The joint characterization of discrete and continuous 'waiting times' through their reciprocal relationships

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Abstract

Let us suppose there is an event C (occurrence of a given defect or disease) that is part of a group of events we are interested in, and whose probability of occurrence in that group is known. The distribution of the waiting times for the r_1-th event in that group, given that we expect r events C, and the distribution of the number of events in the group that not C, given that we waited for a length y, waiting for r_1 events in the group, given that we expected r events C, are derived based on very mild assumptions. Relations of the distributions obtained with known distributions, their expression as mixtures and a limiting case are also studied.

Cases where $r_1 < r$ and $r_1 > r$ are studied in detail, since they correspond to two different situations of interest, the one in which the event C is one of the the rarest ones in its group, is not easy to identify or its occurrence r times kills or disables the observation unit, or the case in which it may be rather common in the group and easy to identify.

Examples of application in epidemiology, industry, transportation and agriculture are used for illustration.

Key words: Waiting times discrete and continuous, mixtures, Negative Binomial, Generalized Gamma distribution, non-central Generalized Gamma distribution.

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1 Introduction

Let us suppose that we know that a given disease or a given defect occurs with a probability p among other diseases or defects which form a well defined 'group', which we are interested in. The questions we raise are: will we be able, under just very mild assumptions on the distribution of such diseases or defects, to compute the waiting times for a given number of events in the whole group, given that we expect a given number r of occurrences of disease or defect C, and the probabilities of the number of overall occurrences of the diseases or defects in that 'group' by just observing the waiting times for a few occurrences of the disease or defect C? Situations where the event C is taken to be the easiest one to spot or correctly identify or otherwise situations where it is one of the rarest in the group, or occurs hidden, or is just not easy to identify or yet its occurrence a given number of times kills or disables the observation unit are studied.

An interpretation of the distributions of the waiting times is given in terms of mixtures for several cases.

As a side result, the Negative Binomial distribution arises as a mixing of generalized Gamma mixtures of discrete confluent hypergeometric distributions.

In Appendix B is settled down the notation used for the distributions in the paper and there is also a brief summary of some of their interrelationships.

2 The main results

2.1 The Negative Binomial mixture of generalized Gammas and the Negative Binomial as a mixing of confluent hypergeometric distributions

Hereon we will call the occurrence of the event C a 'success'. The r.v. X will represent the number of failures till the r-th success, that is the r-th event C, and the r.v. Y will represent the waiting time till the r_1-th event of the group of events we are interested in, given that we expect r events C.

Theorem 1 *Let X be the number of failures till the r-th success, being p the probability of success in the population, that is, let*

$$X \sim NB(r,\, p)\,.$$

Then, if Y given $X{=}x$ has the distribution of the power $1/\beta$ ($\beta{\in}\mathbb{R}\backslash\{0\}$) of the 'waiting time' for the $(r_1{+}x)$-th event ($r_1 > 0$) from a Poisson distribution with rate λ, that is, if

$$(Y|X=x) \sim G\left(r_1+x,\, \lambda,\, \beta\right)\,,$$

then the marginal distribution of Y is a Negative Binomial mixture of generalized Gammas, which has p.d.f. (probability density function)

$$f_Y(y) \;=\; \frac{|\beta|\,\lambda^{r_1}}{\Gamma(r_1)}\;\mathrm{e}^{-\lambda y^\beta}\;y^{\beta r_1-1}\,p^{r_1}\,{}_1F_1\Big(r,\,r_1;\,\lambda y^\beta(1-p)\Big)\;,$$

which for $r_1 > r$ is the p.d.f of the power $1/\beta$ of the sum of two independent r.v.$^{\text{s}}$ *Z_1 and Z_2, where Z_1 has the distribution of the 'waiting time' for the (r_1-r)-th event from a Poisson distribution with rate λ and Z_2 has the distribution of the 'waiting time' for the r-th event from a Poisson distribution with rate λp, that is, where*

$$Z_1 \;\sim\; G(r_1-r,\,\lambda) \qquad \text{and} \qquad Z_2 \;\sim\; G(r,\,\lambda p)\,.$$

Moreover, X given $Y=y$ has a discrete Kummer confluent hypergeometric distribution.

Proof: If

$$X \;\sim\; NB(r,p) \qquad \text{and} \qquad (Y|X=x) \;\sim\; G(r_1+x,\lambda)\,,$$

with $0<p<1$, $\lambda>0$, $r_1, r \in I\!N$ and $x=0,1\ldots$, then

$$\begin{aligned} f_{X,Y}(x,y) &= f_{Y|X=x}(y)\,f_X(x) \\ &= \frac{|\beta|\,\lambda^{r_1+x}}{\Gamma(r_1+x)}\;\mathrm{e}^{-\lambda y^\beta}\;y^{\beta(r_1+x)-1}\binom{r+x-1}{x}p^r\,(1-p)^x\,, \end{aligned} \tag{1}$$

where we should notice that it is required that $r_1>0$, since $x=0,1,\ldots$, and the shape parameter of the conditional Gamma distribution of $Y|X=x$ has to be positive. But then the marginal p.d.f. of Y is obtained, by summing over $x=0,1,\ldots$, giving rise to what we may call a Negative Binomial mixture of generalized Gammas, that is, a mixture of $G(r_1+x,\lambda,\beta)$ distributions $(x=0,1,\ldots)$, with Negative Binomial weights,

$$\begin{aligned} f_Y(y) &= p^r\;\mathrm{e}^{-\lambda y^\beta}\sum_{x=0}^{\infty}\frac{|\beta|\,\lambda^{r_1+x}}{\Gamma(r_1+x)}\,y^{\beta(r_1+x)-1}\,\frac{\Gamma(r+x)}{\Gamma(r)\,x!}\,(1-p)^x \\ &= \frac{p^r\,|\beta|\,\lambda^{r_1}\;\mathrm{e}^{-\lambda y^\beta}\;y^{\beta r_1-1}}{\Gamma(r_1)}\;\underbrace{\frac{\Gamma(r_1)}{\Gamma(r)}\sum_{x=0}^{\infty}\frac{\Gamma(r+x)}{\Gamma(r_1+x)}\,\frac{(\lambda y^\beta(1-p))^x}{x!}}_{{}_1F_1\left(r,r_1;\lambda y^\beta(1-p)\right)} \\ &= \frac{|\beta|\,\lambda^{r_1}}{\Gamma(r_1)}\;\mathrm{e}^{-\lambda y^\beta}\;y^{\beta r_1-1}\,p^r\,{}_1F_1\Big(r,r_1;\lambda y^\beta(1-p)\Big) \end{aligned} \tag{2}$$

