A class of non-local linear operators for vorticity waves

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Abstract

A steady longitudinal current in the nearshore can, in some conditions, support oscillations known as vorticity waves or shear waves. We consider in this paper a family of nonlinear evolution equations derived by Shrira and Voronovitch to describe the dynamics of vorticity waves near the coastal line and make the study of the dispersion and smoothing properties of the associated nonlocal free problems. More precisely, after establishing long and short time uniform estimates for a certain class of oscillatory integrals, we derive "$L^p - L^q$" and Strichartz-type estimates for the solutions of the linearized equations.

Key words : Vorticity waves, Oscillatory Integrals, Strichartz Estimates.

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1 Introduction

In [5], by a multi-scale analysis, Shrira and Voronovich derive a dispersive nonlinear evolution equation ruling the dynamics of vorticity waves near the coastal zone, in the presence of a mean steady current. We begin this introduction by shortly describing their method. In a coordinate system with axes $(0x)$ and $(0y)$ pointing respectively offshore and alongshore, let us denote by $V(x)$ the mean steady current. The total velocity field $\mathbf{u}_*$ is then given by

$$\mathbf{u}_*(x, y, t) = \{u(x, y, t), V(x) + v(x, y, t)\},$$

where $\{u, v\}$ represents the perturbed velocity field.

The standard shallow-water equations read

$$\begin{cases} u_t + Vu_y = -g\zeta_x - (uu_x + vu_y) \\ v_t + Vu_y + uV_x = -g\zeta_y - (uv_x + vv_y) \\ \zeta_t + V\zeta_y + [(\zeta + h)u]_x + [(\zeta + h)v]_y = 0, \end{cases} \tag{1}$$

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where ζ is the free-surface elevation, g the gravity acceleration, and the depth $h(x)$ is assumed to be uniform along the shore.
Rescaling (1) leads to the system (written in non-dimensional units)

$$\left\{\begin{array}{ll} u_t + Vu_y = -\epsilon^2\zeta_x - (uu_x + vu_y) & \text{(a)} \\ v_t + Vu_y + uV_x = -\zeta_y - (uv_x + vv_y) & \text{(b)} \\ (hu)_x + (hv)_y = 0, & \text{(c)} \end{array}\right. \tag{2}$$

$\epsilon = \dfrac{d}{L}$ denoting the ratio of the cross and alonshore spatial scales, respectively d and L.

Note that equation (2c) allows one to define a stream function ψ, such that

$$\psi_x = hv \text{ and } \psi_y = -hu.$$

Then, from (2-a,2-b), one easily gets the nonlinear vorticity equation

$$[\partial_t + V\partial_y]((\frac{\psi_x}{h})_x + \epsilon^2\frac{\psi_{yy}}{h}) - (\frac{V'}{h})' = \partial_x(-\frac{\psi_x}{h}\frac{\psi_{xy}}{h} + \frac{\psi_y}{h}(\frac{\psi_x}{h})_x) + \epsilon^2\partial_y(\frac{\psi_y}{h}(\frac{\psi_y}{h})_x - \frac{\psi_x}{h}\frac{\psi_{yy}}{h}). \tag{3}$$

Finally, for a coastline profile $x = x_o(y)$, the following boundary conditions
are considered :

$$\left\{\begin{array}{ll} \psi_y + (hV + \psi_x)\partial_y x_o = 0 \text{ at } x = x_o(y) & \text{(no mass flux through the coastline)} \\ \lim\limits_{x\to\infty} \psi_y = 0. & \text{(no mass flux at infinity)} \end{array}\right. \tag{4}$$

In order to derive a weakly dispersive nonlinear evolution equation ruling the dynamics of vorticity waves, Shrira and Voronovitch make use of a classical multi-scale analysis. Considering motions with alongshore scale L much larger than the mean-current cross-shore scale d, (i.e. $\epsilon << 1$), two different scales appear naturally in the evolution problem. It is then straightforward to introduce the new “fast” variables

$$\xi = x \ , \ Y = y - ct$$

corresponding to a frame moving in the (Oy) direction, with the celerity c of the vorticity wave (to be determined later), as well as the “slow” variables

$$X = \epsilon x \text{ and } T = \epsilon t.$$

The depht profile is then assumed to depend on both fast and slow variables :

$$h(\xi, X) = H(\xi)D(X).$$

In terms of the new variables, and considering a straight coastline, the boundary conditions (4) read

$$\lim_{X\to\infty}\psi_Y(\xi,X,Y,T)=0\ ,\ \psi_Y(0,X,Y,T)=0\ ,\ \text{ and }\lim_{\xi\to\infty}\psi_{Y\xi}(\xi,0,Y,T)=0. \tag{5}$$

The asymptotic derivation now follows the usual proceeding :
Inserting the ansatz

$$\psi(\xi,X,Y,T)=\sum_{n=1}^{+\infty}\epsilon^n\psi_n(\xi,X,Y,T)$$

into equations (3),(5) and equating the coefficients of ϵ^n to zero leads to partial equations and boundary conditions binding the functions ψ_n. For instance, at first order $n=1$, from (3), one gets an integrable differential equation in ψ_1, with a unique solution compatible with the boundary conditions.The function ψ_1 has the form

$$\psi_1(\xi,X,Y,T)=(V(\xi)-c)(\rho(X,Y)*_Y A(T,Y)),$$

where the functions $A(Y,T)$ and $\rho(X,Y)$ have been introduced to separate the temporal and slow dependences of ψ_1.
Moreover, from (5), one gets the condition $c=0$ and the constraint

$$\lim_{X\to\infty}\rho(X,Y)=0.$$

Considering the second order $n=2$, Schrira and Voronovitch get the nonlinear evolution equation governing the dynamics of weakly dispersive vorticity waves (physical constants have been put equal to the unity)

$$A_T-L[A]+AA_Y=0, \tag{6}$$

where the nonlocal linear operator L is given by its Fourier symbol

$$\widehat{L[A]}(\eta)=+i\phi(\eta)\hat{A}(\eta)$$

$$\phi(\eta)=\eta\frac{\partial_X\rho(X,\eta)}{\rho(X,\eta)}|_{X=0}$$

and $\hat{\rho}$, the Fourier transform of ρ in y, is the solution to the boundary problem

$$\begin{cases}\partial_X(\dfrac{\partial_X\rho(X,\eta)}{D(X)})-\eta^2\dfrac{\rho(X,\eta)}{D(X)}=0\\[2ex] \lim_{X\to\infty}\rho(X,\eta)=0.\end{cases} \tag{7}$$

