A Reformulation–Linearization–Convexification Algorithm for Optimal Correction of an Inconsistent System of Linear Constraints

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Abstract

In this paper, an algorithm is introduced to find an optimal solution for an optimization problem that arises in total least squares with inequality constraints, and in the correction of infeasible linear systems of inequalities. The stated problem is a nonconvex program with a special structure that allows the use of a reformulation–linearization–convexification technique for its solution. A branch–and–bound method for finding a global optimum for this problem is introduced based on this technique. Some computational experiments are included to highlight the efficacy of the proposed methodology.

1 Introduction

The problem we address in this paper arises in the context of correcting infeasible linear systems of inequalities, such as $Ax \leq b, x \in \mathbb{R}^n$, where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$.

In linear programming and in constraint satisfaction, one is often confronted with an empty set of solutions due to many causes such as the lack of communication between different decision makers, update of old models, or integration of partial models. In post-infeasibility analysis, several attempts are made in order to retrieve valuable information regarding the inconsistency of a given model [11],[19],[15],[16],[18] such as the identification of conflicting sets of constraints [10], [12], [14], [1], [2] and irreducible inconsistent sets (IIS) of constraints, for both continuous [13],[17],[29],[22] and mixed-integer [20] problems. This information can be used to reformulate the model, either by removing constraints or slightly changing the coefficients of the constraints. In [21], the authors proposed a method based on an hierarchical classification of constraints to remove constraints in order to obtain a feasible set. This procedure is, however, completely inadequate in cases where the physical situation that the constraints seek to prevent cannot be ignored, or when we are working with only approximate data. In the context of Linear Programming, some theoretical results regarding the distance to infeasibility of a linear system are presented in [23] and [24], and techniques for deciding about the existence of solutions for approximate data are explored in [26].

Although the problem of inconsistency in linear models has attracted some attention, not much has been done concerning the development of exact algorithms for finding minimal corrections (according to some criteria) of the coefficients of an infeasible linear system of inequalities. The perturbation of the vector b alone is a less difficult problem and has been considered in [25]. The correction of both A and b is a more challenging task due to the introduction of nonlinearity, but is more adequate and advisable in practice.

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These considerations have lead to a general formulation that seeks the optimal correction p and H, respectively, of the vector b and the matrix A of the linear system of inequalities $Ax \leq b$. This results in the following optimization problem:

$$\text{Minimize} \quad \varphi(H,p) \tag{1}$$
$$\text{subject to} \quad (A+H)x \leq b+p \tag{2}$$
$$x \in \mathbf{X}, H \in \mathbb{R}^{m\times n}, p \in \mathbb{R}^m, \tag{3}$$

where $\mathbf{X} \subseteq \mathbb{R}^n$ is a convex set and φ is an appropriate matrix norm. For $\varphi = \|\cdot\|_{l_1}$ and $\varphi = \|\cdot\|_{l_\infty}$ (the generalization for matrices of the respective norms l_1 and l_∞), Vatolin [28] proved that it is possible to find an optimal correction by solving a set of linear programming problems. Later in [3] it was shown that this approach is also applicable for $\varphi = \|\cdot\|_\infty$ and that the number of linear programming problems to be solved is $2n+1$ for the l_1 and ∞ norms, and 2^n for the l_∞ norm. Furthermore, it was stated that an optimal correction consisted of changes in only one column of (A,b) in the case of norms l_1 or ∞, while for the norm l_∞, the perturbation of the coefficients of every row would only differ in sign. This introduced a fixed pattern for the correction matrix, which turns out to be quite unnatural in practical situations where a free pattern is more suitable. These conclusions have motivated us to study the case of finding an optimal correction with respect to the Frobenius norm $\|\cdot\|_F$, that is, to consider the following optimization problem:

$$(P) \quad \text{Minimize } \|[H,p]\|_F^2$$
$$\text{subject to} \quad \begin{array}{l}(A+H)x \leq b+p \\ H \in \mathbb{R}^{m\times n}, \quad p \in \mathbb{R}^m, \quad x \in \mathbf{X}.\end{array} \tag{4}$$

It is interesting to note that for $\mathbf{X} = \mathbb{R}^n$ this problem may fail to have a solution and a local minimizer exists iff the correction corresponds to an application of the total least squares (TLS) method to the set of active constraints [6]. Some algorithms for finding such a local minimizer are discussed in [4]. In [7], a tree search procedure based on the enumeration of the active set of constraints was proposed, where some reduction tests were implemented in order to reduce the tree search. Although problems of small size have been efficiently solved, the overall effort required for finding a global minimum is usually too high for medium-scale problems. In practice, it is important to define the set $\mathbf{X}$ such that the existence of an optimal solution is guaranteed, and the solution of the corrected system is in a certain domain of interest. The most general choice for $\mathbf{X}$ corresponds to

$$\mathbf{X} \quad \subseteq \quad \{x \in \mathbb{R}^n : l \leq x \leq u\}, \tag{5}$$

where l and u are fixed vectors. Then, $\mathbf{X}$ is compact and the optimization problem (P) has a global minimum. This is the choice we adopt in this paper.

This paper is organized as follows. In the next section, we present some important results that lead to the development of the main algorithm to be introduced in Section 3. Section 4 includes the report of some experiments with the algorithm for a number of test instances. Some conclusions and recommendations for future research are included in the last section of the paper.

