Study of the quality of several asymptotic and near-exact approximations based on moments for the distribution of the Wilks Lambda statistic

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Abstract

In this paper a measure of proximity of distributions, when moments are known, is proposed. Based on cases where the exact distribution is known, evidence is given that the proposed measure is accurate to evaluate the proximity of quantiles (exact vs. approximated). The measure may be applied to compare asymptotic and near-exact approximations to distributions, in situations where although being known the exact moments, the exact distribution is not known or the expression for its probability density function is not known or too complicated to handle. In this paper the measure is applied to compare newly proposed asymptotic and near-exact approximations to the distribution of the Wilks Lambda statistic when both groups of variables have an odd number of variables. This measure is also applied to the study of several cases of telescopic near-exact approximations to the exact distribution of the Wilks Lambda statistic based on mixtures of Generalized Near-Integer Gamma distributions.

Key words: measure of proximity, Wilks Lambda, likelihood ratio statistics, mixtures, simulation.

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1 Introduction

The Wilks Λ statistic is the well known statistic used to test the independence between two sets of jointly normally distributed variables (Wilks, 1932, 1935).

Let $\underline{x}$ be a $p\times 1$ vector of variables with a joint multivariate Normal distribution, split into two subvectors

$$\underline{x} = [\underline{x}_1', \underline{x}_2']' \sim N_p\left(\underline{\mu}, \Sigma\right) ,$$

with $p = p_1+p_2$, where p_k is the number of variables in $\underline{x}_k$ $(k = 1, 2)$. Further, let $\underline{\mu}$ be the population mean vector

$$\underline{\mu} = \left[\underline{\mu}_1', \underline{\mu}_2'\right]' ,$$

and Σ the population variance-covariance matrix, with

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} .$$

The Wilks Λ statistic is then defined as

$$\Lambda = \frac{|V|}{|V_{11}|\,|V_{22}|}$$

where V is either the MLE of Σ or the sample variance-covariance matrix of $\underline{x}$, and V_{kk} either the MLE of Σ_{kk} or the sample variance-covariance matrix of $\underline{x}_k$ $(k = 1, 2)$. For a sample of size n, the Wilks Λ statistic is the $(2/n)$th power of the likelihood ratio test statistic to test the null hypothesis of independence of the two sets of variables, that is,

$$H_0 : \Sigma = diag\,(\Sigma_{11}, \Sigma_{22}) \; . \tag{1}$$

Under the null hypothesis (1) the Wilks Λ statistic has the same distribution as $\prod_{j=1}^{p_1} Y_j$, where, for a sample of size $n+1$, with $n \geq p$, Y_j are p_1 independent Beta random variables with parameters $(n+1-p_2-j)/2$ and $p_2/2$ (Anderson, 1958, Theorem 9.3.3). Then, for a sample of size $n+1$, we may write the h-th moment of Λ, under (1), as

$$E\left(\Lambda^h\right) = \prod_{j=1}^{p_1} \frac{\Gamma\left(\frac{n+1-j}{2}\right)\,\Gamma\left(\frac{n+1-p_2-j}{2}+h\right)}{\Gamma\left(\frac{n+1-j}{2}+h\right)\,\Gamma\left(\frac{n+1-p_2-j}{2}\right)} \; . \tag{2}$$

Considering the random variable $W = -\log\Lambda$, for a sample of size $n+1$, its characteristic function is given by

$$\phi_W(t) \;=\; E\left(\mathrm{e}^{\mathrm{i}tW}\right) \;=\; E\left(\Lambda^{-\mathrm{i}t}\right) \;=\; \prod_{j=1}^{p_1} \frac{\Gamma\left(\frac{n+1-j}{2}\right)\,\Gamma\left(\frac{n+1-p_2-j}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{n+1-j}{2}-\mathrm{i}t\right)\,\Gamma\left(\frac{n+1-p_2-j}{2}\right)}\,,$$

where $\mathrm{i}=\sqrt{-1}$ and $t \in I\!\!R$.

Taking this characteristic function as a basis we propose several asymptotic and near-exact distributions for W. We then show how their performance and proximity to the exact distribution may be evaluated by using the measure of proximity between distributions proposed in section 3.

2 Approximations based on moments

2.1 Asymptotic distributions

We will consider two different approaches to the problem of approximating the exact characteristic function of $W = -\log\Lambda$. In the first approach $\phi_W(t)$ is taken as a whole, leading to asymptotic distributions. In this case, we will approximate the whole $\phi_W(t)$ by $\phi_{W'}(t)$, the characteristic function of the random variable W', in such a way that their first g derivatives with respect to t at $t=0$, that is, the first moments of both random variables W and W', are the same. If the distribution of W' depends on g parameters, say $\theta_1,\,\theta_2,\,\ldots\,,\theta_g$, the approximation is made by solving the system of g non-linear equations of the type

$$E[W^i] = E[W'^i] \qquad (i = 1, \ldots, g)$$

for $\theta_1,\,\theta_2,\,\ldots\,,\theta_g$.

The asymptotic distributions we propose for the Wilks Λ statistic (and corresponding abbreviations used) are: a Gamma distribution (G), a Generalized Gamma distribution (GG), a Mixture of 2 Gamma distributions with common rate parameter λ (M2Gλ), a Mixture of 2 Generalized Gamma distributions with common λ and common power parameter β (M2GG$\lambda\beta$), a Mixture of 2 Gamma distributions with common shape parameter r (M2Gr), a Mixture of 2 Generalized Gamma distributions with common r (M2GGr) and a Mixture of 3 Gamma distributions with common λ (M3Gλ).

In addition to these distributions we will use, as a reference, the Anderson (1958) and Rao (1948) asymptotic distributions described in subsection 2.3.

In order to clarify the nomenclature used: we say that the random variable Y has a Generalized Gamma distribution, $Y \sim G(r, \lambda, \beta)$, with shape parameter $r > 0$, rate parameter $\lambda > 0$ and power parameter $\beta \in I\!R \backslash \{0\}$, if its p.d.f. may be written as

$$f_Y(y) = \frac{|\,\beta\,|\,\lambda^r}{\Gamma(r)}\; \mathrm{e}^{-\lambda\, y^{\beta}}\; y^{\beta r-1} \quad (y > 0)\,.$$

For $\beta = 1$ the random variable Y has a common Gamma distribution with shape parameter r and rate parameter λ, denoted as $Y \sim G(r, \lambda)$.

