Rate of convergence for correctors in almost periodic homogenization

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Abstract

In the homogenization of second order elliptic equations with periodic coefficients, it is well known that the rate of convergence of the corrector $u_n - u^{\rm hom}$ in the L^2 norm is $1/n$, the same as the scale of periodicity (see Jikov et al [7]). It is possible to have the same rate of convergence in the case of almost periodic coefficients under some stringent structural conditions on the coefficients (see Kozlov [8]). The goal of this note is to construct almost periodic media where the rate of convergence is lower than $1/n$. To that aim, in the one dimensional setting, we introduce a family of random almost periodic coefficients for which we compute, using Fourier series analysis, the mean rate of convergence r_n (mean with respect to the random parameter). This allows us to present examples where we find $r_n \gg 1/n^r$ for every $r > 0$, showing a big contrast with the random case considered by Bourgeat and Piatnitski [3] where $r_n \sim 1/\sqrt{n}$.

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1 Introduction

Let D be a bounded domain in $\mathbb{R}^N$. Let f and g be suitable data. Consider the following sequence of Dirichlet boundary value problems :

$$\begin{cases} -\operatorname{div}(A_n(x)\nabla u_n) &= f \text{ in } D, \\ u_n &= g \text{ on } \partial D \ . \end{cases} \tag{1.1}$$

where $A_n(\cdot)$ is a family of matrices with measurable entries satisfying

$$\lambda\ |\xi|^2\ \le A_n(x)\xi\cdot\xi \le \Lambda\ |\xi|^2 \text{ a.e. } x \in D\ ,\ \forall \xi \in \mathbb{R}^N\ , \tag{1.2}$$

for given constants $0 < \lambda < \Lambda$. Typically the coefficients A_n oscillate at a scale $1/n$ as is the case when $A_n(x) := A(nx)$, and $A(\cdot)$ is either a periodic or an almost periodic function on $\mathbb{R}^N$. So,

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although there is existence and uniqueness of a solution u_n to (1.2), its numerical computation is impracticable due to highly oscillating coefficients. The theory of homogenization provides the asymptotic behaviour of the sequence of problems and it is well known (see [9], for a recent translation of the original work of Murat and Tartar on H-convergence during the 70's) that the sequence u_n converges in $H^1(D)$ towards the solution $u^{\rm hom}$ of a homogenized problem :

$$\left\{ \begin{array}{rcl} -\operatorname{div}(A^{\rm hom}\nabla u^{\rm hom}) & = & f \text{ in } D, \\ u^{\rm hom} & = & g \text{ on } \partial D\ . \end{array} \right. \tag{1.3}$$

The effective matrix $A^{\rm hom}$ is independent of the data f and g, as well as of the domain D, and, in the periodic or almost periodic case, it has constant coefficients. In those special cases the computations required to calculate the matrix $A^{\rm hom}$ and then to calculate $u^{\rm hom}$, knowing $A^{\rm hom}$, are numerically stable. So, $u^{\rm hom}$ can be used as a numerical approximation of u_n if we can estimate for example the mean square of the error $u_n - u^{\rm hom}$, that is $\|u_n - u^{\rm hom}\|_{L^2(D)}$. The error $u_n - u^{\rm hom}$ is also called the corrector of order zero.

When $A(\cdot)$ is periodic it is well known that the following estimate holds (see [7]) :

$$\|u_n - u^{\rm hom}\|_{L^2(D)} \leq C/n \tag{1.4}$$

However, even for very smooth almost periodic coefficients the estimate (1.4) is not always satisfied and additional assumptions are needed. In oder to estimate the corrector in this case, Kozlov (see [8]) proposed a perturbation method derived from a formal asymptotic expansion, $u_n(x) = u^{\rm hom}(x) + \frac{1}{n}u_1(x, nx) + ...$, where u_1 is a sufficiently regular function, almost periodic in the second variable. By this method he could prove an estimate of the type (1.4) provided that the equation :

$$-\operatorname{div}(A(y)\nabla(y_k + \chi_k(y))) = 0 \text{ in } \mathbb{R}^N\,, \quad 1 \leq k \leq N\ . \tag{1.5}$$

has an almost periodic solution χ_k. The existence of such solutions is not always guaranteed. A sufficient condition to have such a solution is that the coordinate projection of the frequencies of $a_{ij}(\cdot)$ (see Definition 2.1), for each coordinate, have a finite basis $\{\omega_{ij}^1, \cdots, \omega_{ij}^d\}$ over $\mathbb{Z}$ and that there exists constants $C > 0$ and $s > 0$ such that

$$|k_1\omega_{ij}^1 + \cdots + k_d\omega_{ij}^d| \geq C|k|^{-s} \quad \forall(k_1, \cdots, k_d) \in \mathbb{Z}^d\,, \forall i, j. \tag{1.6}$$

Furthermore, it is assumed in [8] that the coefficients a_{ij} have a degree of regularity depending on the exponent s. In the purely periodic case the condition (1.6) is trivially verified and no smoothness is required.