2 Preliminary results

In [6] it was shown that for $\mathbf{X} = \mathbb{R}^n$, the problem (4) is equivalent to the unconstrained nonlinear and nonconvex problem:

$$(P) \qquad \min_{x\in\mathbf{X}} \frac{\|(Ax-b)^+\|^2}{1+\|x\|^2}, \tag{6}$$

where $(\cdot)^+$ denotes $\max\{0,\cdot\}$ and $\|\cdot\|$ represents the Euclidian norm. This equivalence still holds for

$$\mathbf{X} \subseteq \{x : l \leq x \leq u\}.$$

Once problem (6) is solved, we can directly obtain the correction matrix $[H, p]$ for Problem (4) by

$$[H, p] = -\lambda^*[x^{*T}, 1], \tag{7}$$

where x^* is the optimal solution found for Problem (6) and

$$\lambda^* = \frac{1}{1 + \|x^*\|^2}(Ax^* - b)^+. \tag{8}$$

The proof can be found in [6]. The next theorem presents another formulation for the original problem (4) that is exploited in this paper.

Theorem 2.1. *The formulation (6) is equivalent to:*

$$\begin{aligned} (P1): \quad & \textit{Minimize} \quad \frac{\beta}{1 + \|x\|^2} && (9)\\ & \textit{subject to} \quad \beta \geq \|v\|^2 && (10)\\ & \qquad v \geq Ax - b && (11)\\ & \qquad v \geq 0 && (12)\\ & \qquad x \in \mathbf{X}. && (13) \end{aligned}$$

Proof: It is sufficient to show that for any feasible solution to one problem, there exists a feasible solution to the other problem having at least as good an objective value.
Given x feasible to (P), it is easy to see that there exists (x, v, β) feasible to $(P1)$ with the same objective value. In fact for $v = (Ax - b)^+$ and $\beta = \|v\|^2$,

$$\frac{\beta}{1 + \|x\|^2} = \frac{\|(Ax - b)^+\|^2}{1 + \|x\|^2}.$$

Now, for (x, v, β) feasible to $(P1)$, let $\overline{v} = (Ax - b)^+$ and $\overline{\beta} = \|\overline{v}\|^2$. Then $(x, \overline{v}, \overline{\beta})$ is feasible to $(P1)$ and

$$\begin{aligned} \frac{\overline{\beta}}{1 + \|x\|^2} &= \frac{\|\overline{v}\|^2}{1 + \|x\|^2} = \frac{\|(Ax - b)^+\|^2}{1 + \|x\|^2} \\ &\leq \frac{\|v\|^2}{1 + \|x\|^2} \leq \frac{\beta}{1 + \|x\|^2}. \end{aligned} \tag{14}$$

In fact, $Ax - b \leq v$ and $v \geq 0$ imply that $0 \leq (Ax - b)^+ \leq v$ and so, $\|(Ax - b)^+\|^2 \leq \|v\|^2$. Obviously x is feasible to (P) and the second inequality in (14) shows that it has at least as good an objective value as (x, v, β) does in (9)-(13). This completes the proof. □

In $(P1)$, upon the substitution

$$\|x\|^2 = \alpha,$$

and considering that $\mathbf{X} \subseteqq \{0 \leq l \leq x \leq u\}$, we immediately obtain the following result.

Corollary 2.1. *If*

$$\alpha_l = \|l\|^2 \textit{ and } \alpha_u = \|u\|^2, \tag{15}$$

then the problem

$$\begin{aligned} (P2) \quad & \textit{Minimize} \quad \frac{\beta}{1 + \alpha} && (16)\\ & \textit{subject to} \quad \beta \geq \|v\|^2 && (17)\\ & \qquad v \geq Ax - b && (18)\\ & \qquad \|x\|^2 = \alpha && (19)\\ & \qquad x \in \mathbf{X} \subseteqq \{x : 0 \leq l \leq x \leq u\} && (20)\\ & \qquad v \geq 0 && (21)\\ & \qquad \alpha_l \leq \alpha \leq \alpha_u, && (22) \end{aligned}$$

is equivalent to $(P1)$ *and, consequently, to Problem (4).*

Now, consider the nonlinear relaxation:

$$
\begin{aligned}
(RP2) \quad & \text{Minimize} \quad && \frac{\beta}{1+\alpha} && (23)\\
& \text{subject to} && \beta \geq \|v\|^2 && (24)\\
& && v \geq Ax - b && (25)\\
& && \|x\|^2 \leq \alpha && (26)\\
& && x \in \mathbf{X} \subseteqq \{x : 0 \leq l \leq x \leq u\} && (27)\\
& && v \geq 0 && (28)\\
& && \alpha_l \leq \alpha \leq \alpha_u. && (29)
\end{aligned}
$$

Then the feasible set of $(RP2)$ is convex and the objective function is pseudoconvex over this set [9]. Therefore, any stationary point of the objective function in this feasible set is also a global minimum of $(RP2)$. Furthermore, any such solution is a global minimum of the problem $(P2)$ if and only if $\|x\|^2 = \alpha$. It might therefore seem that there could exist some cases where the solution of $(P2)$ simply reduces to finding a global minimum for $(RP2)$. Unfortunately, this is not usually the case, as the inequality $\|x\|^2 \leq \alpha$ is often inactive at such global minimum. This is quite understandable as α tends to increase as much as possible in order to minimize the objective function of $(RP2)$. So $(P2)$ needs to be processed by a global optimization algorithm. In this paper we propose a branch-and-bound algorithm for $(P2)$ that is based on the idea of partitioning the set

$$\Omega = \{x : 0 \leq l \leq x \leq u\}.$$

At every node k of the enumeration tree, we consider the proper subset of Ω:

$$\Omega^k = \{x : l_i^k \leq x_i \leq u_i^k, \text{ for } i = 1, \dots, n\}, \qquad (30)$$

along with the following associated problem:

$$
\begin{aligned}
(P2^k) \quad & \text{Minimize} \quad && \frac{\beta}{1+\alpha}\\
& \text{subject to} && \beta \geq \|v\|^2\\
& && v \geq Ax - b\\
& && \|x\|^2 = \alpha && (31)\\
& && x \in \Omega^k\\
& && v \geq 0\\
& && \alpha_l^k \leq \alpha \leq \alpha_u^k.
\end{aligned}
$$

