COMPUTATIONAL METHODS FOR A NONLINEAR VOLTERRA INTEGRAL EQUATION

M.REBELO*, T.DIOGO** AND P.LIMA**

Abstract. In this work we are concerned with the numerical solution of a nonlinear weakly singular Volterra integral equation with a nonsmooth solution. We investigate the application of product integration methods and a detailed analysis of the Trapezoidal method is given. In order to improve the numerical results we consider extrapolation procedures and collocation methods based on graded meshes. Several examples are presented illustrating the performance of the methods.

1. Introduction.

Lighthill [12] derived a nonlinear singular Volterra integral equation which describes the temperature distribution of the surface of a projectile moving through a laminar layer

$$F(z)^4 = -\frac{1}{2\sqrt{z}} \int_0^z \frac{F'(s)}{(z^{\frac{3}{2}} - s^{\frac{3}{2}})^{\frac{1}{3}}} ds, \tag{1.1}$$

where

$$F(0) = 1 \quad F(t) \to 0, \ t \to \infty. \tag{1.2}$$

In principle, equation (1.1) can be solved numerically by a quadrature method, by employing an appropriate approximation for F'. A different approach was considered in [10] and [9], where an inversion formula was applied to (1.1) and the resulting equation was solved by product integration methods. More recently, after applying suitable variable transformations to (1.1), the following nonlinear Volterra integral equation has been considered in [6]

$$y(t) = 1 - \frac{\sqrt{3}}{\pi} \int_0^t \frac{s^{\frac{1}{3}} y(s)^4}{(t-s)^{\frac{2}{3}}} ds, \quad t \in [0,1]. \tag{1.3}$$

Equation (1.3) has an Abel type kernel of the form $p(t, s, y(s))(t-s)^{-\alpha}$, with $\alpha = 2/3$ and $p(t, s, y) = s^{1/3} y^4$. It is straightforward to demonstrate that (1.3) has a unique continuous solution $y(t)$ for $t \in [0, 1]$ (see e.g.[13]). Regularity properties and numerical methods for Abel equations with a sufficiently smooth $p(t, s, y)$ have been considered by many authors. For detailed studies and a list of references we refer to [11], [4], [3]. We note that, since $p(t, s, y) = s^{1/3} y^4$ is not differentiable with respect to s (at $s = 0$), the results of those works are not applicable to equation (1.3).

In [6] a series representation for the solution of equation (1.3) was obtained:

$$y(t) = 1 - 1.461t^{2/3} + 7.252t^{4/3} - 46.460t^2 + 332.9t^{8/3} + \dots \tag{1.4}$$

for values of t satisfying $0 \le t < R^{3/2}$, where $R \simeq 0.106$. We thus see that the derivative $y'(t)$ behaves like $t^{-1/3}$ near the origin. Moreover, using a result from [5], the behaviour of $y(t)$ away from the origin has been analysed in [6].

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Lemma 1.1. The solution of equation (1.3) is such that $y \in C^{1,2/3}(\epsilon, 1]$ and $y \in C^{2,5/3}(\epsilon, 1]$, where $0 < \epsilon < R^{3/2}$. That is, $y(t) \in C^2]\epsilon, 1]$ and for $t \in (\epsilon, 1]$ we have

$$|y'(t)| \leq B_y(t-\epsilon)^{-2/3} \tag{1.5}$$

$$|y''(t)| \leq C_y(t-\epsilon)^{-5/3}, \tag{1.6}$$

for some constants $B_y > 0$ and $C_y > 0$.

In this work we investigate the application of several numerical methods to equation (1.3). The singular behaviour of the solution near the origin is expected to cause a drop in the global convergence orders. Recently, we have proved that the product Euler's method is convergent of order 1/3 and, for t away from the origin, obtained order one. In the present paper we give a detailed convergence analysis of the Trapezoidal method and investigate the use of higher order product integration methods. Spline collocation methods based on graded meshes are also investigated. Extrapolation methods are employed in order to improve the accuracy of low order numerical methods. Numerical results are presented illustrating the performance of the methods.

2. **Two numerical methods.** Here we describe the explicit product Euler's method and the product Trapezoidal method.
We introduce on $I = [0, 1]$ the uniform grid $X_h = \{t_i = ih, \quad 0 \leq i \leq N\}$, with stepsize $h = 1/N$.