which for $r_1 > r$, when compared with the p.d.f. in (B.1) in Appendix B, may be seen as the p.d.f. of the power $1/\beta$ of the sum of two independent Gamma r.v.$^{\text{s}}$, say Z_1 and Z_2, with

$$Z_1 \sim G(r_1 - r, \lambda)\,, \qquad Z_2 \sim G(r, \lambda p)\,.$$

Then it is easy to obtain the conditional p.m.f. (probability mass function) of X given $Y = y$, as

$$\begin{aligned} f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} &= \frac{\frac{|\beta|\lambda^{r_1+x}}{\Gamma(r_1+x)}\ \mathrm{e}^{-\lambda y^\beta}\ y^{\beta(r_1+x)-1}\binom{r+x-1}{x}p^r\,(1-p)^x}{|\beta|\frac{\lambda^{r_1}}{\Gamma(r_1)}\,p^r\ \mathrm{e}^{-\lambda y^\beta}\ y^{\beta r_1-1}\,{}_1F_1\Big(r, r_1; \lambda y^\beta(1-p)\Big)} \\ &= \frac{\frac{\Gamma(r+x)}{\Gamma(r)}\ \frac{\Gamma(r_1)}{\Gamma(r_1+x)}\ \frac{\left(\lambda y^\beta(1-p)\right)^x}{x!}}{{}_1F_1\Big(r, r_1; \lambda y^\beta(1-p)\Big)}\,, \quad \begin{array}{l}(0<p<1,\\ \lambda, y>0; r_1, r\in I\!N\\ x=0,1,\dots)\end{array} \end{aligned} \tag{3}$$

that is a Kummer confluent hypergeometric distribution (see Definition 11 in Appendix B).

We may note that in all the above, mainly if we write $\binom{r_1+x-1}{x}$ as $\frac{\Gamma(r_1+x)}{\Gamma(r_1)\,x!}$ it is indeed not required neither r_1 nor r to be integers, but only positive reals. □

Results concerning the marginal moments of Y and the conditional moments of X, given $Y = y$, may be analyzed in Appendix A.

We may note that for $r_1 > r$ with $r_1, r \in I\!N$ the distribution of Y is also a particular Generalized Integer Gamma distribution (Coelho, 1998) of depth 2 whose p.d.f., using finite sums, may be written as

$$f_Y(y) \;=\; K\left[\mathrm{e}^{-\lambda y}\sum_{k=1}^{r_1-r} c_{1,k}y^{k-1} + \mathrm{e}^{-\lambda p y}\sum_{k=1}^{r} c_{2,k}y^{k-1}\right]$$

and whose c.d.f. (cumulative distribution function) may be written as

$$\begin{aligned} F_Y(y) \;=\; 1 - K\Bigg[&\mathrm{e}^{-\lambda y}\sum_{k=1}^{r_1-r} c_{1,k}(k-1)!\sum_{j=0}^{k-1}\frac{y^j}{j!\lambda^{k-j}} \\ &+ \mathrm{e}^{-\lambda p y}\sum_{k=1}^{r} c_{2,k}(k-1)!\sum_{j=0}^{k-1}\frac{y^j}{j!(\lambda p)^{k-j}}\Bigg] \end{aligned}$$

with

$$K \;=\; \lambda^{r_1-r}(\lambda p)^r \;=\; \lambda^{r_1} p^r\,,$$

and

$$c_{i,r_i^*} \;=\; \frac{1}{\Gamma(r_i^*)}\,(\lambda_{3-i}-\lambda_i)\;, \qquad\qquad i=1,2$$

and, for $k=1,\ldots,r_i^*-1$,

$$c_{i,r_i^*-k} \;=\; \frac{1}{k}\sum_{j=1}^{k}\frac{(r_i^*-k-1+j)!}{(r_i^*-k-1)!}\,r_{3-i}^*\,(\lambda_i-\lambda_{3-i})^{-j}\,c_{i,r_i^*-(k-j)}\,, \qquad i=1,2$$

with

$$r_1^*=r_1-r\,,\quad r_2^*=r\,,\quad \lambda_1=\lambda\,,\quad \lambda_2=\lambda p\,.$$

Taking the reverse result, that is, assuming as known the conditional distribution of X given $Y=y$, given by (3) above, and the marginal distribution of Y, given by (2) above, we may obtain the Negative Binomial distribution as a mixing of confluent hypergeometric distributions. For $\beta=1$ the result is similar to the one from Bhattacharya (1966) and for $\beta=1$ and $r_1=r$ it yields the well known result from Greenwood and Yule (1920) on obtaining the Negative Binomial distribution as a Gamma mixing of Poisson distributions, so that we may see this result as extending both of the above results.

Corollary 2 *If*

$$f_{X|Y=y}(x) \;=\; \frac{\Gamma(r+x)\,\Gamma(r_1)}{\Gamma(r)\,\Gamma(r_1+x)}\;\frac{\left(\lambda y^\beta(1-p)\right)^x}{x!}\;\frac{1}{{}_1F_1\left(r,r_1;\lambda y^\beta(1-p)\right)}$$

and Y is (marginally) a Negative Binomial mixture of $G(r_1+x,\lambda,\beta)$ distributions, or more precisely, if Y is a mixture of $G(r_1+x,\lambda,\beta)$ $(x=0,1,\ldots)$, distributions with weights given by a Negative Binomial distribution with parameters $r\in I\!N$ and p, then X has a (marginal) Negative Binomial distribution with parameters r and p.

In particular, if $r_1=r$, in which case (see Corollary 3 in subsection 2.2)

$$(X|Y=y)\;\sim\;P(\lambda y^\beta(1-p)) \qquad \text{and} \qquad Y\;\sim\;\Gamma(r_1,\lambda p,\beta)\,,$$

we still have

$$X\;\sim\;NB(r,p)\,,$$

what for $\beta=1$ is Greenwood and Yule (1920) result on obtaining the Negative Binomial distribution as a Gamma mixing of Poissons.

Proof: The proof is very easy since in this case we have the joint p.d.f. of X and Y given by

$$f_{X,Y}(x,y) \;=\; f_{X|Y=y}(x)\,f_Y(y)$$

that is (1) in the proof of Theorem 1, so that

$$f_X(x) \;=\; \underbrace{\int_0^\infty \frac{|\beta|\lambda^{r_1+x}}{\Gamma(r_1+x)}\, \mathrm{e}^{-\lambda y^\beta}\; y^{\beta(r_1+x)-1}\,dy}_{=1} \binom{r+x-1}{x} p^r(1-p)^x\,. \qquad \square$$

This result where, as it may be seen in the next subsection, the distribution of Y may be regarded as a mixture of generalized Gamma distributions extends the results of Bhattacharya (1966), which is similar to the result above for $\beta = 1$, and the results of Chukwu and Gupta (1989) who obtained a Negative Binomial distribution through a generalized Gamma mixing of generalized Poissons, where the definitions of such distributions are slightly different from the ones used here.