At every node, instead of solving $(P2^k)$ directly, we obtain a lower bound for the optimal value of $(P2^k)$ by solving a special convex problem. To construct such a program, we can simply replace the equality $\|x\|^2 = \alpha$ by the inequality constraint (26). This nonlinear relaxation is denoted by $(RP2^k)$ and is obtained from $(RP2)$ by forcing x to belong to the set Ω^k and by constraining α accordingly. Alternatively, we can exploit the so–called Reformulation–Linearization–

Convexification Technique (RLT), as described in [27] and consider the following relaxation

$$
\begin{aligned}
LB(P2^k): \quad & \text{Minimize} \quad \frac{\beta}{1+\alpha} && (32)\\
& \text{subject to} \quad \beta \geq \|v\|^2 && (33)\\
& v \geq Ax - b && (34)\\
& \sum_{i=1}^{n} y_i = \alpha && (35)\\
& x_i^2 \leq y_i, \forall i = 1, \dots, n && (36)\\
& \left[(x_i - l_i^k)(u_i^k - x_i)\right]_L \geq 0, \forall i = 1, \dots, n && (37)\\
& x \in \mathbf{X} && (38)\\
& x \in \Omega^k && (39)\\
& \alpha_l^k \leq \alpha \leq \alpha_u^k, && (40)
\end{aligned}
$$

where $[\cdot]_L$ denotes the linearization of the product term $[\cdot]$ under the substitution

$$y_i = x_i^2, \ \forall i = 1, \dots, n. \tag{41}$$

Note that in (37),

$$
\begin{aligned}
\left[(x_i - l_i^k)(u_i^k - x_i)\right]_L &= \left[x_i u_i^k - l_i^k u_i^k + l_i^k x_i - x_i^2\right]_L \\
&= x_i u_i^k - l_i^k u_i^k + l_i^k x_i - y_i.
\end{aligned}
$$

Problem $LB(P2^k)$ is a convex nonlinear program with a very special structure that can be efficiently solved by a nonlinear programming algorithm. It should be added that the constraint $\alpha = \|x\|^2$ has been convexified by introducing new variables y_i and additional constraints (35) and (36). Moreover, instead of imposing $y_i = x_i^2$, we have introduced the relaxed linear bound-factor constraints (37), where, again, x_i^2 is replaced by y_i, $\forall i = 1, \dots, n$. The following result holds in regard to $LB(P2^k)$.

Proposition 2.1. *If $(\overline{x}, \overline{v}, \overline{\beta}, \overline{\alpha}, \overline{y})$ solves Problem $LB(P2^k)$ with objective value $\nu(LB(P2^k))$, then $\nu(LB(P2^k))$ is a lower bound for the optimal value of $(P2^k)$. Moreover, if $\overline{x}_i = l_i^k$ or $\overline{x}_i = u_i^k$, for each $i = 1, \dots, n$, then $\overline{y}_i = \overline{x}_i^2$, $\forall i = 1, \dots, n$.*

Proof: Follows from Sherali and Tuncbilek [27]. □

It is also important to add that the relaxation $LB(P2^k)$ provides tighter lower bounds than the previous one $(RP2^k)$. In fact, since $x_i^2 \leq y_i$ for all $i = 1, \dots, n$, then

$$\|x\|^2 = \sum_{i=1}^{n} x_i^2 \leq \sum_{i=1}^{n} y_i = \alpha.$$

Therefore the feasible region of the problem $LB(P2^k)$ projected onto the (x, v, β, α)-space is included in that for $(P2^k)$. For this reason, in this paper, we use the relaxation $LB(P2^k)$ instead of $(RP2^k)$.

3 Overall algorithm

At every node of the proposed branch-and-bound procedure, starting with $\Omega^0 = \Omega$, the problem $LB(P2^k)$ is solved to derive a lower bound on the node subproblem. Let $(\overline{x}, \overline{v}, \overline{\beta}, \overline{\alpha}, \overline{y})$ be the optimal solution obtained and let $\nu(LB(P2^k))$ be its optimal value. If

$$\nu(LB(P2^k)) \geq UB(1 - \epsilon),$$

for some tolerance $\epsilon \geq 0$, we fathom this node. Otherwise, we partition this node into subproblems $(P2^{k+1})$ and $(P2^{k+2})$, based on the corresponding partition of Ω^k into Ω^{k+1} and Ω^{k+2} as follows:

$$\Omega^{k+1} = \{x : l_i^{k+1} \leq x_i \leq u_i^{k+1}, \text{ for } i = 1, \dots, n\}$$

and

$$\Omega^{k+2} = \{x : l_i^{k+2} \leq x_i \leq u_i^{k+2}, \text{ for } i = 1, \dots, n\},$$

where the bounds describing Ω^{k+1} and Ω^{k+2} are discussed next, within the following branching strategy.