In the Euler's method we approximate the integrand, $s^{\frac{1}{3}}y(s)^4$, by a piecewise constant function, that is, for $j = 0, 1, ..., N-1$,

$$s^{\frac{1}{3}} \; y(s)^4 \approx t_j^{\frac{1}{3}} \; y(t_j)^4, \quad s \in [t_j, t_{j+1}].$$

This yields the algorithm

$$y_i = 1 - \frac{\sqrt{3}}{\pi} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_i - s)^{2/3}} t_j^{1/3} y_j^4, \quad i = 1, 2, ..., N, \tag{2.7}$$

where y_i denotes an approximation to $y(t_i)$.

In the Trapezoidal method we consider a piecewise linear approximation, that is, on each subinterval $[t_j, t_{j+1}]$,

$$s^{1/3}y^4(s) \approx \frac{(s-t_j)}{h} t_{j+1}^{1/3} y^4(t_{j+1}) + \frac{(t_{j+1}-s)}{h} t_j^{1/3} y^4(t_j).$$

We obtain the algorithm

$$y_i = 1 - \frac{\sqrt{3}}{\pi} h \sum_{j=0}^{i} w_{ij} t_j^{1/3} y_j^4, \tag{2.8}$$

with $1 \leq i \leq N$, and where the weights w_{ij} are given by

$$w_{i0} = \frac{1}{h^2} \int_0^{t_1} \frac{(t_1 - s)}{(t_i - s)^{2/3}} ds$$

$$w_{ij} = \frac{1}{h^2} \left(\int_{t_j}^{t_{j+1}} \frac{(t_{j+1} - s)}{(t_i - s)^{2/3}} ds + \int_{t_{j-1}}^{t_j} \frac{(s - t_{j-1})}{(t_i - s)^{2/3}} ds \right)$$

$$1 \leq j \leq i-1, \; 2 \leq i \leq N$$

$$w_{ii} = \frac{1}{h^2} \int_{t_{i-1}}^{t_i} \frac{(s - t_{i-1})}{(t_i - s)^{2/3}} ds, \quad 2 \leq i \leq N$$

It can be shown that there exists $M > 0$ such that

$$0 < w_{ij} < M \frac{h^{-2/3}}{(i-j)^{2/3}}, \quad 1 \leq j \leq i \leq N. \tag{2.9}$$

Taking $y_0 = y(0) = 1$ as a starting value, the above algorithms yield approximate values of $y(t_i)$. In the numerical examples, the non-linear equation (2.8) was solved for y_i by Newton iteration, using y_{i-1} as the initial guess.

3. **Convergence results.** The convergence results for Euler's method, proved in [6], are contained in the next theorem.

Theorem 3.1. Let $y(t)$ be the solution of (1.3) and y_i an approximation to $y(t)$ at $t = t_i$, obtained with (2.7). Then the error $e_i = y(t_i) - y_i, i = 1, 2, \dots, N$, satisfies

$$|e_i| \le D\ h^{1/3} \tag{3.10}$$

$$|e_i| \le D_1 \left(\frac{h^{4/3}}{t_i^{2/3}} + h \right), \tag{3.11}$$

where D, D_1 are positive constants independent of h.

Therefore we can conclude that error of the explicit product Euler's method for equation (1.3) is of order $O(h^{1/3})$. However, at points t_i away from the origin, first order of convergence may be obtained.

In this work we study the product Trapezoidal method and give a summary of its convergence analysis. The total error $e_i = y(t_i) - y_i$ of the approximate solution obtained with the Trapezoidal method satisfies, at $t = t_i$

$$\begin{aligned} e_i &= \frac{\sqrt{3}}{\pi} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{s^{1/3} y(s)^4}{(t_i - s)^{2/3}} ds - h \sum_{j=0}^{i} w_{ij} t_j^{1/3} y_j^4 \right) \\ &= \frac{\sqrt{3}}{\pi} h \left(\sum_{j=0}^{i} \left(t_j^{1/3} y(t_j)^4 - t_j^{1/3} y_j^4 \right) w_{ij} \right) + T_i, \end{aligned} \tag{3.12}$$

where T_i is the quadrature error at $t = t_i$, given by:

$$\begin{aligned} T_i \quad &= \quad \frac{\sqrt{3}}{\pi} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left(s^{1/3} y^4(s) - P_j(s) \right) \frac{1}{(t_i - s)^{2/3}} ds, \\ & \qquad i = 1, ..., N, \end{aligned} \tag{3.13}$$

where, for $s \in [t_j, t_{j+1}]$,

$$P_j(s) = \frac{(t_{j+1} - s)}{h} t_j^{1/3} y^4(t_j) + \frac{(s - t_j)}{h} t_{j+1}^{1/3} y^4(t_{j+1}) \tag{3.14}$$