2.2 Near-exact distributions

The second approach, which we call 'telescopic approach' will lead to near-exact distributions. In this case we will split the characteristic function corresponding to the situation where both p_1 and p_2 are odd in the following way

$$\begin{aligned}
\phi_W(t) &= \prod_{j=1}^{p_1} \frac{\Gamma\!\left(\frac{n+1-j}{2}\right)\,\Gamma\!\left(\frac{n+1-p_2-j}{2} - \mathrm{i}t\right)}{\Gamma\!\left(\frac{n+1-j}{2} - \mathrm{i}t\right)\,\Gamma\!\left(\frac{n+1-p_2-j}{2}\right)} \\
&= \frac{\Gamma\!\left(\frac{n}{2}\right)\,\Gamma\!\left(\frac{n}{2}-\frac{s}{2}-\mathrm{i}t\right)}{\Gamma\!\left(\frac{n}{2}-\frac{s}{2}\right)\,\Gamma\!\left(\frac{n}{2}-\mathrm{i}t\right)}\;\frac{\Gamma\!\left(\frac{n}{2}-\frac{s}{2}\right)\,\Gamma\!\left(\frac{n-p_2}{2}-\mathrm{i}t\right)}{\Gamma\!\left(\frac{n}{2}-\frac{s}{2}-\mathrm{i}t\right)\,\Gamma\!\left(\frac{n-p_2}{2}\right)} \prod_{j=1}^{p_1-1} \frac{\Gamma\!\left(\frac{n-j}{2}\right)\,\Gamma\!\left(\frac{n-p_2-j}{2} - \mathrm{i}t\right)}{\Gamma\!\left(\frac{n-j}{2} - \mathrm{i}t\right)\,\Gamma\!\left(\frac{n-p_2-j}{2}\right)} \\
&= \frac{\Gamma\!\left(\frac{n}{2}\right)\,\Gamma\!\left(\frac{n}{2}-\frac{s}{2}-\mathrm{i}t\right)}{\Gamma\!\left(\frac{n}{2}-\frac{s}{2}\right)\,\Gamma\!\left(\frac{n}{2}-\mathrm{i}t\right)} \prod_{j=0}^{\frac{p_2-s}{2}-1} \left(\frac{n-p_2}{2}+j\right)\left(\frac{n-p_2}{2}+j-\mathrm{i}t\right)^{-1} \\
&\qquad\qquad \prod_{j=1}^{p_1+p_2-3} \left(\frac{n-p_1-p_2+j}{2}\right)^{r_j} \left(\frac{n-p_1-p_2+j}{2} - \mathrm{i}t\right)^{-r_j} \\
&= \underbrace{\frac{\Gamma\!\left(\frac{n}{2}\right)\,\Gamma\!\left(\frac{n}{2}-\frac{s}{2}-\mathrm{i}t\right)}{\Gamma\!\left(\frac{n}{2}-\frac{s}{2}\right)\,\Gamma\!\left(\frac{n}{2}-\mathrm{i}t\right)}}_{\phi_{W^*}(t)} \prod_{j=1}^{p_1+p_2-3} \left(\frac{n-p_1-p_2+j}{2}\right)^{r_j^*} \left(\frac{n-p_1-p_2+j}{2} - \mathrm{i}t\right)^{-r_j^*} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (s = 1, 3, 5, \ldots, p_2)\,, \qquad (3)
\end{aligned}$$

where we used (6) and (21) in Coelho (2004), with

$$r_j^* \;=\; \begin{cases} r_j & j = 1, \ldots, p_1 - 1 \;\text{ and} \\ & j = p_1 + 2n + 1 \;(n = 0, \ldots, (p_2 - s - 4)/2) \\ r_j + 1 & j = p_1 + 2n \;(n = 0, \ldots, (p_2 - s - 2)/2) \end{cases} \qquad (4)$$

with

$$r_j = \begin{cases} h_j & j = 1, 2 \\ r_{j-2} + h_j & j = 3, \ldots, p_1 + p_2 - 3 \end{cases} \tag{5}$$

where

$$h_j = \begin{cases} 1 & j = 1, \ldots, \min(p_1 - 1, p_2) \\ 0 & j = 1 + \min(p_1 - 1, p_2), \ldots, \max(p_1 - 1, p_2) \\ -1 & j = 1 + \max(p_1 - 1, p_2), \ldots, p_1 + p_2 - 3 \end{cases} \tag{6}$$

and then we will replace only $\phi_{W^*}(t)$ by say $\phi_{W''}(t)$, while the remaining part of $\phi_W(t)$ that corresponds to the characteristic function of a sum of $p_1 + p_2 - 3$ independent Gamma random variables with integer shape parameters, remains unchanged. Finally, we incorporate $\phi_{W''}(t)$ with the part of $\phi_W(t)$ that was left unchanged. We call the resulting characteristic function a near-exact characteristic function (see Coelho (2004)). The choice of $\phi_{W''}(t)$ is made with two goals in mind:

- that the final characteristic function corresponds to a known and manageable c.d.f. (cumulative distribution function), so that the computation of near-exact quantiles is possible;
- and that the approximation obtained is the best possible, in the sense that the near-exact quantiles that it generates are the closest possible to the exact ones.

Replacing $\phi_{W^*}(t)$ by the characteristic function of a Gamma distribution that matches the two first moments of W^* or by the characteristic function of a mixture of two (or three) Gamma distributions with the same rate parameter which matches the first four (or six) moments of W^*, we obtain the following near-exact distributions for W: a Generalized Near-Integer Gamma distribution (GNIG), a Mixture of 2 Generalized Near-Integer Gamma distributions (M2GNIG), a Mixture of 3 Generalized Near-Integer Gamma distributions (M3GNIG). See section 4 for an example.

In order to clarify the nomenclature used: we will say that the random variable Z has a Generalized Integer Gamma distribution of depth g, with integer shape parameters $r_1, \ldots, r_g$ and all different rate parameters $\lambda_1, \ldots, \lambda_g \in I\!R^+$, with p.d.f. and c.d.f. given in Coelho (1998) if its distribution is the distribution of the sum of g independent Gamma random variables with integer shape parameters and all different rate parameters; we will denote the fact that Z has this distribution by $Z \sim GIG(r_1, \ldots, r_g; \lambda_1, \ldots, \lambda_g)$. We will say that the random variable W has a Generalized Near-Integer Gamma distribution of depth $g+1$, with integer shape parameters $r_1, \ldots, r_g$, r and rate parameters $\lambda_1, \ldots, \lambda_g$, λ, with p.d.f. and c.d.f. given in Coelho (2004) if its distribution is the distribution of the sum of two independent random variables, say

$$Z \sim GIG\,(r_1\,,\ \ldots\ , r_g;\ \lambda_1\,,\ \ldots\ , \lambda_g\,) \qquad \text{and} \qquad X \sim G(r,\, \lambda)$$

with $\lambda \neq \lambda_j$ $(j = 1, \ldots, g)$.

2.3 Anderson and Rao Chi-square asymptotic distributions

Anderson (1958, Sec. 9.5) derived, under the null hypothesis of independence, an asymptotic Chi-squared series distribution for the Wilks Λ statistic which in the case of two groups with p_1 and p_2 variables may be expressed as

$$\begin{aligned} P\left(-m\,\log\Lambda \leq w\right) = P\left(\chi^2_f \leq w\right) + \\ \frac{\gamma_2}{m^2}\left[P\left(\chi^2_{f+4} \leq w\right) - P\left(\chi^2_f \leq w\right)\right] + O\left(m^{-3}\right), \end{aligned} \tag{7}$$

where χ^2_f is a random variable with a Chi-squared distribution with $f = p_1p_2$ degrees of freedom,

$$\gamma_2 = \frac{p_1p_2}{48}\left(p_1^2 + p_2^2 - 5\right), \qquad \text{and} \qquad m = n - \frac{p_1 + p_2 + 1}{2},$$

for a sample of size $n + 1$.