By choosing some specific profiles for the depht function D, the boundary problem (7) can be solved explicitely.
Namely, for an exponential profile

$$D(X)=e^{qX}\ ,\ q>0,$$

one gets

$$\phi(\eta) = \phi_S(\eta) := \eta \frac{\partial_X \phi(X,\eta)}{\phi(X,\eta)}|_{X=0} = \eta \frac{1}{2}(q - \sqrt{q^2 + 4\eta^2}), \tag{8}$$

which corresponds to the symbol of the Smith operator.

Also, considering a power-law profile

$$D(X) = (1+X)^{2m} \ , \ m > 0,$$

one gets

$$\phi(\eta) = \phi_m(\eta) := \eta \frac{\partial_X \phi(X,\eta)}{\phi(X,\eta)}|_{X=0} = i\eta|\eta|\theta_m(\eta) \tag{9}$$

where θ_m is given in terms of the modified second-order Bessel funcions (McDonald functions) K_μ, by

$$\theta_m(\eta) = \frac{K_{m-\frac{1}{2}}(|\eta|)}{K_{m+\frac{1}{2}}(|\eta|)}.$$

The authors noted that in the limit case $m = \frac{1}{2}$, one gets in the small-η limit,

$$\phi_m(\eta) \sim \phi_L(\eta) := -\eta^3 \log(\eta), \tag{10}$$

which is the well-known Leibovitch operator.

Our aim here is to make the study of the dispersion and smoothing properties for the free evolution problems associated to the non-local operators with symbols ϕ_S, ϕ_m ($m > 0$) and ϕ_L, and consequently fully justify that equation (6) is indeed a dispersive model. For instance, in the case of an exponential profile $D(X) = e^X$, as remarked in [5], we obtain in the long (respectively short) wave limit

$$\begin{cases} \phi_S(\eta) \underset{\eta\to\infty}{\simeq} -|\eta|\eta \\ \phi_S(\eta) \underset{\eta\to 0}{\simeq} -\eta^3 \end{cases} \tag{11}$$

wich correponds to the well-known BO(respectively KdV) equation. Yet, in order for $\partial_t - \phi_S(D)$ to be a dispersive operator, one must garantee that for the intermediate values of η some misbehaviour of ϕ_S and its derivatives will not give rise to a stationnary phase phenomena and consequently provoke a “lack of dispersion”.
The rest of this paper is organized as follows :
In the second section, after establishing sharp uniform in time estimates for oscillatory integrals of the type

$$I_\alpha(y,t) = \int |\eta|^\alpha e^{i(t\phi(\eta)+y\eta)} dt,$$

we derive “$L^p - L^q$” estimates for the above-mentionned operators.
In the third section, we establish the associated Strichartz-type estimates.
Finally, in the Appendix, we prove a few useful technical lemmas concerning the function

ϕ_m.

We end this introduction with a few notations : for all $p \in [1;+\infty]$ and $s \in \mathbb{R}$, we introduce the usual $L^p(\mathbb{R})$ and Sobolev $H^s(\mathbb{R})$ spaces given by

$$L^p(\mathbb{R}) = \{f \ \ / \ \|f\|_p = (\int |f|^p)^{\frac{1}{p}} < +\infty\}$$

and

$$H^s(\mathbb{R}) = \{f \ /\|f\|_{H^s} = \{f \ \ / \ \|\hat{f}(\eta)(1+|\eta|^2)^{\frac{s}{2}}\|_2 < +\infty\}.$$

Also, for $p \geq 1$, $q \geq 1$ and $T > 0$ we define the mixed-space

$$L^q(0,T;L^p) = \{f \ \ / \ \|f\|_{L^q(0,T;L^p)} = (\int_0^T (\|f(.,\tau)\|_p)^q d\tau)^{\frac{1}{q}} < +\infty\}.$$

Finally, we introduce the Riesz potential $|D|^s$, $s \in \mathbb{R}$, given by

$$\widehat{|D|^s f}(\eta) = |\eta|^s \hat{f}(s)$$

for all tempered distribution $f \in \mathcal{D}'(\mathbb{R})$,

2 Dispersion properties

In this section, we will be concerned with the free evolution problem

$$\begin{cases} A_t - L[A] = 0 \\ A(0,y) = f_o(y) \end{cases} \tag{12}$$

for some initial data f_o to be chosen in an adequate space, and where the operator L is given by its Fourier symbol $i\phi(\eta)$.
Here ϕ denotes generically one of the functions defined in (8), (9) and (10).

2.1 Oscillatory integrals

For $t > 0$, we introduce the kernel

$$K(y,t) = \int_{\mathbb{R}} e^{i(t\phi(\eta)+y\eta)} d\eta, \tag{13}$$

defined as the Fourier cotransform of the tempered distribution $e^{it\phi(.)} \in L^\infty(\mathbb{R})$, or, which is the same, as an oscillatory integral.

Solutions for the I.V.P. (12) are generated by the family of operators $W(t) = e^{it\phi(D)}$ given by

$$W(t)[f_o](y) := \frac{1}{2\pi}[K(.,t) * f_o(.)](y).$$

It is clear that $\{W(t)\}_{t\in\mathbb{R}}$ is a unitary one-parameter group acting in $H^s(\mathbb{R})$, for all $s \in \mathbb{R}$. In order to derive "$L^p - L^q$" and Strichartz-type estimates for this family of non-local operators, one must first get uniform in time estimates for $K(y,t)$.