Using (2.9) in (3.12) and the fact that the function $f(x) = x^4$ satisfies a Lipschitz condition, we obtain

$$|e_i| \le |T_i| + M' h^{1/3} \sum_{j=0}^{i} \frac{|e_j|}{(i - j)^{2/3}}$$

Therefore, provided $(1 - M' h^{1/3}) < 1$,

$$|e_i| \le C_1 |T_i| + M'' h^{1/3} \sum_{j=0}^{i-1} \frac{|e_j|}{(i - j)^{2/3}}. \tag{3.15}$$

Lemma 3.1. The quadrature error, T_i, satisfies

$$|T_i| \le C_3\ h^{1/3},\ i = 1, 2, \dots, N, \tag{3.16}$$

where C_3 is a positive constant independent of h.

Proof. In [6] it was proved that the solution y to equation (1.3) satisfies the inequality

$$|y(z) - y(z')| \le C|z - z'|^{1/3}, \ \forall z, z' \in [0,1], \tag{3.17}$$

where C is a positive constant that does not depend on z or z'. Defining

$$\psi_j(s) = \left|\frac{(t_{j+1} - s)}{h}\right| t_j^{1/3} |y^4(s) - y^4(t_j)| + \left|\frac{(s - t_j)}{h}\right| t_{j+1}^{1/3} |y^4(s) - y^4(t_{j+1})|$$

and

$$\phi_j(s) = y^4(s) \left| s^{1/3} - \left(\frac{t_{j+1} - s}{h} t_j^{1/3} + \frac{s - t_j}{h} t_{j+1}^{1/3}\right)\right|,$$

we have

$$|s^{1/3} y^4(s) - P_j(s)| \le \psi_j(s) + \phi_j(s), \ s \in [t_j, t_{j+1}], \tag{3.18}$$

for $j = 0, 1, \cdots, i-1$. Then, making use of (3.17), gives

$$\begin{aligned} \psi_j(s) &\le |y^4(s) - y^4(t_j)| + |y^4(s) - y^4(t_{j+1})| \\ &\le LC(|s - t_j|^{1/3} + |s - t_{j+1}|^{1/3}) \\ &= 2LCh^{1/3}. \end{aligned}$$

On the other hand, it can be shown that, for $s \in (t_j, t_{j+1}], \ j = 0, 1, ..., N-1$,

$$\left| s^{1/3} - \left(\frac{(t_{j+1} - s)}{h} t_j^{1/3} + \frac{(s - t_j)}{h} t_{j+1}^{1/3}\right)\right| \le C_2 h^{1/3}.$$

Let $M_4 = \max_{s \in [0,1]} |y^4(s)|$. Using the above inequalities to bound (3.18), it follows from (3.13)

$$\begin{aligned} |T_i| &\le \max\{2LC + M_4 C_2\} h^{1/3} \int_0^{t_i} \frac{1}{(t_i - s)^{2/3}} ds \\ &\le C_3 h^{1/3}. \end{aligned}$$

□

Using Lemma 2 in (3.15) and applying a standard weakly singular Gronwall lemma (see e.g. [8]) leads to the following theorem.

Theorem 3.2. Let $y(t)$ be the solution of (1.3) and y_i an approximation to $y(t)$ at $t = t_i$ defined by (2.8). Then, the error $e_i = y(t_i) - y_i$ satisfies:

$$|e_i| \le C_4 \ h^{1/3}, \ i = 1, \ldots, N \tag{3.19}$$

where C_4 is a constant independent of h.

By a detailed analysis of the quadrature error, as it was done in [6] for the product Euler's method, we have the following result.

Lemma 3.2. The quadrature error, T_i, satisfies

$$|T_i| \le C_5 \frac{h^{4/3}}{t_i^{2/3}} + C_6 h^2, \quad i = 1, 2, \ldots, N, \tag{3.20}$$

where $C_5, \ C_6$ are positive constants independent of h.

Using (3.20) into (3.15) we obtain

$$|e_i| \le C_7 \frac{h^{4/3}}{t_i^{2/3}} + C_8 h^2 + M'' h^{1/3} \sum_{j=0}^{i-1} \frac{1}{(i-j)^{2/3}} |e_j|.$$

Then a generalized discrete Gronwall lemma from [7] can be applied to yield the following convergence result.