2.2 Particular cases and interpretation of the distribution of Y in terms of mixtures

Particular cases of interest are:

i) If $r_1 > r$, as we saw in the proof of Theorem 1, the marginal distribution of Y may be seen as the distribution of the power $1/\beta$ of the sum of two independent Gamma r.v.^{s}. But in this case, if $r_1, r \in I\!N$, we may write

$$\begin{aligned} {}_1F_1(r, r_1, \theta) \;=\; (-1)^r \frac{(r_1-1)!}{\theta^{r_1-1}} \Bigg[&\sum_{k=0}^{r_1-r-1} \frac{\theta^k}{\Gamma(r)\,k!} \frac{\Gamma(r_1-1-k)}{\Gamma(r_1-r-k)} \\ &- \mathrm{e}^\theta \sum_{k=0}^{r-1} \frac{(-\theta)^k}{\Gamma(r-k)\,k!} \frac{\Gamma(r_1-1-k)}{\Gamma(r_1-r)} \Bigg], \end{aligned}$$

so that, after some small rearrangements, the p.d.f. of Y may be written as

$$\begin{aligned} f_Y(y) \;=\; &\sum_{k=1}^{r_1-r} \underbrace{\frac{|\beta|\lambda^k}{\Gamma(k)}\, \mathrm{e}^{-\lambda y^\beta}\; y^{\beta k-1}}_{\text{p.d.f. of } G(k,\lambda,\beta)} \frac{(-p)^r}{(1-p)^{r_1-k}} \binom{r_1-k-1}{r-1} \\ &+ \sum_{k=1}^{r} \underbrace{\frac{|\beta|(\lambda p)^k}{\Gamma(k)}\, \mathrm{e}^{-\lambda p y^\beta}\; y^{\beta k-1}}_{\text{p.d.f. of } G(k,\lambda p,\beta)} \frac{(-p)^{r-k}}{(1-p)^{r_1-k}} \binom{r_1-k-1}{r_1-r-1} \end{aligned}$$

that is the p.d.f. of a mixture of r_1 generalized Gamma r.v.^{s}, $r_1 - r$ of which are $G(k, \lambda, \beta)$ with weights $\frac{(-p)^r}{(1-p)^{r_1-k}} \binom{r_1-k-1}{r-1}$ (k=1, …, r_1−r) and

r of them are $G(k,\lambda p,\beta)$ with weights $\frac{(-p)^{r-k}}{(1-p)^{r_1-k}}\binom{r_1-k-1}{r_1-r-1}$ $(k=1,\dots,r)$.

We should note that this is actually a mixture with improper weights since although they add up to 1, some of them are always negative while some or all of them may have an absolute value greater than 1, being a noticeable fact that we may express the power $1/\beta$ of the sum of two independent Gamma r.v.s as a mixture of generalized Gamma r.v.s. For $k=1$ such distributions are indeed Weibull distributions, since then they are the distribution of the power $1/\beta$ of Exponential r.v.s. For some examples of weights we may look at Table 1.

Table 1. Examples of 'weights' for the mixture yielding the distribution of Y

r_1	r	p	'weights'	
5	4	1/4	1/81;	-4/81, 4/27, -4/9, 4/3
5	3	1/4	-4/27, -1/27;	16/27, -32/27, 16/9
5	4	1/2	1;	-2, 2, -2, 2
5	3	1/2	-6, -1;	12, -8, 4
5	4	3/4	81;	-108, 36, -12, 4
5	3	3/4	-324, -27;	432, -96, 16

ii) If $r=1$, X has a Geometric marginal distribution with probability parameter $0<p<1$ and X given $Y=y$ has a hyper-Poisson distribution (Bardwell and Crow, 1964), which, assuming $r_1\in I\!N$, is actually a Poisson distribution displaced by r_1-1 (see Appendix B, right after Definition 11). In this case, according to i), the distribution of Y is a mixture of r_1-1 r.v.s $G(k,\lambda,\beta)$ with improper weights $\frac{-p}{(1-p)^{r_1-k}}$ $(k=1,\dots,r_1-1)$ and one $G(1,\lambda p,\beta)$ (that is a Weibull distribution) with weight $\frac{1}{(1-p)^{r_1-1}}$.

iii) If $r_1\le r$ we cannot any longer interpret the distribution of Y as the distribution of the sum of two independent Gamma random variables but we may still see the distribution of Y as a mixture, since now, for $r_1\le r$, with $r_1,r\in I\!N$, we may write

$$ {}_1F_1(r,r_1;\theta) \;=\; (r_1-1)!\;\mathrm{e}^{\theta}\sum_{k=0}^{r-r_1}\frac{\theta^k}{(k+r_1-1)!}\binom{r-r_1}{k}, $$

and thus the p.d.f. of Y may be written as

$$ f_Y(y) \;=\; \sum_{k=0}^{r-r_1}\underbrace{\frac{|\beta|\lambda^{r_1+k}}{\Gamma(r_1+k)}\;\mathrm{e}^{-\lambda p y^{\beta}}\;p^{r_1+k}\,y^{\beta(r_1+k)-1}}_{\text{p.d.f. of } G(r_1+k,\lambda p,\beta)}\binom{r-r_1}{k}p^{r-r_1-k}(1-p)^k $$

that is clearly a Binomial mixture of $r-r_1+1$ $G(r_1+k,\lambda p,\beta)$ r.v.[s] $(k=0,\ldots,r-r_1)$ with weights given by a Binomial distribution with parameters $r-r_1$ and p.

iv) If $r_1=r$, that is, if $r+x$ is the number of events in the group we are waiting for, given that x is the number of failures we know we have to wait for till the r-th success, that is, if we wait for exactly r events C and x non-C events, then the following Corollary is easy to establish.