Let us consider a real odd function ϕ satisfying the following conditions :

$$\begin{cases} \phi \in C^2(\mathbb{R}) \cap C^3(\mathbb{R}/\{0\}) \ , \ \liminf\limits_{\eta\to 0} |\phi'''(\eta)| > 0. & (a) \\ \\ \phi'' \text{ vanishes at most in a unique } \tilde{\eta} \neq 0. \text{ In that case, } \phi'''(\tilde{\eta}) \neq 0 & (b) \\ \\ \text{For some } A > 0,\ \eta_o > 0 \text{ and } p \in [0;1], & (c) \\ \qquad\qquad \forall \eta > \eta_o \ , \ |\phi''(\eta)| \geq A|\eta|^p \end{cases} \tag{14}$$

Note that conditions (14-a) and (14-b) hold for ϕ_L, ϕ_S and ϕ_m for all $m > 0$. Furthermore, (14-c) holds for ϕ_S and ϕ_m ($m > 0$) with $p = 0$ and for ϕ_L with $p = 1$ (see the Appendix).

Following the ideas in [1], which covers the case where ϕ is a polynomial ($\phi(D)$ is a local operator), we prove :

Theorem 2.1 *Long time estimate.*
Let $T > 0$. Let ϕ a real odd function satisfying conditions (14-a), (14-b) and (14-c) for some $p \in [0;1]$.
For $\alpha \in [0;\frac{p}{2}]$, we set

$$I_\alpha(y,t) = \int_{\mathbb{R}} |\eta|^\alpha e^{it(\phi(\eta)+\eta y)}\,d\eta.$$

Then

$$\forall t \in [T;+\infty[\ , \ \sup_{y\in\mathbb{R}} |I_\alpha(y,t)| \leq \frac{C}{t^{\frac{1}{3}}} \tag{15}$$

holds for all $\alpha \in [0;\frac{p}{2}]$, for some $C > 0$ depending exclusively on T and α.

The main ingredient for the proof of Theorem 2.1 is the following lemma :

Lemma 2.2 *(Van Corput)*
Let $k \geq 1$. If $\psi \in C^k([a;b])$ and $\psi^{(k)}$ does not vanish over $[a;b]$, then for all $\alpha \geq 0$,

$$\left| \int_a^b |\eta|^\alpha e^{i\psi(\eta)}\,d\eta \right| \leq \frac{C(|a|^\alpha + |b|^\alpha)}{\min\limits_{\eta\in[a;b]} \left(|\psi^{(k)}(\eta)|\right)^{\frac{1}{k}}}$$

where $C > 0$ depends only on α and k.

For a proof of this lemma, see for instance [7].

Proof of Theorem 2.1 :
Let ϕ a real odd function satisfying conditions (14-a), (14-b) and (14-c) for some $p \in [0;1]$.
Let $t \geq T > 0$, $y \in \mathbb{R}$ and $\alpha \in [0; \frac{p}{2}]$.
In what follows, we will denote by $C > 0$ various constants depending only on α and T.
Let $\delta > 0$ such that

$$m_1 = Inf\{|\phi'''(\eta)| \ / \ \eta \in]0;\delta[\} > 0. \tag{16}$$

Then, for all $\epsilon \in]0;\delta[$, the Van Corput lemma reads :

$$|\int_{\epsilon}^{\delta} \eta^{\alpha} e^{it(\phi(\eta)+y\eta)} d\eta| \leq C\frac{\epsilon^{\alpha}+\delta^{\alpha}}{(tm_1)^{\frac{1}{3}}}.$$

Plainly,

$$|\int_{0}^{\delta} \eta^{\alpha} e^{it(\phi(\eta)+y\eta)} d\eta| \leq C\frac{1+\delta^{\alpha}}{t^{\frac{1}{3}}}. \tag{17}$$

Let $\eta_o > \delta$ such that for $\eta \geq \eta_o$, $|\phi''(\eta)| \geq A|\eta|^p$.
If ϕ'' does not vanish on $[\delta;\eta_o]$, setting $m_2 = Inf\{|\phi''(\eta)| \ , \ \eta \in [\delta;\eta_o]\} > 0$, one obtains by the Van Corput lemma

$$|\int_{\delta}^{\eta_o} \eta^{\alpha} e^{it(\phi(\eta)+y\eta)} d\eta| \leq C\frac{\delta^{\alpha}+\eta_o^{\alpha}}{(tm_2)^{\frac{1}{2}}} \leq C\frac{\delta^{\alpha}+\eta_o^{\alpha}}{t^{\frac{1}{2}}} \leq C\frac{\eta_o^{\alpha}}{t^{\frac{1}{3}}}. \tag{18}$$

If $\phi''(\tilde{\eta}) = 0$ for some $\tilde{\eta} \in [\delta;\eta_o]$, let $\epsilon > 0$ such that for $\eta \in [\tilde{\eta}-\epsilon;\tilde{\eta}+\epsilon]$, $\phi'''(\eta) \neq 0$.
Putting $m_3 = Inf\{|\phi'''(\eta)| \ , \ \eta \in [\tilde{\eta}-\epsilon;\tilde{\eta}+\epsilon]\} > 0$ and
$m_4 = Inf\{|\phi''(\eta)| \ , \ \eta \in [\delta;\eta_o]\backslash[\tilde{\eta}-\epsilon;\tilde{\eta}+\epsilon]\} > 0$

$$\begin{aligned} |\int_{\delta}^{\eta_o} \eta^{\alpha} e^{it(\phi(\eta)+y\eta)} d\eta| &\leq |\int_{[\delta;\eta_o]\backslash[\tilde{\eta}-\epsilon;\tilde{\eta}+\epsilon]} |\eta|^{\alpha} e^{it(\phi(\eta)+y\eta)} d\eta| + |\int_{[\tilde{\eta}-\epsilon;\tilde{\eta}+\epsilon]} |\eta|^{\alpha} e^{it(\phi(\eta)+y\eta)} d\eta| \\ &\leq C\frac{4|\eta_o|^{\alpha}}{(m_4 t)^{\frac{1}{2}}} + \frac{2|\eta_o|^{\alpha}}{(m_3 t)^{\frac{1}{3}}} \leq C\frac{\eta_o^{\alpha}}{t^{\frac{1}{3}}}. \end{aligned}$$