On the other hand, in the case of random media, the rate of convergence results are strikingly different. In the one-dimensional case, Bourgeat and Piatnitski (see [3]) consider the following mixing condition on the random coefficients $\xi(x, \omega)$:

$$\left|\mathbb{E}\Big(\xi(x,\omega)\,\xi(y,\omega)\Big) - \mathbb{E}\xi(x,\omega)\,\mathbb{E}\xi(y,\omega)\right| \leq d^{-\beta}\mathbb{E}(\xi^2(x,\omega))^{\frac{1}{2}}\mathbb{E}(\xi^2(y,\omega))^{\frac{1}{2}}\,, \text{ for all } |x-y| \geq d \tag{1.7}$$

for some $\beta > 1$, where the symbol $\mathbb{E}$ just denotes the expectation, that is, the integral with respect to the random variable ω. Under this condition, they show that for all x, the sequence

$\sqrt{n}(u_n(x,\omega) - u^{\rm hom})$ converges in law to a Gaussian random variable. This means that the typical deviation of u_n from $u^{\rm hom}$ is of the order $1/\sqrt{n}$. This is significantly different from the type of estimate (1.4) obtained in the periodic case or in the almost periodic case so far.

In this note, we take up the problem of determining the rate of convergence of $u_n - u^{\rm hom}$, in the one-dimensional almost periodic problem. We do not make any structural assumptions of the kind (1.6) but we assume as in Guenneau [5], that the almost periodic coefficient is obtained by restricting a periodic function $a(x_1, x_2)$ defined on $\mathbb{R}^2$ along a line of irrational slope α.

We obtain precise estimates, by introducing a random parameter in the problem and by averaging over it. Using Fourier analysis, we derive the average rate of convergence from the behavior of a series involving α and the Fourier coefficients of a^{-1} (Theorem 3.1). As a consequence, we obtain that the rate of convergence cannot be faster than $1/n$ and that it is exactly $1/n$ under a necessary and sufficient condition which is less restrictive than (1.6).

Furthermore, in contrast with the stochastic case considered in [3] where the mixing condition (1.7) is satisfied, we provide examples where the rate is slower than $1/\sqrt{n}$ and in particular can be set to be $1/n^r$ for arbitrary positive $r < 1/2$ (see Corollary 3.4 and Remark 3.5).

Finally we derive an upperbound for r_n taking into account the regularity of $a(x_1, x_2)$ and the irrationality measure of the slope α (Corollary 3.7).

2 Setting of the homogenization problem.

We first recall the notion of almost periodic function and then introduce the problem of almost periodic homogenization in the random setting using the formalism of dynamical systems.

Almost periodic functions. We denote by Trig($\mathbb{R}$) the space of trigonometric polynomials on $\mathbb{R}$, that, is finite sums of the form

$$p(x) = \sum a_k \, e^{i\,\xi_k\,x}\,, \qquad a_k \in \mathbb{C}\,, \xi_k \in \mathbb{R}.$$

There are several choices of norms on the space Trig($\mathbb{R}$) and its completion with respect to any of these leads to different notions of the space of almost periodic functions. The closure with respect to the sup norm $|\cdot|_\infty$ is called the space of almost periodic functions in the sense of Bohr. Obviously, such functions are continuous, and therefore are too restrictive for physical applications, like, for instance, the modeling of quasicrystals. Instead we will use the following norm on Trig($\mathbb{R}$) (see Besicovitch [1] or Jikov et al [7] for details)

$$|p|_2 := \left(\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |p(x)|^2 \, dx \right)^{\frac{1}{2}}. \tag{2.1}$$

Definition 2.1 The closure of Trig($\mathbb{R}$) with respect to the norm $|\cdot|_2$ will be called the space of almost periodic functions in the sense of Besicovitch. We will denote this space by $\mathrm{AP}^2(\mathbb{R})$. The elements of $\mathrm{AP}^2(\mathbb{R})$ are identified as measurable functions f defined on $\mathbb{R}$ for which the limit in the right hand side of (2.1) exists and is finite. For such a function f *the set of frequencies* defined below is an at most countable subset of $\mathbb{R}$

$$\mathrm{Freq}(f) = \{\omega \in \mathbb{R} : \mathrm{M}(f\,e^{-i\,\omega\,x}) \neq 0\}. \tag{2.2}$$

Dynamical systems formalism (see [7]). Recall that a dynamical system on $\mathbb{R}$ with respect to the probability space $(\Omega, \mathcal{A}, \mu)$ is a family $T = \{T(x)\}_{x\in\mathbb{R}} : \Omega \to \Omega$ such that

1. $T(x)$ is measure preserving on $(\Omega, \mathcal{A}, \mu)$, i.e., $\int_\Omega f(T(x)\omega)\, d\mu = \int_\Omega f(\omega)\, d\mu$, for all $f \in L^\infty(\Omega)$ and for each $x \in \mathbb{R}$;
2. $T(x + y) = T(x) \circ T(y)$ for all $x, y \in \mathbb{R}$;
3. for every measurable function f on Ω, the application $(x, \omega) \mapsto f(T(x)\,\omega)$, called *a stationary random field*, is measurable on $(\mathbb{R} \times \Omega, \mathcal{F} \otimes \mathcal{A})$, where $\mathcal{F}$ is the Lebesgue σ-algebra on $\mathbb{R}$.