This result allows us to obtain

$$\phi\,(t) = \left(1 - \frac{\gamma_2}{m^2}\right)\phi_{\chi^2_f}\left(\tfrac{t}{m}\right) + \frac{\gamma_2}{m^2}\,\phi_{\chi^2_{f+4}}\left(\tfrac{t}{m}\right),$$

where $\phi_{\chi^2_f}$ is the characteristic function of a Chi-square random variable with f degrees of freedom, as an approximate characteristic function for $W = -\log\Lambda$.

Rao (1948) proposed for the Wilks Λ distribution the expansion

$$\begin{aligned} P(-m\,\log\Lambda \leq w) = P\Big(\chi^2_f \leq w\Big) + \tfrac{\gamma_2}{m^2}\left[P(\chi^2_{f+4} \leq w) - P(\chi^2_f \leq w)\right] \\ + \tfrac{1}{m^4}\Big\{\gamma_4\left[P\Big(\chi^2_{f+8} \leq w\Big) - P\Big(\chi^2_f \leq w\Big)\right] \\ -\gamma_2^2\left[P\Big(\chi^2_{f+4} \leq w\Big) - P\Big(\chi^2_f \leq w\Big)\right]\Big\} + \ldots\,, \end{aligned} \tag{8}$$

where, again for a sample of size $n+1$, f, γ_2 and m are defined as above, and

$$\gamma_4 = \frac{\gamma_2^2}{2} + \frac{p_1 p_2}{1920}\left[3p_1^4 + 3p_2^4 + 10p_1^2p_2^2 - 50\left(p_1^2 + p_2^2\right) + 159\right].$$

In fact this expansion is equivalent to the one in Anderson (1958, Sec. 8.6.2). We may note that the expansion in (8) is actually the one in (7) with the additional term in m^{-4}.

Therefore, we may approximate the distribution of $W = -\log\Lambda$ by the distribution of a random variable with characteristic function

$$\phi(t) = \left(1 - \frac{\gamma_2}{m^2} - \frac{\gamma_4 + \gamma_2^2}{m^4}\right)\phi_{\chi^2_f}\left(\tfrac{t}{m}\right) + \left(\frac{\gamma_2}{m^2} - \frac{\gamma_2^2}{m^4}\right)\phi_{\chi^2_{f+4}}\left(\tfrac{t}{m}\right) + \frac{\gamma_4}{m^4}\,\phi_{\chi^2_{f+4}}\left(\tfrac{t}{m}\right).$$

These two distributions, the Anderson asymptotic Chi-squared distribution and Rao asymptotic Chi-squared distribution will be ahead denoted respectively by 'And. Chi' and 'Rao Chi'.

2.4 Exact distribution for $p_1 = 3$ and odd p_2

For $p_1 = 3$ and odd p_2, Alberto (1998) obtained the expressions for the exact p.d.f. of Λ under the form

$$f_\Lambda(x) = K\, x^{\frac{n-p_2-4}{2}}\left(\sum_{j=1}^{\frac{p_2+1}{2}} c_{2j-1}\,\Theta_1(2j-1,x)\,x^{j-\frac{1}{2}} + \sum_{j=1}^{\frac{p_2-1}{2}} c_{2j}\,\Theta_2(2j\,,x)\,x^{j}\right)$$

where

$$K = \frac{\prod_{j=1}^{p_2}\lambda_j}{B\left(\frac{n-p_2-2}{2},\,\frac{p_2}{2}\right)}$$

and

$$c_j = \frac{2^{p_2-1}}{\prod_{\substack{i=1\\ i\neq j}}^{p_2}(i-j)}\,,$$

with

$$\lambda_j = \frac{n-p_2-2+j}{2}\,,$$

and, for odd j,

$$\Theta_1(j,x) = \frac{\Gamma\left(\frac{p_2}{2}\right)}{\Gamma\left(\frac{p_2}{2}+1\right)} \frac{(1-x)^{\frac{p_2}{2}}}{x^{\frac{j}{2}}} + \frac{\Gamma\left(\frac{p_2}{2}\right)}{\Gamma\left(\frac{p_2-j}{2}\right)} \left\{ \sum_{i=0}^{\frac{p_2-j-2}{2}} - \frac{\Gamma\left(\frac{p_2-j}{2}-i\right)}{\Gamma\left(\frac{p_2}{2}+1-i\right)} \frac{(1-x)^{\frac{p_2}{2}-i}}{x^{\frac{j}{2}}} \right.$$

$$\left. + \frac{2\,(-1)^{\frac{j+1}{2}}}{\Gamma\left(\frac{j}{2}+1\right)} \left[\sum_{i=0}^{\frac{j-1}{2}} \frac{(-1)^{i+1}}{1+2i} \frac{(1-x)^{\frac{1}{2}+i}}{x^{\frac{1}{2}-i}} + \arcsin\sqrt{1-x} \right] \right\}$$

and, for even j,

$$\Theta_2(j\,,x) = \frac{\Gamma\left(\frac{p_2}{2}\right)}{\Gamma\left(\frac{j}{2}+1\right)} \left\{ \sum_{i=0}^{\frac{j-2}{2}} (-1)^{i+2} \frac{\Gamma\left(\frac{j}{2}-i\right)}{\Gamma\left(\frac{p_2}{2}-i\right)} x^{-\,\frac{j}{2}\,+\,i} (1-x)^{\frac{p_2-2}{2}\,-\,i} \right.$$

$$\left. + \frac{2\,(-1)^{\frac{j+2}{2}}}{\Gamma\left(\frac{p_2-j}{2}\right)} \left[\sum_{i=0}^{\frac{p_2-j-3}{2}} \frac{1}{1+2i} (1-x)^{\frac{1}{2}\,+\,i} - \operatorname{arctanh}\sqrt{1-x} \right] \right\}.$$

The c.d.f. has the form

$$F_\Lambda(x) = 1 + 2\,K \sum_{j=1}^{\frac{p_2+1}{2}} \frac{c_{2j-1}}{n-p_2+2j-3}\; x^{\frac{n-p_2+2j-3}{2}}\, \Theta_1\,(\,2j-1\,,x)$$

$$+2\,K \sum_{j=1}^{\frac{p_2-1}{2}} \frac{c_{2j}}{n-p_2+2j-2}\; x^{\frac{n-p_2+2j-2}{2}}\, \Theta_2\,(2j,x) - \frac{\Theta_{3,4}\,(n,x)}{\Theta_{3,4}(n,0)}\;,$$

with

$$\Theta_{3,4}\,(n,x) = \begin{cases} \Theta_3\,(n,x)\;, & n \text{ odd} \\ \Theta_4\,(n,x)\;, & n \text{ even} \end{cases}$$

where

$$\Theta_3\,(n,x) = -2 \sum_{i=0}^{\frac{n-p_2-4}{2}} \frac{(-1)^i}{p_2+2\,i} \binom{(n-p_2-4)/2}{i} (1-x)^{\frac{p_2}{2}+i}$$

and

$$\Theta_4(n,x) = \frac{\Gamma\left(\frac{p_2}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \left\{ \sum_{i=0}^{\frac{p_2-3}{2}} \frac{\Gamma\left(\frac{n-4}{2}-i\right)}{\Gamma\left(\frac{p_2}{2}-i\right)} \, x^{\frac{n-p_2-2}{2}} \, (1-x)^{\frac{p_2-2}{2}-i} \right.$$

$$- \frac{\Gamma\left(\frac{n-p_2-2}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \left[\sum_{i=1}^{\frac{n-p_2-3}{2}} \frac{(i-1)!}{2\,\Gamma\left(i+\frac{1}{2}\right)} \, x^{i-\frac{1}{2}} \, (1-x)^{\frac{1}{2}} \right.$$

$$\left.\left. + \frac{1}{\Gamma\left(\frac{1}{2}\right)} \arcsin\sqrt{1-x} \right] \right\} .$$

3 The measure of agreement or proximity between distributions

A measure of closeness or agreement between two distributions, developed under the assumption of existence of both moment generating functions, is used to access the quality of several approximations to the exact distribution. Smaller values of this measure will be associated with better approximations or smaller differences among quantiles (exact vs. approximated).