3.1 Branching variable selection scheme

Let

$$p \in \arg \max_{i=1,\dots,n} \{\theta_i\} \quad \text{where} \quad \theta_i = \overline{y}_i - \overline{x}_i^2, \text{ for } i = 1, \dots, n. \tag{42}$$

If $\theta_p > 0$, then

$$\Omega^{k+1} = \{x : l_i^k \leq x_i \leq u_i^k, i = 1, \dots, n, i \neq p, l_p^k \leq x_p \leq \overline{x}_p\}, \tag{43}$$

$$\Omega^{k+2} = \{x : l_i^k \leq x_i \leq u_i^k, i = 1, \dots, n, i \neq p, \overline{x}_p \leq x_p \leq u_p^k\}. \tag{44}$$

If $\max_{i=1,\dots,n} \{\theta_i\} = 0$, then by Proposition 2.1, $(\overline{x}, \overline{v}, \overline{\beta}, \overline{\alpha}, \overline{y})$ is a feasible solution for $(P2^k)$, whose value, $v(LB(P2^k))$, is a lower bound that is achieved, and hence, is the optimal value of $(P2^k)$. Therefore, we update the incumbent solution (along with UB), if necessary, and fathom the node in this case.

3.2 Computing Upper Bounds

Given the solution $\overline{x}$ to $LB(P2^k)$, we compute an upper bound UB by setting

$$\text{UB} \leftarrow \min\{\text{UB}, f(\overline{x}), \phi(\overline{x}, \overline{\beta}, \overline{v})\} \tag{45}$$

where

$$f(\overline{x}) = \frac{\|(A\overline{x} - b)^+\|^2}{1 + \|\overline{x}\|^2},$$

and

$$\phi(\overline{x}, \overline{\beta}, \overline{v}) = \frac{\widehat{\beta}}{1 + \|\widehat{x}\|^2},$$

where $(\widehat{x}, \widehat{\beta}, \widehat{v})$ is a stationary point to $P1$ that is obtained by starting at the initial point $(\overline{x}, \overline{\beta}, \overline{v})$ and applying a nonlinear programming algorithm.

3.3 Computing an initial feasible solution

In order to induce a faster convergence toward optimality for each subproblem $LB(P2^k)$, it is important to start with an initial point, $(\hat{x}, \hat{v}, \hat{\beta}, \hat{\alpha}, \hat{y})$, that is feasible for $LB(P2^k)$. To do this, we solve the following program:

$$\hat{x} = \arg \min_{x \in \mathbf{X} \bigcap \{x : l^k \leq x \leq u^k\}} \|x - x^*\|_{l_1} \tag{46}$$

where x^* is the current best known solution. Note that for $\mathbf{X}$ polyhedral (including $\mathbf{X} = \mathbb{R}^n$), (46) can be solved via the LP:

$$\text{Minimize} \left\{ \sum_{i=1}^{n} z_i : z_i \geq x_i - x_i^*, \;\; z_i \geq x_i^* - x_i, \; \forall i = 1, \ldots, n, \; x \in \mathbf{X}, l^k \leq x \leq u^k \right\}. \tag{47}$$

After obtaining $\hat{x}$, we then compute the remainder of the initial solution as

$$\begin{aligned} \hat{v} &= \max\{0, A\hat{x} - b\}, \\ \hat{\beta} &= \|\hat{v}\|^2, \\ \hat{y}_i &= \hat{x}_i^2, \; \forall i = 1, \ldots, n, \\ \hat{\alpha} &= \|\hat{x}\|^2. \end{aligned} \tag{48}$$

The following result holds.

Proposition 3.1. *The solution $(\hat{x}, \hat{v}, \hat{\beta}, \hat{\alpha}, \hat{y})$ as given by (46-48) is feasible to $LB(P2^k)$.*

Proof Feasibility to (33),(34),(35),(36),(38),(39), and (40) is evident by the construction of the solution (46)-(48) and the bounds derived based on $l^k \leq x \leq u^k$. Moreover, feasibility to the RLT constraints (37) follows by construction, since the product relationship in (41) are satisfied via (48). This completes the proof. □

3.4 Algorithm and Convergence Theorem

In this subsection, we describe the main steps of the algorithm. To do this, we start by describing the following parameters:

k**-** index for the current lower bounding problem under analysis;

UB**-** best known upper bound;

x_{inc}**-** incumbent solution;

L**-** queue of indices of subproblems created but not expanded;

$LB(P2^k)$**-** lower bounding problem as described by (32)-(40);

$(x^k, v^k, y^k, \beta^k, \alpha^k)$**-** optimal solution obtained for $LB(P2^k)$;

Ω^k**-** as defined in (30);

ϵ**-** optimality tolerance;

$\nu(.)$**-** optimal value of problem (.).

Algorithm RLT-BB

0-(Initialization) Let $k = 0$, $L = \emptyset$, UB=∞, and $\epsilon = 0.001$. Solve Problem $LB(P2^k)$. Update UB through (45) and set $x_{inc} = x^k$.

1-(Pick next node) If $L = \emptyset$ then stop; otherwise, find $k \in \arg\min\{\nu(LB(P2^t)) : t \in L\}$.

2-(Dequeue) Set $L \leftarrow L - \{k\}$.

3-(Branching rule) Find a branching index p via (42). If $\theta_p > 0$, go to Step 4. Otherwise, update UB and x_{inc} using the solution to $LB(P2^k)$, remove any indices t from L for which $\nu(LB(P2^t)) \geq UB(1 - \epsilon)$, and go to Step 1.

4-(Solve,Update, and Queue) Set $i = 1$.

4.1 Define Ω^{k+i} according to (43) and (44). Solve Problem $LB(P2^{k+i})$.

4.2 If $\nu(LB(P2^{k+i})) < \text{UB}(1-\epsilon)$ then go to Step 4.3; otherwise, go to Step 4.5.

4.3 Update UB according to (45). If UB was updated remove all indices $t \in L$ for which $\nu(LB(P2^t)) \geq \text{UB}(1-\epsilon)$ and put $x_{inc} = x^{k+i}$.