We now consider the integral over $]\eta_o;+\infty[$. In this set, we assume for example that $\phi'' > 0$, namely

$$\forall \eta > \eta_o \ , \ \phi''(\eta) \geq A|\eta|^p \tag{19}$$

holds. Our main concern here are the critical points η such that

$$\frac{\partial}{\partial \eta}(\phi(\eta)+y\eta) = \phi'(\eta)+y = 0, \tag{20}$$

since they may give rise to the stationnary phase phenomena.
In view of (19), there exists at most one solution $\eta_y \in]\eta_o;+\infty[$ to equation (20), since ϕ' is strictly non-decreasing. Let $B =]\eta_o;+\infty[\cap[\frac{1}{2}\eta_y;2\eta_y]$. By the Van Corput Lemma,

$$|\int_{B} \eta^{\alpha} e^{it(\phi(\eta)+y\eta)} d\eta| \leq C\frac{|\eta_y|^{\alpha}}{\min\limits_{\eta \in B}(|t\phi''(\eta))^{\frac{1}{2}}} \leq C\frac{{\eta_y}^{\alpha}}{A(\frac{1}{2}\eta_y)^{\frac{p}{2}} t^{\frac{1}{2}}} \leq \frac{C}{t^{\frac{1}{2}}}\eta_y^{\alpha-\frac{p}{2}} \leq \frac{C}{t^{\frac{1}{2}}}\eta_o^{\alpha-\frac{p}{2}}. \tag{21}$$

Let $B' = [\eta_o; +\infty[- B$.
Note that, for $\eta \in B'$,

$$|\phi'(\eta) + y| = |\phi'(\eta) - \phi'(\eta_o)| = |\int_{\eta_o}^{\eta} \phi''(\eta) d\eta| \geq \frac{A}{p+1}(\eta^{p+1} - \eta_y^{p+1}) \geq C\eta^{p+1}.$$

Furthermore,

$$\begin{aligned}
|\int_{B'} \eta^\alpha e^{it(\phi(\eta)+y\eta)} d\eta| &= |\frac{1}{it} \int_{B'} \eta^\alpha \frac{d}{d\eta}(e^{it(\phi(\eta)+y\eta)}) \frac{d\eta}{\phi'(\eta) + y}| \\
&\leq \frac{C}{t} |[\frac{\eta^\alpha}{|\phi'(\eta) + y|}]_{\partial B'} + \int_{B'} \frac{d}{d\eta}(\frac{\eta^\alpha}{\phi'(\eta) + y}) e^{it(\phi(\eta)+y\eta)} d\eta| \\
&\leq \frac{C}{t} (\sup_{\eta \in B'} \{\frac{\eta^\alpha}{|\phi'(\eta) - \phi'(\eta_o)|}\} + \int_{B'} \frac{\alpha \eta^{\alpha-1}}{|\phi'(\eta) - \phi'(\eta_o)|} + \frac{\eta^\alpha \phi''(\eta)}{(\phi'(\eta) - \phi'(\eta_o))^2} d\eta) \\
&\leq \frac{C}{t} (\sup_{\eta \in B'} \{\frac{\eta^\alpha}{|\phi'(\eta) - \phi'(\eta_y)|}\} + \int_{B'} \alpha \eta^{\alpha-p-2} d\eta + \int_{B'} \frac{\eta^\alpha \phi''(\eta)}{(\phi'(\eta) - \phi'(\eta_y))^2} d\eta).
\end{aligned}$$

Plainly,

$$\sup_{\eta \in B'} \{\frac{\eta^\alpha}{|\phi'(\eta) - \phi'(\eta_y)|}\} \leq C \sup_{\eta \in B'} \{\eta^{\alpha-p-1}\} \leq C\eta_o^{-1} \quad \text{and} \quad \int_{B'} \alpha \eta^{\alpha-p-2} d\eta \leq C\eta_o^{-1}.$$

Moreover,

$$\begin{aligned}
\int_{B'} \frac{\eta^\alpha \phi''(\eta)}{(\phi'(\eta) - \phi'(\eta_y))^2} d\eta &= -\int_{B'} \eta^\alpha \frac{d}{d\eta}(\frac{1}{\phi'(\eta) - \phi'(\eta_y)}) d\eta \\
&= -[\frac{\eta^\alpha}{\phi'(\eta) - \phi'(\eta_y)}]_{\partial B'} + \int_{B'} \frac{\alpha \eta^{\alpha-1}}{\phi'(\eta) - \phi'(\eta_y)} d\eta \\
&\leq \sup_{\eta \in B'} \{\frac{\eta^\alpha}{|\phi'(\eta) - \phi'(\eta_y)|}\} + \int_{B'} \alpha \eta^{\alpha-p-2} d\eta \\
&\leq C\eta_o^{-1}
\end{aligned}$$

Finally,

$$|\int_{B'} \eta^\alpha e^{it(\phi(\eta)+y\eta)} d\eta| \leq \frac{C}{t}. \tag{22}$$

Since $t \geq T > 0$ and ϕ is an odd function, putting together (17), (18), (21) and (22), we obtain the announced uniform estimate (15). ■