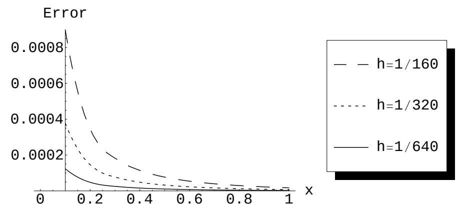

FIGURE 1. Absolute errors for Euler's method with uniform meshes

Theorem 3.3. Let $y(t)$ be the solution of (1.3) and y_i an approximation to $y(t)$ at $t = t_i$ defined by (2.8). Then the error $e_i = y(t_i) - y_i$ satisfies

$$|e_i| = |y(t_i) - y_i| \leq C_9 \left(\frac{h^{4/3}}{t_i^{2/3}} + h^2 \right), \tag{3.21}$$

where C_9 is a positive constant independent of h.

From the above theorem we conclude that the order of the error at the fixed point t_i away from the origin is $4/3$.

4. **Numerical results.** In this section we present some numerical results obtained with the product Euler and Trapezoidal methods considered in the previous section. Table 1 shows the computed experimental convergence rates, defined by

$$p \approx \frac{\log\left(\dfrac{y^{h/2} - y^h}{y^{h/4} - y^{h/2}}\right)}{\log 2}, \tag{4.22}$$

where y^h, $y^{h/2}$ and $y^{h/4}$ denote approximations to $y(t)$ using the mesh spacings h, $h/2$ and $h/4$, respectively. The results of Table I are in agreement with Theorems 1 and 3, confirming the predicted first order of convergence for the Euler's method and $4/3$ for the Trapezoidal method. The absolute errors for $N = 160, 320, 640$ are shown in Figures 1,2.

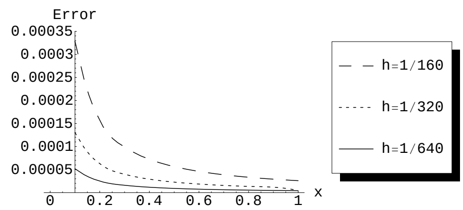

FIGURE 2. Absolute errors for Trapezoidal method

TABLE 1. Convergence rates for several values of N

t_i	Euler's method		Trapezoidal method	
	$80, 160, 320$	$160, 320, 640$	$80, 160, 320$	$160, 320, 640$
0.2	1.108	1.061	1.322	1.320
0.4	1.066	1.041	1.308	1.313
0.5	1.057	1.035	1.305	1.311
0.7	1.045	1.029	1.302	1.309
0.8	1.041	1.026	1.301	1.309
1.0	1.035	1.023	1.300	1.308

Let $|e_N(t_i)| = |y(t_i) - y_i^{1/N}|$. In order to obtain error estimates we have used the formula

$$|e_N(t_i)| \approx \frac{|y_i^{1/N} - y_i^{1/2N}|}{1-(1/2)^p},$$

where the value $p=1$ was taken for Euler's method and $p=1.3$ for the Trapezoidal method, according to the estimate (4.22). The computed error norms, given by:

$$\|e_N\|_\infty = \max_{1\le i\le N} |y(t_i^N) - y_i^N|, \quad t_i^N = \frac{i}{N},$$

are displayed in Table 2. The results show an expected drop in the global convergence orders, due to the nonsmooth behaviour of the solution near the origin.

TABLE 2. Error norms for several values of N

N	Euler's method	Trapezoidal method
40	0.1782	0.1600×10^{-1}
80	0.1122	0.1117×10^{-1}
160	0.7071×10^{-1}	0.7548×10^{-2}
320	0.4454×10^{-1}	0.4981×10^{-2}
640	0.2806×10^{-1}	0.3234×10^{-2}

5. **Other methods.**

5.1. **Higher order product quadrature rules.** Consider the discretised form of (1.3) at $t=t_i$

$$y(t_i) = 1 - \frac{\sqrt{3}}{\pi}\int_0^{t_i} \frac{s^{\frac{1}{3}} y(s)^4}{(t_i - s)^{\frac{2}{3}} ds}, \quad 0 \le i \le N. \tag{5.1}$$