Corollary 3 *For $r_1=r$, Y has a generalized Gamma distribution with shape parameter r, rate parameter λp and power parameter β and X given $Y=y$ has a Poisson distribution with parameter $\lambda y^\beta(1-p)$.*

Proof: For $r_1=r$ we have

$$ {}_1F_1\Big(r,r_1;\lambda y^\beta(1-p)\Big) \;=\; \mathrm{e}^{\lambda y^\beta(1-p)} \tag{4}$$

so that the marginal p.d.f. of Y in (2) is, in this case,

$$f_Y(y) \;=\; \frac{|\beta|\lambda^r}{\Gamma(r)}\,p^r\;\mathrm{e}^{-\lambda y^\beta}\;e^{\lambda y^\beta(1-p)}\,y^{r-1} \;=\; \frac{|\beta|(\lambda p)^r}{\Gamma(r)}\;\mathrm{e}^{-\lambda p y^\beta}\;y^{r-1}$$

which is the p.d.f. of a r.v. with a $G(r,\lambda p,\beta)$ distribution. The conditional distribution of X given $Y=y$ is in this case a Poisson distribution with parameter $\lambda y^\beta(1-p)$, since, given (4) above and the equality of r_1 and r, we have the conditional p.m.f. of X, given $Y=y$, given by

$$f_{X|Y=y}(x) \;=\; \frac{\Big(\lambda y^\beta(1-p)\Big)^x}{x!}\,e^{-\lambda y^\beta(1-p)}\,. \qquad \square$$

In this case it is as if the r.v. Z_1 in Theorem 1 would vanish, remaining only the r.v. Z_2, what is also completely in agreement with the result in iii) on the expression for the distribution of Y as a Binomial mixture of Gamma r.v.[s], since in this case the mixture would have just one generalized Gamma r.v..

We should also note that in this case, taking the reciprocal result stated in the above Corollary, the marginal Negative Binomial distribution of X arises as a generalized Gamma mixing of Poissons with parameter $\lambda Y^\beta(1-p)$ where Y has a generalized Gamma distribution with shape parameter r, rate parameter λp and power parameter β.

In Section 4 we will see under what conditions we get for Y a marginal distribution that is a Poisson mixture of generalized Gamma distributions.

Anyway, we should note that in any of the 4 cases above, the marginal distribution of Y is a Negative Binomial mixture of generalized Gamma r.v.[s], being remarkable the equivalence of this mixture to the mixtures that arise in each one of the 4 cases above, namely the single generalized Gamma distribution in case iv), where a Negative Binomial mixture with parameters r and p of $G(r+x,\lambda,\beta)$ $(x=0,1,\ldots)$ r.v.[s] yields a $G(r,\lambda p,\beta)$ r.v..

3 Interpretation of the results through an example

Let C be a disease in a group of diseases we are interested in, and let p ($0 < p < 1$) be the prevalence of disease C in that group, that is, the proportion of occurrences of disease C among the occurrences of diseases in that group. Let hereon G denote the set of events whose elements are occurrences of diseases in our group of interest, C denote the set of events whose elements are occurrences of disease C and let $\overline{C}$ denote the set of events whose elements are occurrences of diseases in group G that are not cases of disease C. Then,

i) $X \sim NB(r, p)$ — X is the r.v. that represents the number of events in $\overline{C}$, this is, of diseases in the group, that not disease C (say 'failures'), that occur till the r-th occurrence of disease C (X is sometimes called a 'discrete waiting time'),

ii) $(Y|X = x) \sim \Gamma(r_1 + x, \lambda, \beta)$ $(r_1 > 0)$ — given that $X = x$, that is, if we know we have to wait for exactly x occurrences from other diseases that not disease C till the r-th occurrence of disease C, $(Y|X = x)$ is a r.v. that represents the waiting time for the $(r_1 + x)$-th occurrence of a disease in the group of diseases we are interested in; we suppose that this waiting time has the same distribution as the power $1/\beta$ of the waiting time for the $(r_1 + x)$-th event from a Poisson distribution with rate λ.

The marginal distribution of Y represents the distribution of the waiting time for the r_1-th occurrence of any disease in the group, given that we expect r occurrences of disease C. Under the above assumptions,

if $r_1 > r$,

iii) $Y \sim \left(G(r_1 - r, \lambda) + G(r, \lambda p)\right)^{1/\beta}$ — 'in global terms', that is, 'integrating' over or considering all the possible values of x (which are $0, 1, \ldots$) and their corresponding probabilities, the marginal distribution of the waiting time for the r_1-th occurrence of any disease in the group, given that we expect r events C, is the same as the power $1/\beta$ of the sum of the waiting time for the $(r_1 - r)$-th event from a Poisson distribution with rate λ and the waiting time for the r-th event from a Poisson distribution with rate λp.

but if $r_1 = r$ then

iv) $Y \sim \left(\Gamma(r, \lambda p)\right)^{1/\beta} \equiv \Gamma(r, \lambda p, \beta)$ — that is, in this case it is as if the 'first' waiting time above would varnish and Y has just the distribution of the power $1/\beta$ of the waiting time for the r-th event from a Poisson distribution with rate λp, that is a generalized Gamma distribution with shape parameter r, rate parameter λp and power parameter β. In this situation we wait exactly for r events of disease C. Accordingly, in this case the conditional distribution of $Y|X = x$ is exactly the waiting time for the $(r + x)$-th event of disease in the group, where x is the number of occurrences of diseases in the group that not disease C, till r-th occurrence of disease C, so that we waited exactly for r events of disease C to occur. All this is exactly what 'common sense' would tell us;

and if $r_1 < r$,

v) then it is not too clear what 'common sense' would tell us that the distribution of Y will be in this case. Anyway, as shown in the previous section, it is a mixture with Binomial weights with parameters $r_1 - r$ and p, of $G(r_1 + k, \lambda p, \beta)$ $(k = 1, \ldots, r_1 - r)$ distributions. That is, in this case it is as if the distribution of the waiting time for $r_1 < r$ events of disease in the group was the distribution of the power $1/\beta$ of the waiting time for the r_1-th event from a Poisson distribution with rate λp, with probability $p^{r_1 - r}$, plus the power $1/\beta$ of the waiting time for the $(r_1 + 1)$-th event from a Poisson distribution with rate λp, with probability $(r_1 - r)p^{r_1 - r - 1}(1 - p)$, plus, ..., plus the power $1/\beta$ of the waiting time for the r-th event from a Poisson distribution with rate λp, with probability $(1 - p)^{r_1 - r}$.

We may note that in any of the above cases we always have, for $\beta = 1$,

$$
\begin{aligned}
E(Y) = E\left[E(Y|X = x)|X \sim NB(r, p)\right] &= E\left(\tfrac{r_1 + x}{\lambda}\Big| X \sim NB(r, p)\right) \\
= \frac{r_1 + \frac{r(1+p)}{p}}{\lambda} = \frac{r_1}{\lambda} + \frac{r(1 - p)}{\lambda p} &= \frac{r_1 - r}{\lambda} + \frac{r}{\lambda p}\,,
\end{aligned}
$$

with

$$
\begin{aligned}
\tfrac{r_1}{\lambda p} < E(Y) < \tfrac{r}{\lambda p} \qquad &\text{if} \quad r_1 < r \quad \text{and} \\
\max\left(\tfrac{r_1}{\lambda}, \tfrac{r}{\lambda p}\right) < E(Y) < \tfrac{r_1}{\lambda p} \qquad &\text{if} \quad r_1 > r\,.
\end{aligned}
$$

The random variable $(X|Y = y)$ is the number of events in $\overline{C}$ that we have to wait for till the r-th event in C, given that we waited a 'length' y for the r_1-th event in G. The conditional p.m.f. of $X|Y = y$ gives the probability of X assuming a given value $x \in I\!N$, given that we know that we had to wait a 'length' y (of time) for the r_1-th occurrence of a disease in the group, given that we expected r occurrences of disease C.