In order to establish short-time estimates for $I_\alpha(y,t)$, we prove the following lemma :

Lemma 2.3 *Let $T > 0$.*
Let ϕ a real odd function satisfying conditions (14-a), (14-b) and (14-c) for some $p \in [0;1]$.
For $\lambda \geq 1$, we put

$$\phi_\lambda(\xi) = \frac{1}{\lambda^2}\phi(\lambda\eta) \quad \textit{and } I_\alpha(y,t,\lambda) = \int_{-\infty}^{+\infty} |\eta|^\alpha e^{it(\phi_\lambda(\eta)+y\eta)} d\eta.$$

Then

$$\forall t \in]0;T] \ , \ \sup_{\lambda \geq 1} \sup_{\eta \in \mathbb{R}} |I_\alpha(y,t,\lambda)| \leq \frac{C}{t^{\frac{1}{3}}} \tag{23}$$

holds for all $\alpha \in [0; \frac{p}{2}]$*, for some* $C > 0$ *depending exclusively on* T *and* α*.*

Proof of Lemma 2.3:
We begin by noticing that conditions (14-a), (14-b) and (14-c) hold for ϕ_λ.
The choice of $\eta_o > 0$ is clearly independent of $\lambda \geq 1$:

For $\eta \geq \eta_o$, $\phi''_\lambda(\eta) = \phi''(\lambda\eta) \geq A\eta^p$ since $\lambda\eta \geq \eta_o$.
We remark that all the estimates made in the proof of Theorem 2.1 can be done independently of λ.
Indeed, for $\delta > 0$ defined by (16),

$$|\int_0^{\frac{\delta}{\lambda}} \eta^\alpha e^{it(\phi_\lambda(\eta)+y\eta)} d\eta| \leq C \frac{1 + (\frac{\delta}{\lambda})^\alpha}{(t\lambda)^{\frac{1}{3}}} \leq \frac{C}{t^{\frac{1}{3}}},$$

where $C > 0$ does not depend on λ.

Also, if

$$m_2 = Inf\{|\phi''(\eta)| \ , \eta \in [\delta; \eta_o]\} > 0,$$

then for all $\eta \in]\frac{\delta}{\lambda}; \eta_o[$, $|\phi''_\lambda(\eta)| \geq \min\{m_2, A(\lambda\eta_o)^p\} \geq \min\{m_2, A\eta_o^p\}$, and

$$|\int_{\frac{\delta}{\lambda}}^{\eta_o} \eta^\alpha e^{it(\phi_\lambda(\eta)+y\eta)} d\eta| \leq C \frac{\eta_o^\alpha + (\frac{\delta}{\lambda})^\alpha}{(t \min\{m_2, A\eta_o^p\})^{\frac{1}{2}}} \leq \frac{C}{t^{\frac{1}{2}}},$$

where once again $C > 0$ is independent of λ.

In the case where $\phi''(\tilde{\eta}) = 0$ for some $\tilde{\eta} \in]\delta; \eta_o[$, setting as before

$$m_3 = Inf\{|\phi'''(\eta)| \ , \eta \in [\tilde{\eta} - \epsilon; \tilde{\eta} + \epsilon]\} > 0,$$

$$m_4 = Inf\{|\phi''(\eta)| \ , \eta \in [\delta; \eta_o] \backslash [\tilde{\eta} - \epsilon; \tilde{\eta} + \epsilon]\} > 0$$

for some $\epsilon > 0$, and putting

$$M = \inf\{\phi''(\eta) \backslash \eta \in [\delta; \lambda\eta_o] -]\tilde{\eta} - \epsilon; \tilde{\eta} + \epsilon[\} \geq \inf\{m_4; A(\lambda\eta_o)^p\} \geq \inf\{m_4; A\eta_o^p\},$$

we get

$$\begin{aligned} |\int_{\frac{\delta}{\lambda}}^{\eta_o} \eta^\alpha e^{it(\phi_\lambda(\eta)+y\eta)} d\eta| \ &\leq \ |\int_{[\frac{\delta}{\lambda};\eta_o] \backslash [\frac{\tilde{\eta}-\epsilon}{\lambda}; \frac{\tilde{\eta}+\epsilon}{\lambda}]} \eta^\alpha e^{it(\phi_\lambda(\eta)+y\eta)} d\eta| \\ &\quad + |\int_{[\frac{\tilde{\eta}-\epsilon}{\lambda}; \frac{\tilde{\eta}+\epsilon}{\lambda}]} \eta^\alpha e^{it(\phi_\lambda(\eta)+y\eta)} d\eta| \\ &\leq \ \frac{4\eta_o^\alpha}{(tM)^{\frac{1}{2}}} + \frac{(2\frac{\tilde{\eta}+\epsilon}{\lambda})^\alpha}{(t\lambda m_3)^{\frac{1}{3}}} \leq \frac{C}{t^{\frac{1}{3}}} \end{aligned}$$

where $C > 0$ depends only on α and T.
Finally, since for all $\eta \geq \eta_o$, $\lambda\eta \geq \eta_o$ and $|\phi''_\lambda(\eta)| = |\phi''(\lambda\eta)| \geq A(\lambda\eta)^p \geq A\eta^p$, we easily get the following estimate, uniform in λ :

$$|\int_{\eta_o}^{+\infty} \eta^\alpha e^{it(\phi_\lambda(\eta)+y\eta)} d\eta| \leq \frac{C}{t}.$$

$\blacksquare$

From Lemma 2.3 we establish the following short-time estimates for $I_\alpha(y,t)$:

Theorem 2.4 *Short-time estimate*
Let $T > 0$.
Let I_α be defined as in Theorem 2.1.
Then

$$\forall t \in]0;T[\ ,\ \sup_{y\in\mathbb{R}} |I_\alpha(y,t)| \leq \frac{C}{t^{\frac{\alpha+1}{2}}} \tag{24}$$

holds for all $\alpha \in [0; \frac{p}{2}]$, for some $C > 0$ depending exclusively on T and α.