We say that T is ergodic if T-invariant functions are constant that is, if $f(T(x)\omega) = f(\omega)$, μ-a.e. $\omega \in \Omega$, for all $x \in \mathbb{R}$, then f is constant μ a.e. in Ω. In this case, by the Birkhoff Ergodic Theorem (see [7], p.225), for every $f \in L^1(\Omega, \mu)$ and for every compact subset $K \subset \mathbb{R}$ the following convergence holds for almost all $\omega \in \Omega$:

$$\frac{1}{|K|}\int_K f(T(n\,x)\omega)\; dx \overset{n\to+\infty}{\longrightarrow} \langle f\rangle := \int_\Omega f\, d\mu\ . \tag{2.3}$$

Almost periodic random fields. We will focus on a particular stationary random field whose almost all realizations are almost periodic in the sense of Besicovitch. Consider the probability space $([0,1)^2, \mathcal{F}\otimes\mathcal{F}, m)$, where $\mathcal{F}$ is the σ-algebra of Lebesgue measurable sets in $[0, 1)$ and m the restriction of the two-dimensional Lebesgue measure to $[0, 1)^2$. Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and let $T(x) : \Omega \to \Omega$, where $\Omega = [0, 1)^2$, be defined as follows :

$$T(x)\omega := (\omega + (1, \alpha)\, x) \mod [0, 1)^2, \quad \text{for } \omega \in \Omega. \tag{2.4}$$

It is easy to see that $\{T(x)\}_{x\in\mathbb{R}}$ is an ergodic dynamical system with respect to m.

Let now $a(\omega_1, \omega_2)$ be a given function in $L^2(\Omega)$ which we implicitely periodize on all $\mathbb{R}^2$. Then, we claim that, for almost all $\omega \in \Omega$, the function $x \mapsto a(T(x)\omega)$ belongs to the class $\mathrm{AP}^2(\mathbb{R})$. Indeed we may write $a(\omega_1, \omega_2) = \sum_{(k,l)\in\mathbb{Z}\times\mathbb{Z}} a_{kl}\, e^{2\pi i(k\,\omega_1 + l\,\omega_2)}$ and consider its polynomial approximations $p_n := \sum_{|k|,|l|\le n} a_{kl}\, e^{2\pi i(k\,\omega_1+l\,\omega_2)}$. Applying (2.3) with $f = |\, a - p_n|^2$ which is an element of $L^1(\Omega, \mu)$, the following convergence holds for almost all $\omega \in \Omega$

$$\lim_{N\to+\infty}\frac{1}{2\,N}\int_{-N}^{N} |a(T(x)\omega) - p_n(T(x)\omega)|^2\; dx = \int_\Omega |\, a - p_n|^2\; d\omega. \tag{2.5}$$

The claim follows since the right hand side of (2.5) goes to zero as $n \to +\infty$ and $x \to p_n(T(x)\omega)$ is a trigonometric polynomial.

Setting of the homogenization problem. Let α be fixed in $\mathbb{R} \setminus Q$ and consider $T(\cdot)$ defined in (2.4) . Let $a(\omega_1, \omega_2)$ be a bounded measurable function on $\Omega = [0, 1]^2$ satisfying

$$0 < \lambda \le a(\cdot) \le \Lambda \quad \text{ a.e. in } \Omega \tag{2.6}$$

for given constants $0 < \lambda < \Lambda$. Let us define the stationary random field

$$\xi(x, \omega) = a(T(x)\omega). \tag{2.7}$$

We have already noticed that almost all realizations of $\xi(\cdot, \omega)$ are Besicovitch almost periodic and therefore the realizations of the rescaled fields $x \mapsto \xi(nx, \omega)$ are almost periodic at a scale $1/n$, for every $n \in \mathbb{N}$.

We consider the following boundary value problem, where $\omega \in \Omega$ acts like a parameter :

$$\begin{gathered} -\frac{d}{dx}\left(\xi(n\,x, \omega)\frac{du_n(x, \omega)}{dx}\right) = 0 \text{ in } (0,1)\,, \\ u_n(0, \omega) = 0\,,\; u_n(1, \omega) = 1. \end{gathered} \tag{2.8}$$

The solution of (2.8) is given explicitly by

$$u_n(x, \omega) = \frac{\int_0^x\, [\xi(n\,t, \omega)]^{-1}\, dt}{\int_0^1\, [\xi(n\,t, \omega)]^{-1}\, dt} = \frac{\int_0^x\, [a(T(n\,t)\omega)]^{-1}\, dt}{\int_0^1\, [a(T(n\,t)\omega)]^{-1}\, dt} \tag{2.9}$$

Then (2.3), together with the coercivity condition (2.6), yield the weak convergence in $H^1(0,1)$, as $n \to +\infty$, for almost all $\omega \in \Omega$, of the sequence u_n to the function $u(t) = t$. Similarly the sequence $\xi(n\,x, \omega)\,\dfrac{du_n(x, \omega)}{dx} = \dfrac{1}{\int_0^1\, [a(T(n\,t)\omega)]^{-1}\, dt}$, which is a sequence of constant functions in x, converges strongly in $L^2(0,1)$ to $\langle a^{-1}\rangle^{-1}$ for almost all ω in Ω. Thus, for almost all $\omega \in \Omega$, the homogenized equation reads as

$$\begin{gathered} -\frac{d}{dx}(\langle a^{-1}\rangle^{-1}\frac{du}{dx}) = 0\,, \\ u(0) = 0\,,\; u(1) = 1, \end{gathered} \tag{2.10}$$