Instead of the above example we may easily think of examples in industry: the 'success' being to find a given type of defect (say C) on a copper wire, defect that is known to occur with a probability p, among all types of defects, being we interested in the distribution of Y, the length of wire till we find the r_1-th defect, or in the distribution of $X|Y = y$, that is the probabilities that, given that we had to 'wait' y length of wire for the r_1-th defect of type C, what is the probability that the overall number of defects that not of type C is $x \in I\!N$. We may notice that once p is known or assumed as known, we may estimated the probability of a given number of defects just by observing one given type of defects, what is mostly useful in case the defect C is of a more noticeable type or easier to observe or to detect with less error.

The application of the results above to diseases in plants is straightforward, as well as for example to waiting times say for buses, where we consider the line of buses C, with a percentage p among all lines of buses, with the 'success' being the passage at the point we are standing at of a bus from line C.

4 An interesting limit situation

An interesting limit situation occurs when we take $p \to 1$ and $r \to \infty$ with $r(1-p) = \delta \in I\!R^+$ a constant. This would mean that, according to our main example in section 3, we would be waiting for infinitely many buses from line C to pass at our point and that such buses would tend to be the only ones in circulation or that the disease we were interested in recording was far more probable to occur then any other in its group of interest and that we were waiting to record infinitely many of its occurrences.

It is well known that in this situation the marginal Negative Binomial distribution of X tends in distribution to a Poisson distribution with parameter δ. One would then expect that in this case the marginal distribution of Y would tend to a Poisson mixture of generalized Gamma distributions. This is indeed what happens. If we take into account that

$$\lim_{\substack{r\to\infty\\ p\to 1\\ r(1-p)=\delta}} p^r \, {}_1F_1\left(r, r_1; \lambda y^\beta (1-p)\right) = \mathrm{e}^{-\delta} \, {}_0F_1\left(r_1, \delta\lambda y^\beta\right)$$

then, under this limit situation

$$\begin{aligned}\lim_{\substack{r\to\infty\\ p\to 1\\ r(1-p)=\delta}} f_Y(y) &= \lim_{\substack{r\to\infty\\ p\to 1\\ r(1-p)=\delta}} \frac{|\beta|\lambda^{r_1}}{\Gamma(r_1)} \, \mathrm{e}^{-\lambda y^\beta} \, y^{\beta r_1 - 1} \, p^r \, {}_1F_1\left(r, r_1; \lambda y^\beta (1-p)\right) \\ &= \frac{|\beta|\lambda^{r_1}}{\Gamma(r_1)} \, \mathrm{e}^{-\lambda y^\beta} \, y^{\beta r_1 - 1} \, \mathrm{e}^{-\delta} \, {}_0F_1\left(r, \delta\lambda y^\beta\right)\end{aligned}$$

that is the p.d.f. of a non-central generalized Gamma distribution with non-centrality parameter δ and shape parameter r, which is a Poisson mixture of generalized Gamma distributions with shape parameters $r+i$ ($i = 0, 1, \ldots$), rate parameter λ and power parameter β. Under this limit situation the conditional p.m.f. of X given $Y = y$ may be written as

$$\begin{aligned}\lim_{\substack{r\to\infty\\ p\to 1\\ r(1-p)=\delta}} f_{X|Y=y}(x) &= \lim_{\substack{r\to\infty\\ p\to 1\\ r(1-p)=\delta}} \frac{\binom{r+x-1}{x} p^r (1-p)^x \, \frac{\Gamma(r_1)}{\Gamma(r_1+x)} \left(\lambda y^\beta\right)^x}{p^r \, {}_1F_1\left(r, r_1; \lambda y^\beta (1-p)\right)} \\ &= \frac{\Gamma(r_1)}{\Gamma(r_1+x)} \left(\lambda y^\beta\right)^x \frac{\frac{\delta^x}{x!} \, \mathrm{e}^{-\delta}}{\mathrm{e}^{-\delta} \, {}_0F_1\left(r, \delta\lambda y^\beta\right)} \\ &= \frac{\Gamma(r_1)}{\Gamma(r_1+x)} \frac{\left(\delta\lambda y^\beta\right)^x}{x!} \frac{1}{{}_0F_1\left(r, \delta\lambda y^\beta\right)} \,.\end{aligned}$$

We should note that these results are in complete accordance with the previous results, since for $r > r_1$, we obtained, in subsection 2.2, the marginal distribution of Y as a Binomial mixture of generalized Gamma distributions, yielding under these limit conditions a Poisson mixture (with parameter δ) of generalized Gamma distributions, that is the non-central generalized Gamma distribution obtained as the limit distribution for Y.

If we further take $\delta \to 0$ then both the marginal and the conditional distributions of X tend to degenerate distributions with all the probability concentrated at $x = 0$ and the marginal distribution of Y tends to a central generalized Gamma distribution with shape parameter r, rate parameter λ and power parameter β, that was the same as the conditional distribution of Y given $X = x$, that is, in this extreme limit situation, X would be no longer a r.v. and everything would work as if the distribution of Y was the only relevant one. This result makes perfect sense and is like a closure for all these distributions.

5 Conclusions and final remarks

Estimation procedures for the parameters in the marginal distribution of Y and the conditional distribution of X are to be developed and studied, both when p is assumed known and not known.

A crude two step estimation procedure may however be easily designed. One may use, for fixed r, the usual estimator $\hat{p} = \frac{r}{\overline{X}+r}$ for p, based on a random sample from the marginal Negative Binomial distribution for X, of the number of events in $\overline{C}$ till the r-th event C. Then, in a second step one may estimate through the method of moments the parameters r, λ and β from the waiting times y, based on a sample of a given size, by setting $r_1 = 1$, computing the first three raw sample moments and equating them to the expressions for the corresponding population moments of Y given in Appendix A. This procedure seems to work rather well for $r_1 = 1$ and this way, by just waiting for the first event in the group, we will be able to estimate the parameters r, λ and β. Then, with the estimate of r in hands we may think of re-estimating p. We may this way by just observing $r_1 = 1$ event in the group and measuring the waiting time till it happens estimate r, that is, how many events C we were expecting, what is mainly useful when the event C is rare in the group, it happens in a hidden way or a given number of these events kills or disables the observation unit, as it commonly happens with diseases in plants.