In the product Simpson's method we approximate the integral in (5.1) by the product Simpson's rule used repeatedly over $[0, t_i]$ if i is even. When i is odd, we use the product Simpson's rule over $[0, t_{i-3}]$ and the product three-eights rule will be used over $[t_{i-3}, t_i]$.
Then the product Simpson's method for equation (1.3) is defined by

$$y_{2r} = 1 - \frac{\sqrt{3}}{\pi} h^{1/3} \sum_{j=0}^{r-1}\sum_{k=0}^{2} t_{2j+k}^{1/3} y_{2j+k}^4 b_k(2j, 2r) \tag{5.2}$$

$$y_{2r+1} = 1 - \frac{\sqrt{3}}{\pi} h^{1/3} \sum_{j=0}^{r-2}\sum_{k=0}^{2} t_{2j+k}^{1/3} y_{2j+k}^4 b_k(2j, 2r+1) - \frac{\sqrt{3}}{\pi} h^{1/3} \sum_{k=0}^{3} t_{2r-2+k}^{1/3} y_{2r-2+k}^4 \, d_k, \tag{5.3}$$

with

$$b_0(j,i) = \frac{1}{2}\int_0^2 \frac{(\upsilon-2)(\upsilon-1)}{(i-j-\upsilon)^{2/3}} d\upsilon$$

$$b_1(j,i) = -\int_0^2 \frac{\upsilon(\upsilon-2)}{(i-j-\upsilon)^{2/3}} d\upsilon$$

$$b_2(j,i) = \frac{1}{2}\int_0^2 \frac{\upsilon(\upsilon-1)}{(i-j-\upsilon)^{2/3}} d\upsilon$$

and

$$d_0 = -\frac{1}{6}\int_0^3 \frac{(\upsilon-2)(\upsilon-1)(\upsilon-3)}{(3-\upsilon)^{2/3}}d\upsilon$$
$$d_1 = \frac{1}{2}\int_0^3 \frac{\upsilon(\upsilon-2)(\upsilon-3)}{(3-\upsilon)^{2/3}}d\upsilon$$
$$d_2 = -\frac{1}{2}\int_0^3 \frac{\upsilon(\upsilon-1)(\upsilon-3)}{(3-\upsilon)^{2/3}}d\upsilon$$
$$d_3 = \frac{1}{6}\int_0^3 \frac{\upsilon(\upsilon-1)(\upsilon-2)}{(3-\upsilon)^{2/3}}d\upsilon$$

Like in the Trapezoidal method, the nonlinear equations (5.2) and (5.3) were solved by Newton's method and we have used the series (1.4) to compute the starting value $y_1 \approx y(t_1)$.

Again the numerical results show evidence of a drop in the global convergence order (cf. Table 4). The results of Table 3 suggest that, for points away from the origin, the product Simpson's method may exhibit the same convergence order, 4/3, as the Trapezoidal method. We believe that similar conclusions are to be expected for product integration methods based on the application of higher order repeated rules (as main rules).

TABLE 3. Convergence rates for several values of N

	Simpson's method	
t_i	$80, 160, 320$	$160, 320, 640$
0.2	1.379	1.385
0.4	1.388	1.387
0.5	1.389	1.387
0.7	1.390	1.388
0.8	1.390	1.388
1.0	1.391	1.388

TABLE 4. Error norms for several values of N

N	Simpson's method
40	0.4953×10^{-2}
80	0.3500×10^{-2}
160	0.2377×10^{-2}
320	0.1578×10^{-2}
640	0.1030×10^{-2}

5.2. **Collocation methods with uniform meshes.** Here we present numerical results generated by the application of collocation methods with uniform meshes. The following notation and methods were introduced in [2]. Given an integer $N \geq 1$, let $\Pi_N : 0 = t_0 < t_1 < ... < t_N = 1$ be a partition of the interval $[0, 1]$. Consider the associated subintervals $\sigma_0 := [t_0, t_1]$, $\sigma_n := (t_n, t_{n+1}]$, $1 \leq n \leq N-1$, and define $Z_N := \{t_n : n = 1, \ldots, N-1\}$. The collocation methods use elements of the polynomial spline space $S^{(d)}_{m-1}(Z_N)$, that is, functions $u \in C^{(d)}([0, 1]) : u|_{\sigma_n} = u_n \in \pi_{m-1}$, $1 \leq n \leq N-1$. Here π_{m-1} is the set of polynomials of degree not exceeding $m-1$ (with $m \geq 1$). In the case $d = -1$, no continuity conditions are imposed at the mesh points and u will in general possess jump discontinuities at the knots Z_N. The desired approximation to the solution of equation (1.3) is an element $u \in S^{(d)}_{m-1}(Z_N)$ satisfying

$$u(t) = 1 - \frac{\sqrt{3}}{\pi}\int_0^t \frac{s^{1/3}u^4(s)}{(t-s)^{2/3}}ds,\ t \in X(N), \tag{5.1}$$

where $X_N = \bigcup_{n=0}^{N-1} X_n$ with

$$X_n = \left\{ t_{nj} = t_n + c_j h : 0 \leq c_1 < ... < c_m \leq 1, h = \frac{1}{N} \right\}.$$