where $\langle a^{-1}\rangle = \int_\Omega\, a(\omega)^{-1}\, d\omega$. Note that the equation (2.10) is independent of the parameter ω. Due to the compact inclusion $H^1(0,1) \subset L^2(0,1)$, the following convergence takes place :

$$r_n^2(\omega) := \int_0^1\, |u_n(x, \omega) - x|^2\; dx \to 0, \tag{2.11}$$

for almost all $\omega \in \Omega$. By (2.6) the sequence $\{u_n\}$ given by (2.9) is uniformly bounded for all $x \in [0,1]$ and all $\omega \in \Omega$ and therefore, by Lebesgue's Dominated Convergence Theorem we conclude from (2.11) that

$$r_n^2 := \int_\Omega \int_0^1\, |u_n(x, \omega) - x|^2\; dx\; d\omega \to 0.$$

The main question, now, is to find the rate of the convergence of the error r_n.

Remark 2.2 Actually it is easy to deduce a rate of convergence for individual realizations (that is for $r_n(\omega)$ given in (2.11)) if we know the behavior of the quadratic average r_n. More precisely assume that $r_n \ll \varepsilon_n$ (which means that $\frac{r_n}{\varepsilon_n}$ converges to zero as n tends to $+\infty$) for a suitable sequence $\varepsilon_n \searrow 0$, then $\dfrac{r_n(\omega)}{\varepsilon_n}$ converges to 0 in $L^2(\Omega)$ and therefore by Egoroff's Theorem, there exists for every $\delta > 0$ a subset $\Omega_\delta \subset \Omega$ such that $P(\Omega \setminus \Omega_\delta) < \delta$ and $\sup\limits_{\omega \in \Omega_\delta} r_n(\omega) \ll \varepsilon_n$.

Remark 2.3 We conclude this section by emphasizing that our problem is not a particular case of the one studied in [3], since the mixing condition (1.7) is not satisfied, in general. In fact we claim that it is never the case unless $a(\omega)$ is constant. Indeed in view of (1.7), saying that $\xi(x,\omega) := a(T(x)\omega)$ satisfies a mixing condition implies that there exists $\beta > 1$ such that

$$\left| \int_\Omega a(T(x)\omega)\ a(T(y)\omega)\ d\omega - \left[\int_\Omega a(\omega)\ d\omega \right]^2 \right| \leq \frac{1}{d^\beta} \int_\Omega a(\omega)^2\ d\omega, \text{ whenever } |x-y| \geq d. \tag{2.12}$$

Since α is irrational, we may choose a subsequence n_k of n such that the fractional part $\{n_k\,\alpha\}$ of $n_k\,\alpha$ converges to 0, as $k \to +\infty$. Applying (2.12) with $x = 0$, $y = n_k$, we would obtain

$$\lim_{k\to+\infty} \int_0^1 \int_0^1 a(\omega_1,\omega_2)\, a(\omega_1 + n_k, \omega_2 + n_k\,\alpha)\, d\omega_1\, d\omega_2 = \langle a \rangle^2, \tag{2.13}$$

Now, using some approximation arguments and the periodicity of $a(\omega)$, it can be easily shown that $a_k(\omega_1,\omega_2) := a(\omega_1 + n_k, \omega_2 + n_k\,\alpha)$ converges strongly to $a(\omega)$ in $L^2(\Omega)$. Thus, by (2.13), $\int_\Omega (a(\omega_1,\omega_2))^2\ d\omega_1\, d\omega_2 = \langle a \rangle^2$ and $a(\omega) = \langle a \rangle$ for almost all $\omega \in \Omega$ as claimed.

3 Main result

In this section we state and prove our main result on the rate of convergence of correctors for the problem (2.8). More precisely we give an equivalent of r_n as $n \to +\infty$ where

$$r_n\ := \|u_n - u\|_{L^2((0,1)\times\Omega)}\ . \tag{3.1}$$

In the following we will write $r_n \sim \epsilon_n$ if the ratio $\dfrac{r_n}{\epsilon_n}$ ranges between two positive constants for large n.
We consider a non-constant measurable periodic function $a(\omega_1,\omega_2)$ satisfying (2.6) and we represent its inverse in terms of Fourier series

$$a^{-1}(\omega_1,\omega_2) = \sum_{(k,l)\in\mathbb{Z}\times\mathbb{Z}} b_{kl}\, e^{2\,\pi\, i(k\,\omega_1 + l\,\omega_2)}. \tag{3.2}$$

We will say that $a^{-1} \in H^s_{\text{per}}(\mathbb{R}^2)$ if

$$\sum_{k,l} |b_{kl}|^2\ (k^2 + l^2)^s < \infty. \tag{3.3}$$

We introduce the non increasing function $S : \mathbb{R}_+ \to (0,+\infty]$ defined by

$$S(t)\ :=\ \sum_{|k+\alpha\, l|\geq t} \frac{|b_{kl}|^2}{(k+\alpha\, l)^2}\ . \tag{3.4}$$