An interesting fact is that whatever is the relation between r_1 and r, we always have for $\beta = 1$, $E(Y) = \frac{r_1 - r}{\lambda} + \frac{r}{\lambda p}$, a result that may be easily obtained from the conditional expectation of Y given $X = x$ or even directly from the expression for $E(Y^h)$ in Appendix A, with some more calculation. We may then note that for $r_1 < r$ it is as if the waiting time was right censored.

Appendices

A Expressions for the moments of Y and of $X|Y=y$

Given the marginal distribution of Y and the conditional distribution of X given $Y=y$, in Theorem 1, after some little calculation the h-th raw moment of Y may be written as

$$E\left(Y^h\right) = \underbrace{\frac{\Gamma\left(r_1+\frac{h}{\beta}\right)}{\Gamma\left(r_1\right)}\,\frac{1}{\lambda^{h/\beta}}}_{h\text{-th moment of } G(r_1,\lambda,\beta)}\, p^r\ {}_2F_1\left(r, r_1+\frac{h}{\beta}; r_1; 1-p\right)$$

where we may note that the first two terms define the h-th raw moment of a generalized Gamma distribution with shape parameter r_1, rate parameter λ and power parameter β. For $r, r_1 \in I\!N$ we may use the fact that the distribution of Y may be expressed as a mixture, and using the results in subsection 2.2 and the fact that the h-th raw moment of a $G(r,\lambda,\beta)$ distribution is given by

$$\frac{\Gamma\left(r+\frac{h}{\beta}\right)}{\Gamma(r)}\,\lambda^{-h/\beta}\,,$$

to express the h-th raw moment of Y as

$$E\left(Y^h\right) = \sum_{k=1}^{r_1-r} \frac{(-p)^r}{(1-p)^{r_1-k}}\binom{r_1-k-1}{r-1}\frac{\Gamma\left(k+\frac{h}{\beta}\right)}{\Gamma(k)}\,\lambda^{-h/\beta}$$
$$+\sum_{k=1}^{r} \frac{(-p)^{r-k}}{(1-p)^{r_1-k}}\binom{r_1-k-1}{r_1-r-1}\frac{\Gamma\left(k+\frac{h}{\beta}\right)}{\Gamma(k)}\,(\lambda p)^{-h/\beta}\,, \qquad \text{for } r_1>r,$$

or as

$$E\left(Y^h\right) = \sum_{k=0}^{r-r_1}\binom{r-r_1}{k}p^{r-r_1-k}(1-p)^k\,\frac{\Gamma\left(r_1+k+\frac{h}{\beta}\right)}{\Gamma(r_1+k)}\,(\lambda p)^{-h/\beta}\,, \qquad \text{for } r\geq r_1\,.$$

On the other hand, after some calculations, it is possible to express the h-th raw moment of X, given $Y=y$ as

$$E\left(X^h|Y=y\right) = \frac{1}{{}_1F_1\left(r, r_1; \lambda y^\beta(1-p)\right)}$$
$$\sum_{k=1}^{h} t_{h,k}\left(\lambda y^\beta(1-p)\right)^k\frac{\Gamma(r+k)\,\Gamma(r_1)}{\Gamma(r)\,\Gamma(r_1+k)}\ {}_1F_1\left(r+k, r_1+k; \lambda y^\beta(1-p)\right)$$

where, for $h = 2, 3, \dots$ and $k \in \{1, \dots, h\}$,

$$t_{h,k} = k\, t_{h-1,k} + t_{h-1,k-1} \qquad \text{with} \qquad t_{1,1} = 1 \;\text{ and }\; t_{h,k} = 0 \begin{cases} k > h \\ h = 0 \text{ or } k = 0 \end{cases}$$

and where for $r, r_1 \in I\!N$ we may use the expressions for ${}_1F_1(\cdot, \cdot; \cdot)$ in subsection 2.2. If one wants to get a look at the structure of the values of $t_{h,k}$, such values are displayed in Table 2, for $h = 1, \dots, 10$. We may note that in fact for any $h \in I\!N$, $t_{h,1} = t_{1,1} = 1$.

Table 2. Values of $t_{h,k}$ for $h = 1, \dots, 10$, with $k \in \{1, \dots, h\}$ running from the left to the right.

h										
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

B Definition of some distributions used in the paper

Since notation for some distributions may vary slightly, in this Appendix we will establish some notation and a couple of results needed in the paper.

Definition 4 *We will say that the discrete r.v. X has a Negative Binomial distribution with parameters $r \in I\!N$ and $p \in]0, 1[$ if and only if the p.m.f. (probability mass function) of X is*

$$f_X(x) \;=\; P(X = x) \;=\; \binom{x + r - 1}{x} p^r \,(1 - p)^x\,, \quad x = 0, 1, \dots\;.$$

We will denote this fact by

$$X \;\sim\; NB(r, p)\,.$$

The r.v. X represents the number of failures till the r-th success in a sequence of independent Bernoulli trials with probability of success p.

The above Negative Binomial distribution is sometimes called a discrete 'waiting time' distribution. We may also notice that in this distribution if we write $\Gamma(x+r)/(\Gamma(r)\,\Gamma(x+1))$ instead of $\binom{x+r-1}{x}$, r is no longer required to be an integer.

Definition 5 *We will say that the discrete r.v. Y has a Poisson distribution with parameter $\lambda(>0)$ if and only if the p.m.f. of Y is*

$$f_Y(y) \;=\; P(Y=y) \;=\; \frac{\mathrm{e}^{-\lambda}\,\lambda^y}{y!}\,, \qquad y=0,1,\dots\;.$$

We will denote this fact by

$$Y \;\sim\; P(\lambda)\,.$$

It is a well known fact that if $X \sim NB(r,p)$ and if $r \to \infty$ and $p \to 1$, such that $r(1-p)=\delta \in I\!R^+$, then the distribution of X tends weakly to a Poisson distribution with parameter δ.

Let us consider the event A and let Y be the number of events A in a given interval of length t. Then, a well-known fact is that if $Y \sim P(\lambda t)$ and X is the 'waiting time' for the r-th event A, then X has a Gamma distribution with shape parameter r and rate parameter λ.