The collocation equation (5.1) has the form

$$\begin{aligned} &u_i(t_{ik}) \\ &= 1 - \frac{\sqrt{3}}{\pi}h^{1/3}\sum_{j=0}^{i-1}\int_0^1 \frac{(t_j+sh)^{1/3}u_j^4(t_j+sh)}{(i+c_k-j-s)^{2/3}}ds \\ &-\frac{\sqrt{3}}{\pi}h^{1/3}\int_0^{c_k}\frac{(t_i+sh)^{1/3}u_i^4(t_i+sh)}{(c_k-s)^{2/3}}ds \end{aligned} \tag{5.2}$$

Approximating the integrals in (5.2) by product integration formulae, we obtain the following discretised version of (5.1)

$$\begin{aligned} u_{ik} = 1 - &\frac{\sqrt{3}}{\pi}h^{1/3}\sum_{j=0}^{i-1}\sum_{l=1}^{m} w_{kl}^{(ij)}(t_j+c_l h)^{1/3}u_{jl}^4 \\ -&\frac{\sqrt{3}}{\pi}h^{1/3}\sum_{l=1}^{m} w_{kl}(t_i+c_l c_k h)^{1/3}\sum_{r=1}^{m} L_r(c_k c_l)u_{ir}^4 \end{aligned} \tag{5.3}$$

for $k = 1, ..., m$ and $0 \leq i \leq N-1$. $L_r,\ r = 1, ..., m$, are the Lagrange polynomials associated with c_r.
The quadratures weights in (5.3) are given by

$$\begin{aligned} w_{kl}^{(ij)} &= \int_0^1 \frac{\lambda_l(s)}{(i+c_k-j-s)^{2/3}}ds \\ w_{kl} &= \int_0^{c_k}\frac{\lambda_{kl}(s)}{(c_k-s)^{2/3}}ds, \end{aligned}$$

where

$$\lambda_l(s) = \prod_{j=1, j\neq l}^{m} \frac{(s-c_j)}{(c_l-c_j)}$$

and

$$\lambda_{kl}(s) = \prod_{j=1, j\neq l}^{m} \frac{(s-c_k c_j)}{(c_k c_l - c_j c_k)}.$$

In order to approximate the solution of equation (1.3) we have considered several choices of m and of the collocation parameters:

1. $m = 2$ ($u \in S_1^{(-1)}(Z_N)$), with collocation parameters
   - $c_1 = 1/2$ and $c_2 = 1$,
   - $c_1 = 2/3$ and $c_2 = 1$,
2. $m = 3$ ($u \in S_2^{(-1)}(Z_N)$), with collocation parameters
   - Gauss points:
     $c_1 = \frac{3-\sqrt{3}}{6}, c_2 = \frac{3+\sqrt{3}}{6}, c_3 = 1$;
   - Radau II points:
     $c_1 = \frac{4-\sqrt{6}}{10}, c_2 = \frac{4+\sqrt{6}}{10}, c_3 = 1$;

In Tables 5 and 6 we show the computed experimental rates of convergence. The results suggest that the convergence order of the collocation methods is approximately 1.3 and, similarly to product integration methods, we see a reduction in the global orders of convergence (cf. Tables 7). However, let us compare, for example, the error norms of Tables 7 and 8 with the ones in Tables 2 and 4. It would appear that collocation methods give more accurate approximations than the ones obtained with product integration methods using quadrature rules of the same orders.

TABLE 5. Convergence rates for several values of N for collocation method in $S_1^{(-1)}(Z_N)$

t_i	$c_1 = 1/2, c_2 = 1$		$c_1 = 2/3, c_2 = 1$	
	$80, 160, 320$	$160, 320, 640$	$80, 160, 320$	$160, 320, 640$
0.2	1.314	1.324	1.285	1.302
0.4	1.306	1.317	1.270	1.291
0.5	1.302	1.315	1.266	1.288
0.7	1.298	1.312	1.260	1.284
0.8	1.296	1.311	1.258	1.282
1.0	1.294	1.309	1.254	1.281