Theorem 3.1 *Let u_n and u be the solutions of the equations (2.8) and (2.10) respectively, where $a(\omega)$ is given by (3.2). Then we have*

$$r_n\ \sim\ n\ \ \sqrt{\int_0^{\frac{1}{n}} t^3\, S(t)\, dt}\ , \tag{3.5}$$

Remark 3.2 A trivial consequence of (3.5) is that the rate of convergence of r_n cannot be faster than $\dfrac{1}{n}$. Indeed, since $S(t)$ is non increasing, we have

$$\lim_{n\to+\infty} n^4 \int_0^{\frac{1}{n}} t^3\, S(t)\, dt \;=\; \frac{S(0_+)}{4} > 0\ .$$

Therefore, by (3.5), the following equivalence holds

$$r_n \sim \frac{1}{n} \qquad \Longleftrightarrow \qquad S(0_+) = \sum_{(k,l)\neq(0,0)} \frac{|b_{kl}|^2}{(k+\alpha\, l)^2} < +\infty\ . \tag{3.6}$$

Notice that the right-hand side condition holds when the function a^{-1} belongs to $\mathrm{Trig}(\mathbb{R}^2)$.

In the case where the previous series diverges (i.e. $S(0_+) = +\infty$), the speed of convergence of r_n can very often be deduced from the behavior of $S(t)$ near $t = 0$. For example, it is easy to check that, for every real $\beta \in [0, 2]$, one has

$$S(t) \sim \frac{1}{t^\beta} \quad \text{as } t \to 0 \implies r_n \sim \frac{1}{n^{1-\frac{\beta}{2}}}\ . \tag{3.7}$$

The proof of the theorem is based on the following lemma which converts the initial problem into a more explicit one, involving Fourier series. Let $g : \mathbb{R} \to \mathbb{R}_+$ be the function defined by

$$g(t) \;:=\; \begin{cases} \dfrac{1}{3}\left[5 - \dfrac{3}{\pi^2 t^2} + \left(1 + \dfrac{3}{\pi^2 t^2}\right)\cos 2\pi t\right] & t \neq 0, \\[2ex] 0 & t = 0\ . \end{cases} \tag{3.8}$$

Lemma 3.3 *Let a^{-1} be given by (3.2). Then*

$$r_n^2 \quad \sim \quad \frac{1}{n^2} \sum_{(k,l)\neq(0,0)} \frac{|b_{kl}|^2}{(k+\alpha\, l)^2}\; g(n\,(k+\alpha\, l))\ . \tag{3.9}$$

Proof: From (2.9), we obtain

$$r_n^2 = \int_0^1 \int_\Omega \frac{|\int_0^x\, [a(T(nt)\omega)]^{-1}\, dt - x \int_0^1\, [a(T(nt)\omega)]^{-1}\, dt|^2}{|\int_0^1\, [a(T(nt)\omega)]^{-1}\; dt|^2}\, d\omega\, dx\,.$$

The assumption (2.6) implies that $\dfrac{1}{|\int_0^1\, [a(T(nt)\omega)]^{-1}\; dt|^2}$ is bounded from below and from above by positive constants λ^2 and Λ^2, respectively, independently of n. So, we have

$$r_n^2 \sim \int_0^1 \int_\Omega \left|\int_0^x\, [a(T(nt)\omega)]^{-1}\, dt - x \int_0^1\, [a(T(nt)\omega)]^{-1}\, dt\right|^2\, d\omega\, dx\,.$$

Using the Fourier representation (3.2), we may write for every fixed x :

$$\int_0^x [a(T(nt)\omega)]^{-1}\,dt - x\int_0^1 [a(T(nt)\omega)]^{-1}\,dt$$
$$= \sum_{(k,l)\neq(0,0)} b_{kl}\frac{e^{2\pi i n(k+\alpha l)x}-1-x(e^{2\pi i n(k+\alpha l)}-1)}{2\pi i n(k+\alpha l)}\, e^{2\pi i(k\omega_1+l\omega_2)}\,,$$

and compute the $L^2(\Omega)$ norm of the right hand side (as a function of (ω_1,ω_2)) thanks to Parseval identity. The relation (3.3) follows by integrating in x and after checking that g defined in (3.8) satisfies the equality

$$g(t) = \int_0^1 |e^{2\pi i t x} - 1 - x(e^{2\pi i t}-1)|^2\,dx.$$

■

Proof of Theorem 3.1 Recalling (3.8) we can check, by developing the cosine function in a Taylor series near 0 and also by noticing that $g(t)$ is $O(1)$ away from 0, that there exist two positive constants c_0 and c_1 such that,

$$c_0\; h(t) \le g(t) \le c_1\; h(t),\;\text{ where } h(t) := \begin{cases} t^4 & t\le 1,\\ 1 & t>1\,. \end{cases} \tag{3.10}$$

Define on $\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$ the measure μ and, for each n, the function Φ_n as follows

$$\mu := \sum_{(k,l)\neq(0,0)} \frac{|b_{kl}|^2}{(k+\alpha\,l)^2}\,\delta_{kl}\,,\quad \Phi_n(k,l) := \inf\left\{\frac{1}{n},\; n\,(k+\alpha l)^2\right\}.$$

Then we can write

$$\frac{1}{n^2}\sum_{(k,l)\neq(0,0)} \frac{|b_{kl}|^2}{(k+\alpha\,l)^2}\; g(n\,(k+\alpha\,l)) = \int \frac{g(n\,(k+\alpha\,l))}{n^2}\,d\mu \tag{3.11}$$

and, in view of (3.10),

$$c_0\;\Phi_n^2(k,l) \le \frac{g(n\,(k+\alpha\,l))}{n^2} \le c_1\;\Phi_n^2(k,l)\,. \tag{3.12}$$