Definition 6 *We will say that the continuous r.v. X has a Gamma distribution with shape parameter r and rate parameter λ if and only if the p.d.f. (probability density function) of X is*

$$f_X(x) \;=\; \frac{\lambda^r}{\Gamma(r)}\;\mathrm{e}^{-\lambda x}\,x^{r-1}\,, \qquad r,\lambda>0;\;\; x>0\,.$$

We will denote this fact by

$$X \;\sim\; G(r,\lambda)\,.$$

Lemma 7 *If*

$$X_1 \;\sim\; G(r_1,\lambda_1) \qquad \text{and} \qquad X_2 \;\sim\; G(r_2,\lambda_2)$$

are two independent r.v.s, where $\lambda_1 \neq \lambda_2$, then

$$Z \;=\; X_1+X_2$$

has p.d.f.

$$f_Z(z) = \frac{\lambda_1^{r_1}\,\lambda_2^{r_2}}{\Gamma(r_1+r_2)}\ \mathrm{e}^{-\lambda_1 z}\ z^{r_1+r_2-1}\ {}_1F_1\left(r_2, r_1+r_2; (\lambda_1-\lambda_2)z\right)$$
$$= \frac{\lambda_1^{r_1}\,\lambda_2^{r_2}}{\Gamma(r_1+r_2)}\ \mathrm{e}^{-\lambda_2 z}\ z^{r_1+r_2-1}\ {}_1F_1\left(r_1, r_1+r_2; (\lambda_2-\lambda_1)z\right) \tag{B.1}$$

where

$${}_1F_1(a,b;x) = \frac{\Gamma(b)}{\Gamma(b-a)\,\Gamma(a)} \int_0^1 \mathrm{e}^{xt}\ t^{a-1}(1-t)^{b-a-1}\,dt$$
$$= \frac{\Gamma(b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{\Gamma(b+i)}\,\frac{x^i}{i!}$$

is the Kummer confluent hypergeometric function. Then $Y = Z^{1/\beta}$, $(\beta \neq 0)$, *has p.d.f.*

$$f_Y(y) = \frac{|\beta|\,\lambda_1^{r_1}\,\lambda_2^{r_2}}{\Gamma(r_1)\,\Gamma(r_2)}\ \mathrm{e}^{-\lambda_1 y^\beta}\ y^{\beta(r_1+r_2-1)}\ {}_1F_1\Big(r_2, r_1+r_2; (\lambda_1-\lambda_2)y^\beta\Big)$$
$$= \frac{|\beta|\,\lambda_1^{r_1}\,\lambda_2^{r_2}}{\Gamma(r_1)\,\Gamma(r_2)}\ \mathrm{e}^{-\lambda_2 y^\beta}\ y^{\beta(r_1+r_2-1)}\ {}_1F_1\Big(r_1, r_1+r_2; (\lambda_2-\lambda_1)y^\beta\Big)\ . \tag{B.2}$$

Proof: Given the independence of X_1 and X_2, their joint p.d.f. is simply

$$f_{X_1,X_2}(x_1,x_2) \ = \ \frac{\lambda_1^{r_1}\,\lambda_2^{r_2}}{\Gamma(r_1)\,\Gamma(r_2)}\ \mathrm{e}^{-\lambda_1 x_1 - \lambda_2 x_2}\ x_1^{r_1-1} x_2^{r_2-1}$$

and thus, if $Z = X_1 + X_2$,

$$f_Z(z) \ = \ \frac{\lambda_1^{r_1}\,\lambda_2^{r_2}}{\Gamma(r_1)\,\Gamma(r_2)}\ \mathrm{e}^{-\lambda_1 z} \int_0^z \mathrm{e}^{-(\lambda_2-\lambda_1)x_2}\ x_2^{r_2-1}(z-x_2)^{r_1-1}\,dx_2$$

where, using (2) and the change of variable $y = x_2/z$,

$$\int_0^z \mathrm{e}^{-(\lambda_2-\lambda_1)x_2}\ x_2^{r_2-1}(z-x_2)^{r_1-1}\,dx_2$$
$$= \int_0^1 \mathrm{e}^{(\lambda_1-\lambda_2)zy}\ z^{r_2-1}\,y^{r_2-1}\,(1-y)^{r_1-1}\,z^{r_1-1}\,z\,dy$$
$$= z^{r_1+r_2-1}\,\frac{\Gamma(r_1)\,\Gamma(r_2)}{\Gamma(r_1+r_2)}\ {}_1F_1\Big(r_2, r_1+r_2; (\lambda_1-\lambda_2)z\Big)$$

so that $f_Z(z)$ turns out given by the first expression in (1). Using the identity

$${}_1F_1(a,b;kx) \ = \ \mathrm{e}^{kx}\ {}_1F_1(b-a,b;-kx)$$

we obtain the second expression in (1). Making in (1) the transformation of variable $y = z^{1/\beta} \Leftrightarrow z = y^\beta$, with $\frac{d}{dy} z = \beta z^{\beta-1}$, we obtain the p.d.f.s in (2). □

Definition 8 *We will say that the continuous r.v. Y has a generalized Gamma distribution with shape parameter r, rate parameter λ and power parameter β if and only if the p.d.f. of Y is*

$$f_Y(y) \;=\; \frac{|\beta|\,\lambda^r}{\Gamma(r)}\; \mathrm{e}^{-\lambda y^\beta}\; y^{\beta r-1}\,, \qquad \beta \in I\!R\backslash\{0\}\,,\; r,\lambda>0;\; y>0$$

(that is, Y^β has a $G(r,\lambda)$ distribution).

We will denote this fact by

$$Y \;\sim\; G(r,\lambda,\beta)\,.$$

We should note that if $X \sim G(r,\lambda)$, then $Y = X^{1/\beta}$ has a $G(r,\lambda,\beta)$ distribution, or vice-versa, if $Y \sim G(r,\lambda,\beta)$ then $Y^\beta \sim G(r,\lambda)$.

We should also note that we use a slightly different notation and nomenclature from the one used by Stacy and Mihran (1965) and Lawless (1982).

Definition 9 *We will say that the continuous r.v. X has a non-central Gamma distribution with shape parameter r, rate parameter λ and non-centrality parameter δ if and only if the p.d.f. of X is*

$$f_X(x) \;=\; \frac{\lambda^r}{\Gamma(r)}\; \mathrm{e}^{-\lambda x}\; x^{r-1}\; \mathrm{e}^{-\delta\lambda}\; {}_0F_1(r;\delta\lambda^2 x)$$

where

$${}_0F_1(a;x) \;=\; \sum_{i=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a+i)}\,\frac{x^i}{i!}\,.$$

