f. and the c.d.f. of the r.v. Y_k in section 1.

```
Momwt[alpha_,lambda_,h_]:=Module[{k,w},
  Needs["DiscreteMath`Combinatorica`"];
  k=Length[lambda]; w=Weights[alpha];
  h!*Sum[ w[[j]]*Sum[ Product[lambda[[i]]^(-Compositions[h, j][[m]][[i]]), {i,1,j}],
                           {m,1,Length[Compositions[h,j]]}],{j,1,k}] ]
```

Figure 6 – Mathematica module for the computation of the moments in (4).

An example of the use of both modules Distwt and Momwt may be the following, in which we compute the second moment of a waiting time Y_3 with probabilities $\alpha_3 = .2$, $\alpha_2 = .5$ and rate parameters $\lambda_1 = 1.23, \lambda_2 = 3.4, \lambda_1 = 2.3$,

```
NIntegrate[x^2*Distwt[0, {.2, .5}, {1.23, 3.4, 2.3}, x], {x, 0, Infinity}]
```

and

```
Momwt[{.2, .5}, {1.23, 3.4, 2.3}, 2]
```

either call with output, 2.37928.

The modules provided allow only for the computation of the distribution of Y_k for the usual case in section 1, but based on these modules it is not hard to build other modules to compute the distributions of Y_k for the generalizations in section 2.

4 Conclusions

The GIG distribution is a much useful tool in representing and studying the exact distribution of the waiting time till absorption in state 0 both for the usual Cox distribution and its proposed generalizations.

The use of the GIG distribution allows us to easily generalize the Cox-Markov process to several other situations, including situations where the waiting times in each state are generated by non-homogeneous Poisson processes and still be able to quite easily devise the exact distribution (and moments) of the waiting time till absorption in state 0. The usual Cox distribution may thus be seen as a particular case of the mixture of GIG distributions presented as the distribution of the waiting times till absorption in state 0.

This way we have easily at hand the exact distribution for waiting times in more general models for queues where the time in each state may be generated either by homogeneous or non-homogeneous Poisson processes as it may be the case when the waiting time in each state depends on the occurrence of multiple actions as for example not only the time each element ahead in the queue may take to leave the queue once it reaches its end but also the rate at which the elements in each state abandon the queue. With the generalizations proposed for the Cox-Markov model we will be able to model more precisely such situations.

Appendix A

A number written in the multiple basis $(N_1+1),(N_2+1),\ldots,(N_m+1)$ is a number with m digits whose j-th digit $(j=1,\ldots,m)$ d_j is such that $0 \le d_j \le N_j$, $(j=1,\ldots,m)$. There are a total of $\prod_{j=1}^{m}(N_j+1)$ different numbers in this basis.

The natural order of these numbers is the one imposed by the evolving of the values once the multiple basis system is taken into account. That is, for example, for $m=3$ with $N_1=2$, $N_2=4$ and $N_3=3$, the number 200 is the one that follows immediately the number 143, the number 100 is the one that follows immediately the number 43, 40 is the number that follows 33, and 10 the one that follows 3. The highest number in this multiple basis system would be 243 and there are 60 different numbers in this multiple basis system, their ordered sequence being

$$000,\ 001,\ 002,\ 003,\ 010,\ 011,\ 012,\ 013,\ 020,\ \ldots,\ 033,\ 040,\ \ldots,\ 043,\ 100,\ \ldots,\ 143,\ 200,\ \ldots,\ 243.$$

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