TABLE 6. Convergence rates for several values of N for collocation method in $S_2^{(-1)}(Z_N)$

t_i	Gauss points		Radau II points	
	$80, 160, 320$	$160, 320, 640$	$80, 160, 320$	$160, 320, 640$
0.2	1.386	1.375	1.367	1.361
0.4	1.388	1.377	1.370	1.362
0.5	1.388	1.379	1.370	1.360
0.7	1.389	1.393	1.371	1.353
0.8	1.390	1.340	1.371	1.347
1.0	1.392	1.415	1.369	1.335

TABLE 7. Error norms for several values of N

N	$S_1^{(-1)}(Z_N)$ $c_1 = 1/2,\ c_2 = 1$	$S_2^{(-1)}(Z_N)$ Gauss Points
40	0.1177×10^{-2}	0.2947×10^{-3}
80	0.8296×10^{-3}	0.2013×10^{-3}
160	0.5648×10^{-3}	0.1342×10^{-3}
320	0.3745×10^{-3}	0.8787×10^{-4}
640	0.2437×10^{-3}	0.5684×10^{-4}

5.3. **Collocation methods with graded meshes.** One way to recover the optimal convergence rates of collocation methods for weakly singular equations, when the solution is not smooth, is to use a graded mesh (see e.g. [4]). We have considered collocation in the polynomial spline space $S_0^{(-1)}(\mathbf{Z}_N)$, using the grid:

$$\Delta^N = \left\{ t_i \in [0,1] : t_i = (i/N)^3,\ i = 0, 1, ..., N \right\}.$$

The numerical results of Table 8 indicate first order of convergence for the piecewise constant approximation, that is, the optimal order seems to have been recovered.

TABLE 8. Error norms for several values of N, for collocation with graded meshes

$S_0^{(-1)}(\mathbf{Z}_N)$ $(c_1 = 1)$		
N	$\|\mathbf{e}\|_\infty$	rate
10	0.2135×10^{-1}	–
20	0.7969×10^{-2}	1.422
40	0.3352×10^{-2}	1.250
80	0.1502×10^{-2}	1.158
160	0.6935×10^{-3}	1.115
320	0.3269×10^{-3}	1.085
640	0.1563×10^{-3}	1.065

5.4. **Extrapolation.** Here we present some results concerning the use of extrapolation procedures in order to accelerate the convergence of the numerical results obtained with the two low order methods considered in Section II. In the case of the explicit product Euler's method, we have assumed that the approximate solution y^{h_n} has an asymptotic error expansion with the form

$$y_j^{h_n} - y(t_j) = a_1 h_n + a_2 h_n^2 + O(h^3),$$

for points t_j away from the origin. Using an algorithm based on Richardson's extrapolation, we started with an inicial approximation $E_{0j}^{(n)} = y_j^{h_n}$ and the new approximations were computed recursively by

$$E_{kj}^{(n)} = \frac{h_{n+k} E_{k-1j}^{(n)} - h_n E_{k-1j}^{(n+1)}}{h_{n+k} - h_n},$$

$$k = 1, 2; \;\; n = 0, 1, 2, 3$$

The results of Tables 9 and 10 illustrate this extrapolation process at points 0.3 and 0.5, where we have taken $E_2^{(1)}$ as the exact solution of equation (1.3) at those points. We see that the convergence is accelerated only in the first step of the process, that is, there is an improvement in the accuracy from the first to the second column but not to the third column. This seems to confirm the $O(h)$ order of the first term of the error expansion, but no conclusions can be drawn about the order of the next term.

TABLE 9. Absolute error of the entries of the E-array ($t = 0.3$) for Euler's method

n	$\|E_0^{(n)} - y(0.3)\|$	$\|E_1^{(n)} - y(0.3)\|$	$\|E_2^{(n)} - y(0.3)\|$
0	2097×10^{-3}	0.3337×10^{-5}	0.1427×10^{-5}
1	1032×10^{-3}	0.1904×10^{-5}	
2	5064×10^{-4}	0.4761×10^{-6}	
3	2508×10^{-4}		

TABLE 10. Absolute error of the entries of the E-array ($t = 0.5$) for Euler's method

n	$\|E_0^{(n)} - y(0.5)\|$	$\|E_1^{(n)} - y(0.5)\|$	$\|E_2^{(n)} - y(0.5)\|$
0	0.8704×10^{-4}	0.1737×10^{-5}	0.7749×10^{-6}
1	0.4265×10^{-4}	0.1015×10^{-5}	
2	0.2082×10^{-4}	0.2539×10^{-6}	
3	0.1028×10^{-4}		