We will denote this fact by

$$X \;\sim\; \Gamma(r,\lambda;\delta)\,.$$

The p.d.f. of X may be seen either as a Poisson mixture of Gamma distributions or as the posterior distribution of X when

$$(X|Y=y) \;\sim\; \Gamma(r+y,\lambda) \qquad \text{and} \qquad Y \;\sim\; P(\delta\lambda)\,,$$

since then we would have

$$\begin{aligned} f_{X,Y}(x,y) &= f_{X|Y=y}(x)\, f_Y(y) \\ &= \frac{\lambda^{r+y}}{\Gamma(r+y)}\, e^{-\lambda x}\, x^{r+y-1}\, e^{-\delta\lambda}\, \frac{(\delta\lambda)^y}{y!}\,, \qquad \lambda, r>0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y=0,1,\ldots \end{aligned}$$

and thus

$$f_X(x) \;=\; \sum_{y=0}^{\infty} e^{-\delta\lambda}\,\frac{(\delta\lambda)^y}{y!}\;\underbrace{\frac{\lambda^{r+y}}{\Gamma(r+y)}\,e^{-\lambda x}\,x^{r+y-1}}_{p.d.f.\ of\ \Gamma(r+y,\lambda)} \tag{B.3}$$

what is the p.d.f. of a mixture of distributions $\Gamma(r{+}y,\lambda)$, $(y=0,1,2,\ldots)$, *with Poisson weights given by a Poisson distribution with parameter* $\delta\lambda$. *Expression (3) above may then be written as*

$$f_X(x) \;=\; \frac{\lambda^r}{\Gamma(r)}\,e^{-\lambda x}\,x^{r-1}\,e^{-\delta\lambda}\,\underbrace{\sum_{y=0}^{\infty}\frac{\Gamma(r)}{\Gamma(r+y)}\,\frac{(\delta\lambda^2)^y\,x^y}{y!}}_{={}_0F_1(r;\delta\lambda^2 x)}\,.$$

If $\delta=0$, *then* X *will have a central or usual Gamma distribution with shape parameter* r *and rate parameter* λ.

We will for short say that Y is a mixture of X_i $(i=1,2,\ldots)$ if and only if

$$f_Y(y) \;=\; \sum_{i=0}^{\infty} f_{X_i}(y)$$

where $f_Y(\cdot)$ is the p.d.f. or the p.m.f. of Y and $f_{X_i}(\cdot)$ is correspondingly the p.d.f. or p.m.f. of X_i $(i=0,1,2,\ldots)$.

But then, if $Z=Y^{1/\beta}$ and $W_i=X_i^{1/\beta}$,

$$\begin{aligned} f_Z(z) &= f_Y\left(z^\beta\right)\beta z^{\beta-1} \\ &= \beta z^{\beta-1}\sum_{i=0}^{\infty} f_{X_i}\left(z^\beta\right) \;=\; \sum_{i=0}^{\infty} f_{X_i}\left(z^\beta\right)\beta z^{\beta-1} \;=\; \sum_{i=0}^{\infty} f_{W_i}(z) \end{aligned}$$

that is, Z will be a mixture of $W_i=X_i^\beta$, or in other words, the distribution of the power of a mixture is the mixture of the powers.

Definition 10 *We will say that the continuous r.v.* Y *has a non-central generalized Gamma distribution with shape parameter* r, *rate parameter* λ, *power parameter* β *and non-centrality parameter* δ *if and only if the p.d.f. of* Y *is*

$$f_Y(y) \;=\; \frac{|\beta|\,\lambda^r}{\Gamma(r)}\,e^{-\lambda y^\beta}\,y^{\beta r-1}\,e^{-\delta\lambda}\,{}_0F_1(r;\delta\lambda^2 y^\beta)\,.$$

We will denote this fact by

$$Y \;\sim\; \Gamma(r,\lambda,\beta;\delta)\,.$$

According to the remark above, the distribution of Y may be seen either as a Poisson mixture of generalized Gamma distributions $\Gamma(r+i,\lambda,\beta)$, $(i = 0,1,2,\ldots)$, or as the p.d.f. of the the r.v. $Y = X^{1/\beta}$ where $X \sim \Gamma(r,\lambda;\delta)$.

If $\delta = 0$, then Y will have a central generalized Gamma distribution with shape parameter r, rate parameter λ and power parameter β.

Definition 11 *We will say that the discrete r.v. X has a Kummer confluent hypergeometric distribution with parameters r, r_1 and θ if and only if its c.d.f. (cumulative distribution function) is*

$$F_X(x) \;=\; P\,[X \le x] \;=\; \frac{1}{{}_1F_1(r,r_1;\theta)} \sum_{i=0}^{x} \frac{\Gamma(r+i)}{\Gamma(r)}\,\frac{\Gamma(r_1)}{\Gamma(r_1+i)}\,\frac{\theta^i}{i!} \qquad \begin{array}{l} x = 0,1,\ldots \\ r_1, r, \theta > 0, \end{array}$$

its p.m.f. being

$$f_X(x) \;=\; P\,[X = x] \;=\; \frac{1}{{}_1F_1(r,r_1;\theta)}\,\frac{\Gamma(r+x)}{\Gamma(r)}\,\frac{\Gamma(r_1)}{\Gamma(r_1+x)}\,\frac{\theta^x}{x!}\,.$$

These distributions which are in fact particular cases of the wider Kemp family of generalized hypergeometric probability distributions (Kemp, 1968a,b), were first studied by Hall (1956) and Bhattacharya (1966) respectively in the contexts of birth-and-death processes at equilibrium and accident proneness. Particular cases of the confluent hypergeometric distribution are:

i) for $r_1 = r$, the Poisson distribution with parameter θ, since then

$$ {}_1F_1(r,r_1;\theta) \;=\; {}_1F_1(r,r;\theta) \;=\; e^{\theta}$$

and thus

$$f_X(x) \;=\; \frac{\theta^x}{x!}\,e^{-\theta}\,;$$

ii) for $r = 1$, the hyper-Poisson distribution of Bardwell and Crow (1964) with p.m.f.

$$f_X(x) \;=\; \frac{1}{{}_1F_1(1,r_1;\theta)}\,\frac{\Gamma(r_1)}{\Gamma(r_1+x)}\,\theta^x$$

which is a generalization of the Poisson distribution since for $r_1 = 1$ it gives, according to the stated in i) above, the Poisson distribution with parameter θ, while for $r_1 \in \mathbb{N}$ it gives what is sometimes called a left truncated Poisson distribution but should be rather called a right shifted Poisson distribution, with a shift of $r_1 - 1$, since for $r_1 \in \mathbb{N}$,

$$ {}_1F_1(1,r_1;\theta) \;=\; \frac{(r_1-1)!\,e^{\theta}}{\theta^{r_1-1}} \left[1 - \sum_{j=0}^{r_1-2} \frac{\theta^j\,e^{-\theta}}{j!}\right]$$

so that we have in this case

$$f_X(x) = \frac{\dfrac{\theta^{x+r_1-1}}{(x+r_1-1)!}\, e^{-\theta}}{1-\displaystyle\sum_{j=0}^{r_1-2} \frac{\theta^j\, e^{-\theta}}{j!}}, \qquad \begin{array}{l} x=0,1,\ldots \\ (x+r_1-1=r_1-1, r_1, r_1+1, \ldots)\,. \end{array}$$

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