This measure is particularly useful when the moments are the only available information on the exact distribution, in the sense that the p.d.f. and c.d.f. expressions are not known or available.

We will take

$$M_X(t) = \sum_{i=0}^{\infty} M_X^{(i)}(0) \, \frac{(t-0)^i}{i!}$$

as the exact m.g.f. of the random variable X and

$$N_X(t) = \sum_{i=0}^{\infty} N_X^{(i)}(0) \, \frac{(t-0)^i}{i!}$$

as the approximate m.g.f. of X where $M_X^{(i)}(0)$ and $N_X^{(i)}(0)$ are respectively the exact and approximate moments of order i of X.

Then

$$\begin{aligned}
\int_{-1}^{1} |\, M_X(t) - N_X(t) \,|\, dt &= \int_{-1}^{1} \left| \sum_{i=0}^{\infty} \left(M_X^{(i)}(0) - N_X^{(i)}(0) \right) \frac{(t-0)^i}{i!} \right| dt \\
&\leq \int_{-1}^{1} \sum_{i=0}^{\infty} \left| \left(M_X^{(i)}(0) - N_X^{(i)}(0) \right) \frac{(t-0)^i}{i!} \right| dt \\
&= \sum_{i=0}^{\infty} \frac{\left| M_X^{(i)}(0) - N_X^{(i)}(0) \right|}{i!} \underbrace{\int_{-1}^{1} |\, t \,|^i \, dt}_{=\frac{2}{i+1}} \\
&= 2 \sum_{i=0}^{\infty} \frac{\left| M_X^{(i)}(0) - N_X^{(i)}(0) \right|}{(i+1)!}
\end{aligned}$$

so that the measure of proximity between the two distributions corresponding to the moment generating functions $M_X(t)$ and $N_X(t)$ we propose is

$$C_{M,N} = 2 \sum_{i=1}^{12} \frac{\left| M_X^{(i)}(0) - N_X^{(i)}(0) \right|}{(i+1)!} \,.$$

The use of the value 12, as upper limit in the summation, is somewhat arbitrary, but it seems to work rather well with most distributions. Moreover, this value has to be larger than the highest moment order used in any of the approximation processes (that is, larger than the highest moment order that is set equal in both the exact and the approximate distributions), while on the other hand this value has also to be kept at manageable levels. Also, even somehow large deviations in the value of higher moments, for two distributions that match the previous moments, have little weight in characterizing the difference between the two distributions as it is a fact that in the series expansion of the moment generating function itself, the moments are weighted by the inverse of the factorial of their order.

One could then think that since there are distributions that although being far different have all moments equal, this measure may be not adequate. However, in those situations at least one of the moment generating functions does not exist, while the above measure was developed to be used only in situations where both moment generating functions exist. We may note that these are also the situations where we use it in this paper.

4 The telescopic approach – an example

Let us considerer the particular case of two groups of variables, both with an odd number of variables, say $p_1 = 3$, $p_2 = 5$, for a sample size $n + 1 = 20$. Then from (3) we may write

$$\phi_W(t) = \underbrace{\frac{\Gamma\left(\frac{19}{2}\right)\Gamma\left(\frac{19}{2}-\frac{s}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{19}{2}-\frac{s}{2}\right)\Gamma\left(\frac{19}{2}-\mathrm{i}t\right)}}_{\phi_{W_1}(t)} \underbrace{\prod_{j=1}^{5}\left(\frac{11+j}{2}\right)^{r_j^*}\left(\frac{11+j}{2}-\mathrm{i}t\right)^{-r_j^*}}_{\phi_{W_2}(t)} \qquad (s{=}1,3,5)$$

where W_1 and W_2 are independent random variables, W_1 with a Logbeta distribution and W_2 with a Generalized Integer Gamma distribution.

Now, for example, setting $s = 1$, we approximate the distribution of W_1 by a mixture of three Gamma distributions with common rate parameter. For this purpose we take the random variable X

$$f_X(x) = \pi_1\, f_{X_1}(x) + \pi_2\, f_{X_2}(x) + \pi_3\, f_{X_3}(x),$$
$$0 < \pi_1, \pi_2, \pi_3 < 1; \qquad \pi_1 + \pi_2 + \pi_3 = 1$$

where $X_i \sim G\,(\,r_i\,, \lambda)$, $(i = 1, 2, 3)$.

The characteristic function of X is then

$$\phi_X(t) = \pi_1\, \phi_{X_1}(t) + \pi_2\, \phi_{X_2}(t) + \pi_3\, \phi_{X_3}(t),$$

where $\phi_{X_i}(t) = \lambda^{r_i}\,(\lambda - \mathrm{i}t)^{-r_i}$, $(i = 1, 2, 3)$.

The approximation process involves the numerical solution of the non-linear system of equations

$$\left\{ E\left[W_1^i\right] = E\left[X^i\right]\right\}, \qquad i = 1, \ldots, 6$$

where

$$E(X^h) \;=\; \pi_1\, \frac{\Gamma(r_1{+}h)}{\Gamma(r_1)}\,\lambda^{-h} + \pi_2\, \frac{\Gamma(r_2{+}h)}{\Gamma(r_2)}\,\lambda^{-h} + (1{-}\pi_1{-}\pi_2)\,\frac{\Gamma(r_3{+}h)}{\Gamma(r_3)}\,\lambda^{-h}$$

and

$$
\begin{array}{ll}
E\left[W_1\right]=0.0570963984 & E\left[W_1^4\right]=0.0011123085 \\
E\left[W_1^2\right]=0.0097747249 & E\left[W_1^5\right]=0.0005703606 \\
E\left[W_1^3\right]=0.0027875098 & E\left[W_1^6\right]=0.0003572704\,,
\end{array}
$$

in order to the variables r_1, r_2, r_3, λ, π_1 and π_2. We found the solution

$$
\begin{array}{ll}
r_1=0.5000007019 & \lambda=9.1226850665 \\
r_2=1.5003310118 & \pi_1=0.9795627920 \\
r_3=2.5151190734 & \pi_2=0.0004215669\,.
\end{array}
$$

From (4) through (6), the parameters for the distribution of W_2 are $[r_1^*, r_2^*, r_3^*, r_4^*, r_5^*]=[1,1,2,1,2]$ and $[\lambda_1,\ \lambda_2, \lambda_3, \lambda_4, \lambda_5]=\left[6, \frac{13}{2}, 7, \frac{15}{2}, 8\right]$.