We now derive the associated "$L^p - L^q$" estimates.

We set, for $t \in \mathbb{R}$, $W(t) = e^{it\phi(D)}$. For K the associated kernel defined in (13),

$$|D|^\alpha W(t) f_o(y) = |D|^\alpha (\frac{1}{2\pi} K(.,t) * f_o(.)[y]) = \frac{1}{2\pi}(|D|^\alpha K(.,t)) * f_o = \frac{1}{2\pi} I_\alpha(\frac{\cdot}{t}, t) * f_o(.)[y],$$

for all tempered distribution f_o, $\alpha \in [0; \frac{p}{2}]$.
In view of Theorem 2.4 one gets for all $T > 0$ the existence of $C > 0$ such that, for all $f \in L^1(\mathbb{R})$, and for all $t \in]0;T]$

$$\||D|^\alpha W(t) f_o\|_\infty \leq \frac{C}{t^{\frac{\alpha+1}{2}}} \|f_o\|_1. \tag{25}$$

Moreover, since for $f_o \in L^2(\mathbb{R})$,

$$|W(t) f_o\|_2 = \|f_o\|_2, \tag{26}$$

we claim

Lemma 2.5 *"$L^p - L^q$" estimates*
Let ϕ a real odd function satisfying conditions (14-a),(14-b) and (14-c) for some $p \geq 0$.
Let $W(t) = e^{it\phi(D)}$, $T > 0$ and $\alpha \in [0; \frac{p}{2}]$.
Then there exists $C > 0$ such that for all $\theta \in [0;1]$ and $f_o \in L^{\frac{2}{(1+\theta)}}(\mathbb{R})$,

$$\forall t \in]0;T]\ ,\ \||D|^{\theta\alpha} W(t) f_o(y)\|_{\frac{2}{(1-\theta)}} \leq \frac{C}{t^{\frac{\theta(\alpha+1)}{2}}} \|f_o\|_{\frac{2}{(1+\theta)}}.$$

Following equation (25), the proof of this lemma is standard :
Let $\alpha \in [0; \frac{p}{2}]$. We introduce the analytic family of operators

$$W_{a+ib}(t) = |D|^{\alpha a+ib} W(t) \ , \ a \in [0;1] \ , \ b \in \mathbb{R}.$$

For all $f_o \in L^2(\mathbb{R})$, $\beta \in \mathbb{R}$, $\|W_{0+ib} f_o\|_2 = \||D|^{ib} f_o\|_2 = \|f_o\|_2$.
Also, a slight modification in the proof of Theorem 2.4 yields for $f_o \in L^1(\mathbb{R})$,

$$\|W_{1+ib} f_o\|_\infty \leq (1+|b|)\||D^\alpha|W(t)f_o\|_\infty \leq \frac{C(1+|b|)}{t^{\frac{\alpha+1}{2}}}\|f_o\|_1.$$

Finally by the complex interpolation theorem (see [6]), we get the announced result.

3 Strichartz estimates

As remarked before, the functions ϕ_S and ϕ_m $(m > 0)$ defined respectively in (9), (8) satisfy conditions (a), (b) and (c) with $p = 0$. Therefore, the dispersion estimate in Lemma 2.5 holds for the operators $e^{it\phi_S(D)}$ and $e^{it\phi_m(D)}$ with $\alpha = 0$. As in the case of the free Schrödinger operator $e^{it\Delta}$,it is known (see [2]) that this estimate implies the following Strichartz-type estimates :

Corollary 3.1 *Let $\phi = \phi_m$, $m > 0$ or $\phi = \phi_S$, defined in (9,8) and $W(t) = e^{i\phi}$. Let $T > 0$. Then there exists $C > 0$ depending exclusively on T such that for all $f_o \in L^2(\mathbb{R})$, and for all $(p,q) = (\frac{2}{1-\theta}, \frac{4}{\theta})$, $\theta \in [0;1]$,*

$$\|W(t)f_o\|_{L^q(0,T;L^p(\mathbb{R}))} \leq C\|f_o\|_{L^2(\mathbb{R})} \ , \ t \in]0;T] \tag{27}$$

and, for all $g \in L^{q'}(0,T;L^{p'}(\mathbb{R}))$,

$$\| \int_0^t W(t-\tau)g(.,\tau)d\tau\|_{L^q(0,T;L^p(\mathbb{R}))} \leq C\|g\|_{L^{q'}(0,T;L^{p'}(\mathbb{R}))} \ , \ t \in]0;T] \tag{28}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Furthermore, ϕ_L defined in (10) satisfies conditions (a), (b), (c) for $p = 1$. This will allow us to derive Strichartz-type estimates with smoothing for the operator $e^{it\phi_L}$. We claim :

Corollary 3.2 *Let $W(t) = e^{i\phi_L}$, ϕ_L defined in (10).
Let $T > 0$.
Then there exists $C > 0$ depending exclusively on T such that for all $f_o \in L^2(\mathbb{R})$, and for all $(p,q) = (\frac{2}{1-\theta}, \frac{8}{3\theta})$, $\theta \in [0;1]$,*

$$\| \ |D|^{\frac{\theta}{4}} W(t)f_o\|_{L^q(0,T;L^p(\mathbb{R}))} \leq C\|f_o\|_{L^2(\mathbb{R})} \ , \ t \in]0;T] \tag{29}$$

and, for all $g \in L^{q'}(0,T;L^{p'}(\mathbb{R}))$,

$$\| \int_0^t |D|^{\frac{\theta}{2}} W(t-\tau)g(.,\tau)d\tau\|_{L^q(0,T;L^p(\mathbb{R}))} \leq C\|g\|_{L^{q'}(0,T;L^{p'}(\mathbb{R}))} \ , \ t \in]0;T] \tag{30}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Proof:
We prove (30), the standard Tomas duality argument ([8]) will then yield the homogeneous estimate (29).