4.4 Set $L \leftarrow L \cup \{k+i\}$.

4.5 If $i = 2$ go to Step 1; otherwise, let $i = 2$ and go to Step 4.1.

To complete this section we present the convergence theorem for this algorithm.

Proposition 3.2. *The algorithm RLT-BB either terminates finitely with a global optimum to the problem* (*when* $\epsilon \equiv 0$)*, or else, an infinite branch-and-bound tree is generated, which is such that any accumulation point of the relaxation lower-bounding problem solution along any infinite branch of the enumeration tree is a global optimum for problem* $(P2)$.

Proof The case of finite convergence is obvious from the validity of the bounds derived by the algorithm. Now, suppose that an infinite branch-and-bound tree is generated. Then there exists a branching index p that is selected infinitely often. Let K be an infinite sequence of nodes generated by a sequence of branchings over p and let $LB(P2^k)$ and $(x^k, v^k, \beta^k, \alpha^k, y^k)$ for $k \in K$ be the corresponding lower bounding problems and optimal solutions, respectively. Over some convergent subsequence ($k \in K_1 \subseteq K$) suppose that

$$(x^k, v^k, \beta^k, \alpha^k, y^k, l^k, u^k) \underset{k \in K_1}{\longrightarrow} (x^*, v^*, \beta^*, \alpha^*, y^*, l^*, u^*).$$

Then, using the proof in [27], we see that in the limit $x_p^* = l_p^*$ or $x_p^* = u_p^*$. Moreover, by Proposition 2.1, this yields $\theta_p = 0$, in the limit, where θ_i is defined in (42). Furthermore, again by (42), this gives $\theta_i = 0, \forall i = 1, \ldots, n$, in the limit. Hence, since the limiting solution $(x^*, v^*, \beta^*, \alpha^*, y^*)$ is feasible with objective value V^*, we get

$$V^* \geq \nu(P2). \tag{49}$$

However, the least lower bound node selection criteria ensures that $\nu(LB(P2^k)) \leq \nu(P2), \forall k \in K_1$. In the limit, this yields $V^* \leq \nu(P2)$. This, together with (49), yields $V^* = \nu(P2)$ and so, x^* solves Problem $P2$. This completes the proof. □

3.5 An example

Consider the following inconsistent system of inequalities, illustrated in Figure (1):

$$\begin{cases} -x_1 & - & x_2 & \leq & -7 \\ & & x_2 & \leq & 3 \\ 2x_1 & - & x_2 & \leq & -2. \end{cases}$$

Suppose that we seek an optimal correction on the domain $\mathbf{X}$ defined by

$$1 \leq x_i \leq 5 \text{ for } i = 1, 2.$$

To apply the algorithm, we have:

$$l = [1, 1] \text{ and } u = [5, 5].$$

Hence,

$$\Omega^0 = \{x : 1 \leq x_1 \leq 5, 1 \leq x_2 \leq 5\},$$

$$\alpha_l^0 = 2.0000 \, , \alpha_u^0 = 50.0000.$$

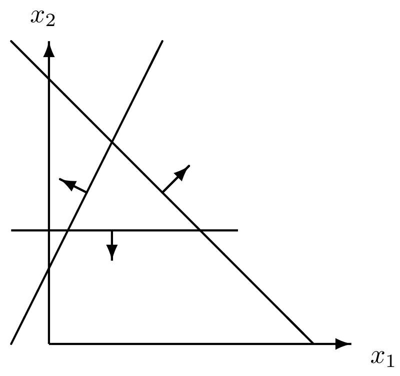

Figure 1: Inconsistent system for the illustrative example.

The solution to $LB(P2^0)$ is as follows:

$$\begin{array}{lll} x^0 = (1.6079, 4.6618) & & y^0 = (4.6473, 22.9709) \\ \alpha^0 = 27.6182 & & \beta^0 = 3.6018 \\ v^0 = (0.7303, 1.6618, 0.5539), & & \end{array}$$

and the lower bound is given by

$$v(LB(P2^0)) = 0.1259.$$

The upper bound can be updated to 0.1423 according to (45), and the corresponding incumbent solution is $x_{inc} = x^0$. In order to apply the branching rule to partition Ω^0 we get:

$$\theta_1 = \overline{y}_1 - \overline{x}_1^2 = 2.0620,$$
$$\theta_2 = \overline{y}_2 - \overline{x}_2^2 = 1.2385,$$
$$\max_{i=1,2}\{\theta_i\} = 2.0620 \text{ and } p = \arg\max_{i=1,2}\{\theta_i\} = 1.$$

We thus obtain Ω^1 and Ω^2 based on the partition of $[l_1, u_1] = [1, 5]$ into $[l_1^1, u_1^1] = [1, 1.6079]$ and $[l_1^2, u_1^2] = [1.6079, 5]$:

$$\Omega^1 = \{x : 1 \leq x_1 \leq 1.6079, 1 \leq x_2 \leq 5\},$$
$$\Omega^2 = \{x : 1.6079 \leq x_1 \leq 5, 1 \leq x_2 \leq 5\}.$$

Now, solving $LB(P2^1)$, we get:

$$\begin{array}{lll} x^1 = (1.5668, 4.6576) & & y^1 = (2.4782, 22.9457) \\ \alpha^1 = 25.4239 & & \beta^1 = 3.5758 \\ v^1 = (0.7756, 1.6576, 0.4760) & & v(LB(P2^1)) = 0.1353 \end{array}$$

and the upper bound UB is updated to 0.1422, with $x_{inc} = x^1$.