For the product Trapezoidal method we have assumed that the approximate solution y^{h_n} has an asymptotic error expansion with the form

$$y_j^{h_n} - y(t_j) = a_1 h_n^{4/3} + a_2 h_n^2 + O(h^3),$$

for points t_j away from the origin. In this case we have used the E-algorithm in order to accelerate the convergence of the Trapezoidal method. The computation of $E_k^{(n)}$ begins with

$$E_0^{(n)} = y_j^{h_n}, \quad n = 0, 1, 2$$
$$g_{0,i}^{(n)} = g_i(n), \quad i = 1, 2 \; n = 0, 1, 2$$

where $g_1(n) = h_n^{4/3}$ and $g_2(n) = h_n^2$.

For $k = 1, 2, 3$ and $n = 0, 1, ..., 3 - k$ the new approximations are computed recursively by

$$E_k^{(n)} = \frac{E_{k-1}^{(n)} g_{k-1,k}^{(n+1)} - E_{k-1}^{(n+1)} g_{k-1,k}^{(n)}}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}}$$
$$g_{k,i}^{(n)} = \frac{g_{k-1,i}^{(n)} g_{k-1,k}^{(n+1)} - g_{k-1,i}^{(n+1)} g_{k-1,k}^{(n)}}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}}, \; i = k+1, k+2, ...$$

We have taken $E_2^{(1)}$ as the exact solution of equation (1.3) and the absolute errors, for points 0.3 and 0.5, of the entries of the E-array are displayed in Tables 11 and 12. The results show that the accuracy is improved in each step of the extrapolation process. This seems to confirm the $O(h^{4/3})$ order of the first term of the error expansion and the $O(h^2)$ order of the second term.

TABLE 11. Absolute error of the entries of the E-array ($t = 0.3$) for Trapezoidal method

n	$\|E_0^{(n)} - y(0.3)\|$	$\|E_1^{(n)} - y(0.3)\|$	$\|E_2^{(n)} - y(0.3)\|$
0	9875×10^{-4}	0.7041×10^{-6}	0.4993×10^{-7}
1	3961×10^{-4}	0.2135×10^{-6}	
2	1585×10^{-4}	0.5337×10^{-7}	
3	6322×10^{-5}		

TABLE 12. Absolute error of the entries of the E-array ($t = 0.5$) for Trapezoidal method

n	$\|E_0^{(n)} - y(0.5)\|$	$\|E_1^{(n)} - y(0.5)\|$	$\|E_2^{(n)} - y(0.5)\|$
0	0.5608×10^{-4}	0.4869×10^{-6}	0.2736×10^{-7}
1	0.2255×10^{-4}	0.1423×10^{-6}	
2	0.9034×10^{-5}	0.3556×10^{-7}	
3	0.3606×10^{-5}		

6. **Conclusions.** This work has been concerned with the numerical analysis of the nonlinear Volterra integral equation (1.3), which has a weakly singular kernel of the form $s^{1/3}y(s)^4\,(t-s)^{-2/3}$. The derivative $y'(t)$ of the solution of this equation behaves like $t^{-1/3}$ near the origin and this is expected to cause a loss in the global convergence order of product integration and collocation methods. This has been shown theoretically for the explicit product Euler's method and the product Trapezoidal method, where the errors are of orders $O(h^{1/3})$; for points t away from the origin, the convergence order is one for Euler's method and 4/3 for the Trapezoidal method. These results were confirmed by some numerical examples. We have also implemented a product integration method based on Simpson's rule as well as collocation methods using polynomial splines of degrees 0,1,2. The numerical experiments of Section V suggest that general product integration and collocation methods applied to equation (1.3) have 1/3 global order of convergence, independently of the degree of the approximating polynomials used; as t increases the errors seem to be of order 4/3. On the other hand, the use of collocation methods based on graded meshes suggest that the optimal orders can be recovered. The theoretical study of these methods will be done elsewhere. Finally, Richardson's extrapolation procedure was used in conjunction with the product Euler's method and some improvement in the accuracy was observed. In order to accelerate the convergence of the Trapezoidal method the E-algorithm was applied, indicating that the first and the second terms of the error expansion are of orders $O(h^{4/3})$ and $O(h^2)$, respectively.

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* Departamento de Matemática, Faculdade de Ciências e Tecnologia, Monte da Caparica, Portugal

**CEMAT/Dep. de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

E-mail address: msjr@fct.unl.pt
E-mail address: tdiogo@math.ist.utl.pt
E-mail address: plima@math.ist.utl.pt