From Lemma 2.5 with $\alpha = \frac{1}{2}$ we get the existence of $C > 0$ such that

$$\forall t \in]0;T] \ , \ \|D^{\frac{\theta}{2}} W(t) f_o(y)\|_{\frac{2}{(1-\theta)}} \leq \frac{C}{t^{\frac{3\theta}{4}}} \|f_o\|_{\frac{2}{(1+\theta)}}.$$

Hence,

$$\|\int_0^t |D|^{\frac{\theta}{2}} W(t-\tau) g(.,\tau) d\tau\|_{L^q(0,T;L^p(\mathbb{R}))} \leq \|\int_0^T \| \ |D|^{\frac{\theta}{2}} W(t-\tau) g(.,\tau)\|_p d\tau\|_{L^q(0,T)}$$

$$\leq C\|\int_0^T \frac{1}{|t-\tau|^{\frac{3\theta}{4}}} \|g(.,\tau)\|_{p'} d\tau\|_{L^q(0,T)}.$$

Moreover, by the classical results on fractional integration,

$$\|\int_0^T \frac{1}{|t-\tau|^{\frac{3\theta}{4}}} \|g(.,\tau)\|_{p'} d\tau\|_{L^q(0,T)} \leq \|g\|_{L^{q'}(0,T;L^{p'}(\mathbb{R}))}$$

provided that $\frac{1}{q} = \frac{1}{q'} - (1 - \frac{3\theta}{4})$, which is the case. ■

4 Appendix

For $\eta > 0$, and $m > 0$, we put $\theta_m(\eta) = \frac{K_{m-\frac{1}{2}}(\eta)}{K_{m+\frac{1}{2}}(\eta)}$, where, for $\nu \in \mathbb{R}$,

$$K_\nu(\eta) = \int_0^{+\infty} e^{-\eta cosh(x)} cosh(\nu x) dx$$

are the McDonald's functions.
Finally, let

$$\forall \eta \in \mathbb{R} \ , \ \phi_m(\eta) = \eta|\eta|\theta_m(|\eta|).$$

The aim of this appendix is to prove the following lemma :

Lemma 4.1 *For all* $\eta > 0$, $\phi''_m(\eta) > 0$ *and for all* $m > 0$ *there exists* $\eta_o > 0$ *and* $M > 0$ *such that*

$$\phi''_m(\eta) \geq M \ for \ |\eta| \geq |\eta_o|.$$

We begin by stating some well-known facts :

Lemma 4.2 *For all $\nu \in \mathbb{R}$ and $\eta > 0$,*

$$\eta K_\nu'(\eta) = -\nu K_\nu(\eta) - \eta K_{\nu-1}(\eta). \tag{31}$$

Furthermore, the next asymptotic expansions holds :

$$K_\nu(\eta) = \sqrt{\frac{\pi}{2\eta}} e^{-\eta}(1 + \frac{4\nu^2 - 1}{8\eta} + \frac{(4\nu^2-1)(4\nu^2-9)}{2(8\eta)^2} + \frac{1}{\eta^2}\epsilon(\eta)) \ , \ \lim_{\eta\to\infty} \epsilon(\eta) = 0, \tag{32}$$

$$\begin{cases} \text{if } \nu > 0 \ , & K_\nu(\eta) \underset{\eta\to 0}{\simeq} \frac{1}{2^{1-\nu}}\Gamma(\nu)\eta^{-\nu} \\ \\ \text{and} & K_o(\eta) \underset{\eta\to 0}{\simeq} -\log(\eta). \end{cases} \tag{33}$$

For a proof, see for instance [4].
From Lemma 4.2, by some elementary computations, one obtains :

Lemma 4.3 *For all $m > 0$ and $\eta > 0$,*

$$\theta_m'(\eta) = \frac{2m}{\eta}\theta_m(\eta) - 1 + \theta_m^2(\eta). \tag{34}$$

Moreover,

$$\theta_m(\eta) = 1 - \frac{m}{\eta} + \frac{m(m+1)}{2\eta^2} + \frac{1}{\eta^2}\epsilon(\eta) \ , \ \lim_{\eta\to\infty} \epsilon(\eta) = 0, \tag{35}$$

and, near the origin,

$$\begin{cases} \text{if } m > \dfrac{1}{2} \ , & \theta_m(\eta) \underset{\eta\to 0}{\simeq} \frac{\eta}{2m+1}, \\ \\ \text{if } m < \dfrac{1}{2} \ , & \theta_m(\eta) \underset{\eta\to 0}{\simeq} C(m)\eta^{2m} \text{for some } C(m) > 0 \\ \\ \text{and} & \theta_{\frac{1}{2}}(\eta) \underset{\eta\to 0}{\simeq} -\eta\log(\eta). \end{cases} \tag{36}$$

We now claim

Lemma 4.4 *For all $m > 0$ and $\eta > 0$,*

$$\theta_m'(\eta) > 0. \tag{37}$$

Proof : Assume that for some $\eta_o > 0$, $\theta_m'(\eta_o) = 0$. Then,

$$\theta_m''(\eta_o) = -\frac{2m}{\eta_o^2}\theta_m(\eta_o) + \theta_m'(\eta_o)(2\theta(\eta_o) + \frac{2m}{\eta_o}) = -\frac{2m}{\eta_o^2}\theta_m(\eta_o) < 0.$$

Yet η_o is a local maximum, which is of course absurd since

$$0 \le \theta_m < 1 \ , \ \theta_m(0) = 0 \text{ and } \lim_{\eta\to\infty} \theta_m(\eta) = 1 \quad \blacksquare.$$