Likewise, $LB(P2^2)$ leads to the following solution:

$$\begin{array}{lll} x^2 = (1.6249, 4.6790) & & y^2 = (2.6979, 23.0738) \\ \alpha^2 = 25.7718 & & \beta^2 = 3.6294 \\ v^2 = (0.6961, 1.6790, 0.5709) & & v(LB(P2^1)) = 0.1356. \end{array}$$

The Upper Bound is updated to 0.1421, with $x_{inc} = x^2$. Now, $L = \{1, 2\}$ and since

$$k \in argmin\{v(LB(P2^t)) : t \in L\} = 1,$$

the first subproblem is chosen to apply the branching rule and to partition the corresponding hyperrectangle Ω^1. This yields:

$$\max_{i=1,2}\{\theta_i\} = 1.2523 \text{ and } p = \arg\max_{i=1,2}\{\theta_i\} = 2.$$

Accordingly, Ω^3 and Ω^4 are constructed based on the partition of $[l_2^1, u_2^1] = [1, 5]$ into $[l_2^3, u_2^3] = [1, 4.6576]$ and $[l_2^4, u_2^4] = [4.6576, 5]$, respectively:

$$\Omega^3 = \{x : 1 \le x_1 \le 1.6079, 1 \le x_2 \le 4.6576\},$$

$$\Omega^4 = \{x : 1 \le x_1 \le 1.6079, 4.6576 \le x_2 \le 5\}.$$

The solution to $LB(P2^3)$ is as follows:

$$\begin{array}{lllllll} x^3 & = & (1.5686, 4.6575) & & y^3 & = & (2.4830, 21.6929) \\ \alpha^1 & = & 24.1759 & & \beta^1 & = & 3.5764 \\ v^3 & = & (0.7738, 1.6575, 0.4797) & & v(LB(P2^3)) & = & 0.1421 = UB. \end{array}$$

Consequently, the node corresponding to subproblem $LB(P2^3)$ is fathomed. Hence, L is updated to $\{2\}$ and $LB(P2^4)$ is solved, producing the following solution:

$$\begin{array}{lllllll} x^4 & = & (1.5880, 4.7563) & & y^4 & = & (2.5335, 22.6460) \\ \alpha^4 & = & 25.1795 & & \beta^4 & = & 3.6906 \\ v^4 & = & (0.6557, 1.7563, 0.4198) & & v(LB(P2^4)) & = & 0.1410. \end{array}$$

The upper bound is updated to 0.1412 with $x_{inc} = x^4$. Thus, $L = \{2, 4\}$ and $LB(P2^2)$ is the next subproblem that is selected to continue the search, because

$$\begin{aligned} v(LB(P2^2)) &= \min\{v(LB(P2^2)), v(LB(P2^4))\} \\ &= \min\{0.1356, 0.1410\} \\ &= 0.1356. \end{aligned}$$

Now, $\max\{\theta_i\} = 1.1810$ and $p = 2$, and proceeding as above, the partition of Ω^2 results in

$$\Omega^5 = \{x : 1.6079 \le x_1 \le 5, 1 \le x_2 \le 4.6790\},$$

$$\Omega^6 = \{x : 1.6079 \le x_1 \le 5, 4.6790 \le x_2 \le 5\}.$$

Since $v(LB(P2^5)) = 0.1418$ is greater than the upper bound, this node is fathomed. The optimal value to $LB(P2^6)$ is $v(LB(P2^6)) = 0.1408$. Hence, L is updated to $L = \{4, 6\}$. The next subproblem to be picked for branching is $LB(P2^6)$. Now $\max\{\theta_i\} = 0.1356$, $p = 1$ and from the partition of Ω^6, we obtain:

$$\Omega^7 = \{x : 1.6079 \le x_1 \le 1.6483, 4.6790 \le x_2 \le 5\},$$

$$\Omega^8 = \{x : 1.6483 \le x_1 \le 5, 4.6790 \le x_2 \le 5\}.$$

The optimal values of $v(LB(P2^7))$ and $v(LB(P2^8))$ are respectively 0.1411 and 0.1415. Since this last value is greater than the upper bound, we only include node 7 in L, which yields $L = \{4, 7\}$. This process continues until a global minimum is found. The search tree in Figure 2 indicates how the algorithm has performed in order to find such a global minimum. For each node, the optimal value of the lower bound problem (LB), the upper bound (UB), when it is updated in that node, and the optimal solution $x = (x_1, x_2)$ of the corresponding relaxation are shown. In the right-upper corner of each box (node), we indicate the order in which each node is selected from the queue L. The number appearing at the top of each box shows the order in which each node is inspected. The value of θ is given presented on the right of each box.

The search inspects 16 nodes, but only 7 are introduced in the queue L for branching. The variable that induced the partition of Ω^k for each branch is also depicted in Figure 2. The optimal

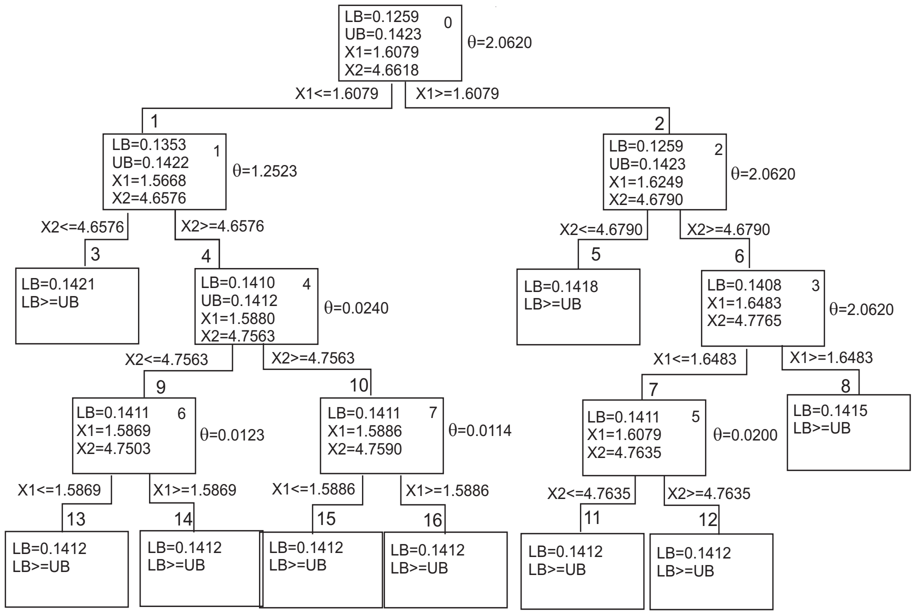

Figure 2: Graphical representation for the solution of the illustrative example.