On the other hand, since μ is a Radon measure, one has

$$\int \Phi_n^2\,d\mu = 2\int_0^{+\infty} t\mu(\{(k,l):\Phi_n(k,l)>t\})\,dt\,. \tag{3.13}$$

Computing the measure of

$$\{(k,l):\Phi_n(k,l)>t\} = \begin{cases} \emptyset\,, & t\ge \frac{1}{n}\,,\\ \left\{(k,l): |k+\alpha\,l| > \sqrt{\frac{t}{n}}\right\}\,, & t<\frac{1}{n}\,, \end{cases}$$

we obtain

$$\mu(\{(k,l) : \Phi_n(k,l) > t\}) = \begin{cases} 0\,, & t \geq \frac{1}{n}\,, \\ S\left(\sqrt{\frac{t}{n}}\right)\,, & t < \frac{1}{n}\,. \end{cases} \tag{3.14}$$

The relation (3.5) immediately follows from Lemma 3.3 and equations (3.11)-(3.14) after a simple change of variables. ∎

Corollary 3.4 *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and let $\{\varepsilon_n\}$ be a sequence such that*

$$\varepsilon_n \searrow 0_+ \quad , \quad \{n\,\varepsilon_n\} \quad \textit{is increasing} \quad , \quad \sum(\varepsilon_n/n) < +\infty\ .$$

Then there exists a bounded periodic function $a(\omega_1,\omega_2)$ satisfying (2.6) such that $r_n \sim \varepsilon_n$.

Proof: Given any $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, the set $\{k-\alpha\, l : k,l \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. So given any real number $0<\lambda<1$ we can choose, for every $j\in\mathbb{N}$, a pair $(k_j,l_j)\in\mathbb{Z}^2$ (among an infinite number of such pairs) such that

$$\lambda^j - \lambda^{2j} \leq k_j - \alpha\, l_j \leq \lambda^j + \lambda^{2j}.$$

Then we set

$$\alpha_j := |k_j - \alpha\, l_j| \quad , \quad n_j := \left[\frac{1}{\alpha_j}\right] \quad , \quad j_n := \sup\left\{j \ :\ \alpha_j > \frac{1}{n}\right\}\,,$$

where $[x]$ stands for the integer part of x. We observe that α_j decreases exponentially to zero as $j\to+\infty$ and therefore,

$$j_n \sim \ln(n)\ , \qquad \lim_{j\to+\infty}\frac{n_{j+1}}{n_j} = \lambda^{-1}\ . \tag{3.15}$$

We claim that, given any δ with $0<\delta<1$, we have

$$\frac{\delta\,\lambda}{2}\,n \ \leq\ n_{j_n} < n \qquad \text{for large } n. \tag{3.16}$$

Indeed, by the definition of j_n, we have $\alpha_{j_n} > \dfrac{1}{n}$ which implies the second inequality. On the other hand, as j_n tends increasingly to infinity, there holds for n sufficiently large

$$\begin{aligned} \alpha_{j_n+1} &\geq \lambda^{j_n+1} - \lambda^{2(j_n+1)} \\ &\geq \lambda\left(\lambda^{j_n} - \lambda^{2\,j_n}\right) \\ &\geq \delta\,\lambda\,\left(\lambda^{j_n} + \lambda^{2\,j_n}\right) \geq \delta\,\lambda\,\alpha_{j_n} \end{aligned}$$

Then, since $\alpha_{j_n+1} \leq \dfrac{1}{n}$, we conclude that for large n

$$n_{j_n} \geq \frac{1}{\alpha_{j_n}} - 1 \geq \frac{\delta\,\lambda}{\alpha_{j_n+1}} - 1 \geq \delta\,\lambda\,n - 1 \geq \frac{\delta\,\lambda}{2}n.$$

This proves (3.16). Recalling that ε_n is non increasing whereas $n\varepsilon_n$ is increasing, we deduce

$$\varepsilon_n \sim \varepsilon_{n_{j_n}} \ . \tag{3.17}$$

Now we define a^{-1} by specifying its Fourier coefficients b_{kl}. We set

$$b_{kl} := \begin{cases} f_j & \text{if } (|k|,|l|) = (a_{j+1}, a_j),\ kl \leq 0\ , (k,l) \neq (0,0) \\ c_0 & \text{if } (k,l) = (0,0) \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

where, recalling that $\{n\, \varepsilon_n\}$ is increasing, we define f_j for $j \geq 1$ by

$$f_j := \alpha_j \sqrt{n_j^2\, \varepsilon_{n_j}^2 - n_{j-1}^2\, \varepsilon_{n_{j-1}}^2}. \tag{3.19}$$

Since $n_j \alpha_j \leq 1$, it is easy to check that

$$f_j \leq \varepsilon_{n_j} \ . \tag{3.20}$$

On the other hand, since ε_n is non increasing, we have

$$\varepsilon_{n_j} \ \leq \ \frac{1}{(n_j - n_{j-1})} \sum_{k=n_{j-1}+1}^{n_j} \varepsilon_k \ \leq \ \frac{1}{(1 - \frac{n_{j-1}}{n_j})} \sum_{k=n_{j-1}+1}^{n_j} \frac{\varepsilon_k}{k} \ .$$