This result allows us to prove that $\phi''_m(\eta)$ does not vanish for $\eta > 0$ in the case where $m < 1$. Indeed, note that for $\eta > 0$, by (34),

$$\phi''_m(\eta) = 2(1-m)\theta_m(\eta) + 2\eta^2\theta'_m(\eta)\theta_m(\eta) + (2m+4)\eta\theta_m(\eta) \geq 2(1-m)\theta_m(\eta) > 0.$$

In order to extend this result for $m \geq 1$, sharper estimates are needed :

Lemma 4.5 *Let $m \geq 1$. Then, for all $\eta > 0$,*

$$\eta\theta'_m(\eta) < \theta_m(\eta) \tag{38}$$

and

$$\frac{\theta_m(\eta)}{\eta} \leq \frac{1}{2m-1}. \tag{39}$$

Proof :
Let $f(\eta) = \eta\theta'_m(\eta) - \theta(\eta)$.
Assume that for some $\eta_o > 0$, $f(\eta_o) = 0$. Then

$$f'(\eta_o) = \eta_o\theta''_m(\eta_o) = \eta_o[2\theta_m(\eta_o)\theta'_m(\eta_o) + \frac{2m}{\eta_o}\theta'_m(\eta_o) - \frac{2m}{\eta_o^2}\theta_m(\eta_o)] = 2\eta_o\theta_m(\eta_o)\theta'_m(\eta_o) > 0$$

which is absurd, since, from Lemma 4.2, one gets

$$f(0) = 0 \text{ and } f(\eta) = -1 + \frac{2m}{\eta} + \frac{1}{\eta}\epsilon(\eta) \ , \ \lim_{\eta\to+\infty} \epsilon(\eta) = 0.$$

The proof of (39) is then straightforward :

Putting $g(\eta) = \dfrac{\theta_m(\eta)}{\eta}$, $g'(\eta) = \dfrac{f(\eta)}{\eta^2} < 0$, and for all $\eta > 0$, $g(\eta) \leq \lim_{\eta\to 0} \dfrac{\theta_m(\eta)}{\eta} = \dfrac{1}{2m-1}$. ∎

Finally, we end the proof of Lemma 4.1 :
For all $\eta > 0$,

$$\phi''_m(\eta) = 2\theta_m(\eta) + 4\eta\theta'_m(\eta) + \eta^2\theta''_m(\eta) = 2(1-m)\theta_m(\eta) + 2\eta^2\theta'_m(\eta)\theta_m(\eta) + 2(m+2)\eta\theta'_m(\eta).$$

From Lemma 4.3, one gets

$$\phi''_m(0) = 0 \text{ and } \phi''_m(\eta) = 2 - \frac{m(m+1)}{\eta}^2 + \epsilon(\eta)\frac{1}{\eta^2} \ , \ \lim_{\eta\to\infty} \epsilon(\eta) = 0. \tag{40}$$

As before, assume that for some $\eta > 0$, $\phi''_m(\eta_o) = 0$, i.e.

$$2(m+2)\eta\theta'_m(\eta_o) = 2(m-1) - 2\eta_o^2\theta'_m(\eta_o)\theta_m(\eta_o). \tag{41}$$

Then,

$$\begin{aligned}\phi'''_m(\eta_o) &= 2(1-m)\theta'_m(\eta_o) + 2\eta_o^2\theta'^2_m(\eta_o) + 4\eta_o\theta'_m(\eta_o)\theta_m(\eta_o) \\ &\quad +(2m+4)\theta'_m(\eta_o) + \theta''_m(\eta_o)((2m+4)\eta_o + 2\eta_o^2\theta_m(\eta_o)).\end{aligned}$$

Therefore,

$$\begin{aligned}\theta_m(\eta_o)\phi'''_m(\eta_o) &= \theta'_m(\eta_o)[2(m+2)\theta_m(\eta_o)+4\eta_o\theta_m^2(\eta_o)+2(1-m)\theta_m(\eta_o)+2\eta_o^2\theta'_m(\eta_o)\theta_m(\eta_o)] \\ &\quad +\theta''_m(\eta_o)[(2m+4)\eta_o+2\eta_o^2\theta_m(\eta_o)].\end{aligned}$$

By (41), and since $\theta''_m(\eta_o)\leq 0$,

$$\begin{aligned}\theta_m(\eta_o)\phi'''_m(\eta_o) &\leq 2\theta'_m(\eta_o)[(m+2)(\theta_m(\eta_o)-\eta_o)+2\eta\theta_m^2(\eta_o)] \\ &\leq 2\theta'_m(\eta_o)[2\eta_o(\theta_m^2(\eta_o)-1)-m\eta_o+(m+2)\theta_m(\eta_o)] \\ &\leq 2\theta'_m(\eta_o)[2\eta_o(\theta'_m(\eta_o)-\frac{2m}{\eta_o}\theta_m(\eta_o)-m\eta_o+(m+2)\theta_m(\eta_o)] \text{ by (34)} \\ &\leq 2\theta'_m(\eta_o)[2\theta_m(\eta_o)-4m\theta_o(\eta)-m\eta_o+(m+2)\theta_m(\eta_o)] \text{ by by (39)} \\ &\leq 2\theta'_m(\eta_o)[(4-3m)\theta_m(\eta_o)-m\eta_o].\end{aligned}$$

If $m\leq\frac{4}{3}$, then $\phi'''_m(\eta_o)<0$. If $m>\frac{4}{3}$:

$$\theta_m(\eta_o)\phi'''_m(\eta_o)\leq 2\theta'_m(\eta_o)\eta_o[(4m-3)\frac{\theta_m(\eta_o)}{\eta_o}-m]\leq 2\theta'_m(\eta_o)\eta_o\frac{-2m^2-2m+4}{2m-1}<0 \text{ by (39)}$$

Therefore, if ϕ''_m vanishes at some point, it is a local maximum, which contradicts (40). The second part of Lemma 4.1 clearly follows from the asymptotic expansion of ϕ''_m.

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