solution obtained is the incumbent solution corresponding to the last update of the upper bound (0.1412), which is given by $x^4 = (1.5880, 4.7563)$. The optimal correction of the matrix $[A, -b]$, as given by (7) and (8), is

$$[H,p] = -\frac{1}{26.1441}\begin{bmatrix} 0.0251 \\ 0.0672 \\ 0.0161 \end{bmatrix}\begin{bmatrix} 1.5880 & 4.75633 & 1 \end{bmatrix} = \begin{bmatrix} -0.03983 & -0.1193 & 0.0251 \\ -0.1067 & -0.3195 & 0.0672 \\ -0.0255 & -0.0764 & 0.0161 \end{bmatrix}.$$

Thus, the corrected linear system is

$$\left\{\begin{array}{rcrcrcr} -1.0398 & x_1 & - & 1.1193 & x_2 & \leq & -6.9749 \\ -0.1067 & x_1 & + & 0.6805 & x_2 & \leq & 3.0672 \\ 1.9745 & x_1 & - & 1.0764 & x_2 & \leq & -1.9839. \end{array}\right.$$

As expected, upon the substitution of $(x_1, x_2) = (1.5880, 4.7563)$ the inequalities are verified to be satisfied as equalities.

4 Computational experience

In order to test the performance of the algorithm we report some computational results for a set of infeasible linear systems of the type $\{x \in \mathbb{R}^n : Ax \leq b, l \leq x \leq u\}$, where A and b are respectively a matrix having m rows and n columns and a vector of size m, both with real coefficients that were randomly generated. Column "total-vars" represents the actual number of variables in the subproblem at each iteration. Table 1 summarizes the main characteristics of the test problems. All the tests have been performed on a Pentium(R) 4, CPU 2.60 Ghz, 512 Mb RAM computer. The optimization toolbox of Matlab was used to find the lower bounds. The tolerance parameter ϵ defined in subsection 3.4 was set to 10^{-5}.

Table 2 reports the following information for each test problem:

Optimum: the optimal value;

Problems	n	m	total-vars	l_i	u_i
Pro4 to Pro5	5	10	22	1	5
Pro6 to Pro10	10	20	42	1	5
Pro11 to Pro15	15	30	62	1	5
Pro16 to Pro20	20	40	82	1	5

Table 1: Dimension of test problems.

$Best - iter$**:** iteration at which the optimal value was obtained;

$Max - queue$**:** the maximal size of queue L;

$Nodes$**:** the total number of problems analyzed;

$Time$**:** the total CPU time in seconds.

Problem	Optimum	Best-iter	Max-queue	Nodes	Time
Pro4	157.6815	7	2	7	27.1023
Pro5	262.8295	8	3	8	48.1560
Pro6	315.1673	8	2	8	116.4530
Pro7	214.8733	35	8	36	749.8290
Pro8	2.1870e+003	1	1	1	12.7500
Pro9	149.3485	24	5	24	267.2040
Pro10	250.8688	20	5	21	438.7810
Pro11	396.0443	21	6	21	596.1250
Pro12	1.1305e+003	6	2	6	292.6710
Pro13	679.4864	15	5	15	1237.9010
Pro14	756.9513	11	4	11	792.0940
Pro15	364.4506	358	87	358	4.0702e+004
Pro16	1.1215e+003	40	17	40	8.4770e+003
Pro17	1.0930e+003	45	18	45	6.0519e+003
Pro18	1.7509e+003	15	5	15	3.0759e+003
Pro19	1.1046e+003	34	13	34	5.3618e+003
Pro20	944.7827	249	80	249	4.3561e+003

Table 2: Computational results.

The analysis of the results shows that we have been able to solve problems of medium size with a reasonable computational effort. Although we address in this paper a problem different from the one solved in [5], as no lower and upper limits were included in the set $\mathbf{X}$ in [5], we can still make some comparisons with the enumerative procedure discussed there. The algorithm introduced in this paper is far better in terms of computational effort to find a global optimum.

5 Conclusions and future work

In this paper, we have proposed a method for obtaining an optimal solution for a nonlinear nonconvex program that arises in a total least squares approach for finding a correction of an infeasible linear system of inequalities. Using its KKT conditions, we have obtained an equivalent formulation that was processed by a new global optimization branch-and-bound algorithm. This procedure exploits the Reformulation–Linearization Convexification Technique (RLT) to convexify a relaxation for deriving lower bounds on the optimal value of the original problem. Together with a framework to obtain upper bounds, we developed a tree search procedure based on a partitioning of the domain of the original variables. Computational experience showed that the approach was successful for handling medium-scale problems. This is not too restrictive, especially considering the application of this theory in the context of infeasible problem corrections. In many of these cases, we are required to maintain some constraints unchanged, and so, the nonlinear problem we formulate is defined only over the remaining constraints. Also, this formulation could be applied to Irreducible Inconsistent Systems (IIS) as identified in postoptimality analyses [8], where each IIS might involve only a relatively small subset of the original LP constraints.