Therefore, by (3.20) and (3.15) there exists a positive constant C such that

$$\sum_{j\geq 1} f_j \ \leq \ \sum_{j \geq 1} \varepsilon_{n_j} \ \leq C \sum_{k \geq 1} \frac{\varepsilon_k}{k} \ < +\infty \ . \tag{3.21}$$

Writing a^{-1} in the form

$$a^{-1}(\omega_1, \omega_2) = c_0 + \sum_{j \geq 1} f_j \cos[2\pi(a_{j+1}\omega_1 - a_j \omega_2)],$$

we see that, possibly after multiplying the f_j's by a suitable constant, we may choose c_0 in (3.18) so that condition (2.6) is fulfilled.
Now we may compute $S(t)$ given in (3.4) taking into account (3.18)). We have, for all $t \in \mathbb{R}_+$,

$$S(t) \ = \ \sum_{\alpha_j > t} \frac{f_j^2}{\alpha_j^2} \ .$$

Thus, by Fubini's Theorem :

$$n^2 \int_0^{1/n} t^3 S(t)\, dt = \sum_{j \geq 0} \frac{f_j^2}{\alpha_j^2} \left[\inf \left\{ \alpha_j, \frac{1}{n} \right\} \right]^4 = a_n + b_n \ , \tag{3.22}$$

where we have set

$$a_n = \frac{1}{n^2} \sum_{\alpha_j > \frac{1}{n}} \frac{f_j^2}{\alpha_j^2} \quad , \quad b_n = n^2 \sum_{\alpha_j \leq \frac{1}{n}} f_j^2 \alpha_j^2 \ .$$

We can calculate a_n with the help of the definition (3.19), using (3.16) and (3.17), whereby it follows immediately that

$$a_n = \frac{1}{n^2} \sum_{j \leq j_n} \frac{f_j^2}{\alpha_j^2} = \frac{1}{n^2}\, n_{j_n}^2\, \varepsilon_{n_{j_n}}^2 \sim \varepsilon_n^2. \tag{3.23}$$

On the other hand, using the fact that $\alpha_j \sim \lambda^j$ and also the definition of j_n, the following calculation yields, for some positive constant C, that

$$\sum_{\alpha_j \leq \frac{1}{n}} \alpha_j^2 = \sum_{j > j_n} \alpha_j^2 = C\ \alpha_{j_n+1}^2 \leq C \frac{1}{n^2}\,. \tag{3.24}$$

From (3.20), (3.24) and since the sequence ε_n is nonincreasing, we infer that

$$b_n\ \leq n^2\, \varepsilon_{n_{j_n}}^2 \sum_{j > j_n} \alpha_j^2\ \leq C\varepsilon_n^2. \tag{3.25}$$

The conclusion follows from (3.22), (3.23) and (3.25). ■

Remark 3.5 The conditions of Corollary 3.4 are satisfied if we take $\varepsilon_n = \dfrac{1}{n^r}$ for any $r \in (0,1)$. We may also choose $\varepsilon_n = \left\{(\log n)\dots(\log^{m-1} n)(\log^m n)^s\right\}^{-1}$ for every $m \in \mathbb{N}$ and $s > 1$, where $\log^m$ denotes the log function composed with itself m times. These examples show that the rate can be much weaker than that previously exhibited by Bourgeat and Piatnitski in the random case (see [3]) without the mixing condition (1.7).

We now wish to study how the regularity of $a^{-1}(\omega_1, \omega_2)$ and the degree of irrationality of α influence the behavior of r_n. It is well known in Number Theory that to each real number α we can assign a value $\xi_0(\alpha)$, the *irrationality measure of* α, defined by $\xi_0(\alpha) := \sup_{\xi \in \chi(\alpha)} \xi$, where $\chi(\alpha)$ is the set of all $\xi \in \mathbb{R}_+$ is such that the inequality $0 < \left|\alpha - \dfrac{k}{l}\right| \leq \dfrac{1}{l^\xi}$ has an infinite number of solutions $(k,l) \in \mathbb{Z}, l > 0$. It can be seen easily that $\xi_0(\alpha) = 1$ if x is rational, and $\xi_0(\alpha) \geq 2$ if α is irrational. In fact it is a difficult result due to K.F. Roth that $\xi_0(\alpha) = 2$ for every algebraic number. In view of Lemma 3.6 below this equality extends to almost all irrational numbers. If $\xi_0(\alpha) = +\infty$, then α is called a Liouville number. For further details and results on this topic we refer to Hardy and Wright [6] or Cassels [4].