Finally, the distribution of W may be approximated by the distribution of W'', which is a mixture of three Generalized Near-Integer Gamma distributions on the above parameters

$$
f_{W''}(w)=\pi_1 f_{W_1''}(w)+\pi_2 f_{W_2''}(w)+(1-\pi_1-\pi_2) f_{W_3''}(w)
$$

where,

$$
W_i'' \sim GNIG\left(r_1^*, r_2^*, r_3^*, r_4^*, r_5^*, r_i\ ;\ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda\right) \qquad (i=1,2,3)\,.
$$

The first 12 moments of W and W' are given in Table 1

Table 1
First 12 moments of W and W''

h	W	W''
1	1.046656838007779	1.046656838007779
2	1.243295783912797	1.243295783912797
3	1.652949218731703	1.652949218731703
4	2.432329963684224	2.432329963684224
5	3.925619475588833	3.925619475588833
6	6.896395675726477	6.896395675726477
7	13.103213672642548	13.103213672643382
8	26.778571336764475	26.778571336772210
9	58.584411153518648	58.584411153556672
10	136.631861993942157	136.631861994045039
11	338.454697223175999	338.454697223039265
12	887.582559644207445	887.582559640090829

and therefore the calculated value of the proposed measure is

$$
C_{W,W''}=3.08\times 10^{-7}\,.
$$

5 Analysis of the quality of several approximations

We will start with a few cases where the exact distribution is known in order to assess the adequability of the proposed measure. These cases correspond to situations where either at most one of the two groups of variables has an odd number of variables, in which case the exact distribution is known to be a Generalized Integer Gamma distribution (Coelho, 1998), or both groups of variables have an odd number of variables, one of them with three variables, situation where the exact p.d.f. and c.d.f. were given in subsection 2.4.

On Tables 3 through 8 are shown the results for cases where the exact distribution is known to be a Generalized Integer Gamma distribution. In all these cases the proposed measure of closeness between the exact and each of the asymptotic distributions studied shows a very much adequate behaviour, with smaller values corresponding to situations where the asymptotic quantiles lay closer to the exact ones. Also, as expected, for a given number of variables, higher values for the sample size, which is taken to be $n+1$, correspond to situations where the asymptotic approximations behave remarkably better, with the proposed measure exhibiting remarkably lower values.

Thus, the proposed measure seems to be much adequate to evaluate and study the closeness between two distributions, namely the exact and some approximate distribution.

The number of first moments that are equal between the exact and each asymptotic distribution, varies among the asymptotic distributions used. These numbers may be analyzed on Table 2.

Table 2
Number of moments equated for each asymptotic distribution

Distribution	And.Chi	Rao Chi	G	GG	M2Gλ	M2G$\lambda\beta$	M2GGλ	M2Gr	M2GGr	M3Gλ
Mom.^{s} equated	0	0	2	3	4	5	6	4	6	6

We should also notice that the measure penalizes a bit unfairly approximations that, like Rao's asymptotic approximation, although having a very good behaviour in terms of quantiles do not exactly match the first few moments. However, since our main aim is to use the measure in situations where the first moments of the two distributions (exact and approximate) match the first moments, this is seen as a minor drawback.

On Tables 3 through 12, a practical measure of quantile comparison, defined as $\Delta = -\log_{10}|\text{exact} - \text{approx.}|$ is used to assess the closeness between the exact q-quantile and a given approximate q-quantile giving an accurate measure of the 'number' of decimal places of agreement between them.

We may notice that the new proposed asymptotic approximations based on mixtures perform much better than all the other ones.

Table 3
Exact and asymptotic 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ, for $n+1$=20, p_1=3, p_2=4.

Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
exact	——	1.41496041	—	1.76677002	—
And. Chi	2.4E-04	1.41458589	3.4	1.76592575	3.1
Rao Chi	5.2E-06	1.41495369	5.2	1.76674565	4.6
G	4.9E-05	1.41477648	3.7	1.76523729	2.8
GG	1.5E-06	1.41505223	4.0	1.76675254	4.8
M2Gλ	3.1E-08	1.41496102	6.2	1.76678996	4.7
M2G$\lambda\beta$	1.3E-09	1.41495826	5.7	1.76677059	6.2
M2GGλ	2.6E-11	1.41496046	7.3	1.76677050	6.3
M2Gr	1.2E-08	1.41496033	7.1	1.76677758	5.1
M2GGr	9.9E-11	1.41496056	6.8	1.76677174	5.8
M3Gλ	1.6E-11	1.41496048	7.2	1.76676977	6.6

Table 4
Exact and asymptotic 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ, for $n+1$=100, p_1=3, p_2=4.

Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
exact	——	0.2213782024775	—	0.2760413132696	—
And. Chi	1.0E-08	0.2213781662983	7.4	0.2760412303738	7.1
Rao Chi	4.4E-12	0.2213782024616	10.8	0.2760413132107	10.2
G	2.0E-09	0.2213774878849	6.1	0.2760354835173	5.2
GG	8.5E-12	0.2213785451225	6.5	0.2760412390316	7.1
M2Gλ	6.8E-16	0.2213782025958	9.9	0.2760413154297	8.7
M2G$\lambda\beta$	6.0E-18	0.2213782021284	9.5	0.2760413134324	9.8
M2GGλ	8.3E-21	0.2213782024785	12.0	0.2760413132332	10.4
M2Gr	2.3E-16	0.2213782024766	12.0	0.2760413139894	9.1
M2GGr	2.6E-18	0.2213781971718	8.3	0.2760413073619	8.2
M3Gλ	1.5E-22	0.2213782024777	12.7	0.2760413132690	12.2

Table 5
Exact and asymptotic 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ, for $n+1$=20, p_1=3, p_2=6.

Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
exact	——	2.10668334	—	2.54662794	—
And. Chi	3.4E-03	2.10347794	2.5	2.54010972	2.2
Rao Chi	1.9E-04	2.10653010	3.8	2.54613372	3.3
G	3.6E-04	2.10599489	3.2	2.54252922	2.4
GG	1.3E-05	2.10686938	3.7	2.54652686	4.0
M2Gλ	7.3E-07	2.10669749	4.8	2.54672620	4.0
M2G$\lambda\beta$	2.8E-08	2.10667626	5.1	2.54663326	5.3
M2GGλ	9.4E-10	2.10668421	6.1	2.54662938	5.8
M2Gr	3.3E-07	2.10668858	5.3	2.54667185	4.4
M2GGr	2.3E-09	2.10668513	5.7	2.54663153	5.4
M3Gλ	1.3E-09	2.10668359	6.6	2.54662492	5.5

Table 6
Exact and asymptotic 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ,for $n+1$=100, p_1=3, p_2=6.

Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
exact	——	0.3072618781222	—	0.3704607746731	—
And. Chi	6.6E-08	0.3072616498004	6.6	0.3704602899400	6.3
Rao Chi	6.2E-11	0.3072618778900	9.6	0.3704607738824	9.1
G	6.6E-09	0.3072597179070	5.7	0.3704482545381	4.9
GG	2.9E-11	0.3072624240150	6.3	0.3704604452252	6.5
M2Gλ	6.1E-15	0.3072618797337	8.8	0.3704607830926	8.1
M2G$\lambda\beta$	5.1E-17	0.3072618771297	9.0	0.3704607758032	8.9
M2GGλ	3.5E-20	0.3072618781110	11.0	0.3704607747650	10.0
M2Gr	2.0E-15	0.3072618785338	9.4	0.3704607775950	8.5
M2GGr	8.9E-18	0.3072618714363	8.2	0.3704607711509	8.5
M3Gλ	3.4E-21	0.3072618781224	12.7	0.3704607746682	11.3

Table 7
Exact and asymptotic 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ,for $n+1$=30, p_1=11, p_2=10.

Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
exact	——	8.07206786	—	8.81819450	—
And. Chi	2.8E+01	7.86932203	0.7	8.53716372	0.6
Rao Chi	8.5E+00	8.00534943	1.2	8.69901630	0.9
G	5.8E-01	8.06760389	2.4	8.80247348	1.8
GG	2.3E-02	8.07235216	3.5	8.81759167	3.2
M2Gλ	5.7E-03	8.07235326	3.5	8.81869267	3.3
M2G$\lambda\beta$	1.3E-04	8.07205853	5.0	8.81823290	4.4
M2GGλ	4.3E-06	8.07206531	5.6	8.81819372	6.1
M2Gr	4.0E-03	8.07226426	3.7	8.81853797	3.5
M2GGr	2.9E-06	8.07206621	5.8	8.81819354	6.0
M3Gλ	3.4E-05	8.07204730	4.7	8.81817507	4.7

Table 8
Exact and asymptotic 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ,for $n+1$=110, p_1=11, p_2=10.

Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
exact	——	1.38532752850	—	1.50750956254	—
And. Chi	4.9E-05	1.38528205652	4.3	1.50743489965	4.1
Rao Chi	1.6E-07	1.38532709301	6.4	1.50750844529	6.0
G	3.9E-07	1.38530918563	4.7	1.50744823089	4.2
GG	1.7E-09	1.38532819342	6.2	1.50750775333	5.7
M2Gλ	6.4E-12	1.38532759861	7.2	1.50750964966	7.1
M2G$\lambda\beta$	3.3E-14	1.38532752779	9.2	1.50750957412	7.9
M2GGλ	1.2E-16	1.38532752884	9.5	1.50750956235	9.7
M2Gr	2.7E-12	1.38532755750	7.5	1.50750960055	7.4
M2GGr	3.4E-16	1.38532752944	9.0	1.50750956208	9.3
M3Gλ	7.2E-17	1.38532752830	9.7	1.50750956251	10.5

Cases where both groups of variables have an odd number of variables are shown on Tables 9 through 14. On Tables 9 through 12 we have $p_1 = 3$, situation in which the exact distribution of the Wilks Λ statistic is known (see subsection 2.4), being thus possible to compute the exact quantiles.

For all the cases it was also computed an estimate and a 0.95 confidence interval for the true value of the quantile, by simulation, based on sample sizes of 40000. For the computation of the confidence intervals we took as references the works from David (1981) and Juritz, Juritz and Stephens (1983). Let $X_{(i)}$ denote the ith order statistic or the ith smallest observation drawn from a random sample of size n of a continuous random variable X. Then the usual estimator of the pth quantile of X is given by $X_{(k)}$, with $k = \lfloor np \rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes the largest integer less than or equal to the argument. A confidence interval for the pth quantile of X at level $1 - \alpha$ may then be given by $\left[X_{(r)}, X_{(s)}\right]$, with $r = -wz_{1-\alpha/2} + np + 1/2$ and $s = wz_{1-\alpha/2} + np + 1/2$, where $w = (np(1-p))^{1/2}$ and $z_{1-\alpha/2}$ denotes the $1 - \alpha/2$ quantile of the standard Normal distribution. Simulations were carried out using programs in Fortran language while in all cases quantile calculations were performed with the software Mathematica.

On tables 9 through 14 we consider, besides the asymptotic distributions we considered before, also the three near-exact distributions proposed, that is, the GNIG, M2GNIG and M3GNIG distributions, which match respectively the first 2, 4 and 6 exact moments. We may see that for the same number of moments equated, the near-exact distributions perform better than the new proposed asymptotic distributions. This behaviour is even better for smaller samples sizes. Also, as expected, better results are obtained for smaller values of s, since in these cases more Exponential distibutions are considered in the exact part of the characteristic function (see expression (3)).

Table 9
Exact and approximate (asymptotic and near-exact) 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ, for $n+1$=10, p_1=3, p_2=5.

	Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
	exact	——	6.7089911416542	—	8.4582644671729	—
	simulated	——	6.705	—	8.445	—
	C.I. (95%)	——	[6.656,6.755]	—	[8.355,8.593]	—
asymptotic	And. Chi	4.8E+01	6.2085637108532	0.3	7.5638792765798	0.0
	Rao Chi	3.4E+01	6.5162914259520	0.7	8.0018418051190	0.3
	G	1.8E+01	6.6909840655388	1.7	8.3038721176926	0.8
	GG	2.4E+00	6.7165234516414	2.1	8.4556415145674	2.6
	M2Gλ	3.6E+00	6.7070878517524	2.7	8.4828207141137	1.6
	M2G$\lambda\beta$	3.5E-01	6.7078120677218	2.9	8.4580613998845	3.7
	M2GGλ	1.3E-01	6.7095621212345	3.2	8.4593013306554	3.0
	M2Gr	2.3E+00	6.7081883014781	3.1	8.4713370382770	1.9
	M2GGr	1.3E-01	6.7095729822655	3.2	8.4592934009315	3.0
	M3Gλ	3.6E-01	6.7111294136457	2.7	8.4538438386052	2.4
near-exact	GNIG s 1	9.9E-04	6.7090003202983	5.0	8.4582812314883	4.8
	GNIG s 3	1.2E-02	6.7088917346415	4.0	8.4580649599037	3.7
	GNIG s 5	3.2E-01	6.7070737420589	2.7	8.4533444311154	2.3
	M2GNIG s 1	8.3E-07	6.7089911318020	8.0	8.4582644742046	8.2
	M2GNIG s 3	1.0E-05	6.7089912746708	6.9	8.4582644553209	7.9
	M2GNIG s 5	5.6E-03	6.7090472886933	4.3	8.4582975581782	4.5
	M3GNIG s 1	8.0E-10	6.7089911416537	12.3	8.4582644672155	10.4
	M3GNIG s 3	4.3E-09	6.7089911414853	9.8	8.4582644673813	9.7
	M3GNIG s 5	6.1E-05	6.7089890549665	5.7	8.4582648088488	6.5

Table 10
Exact and approximate (asymptotic and near-exact) 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ, for $n+1$=100, p_1=3, p_2=5.

	Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
	exact	——	0.264594184683306	—	0.323698431562899	—
	simulated	——	0.264	—	0.323	—
	C.I. (95%)	——	(0.263, 0.266)	—	(0.320, 0.326)	—
asymptotic	And. Chi	2.8E-08	0.264594087044116	7.0	0.323698217427613	6.7
	Rao Chi	1.9E-11	0.264594184614804	10.2	0.323698431322362	9.6
	G	3.8E-09	0.264592850646964	5.9	0.323689629446951	5.1
	GG	1.6E-11	0.264594624278909	6.4	0.323698243894043	6.7
	M2Gλ	2.2E-15	0.264594185277205	9.2	0.323698436071916	8.3
	M2G$\lambda\beta$	2.3E-17	0.264594184523652	9.8	0.323698429226675	8.6
	M2GGλ	7.5E-19	0.264594183960291	9.1	0.323698430462757	9.0
	M2Gr	7.4E-16	0.264594184805260	9.9	0.323698433101771	8.8
	M2GGr	3.6E-19	0.264594184496738	9.7	0.323698430856647	9.2
	M3Gλ	8.1E-22	0.264594184683557	12.6	0.323698431560969	11.7
near-exact	GNIG s 1	2.2E-11	0.264594193358092	8.1	0.323698483353111	7.3
	GNIG s 3	1.2E-10	0.264594140389784	7.4	0.323698158746323	6.6
	GNIG s 5	8.7E-10	0.264593868497863	6.5	0.323696417963287	5.7
	M2GNIG s 1	2.0E-17	0.264594184672349	11.0	0.323698431526710	10.4
	M2GNIG s 3	1.2E-18	0.264594184683695	12.4	0.323698431565334	11.6
	M2GNIG s 5	1.6E-16	0.264594184730511	10.3	0.323698431885880	9.5
	M3GNIG s 1	1.3E-23	0.264594184683314	14.1	0.323698431562926	13.6
	M3GNIG s 3	1.3E-25	0.264594184683311	14.3	0.323698431562853	13.3
	M3GNIG s 5	1.9E-23	0.264594184683311	14.3	0.323698431562853	13.3

Table 11
Exact and approximate (asymptotic and near-exact) 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ, for $n+1$=12, p_1=3, p_2=7.

	Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
	exact	——	7.5563906371236	—	9.3209508307624	—
	simulated	——	7.552	—	9.350	—
	C.I. (95%)	——	[7.512,7.598]	—	[9.269,9.471]	—
asymptotic	And. Chi	1.2E+02	6.6969820614062	0.1	7.9149418104934	-0.1
	Rao Chi	9.1E+01	7.1201213804645	0.4	8.4457891880740	0.1
	G	3.7E+01	7.5282553706282	1.6	9.1282511671657	0.7
	GG	4.7E+00	7.5643363938111	2.1	9.3167705797482	2.4
	M2Gλ	7.9E+00	7.5559428347287	3.3	9.3568933666326	1.4
	M2G$\lambda\beta$	6.5E-01	7.5549645992545	2.8	9.3210779012135	3.9
	M2GGλ	2.4E-01	7.5571869553321	3.1	9.3219971962984	3.0
	M2Gr	5.4E+00	7.5565165709236	3.9	9.3419788564664	1.7
	M2GGr	2.4E-01	7.5571905650425	3.1	9.3219937240073	3.0
	M3Gλ	8.2E-01	7.5595393947748	2.5	9.3124081029349	2.1
near-exact	GNIG s 1	5.3E-04	7.5563933232183	5.6	9.3209555200579	5.3
	GNIG s 3	5.1E-03	7.5563660209249	4.6	9.3209058907006	4.3
	GNIG s 5	8.6E-02	7.5560173936531	3.4	9.3202081381260	3.1
	GNIG s 7	9.5E-01	7.5532577822142	2.5	9.3131519511081	2.1
	M2GNIG s 1	2.5E-07	7.5563906357339	8.9	9.3209508321093	8.9
	M2GNIG s 3	1.1E-06	7.5563906446707	8.1	9.3209508270416	8.4
	M2GNIG s 5	3.6E-04	7.5563931709283	5.6	9.3209502991137	6.3
	M2GNIG s 7	2.2E-02	7.5565188004597	3.9	9.3209502991137	6.3
	M3GNIG s 1	7.5E-11	7.5563906371242	12.2	9.3209508307644	11.7
	M3GNIG s 3	2.9E-10	7.5563906371214	11.7	9.3209508307715	11.0
	M3GNIG s 5	9.3E-07	7.5563906183723	7.7	9.3209508577397	7.6
	M3GNIG s 7	3.3E-04	7.5563833772081	5.1	9.3209536184152	5.6

Table 12
Exact and approximate (asymptotic and near-exact) 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ, for $n+1$=100, p_1=3, p_2=7.

	Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
	exact	——	0.34963177881533837	—	0.41667044442505882	—
	simulated	——	0.349	—	0.417	—
	C.I. (95%)	——	(0.347, 0.351)	—	(0.413, 0.422)	—
asymptotic	And. Chi	1.4E-07	0.34963129839497500	6.3	0.41666945052869902	6.0
	Rao Chi	1.8E-10	0.34963177814536319	9.2	0.41667044219064030	8.7
	G	1.1E-08	0.34962856829230487	5.5	0.41665342543524892	4.8
	GG	4.7E-11	0.34963243933719237	6.2	0.41666994444668912	6.3
	M2Gλ	1.5E-14	0.34963178229615840	8.5	0.41667045889982562	7.8
	M2G$\lambda\beta$	1.2E-16	0.34963177734022679	8.8	0.41667044653196839	8.7
	M2GGλ	3.9E-15	0.34963389197517162	5.7	0.41667183585979366	5.9
	M2Gr	5.0E-15	0.34963177978962683	9.0	0.41667044952245305	8.3
	M2GGr	3.0E-19	0.34963177873110812	10.1	0.41667044423447363	9.7
	M3Gλ	1.2E-20	0.34963177881472260	12.2	0.41667044441412605	11.0
near-exact	GNIG s 1	2.4E-11	0.34963178643439657	8.1	0.41667048125110169	7.4
	GNIG s 3	1.2E-10	0.34963173946375277	7.4	0.41667025015084576	6.7
	GNIG s 5	9.2E-10	0.34963149436601338	6.5	0.41666900823776035	5.8
	GNIG s 7	2.9E-09	0.34963089801992367	6.1	0.41666588983813641	5.3
	M2GNIG s 1	2.1E-17	0.34963177880795905	11.1	0.41667044440804687	10.8
	M2GNIG s 3	1.3E-18	0.34963177881568070	12.4	0.41667044442625868	11.9
	M2GNIG s 5	1.7E-16	0.34963177885880965	10.4	0.41667044458536435	9.8
	M2GNIG s 7	1.5E-15	0.34963177919159566	9.4	0.41667044589328374	8.8
	M3GNIG s 1	1.1E-25	0.34963177881533839	16.7	0.41667044442505892	16.0
	M3GNIG s 3	1.4E-25	0.34963177881533836	17.0	0.41667044442505870	15.9
	M3GNIG s 5	2.0E-23	0.34963177881533659	14.7	0.41667044442504109	13.8
	M3GNIG s 7	5.4E-22	0.34963177881530098	13.4	0.41667044442458288	12.3

Tables 13 and 14, relating to cases where the exact distribution is not available in a manageable form, adequate to the expedit computation of quantiles, the exact value of the quantile, in the original definition of Δ, was replaced by the most accurate available value for the quantile, that is the one from M3GNIG with $s = 1$.

Table 13
Approximate (asymptotic and near-exact) 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ, for $n+1$=14, p_1=7, p_2=5.

	Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
	exact	——	——	—	——	—
	simulated	——	9.675	—	11.439	—
	C.I. (95%)	——	[9.624,9.717]	—	[11.350,11.561]	—
asymptotic	And. Chi	6.8E+02	8.4890336141875	-0.1	9.6921679433633	-0.3
	Rao Chi	5.4E+02	8.9768080433686	0.2	10.2573461507777	-0.1
	G	1.8E+02	9.6342770548545	1.4	11.2711555566127	0.7
	GG	2.7E+01	9.6825992578045	2.0	11.4817255832348	2.2
	M2Gλ	3.1E+01	9.6748133460895	2.6	11.5282852965651	1.4
	M2G$\lambda\beta$	2.8E+00	9.6703072675472	2.7	11.4887470412264	3.3
	M2GGλ	1.1E-01	9.6720706945677	4.1	11.4879235086357	3.5
	M2Gr	2.3E+01	9.6741328553340	2.7	11.5141768190870	1.6
	M2GGr	5.6E-02	9.6721075790828	4.3	11.4880345293060	3.7
	M3Gλ	2.6E+00	9.6746726200182	2.6	11.4776927059223	2.0
near-exact	GNIG s 1	8.6E-04	9.6721566375964	6.1	11.4882245589705	5.8
	GNIG s 3	7.2E-03	9.6721487101512	5.2	11.4882098297589	4.9
	GNIG s 5	9.7E-02	9.6720657949987	4.0	11.4880480769284	3.8
	GNIG s 7	6.8E-01	9.6715752197697	3.2	11.4870003775728	2.9
	M2GNIG s 1	2.1E-07	9.6721557721294	9.6	11.4882229888862	9.6
	M2GNIG s 3	4.9E-07	9.6721557730395	9.2	11.4882229882286	9.4
	M2GNIG s 5	1.2E-04	9.6721559469302	6.8	11.4882229087306	7.1
	M2GNIG s 7	3.7E-03	9.6721612360931	5.3	11.4882221081721	6.1
	M3GNIG s 1	1.8E-11	9.6721557723783	–	11.4882229886465	–
	M3GNIG s 3	6.3E-11	9.6721557723782	13.0	11.4882229886469	12.4
	M3GNIG s 5	8.7E-08	9.6721557721712	9.7	11.4882229893294	9.2
	M3GNIG s 7	1.3E-05	9.6721557147671	7.2	11.4882230804218	7.0

Table 14
Approximate (asymptotic and near-exact) 0.95 and 0.99 quantiles for the negative logarithm of the Wilks Λ, for $n+1$=100, p_1=7, p_2=5.

	Distribution	measure	quant. 0.95	Δ	quant. 0.99	Δ
	exact	——	——	—	——	—
	simulated	——	0.538	—	0.624	—
	C.I. (95%)	——	(0.536, 0.541)	—	(0.619, 0.628)	—
asymptotic	And. Chi	5.5E-07	0.538821905041529	5.8	0.620443393194623	5.5
	Rao Chi	1.1E-09	0.538823517515221	8.4	0.620446420192520	7.9
	G	2.8E-08	0.538818092966970	5.3	0.620423154273619	4.6
	GG	1.3E-10	0.538824133065512	6.2	0.620445638656719	6.1
	M2Gλ	7.9E-14	0.538823530494207	8.0	0.620446454153050	7.7
	M2G$\lambda\beta$	6.1E-16	0.538823520143052	8.9	0.620446435582440	8.4
	M2GGλ	8.0E-18	0.538823520703127	9.1	0.620446431772011	10.1
	M2Gr	2.8E-14	0.538823524453608	8.5	0.620446440157979	8.1
	M2GGr	1.8E-17	0.538823519664298	8.7	0.620446432472145	9.2
	M3Gλ	1.3E-19	0.538823521501927	11.1	0.620446431828845	10.7
near-exact	GNIG s 1	2.7E-11	0.538823526980504	8.3	0.620446453633724	7.7
	GNIG s 3	1.4E-10	0.538823493002319	7.5	0.620446316804102	6.9
	GNIG s 5	1.0E-09	0.538823313489890	6.7	0.620445580426555	6.1
	GNIG s 7	3.3E-09	0.538822870754379	6.2	0.620443728370592	5.6
	M2GNIG s 1	2.4E-17	0.538823521506041	11.5	0.620446431842974	11.3
	M2GNIG s 3	1.4E-18	0.538823521509501	12.7	0.620446431848493	12.4
	M2GNIG s 5	1.9E-16	0.538823521532625	10.6	0.620446431899325	10.3
	M2GNIG s 7	1.7E-15	0.538823521716441	9.7	0.620446432320557	9.3
	M3GNIG s 1	1.3E-25	0.538823521509323	–	0.620446431848112	–
	M3GNIG s 3	1.6E-25	0.538823521509312	14.0	0.620446431848208	13.0
	M3GNIG s 5	2.3E-23	0.538823521509318	14.3	0.620446431848110	14.7
	M3GNIG s 7	6.2E-22	0.538823521509279	13.4	0.620446431848028	13.1

6 Conclusions and Final Remarks

We have to stress the outstanding performance of the near-exact distributions for small values of n, or should we rather say, for small values of $n - p_1 - p_2$, namely for larger values of $p_1 + p_2$. Also, for smaller values of $n - p_1 - p_2$, the proposed asymptotic distributions based on mixtures perform better than any of the other asymptotic distributions, while for large values of n the extraordinary performance of the Rao asymptotic distribution is only supperated by the near-exact distributions based on mixtures.

Considering asymptotic and near-exact distributions that equate a given number of exact moments, the near-exact distributions always have a much better performance than the asymptotic distributions.

All the near-exact distributions perform better for smaller values of s, what was really expected, given the methodology that supports the building of the near-exact distributions.

We may also notice that in the near-exact approximations, $\phi_{W^*}(t)$ in (3) does not depend on the values of either p_1 or p_2, but only on the values of n, that is, the sample size. This way, all the parameters of the near-exact distribution corresponding to this part of the characteristic function, computed for a given value of n, are valid for any combination of values of p_1 and p_2 such that $n > p_1 + p_2$.

As already stated in the previous section, a minor drawback of the proposed measure is that it penalizes a bit unfairly approximations that, like the Anderson and Rao asymptotic distributions, do not equate moments. Anyway, this is seen as a minor undesirable or objectionable feature since not only this penalization seems to be rather moderate but also because we intend to use this measure mainly in situations where the approximate (asymptotic or near-exact) distributions equate a few of the first exact moments.

As an overall conclusion we may say that:

i) the newly proposed asymptotic and near-exact distributions, namely the ones based on mixtures, are particularly adequate and useful for cases where $n - p_1 - p_2$ is rather small;
ii) the proposed measure of closeness or agreement between two distributions shows a good performance, generally with smaller values corresponding to better approximations between the quantile values (exact vs. approximated); this way, in situations where the exact distribution is not known (or the expressions for its p.d.f. or c.d.f. are not known, or being known are too complicated for practical use), under the assumption of existence of both the exact and approximate moment generating functions, this mea-

sure may be a useful tool for evaluating the performance of asymptotic and near-exact distributions.

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