We would like to point out the importance of finding global optimal corrections, for instance, in the framework of Constraint Satisfaction Techniques, as in other contexts. When dealing with real models, it is essential to make as small changes as possible, in order to mitigate the risk of invalidating the corrected model.

The solution of problems of this type with other definitions for the compact set $\mathbf{X}$ is a useful topic for future investigation. We also recommend considering a partitioning of the constraints into two groups, namely, soft and hard constraints, where the set of hard constraints is assumed to be invariant and cannot be corrected. This situation is typical in the analysis of problems that arise in real-life applications.

References

[1] E. Amaldi and V. Kann, *The complexity and approximability of finding maximum feasible subsystems of linear relations*, Theoretical Computer Science, 147 (1995), pp. 181–210.

[2] E. Amaldi, M. E. Pfetsch, and J. L. E. Trotter, *On the maximum feasible subsystem problem, iiss and iis-hypergraphs*, Mathematical Series A, 95 (2003), pp. 533–554.

[3] P. Amaral, *Contribuições para o estudo de sistemas lineares inconsistentes*, PhD thesis, Faculty of Science and Technology, UNL, Lisbon, 2001.

[4] P. Amaral and P. Barahona, *About infeasibility in the constraints of a linear model*, Ricerca Operativa, 92 (1999), pp. 49–67.

[5] ——, *On optimal correction of inconsistent linear constraints*, in Principles and Practice of Constraint Programming, CP'2002 (Procs.), P. V. Hentenryck, ed., vol. 2470 of Lecture Notes in Computer Science, 2002, pp. 33–46.

[6] ——, *Connections between the total least squares and the correction of an infeasible system of linear inequalities*, Linear Algebra and Applications, 395 (2005), pp. 191–210.

[7] ——, *A framework for optimal correction of inconsistent linear constraints*, Constraints, 10 (2005), pp. 67–86.

[8] M. S. Bazaraa, J. J. Jarvis, and H. D. Sherali, *Linear Programming and Network Flows*, John Wiley & Sons, New York, NY, 1990.

[9] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear Programming: Theory and Algorithms*, John Wiley & Sons, New York, NY, 1993.

[10] N. Chakravarti, *Some results concerning post-infeasibility analysis*, EJOR, 73 (1994), pp. 139–143.

[11] J. W. Chinneck, *Minos(iis) : Infeasibility analysis using minos*, Computers and Operations Research, 21 (1994), pp. 1–9.

[12] ——, *Finding a useful subset of constraints for analysis in an infeasible linear program*, INFORMS J. Comput., 9 (1997), pp. 164–174.

[13] J. W. Chinneck and E. W. Dravnieks, *Locating minimal infeasible constraint sets in linear programs*, ORSA Journal on Computing, 3 (1991), pp. 157–168.

[14] J. W. Chinneck and M. A. Saunders, *Minos(iis) version 4.2: Analyzing infeasibilities in linear programs*, European Journal of Operations Research, 81 (1995), pp. 217–218.

[15] H. J. Greenberg, *Computer-assisted analysis for diagnosing infeasible or unbounded linear programs*, Mathematical Programming, Study 31 (1987), pp. 79–97.

[16] ——, *Enhancements of analyze: A computer-assisted analysis system for mathematical programming models and solutions*, ACM Transactions On Mathematical Software, 19 (1993), pp. 233–256.

[17] ——, *How to analyze the results of linear programs- part 3: Infeasibility diagnoses*, Interfaces, 23 (1993), pp. 120–139.

[18] ——, *Consistency, redundancy and implied equalities in linear systems*, Annals of Mathematics and artificial Intelligence, 17 (1996), pp. 37–83.

[19] H. J. Greenberg and F. H. Murphy, *Approaches to diagnosing infeasible linear programs*, Mathematical Programming, (1987).

[20] O. Guieu and J. W. Chinneck, *Analyzing infeasible mixed-integer and integer linear programs*, INFORMS J. Comput., (1999).

[21] C. Holzbaur, F. Menezes, and P. Barahona, *Defeasibility in clp(q) through generalised slack variables*, in Proceedings of CP´96, 2nd Int. Conf. in Principles and Practice of Constraint Programming, E. C. Freuder, ed., vol. 1118 of Lecture Notes in Computer Science, Springer–Verlag, 1996, pp. 209–223.

[22] J. N. M. V. Loon, *Irreducibly inconsistent systems of linear inequalities*, EJOR, 8 (1981), pp. 282–288.

[23] J. Pena, *Understanding the geometry of infeasible perturbations of a conic linear system*, SIAM Journal on Optimization, 10 (2000), pp. 534–550.

[24] J. Renegar, *Some perturbation theory for linear programming*, Mathematical Programming, 65 (1994), pp. 73–91.

[25] G. M. Roodman, *Post-infeasibility analysis in linear programming*, Journal of Global Optimization, 2 (1992), pp. 101–112.

[26] J. R.Vera, *Ill-posedness and the complexity of deciding existence of solutions to linear programs*, SIAM Journal Optimization, 6 (1996), pp. 549–569.

[27] H. D. Sherali and C. H. Tuncbilek, *A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique*, Journal of Global Optimization, 2 (1992), pp. 101–112.

[28] A. A. Vatolin, *An lp-based algorithm for the correction of inconsistent linear equation and inequality systems*, Optimization, 24 (1992), pp. 157–164.

[29] H.-F. Wang and C.-S. Huang, *Inconsistent structures of linear systems*, Int. J. General Systems, 21 (1992), pp. 65–81.