Lemma 3.6 *There exists a Lebesgue negligible subset $D \subset \mathbb{R}$ such that $\mathbb{Q} \subset D$ and $\xi_0(\alpha) \leq 2$ for every $\alpha \in \mathbb{R} \setminus D$.*

Proof: Since $\xi_0(\alpha + n) = \xi_0(\alpha)$ for all $n \in \mathbb{Z}$, it is enough to specify D as a subset of $[0,1)$ we will implicitely extend by periodicity. Define, for any $A > 1$ and $\varepsilon > 0$,

$$E_{A,\varepsilon} := \bigcup_{\frac{p}{q} \in \mathbb{Q} \cap [0,1]} \left[\frac{p}{q} - \frac{1}{A\, q^{2+\epsilon}}, \frac{p}{q} + \frac{1}{A\, q^{2+\epsilon}}\right] \cap \Big[0,1\Big).$$

Then we set

$$D := \bigcup_{\varepsilon>0} \left(\bigcap_{A>1} E_{A,\varepsilon} \right) .$$

Clearly we have $m(D) = 0$ since

$$m(E_{A,\varepsilon}) \leq \sum_{q\in\mathbb{Q}} \frac{2\,q}{A\,q^{2+\varepsilon}} < \frac{C_\varepsilon}{A} .$$

Let $\alpha \in [0,1)\backslash D$. Then, for every $\varepsilon > 0$, there exists a contant A_ε such that $\left|\frac{p}{q} - \alpha\right| > \frac{1}{A_\varepsilon q^{2+\epsilon}}$ for all $\frac{p}{q} \in [0,1]$. The latter inequality extends to all $\frac{p}{q} \in \mathbb{Q}$ provided q is so large that $\frac{1}{A_\varepsilon q^{2+\epsilon}} < \min\{\alpha, 1-\alpha\}$. It follows that $\xi_0(\alpha) \leq 2+\varepsilon$. Hence the conclusion by sending ε to zero. ∎

Let us consider an irrational number α with finite irrationality measure $\xi_0 = \xi_0(\alpha)$. Then, by the definition, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $|k|, |l| \geq n_0$, $\left|\alpha - \frac{k}{l}\right| > \frac{1}{|l|^{\xi_0+\varepsilon}}$ and, consequently,

$$|k + \alpha\, l| \geq \frac{1}{(k^2+l^2)^{\frac{\xi_0-1+\varepsilon}{2}}} \tag{3.26}$$

Corollary 3.7 *Let* $a^{-1} \in H^s_{\mathrm{per}}(\mathbb{R}^2)$ *and suppose that* α *has irrationality measure* $\xi_0 = \xi_0(\alpha) < +\infty$*, then,*

$$r_n \;\leq\; C\,\frac{1}{n^r}\,, \quad \text{for all} \quad r < \min\left\{ \frac{s}{(\xi_0-1)}\,, 1 \right\}.$$

Proof: From Theorem 3.1, if $t^\beta S(t)$ is bounded then $\beta \geq 0$ and $r_n \leq C\frac{1}{n^{1-\frac{\beta}{2}}}$ for some constant C. Recalling (3.4) and (3.26) we obtain the following estimate

$$t^\beta S(t) \leq \sum_{k,l\in\mathbb{Z}} |b_{kl}|^2 \;(k^2+l^2)^{(\xi_0-1+\varepsilon)\left(1-\frac{\beta}{2}\right)} . \tag{3.27}$$

Since $a^{-1} \in H^s(\Omega)$, for all β such that $(\xi_0 - 1 + \varepsilon)\left(1 - \frac{\beta}{2}\right) \leq s$ or, equivalently, such that $\left(1 - \frac{\beta}{2}\right) < \frac{s}{\xi_0 - 1}$, we conclude from (3.27) that $t^\beta S(t)$ is bounded. The result follows by setting $r = 1 - \frac{\beta}{2}$. ∎

Remark 3.8 By Corollary 3.7 and Lemma 3.6, it turns out that, in a realistic situation like when $a(\omega_1,\omega_2)$ is a piecewise constant periodic function discontinuous along Lipschitz curves in Ω, the error r_n satisfies $r_n \leq \frac{C}{n^r}$ for every $r < 1/2$, provided α is either algebric or it does not belong to the negligible subset D defined in Lemma 3.6. Indeed in this case a^{-1} belongs to $H^s_{\mathrm{per}}(\mathbb{R}^2)$ for all $s < 1/2$. In a very similar way, keeping the same assumption on α, we obtain $r_n \sim \dfrac{1}{n}$ if a^{-1} is assumed to have the regularity $H^s_{\mathrm{per}}(\mathbb{R}^2)$ for some $s > 1$.

Remark 3.9 The upper bound obtained in Corollary 3.7 is not optimal in general. For example, let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $\xi_0(\alpha) = 2$ and let a be the characteristic function of periodically distributed squares in $\mathbb{R}^2$. Then, Corollary 3.7 yields that $r_n \leq C\dfrac{1}{n^r}$ for every $r < \dfrac{1}{2}$. Whereas, we can make a more refined estimate using the fact that $|b_{kl}| \sim \dfrac{1}{k\,l}$. For that we split the series given by $S(0_+)$ into two parts: the sum over (k,l) such that $|k - \alpha\, l| > \dfrac{1}{2}$ which results in a finite value and the sum over (k,l) with $\left|\dfrac{k}{l} - \alpha\right| < \dfrac{1}{2\,l}$ which is of the order $\displaystyle\sum_{l>>M} \frac{1}{l^6}\, l^{2(2+\varepsilon)} < +\infty$ (notice that for any given l a unique $k(l) \sim \alpha\, l$ is associated in the latter sum, besides the fact that for given $\varepsilon > 0$, one has $|k - \alpha\, l| > \dfrac{1}{l^{2+\varepsilon}}$ for k,l large enough). Eventually, by Remark 3.2, we find that $r_n \sim \dfrac{1}{n}$ which improves vastly the behaviour predicted by the above corollary in this special case. ■

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