Decomposable critical tensors

J. A. Dias da Silva [*‡] Fátima Rodrigues [†‡]

March, 2005

Abstract

Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be a partition of m and $\lambda' = (\lambda'_1, \dots, \lambda'_{\lambda_1})$ its conjugate partition. Denote also by λ the irreducible $\mathbb{C}$-character of S_m associated with λ. Let V be a finite dimensional vector space over $\mathbb{C}$.

The reach of an element of the symmetry class of tensors V_λ (symmetry class of tensors associated with λ) is defined. The concept of critical element is introduced, as an element whose reach has dimension equal to λ'_1. It is observed the coincidence, in $\wedge^m V$, of the notions of critical element and decomposable element. Known results for decomposable elements of $\wedge^m V$ are extended to critical elements of V_λ. In particular, for a basis of $\otimes^m V$ induced by a basis of V, generalized Plücker polynomials are constructed in a way that the set of their common roots contains the set of the families of components of decomposable critical elements of V_λ.

[*]Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, Campo Grande, 1749-016 Lisboa, Portugal. (perdigao@hermite.cii.fc.ul.pt)

[†]Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829-516 Monte de Caparica, Portugal. (fatima@ptmat.fc.ul.pt)

[‡]This work was partially supported by *Fundação para a Ciência e Tecnologia* and was done within the activities of the *Centro de Estruturas Lineares e Combinatórias*.

1 Introduction

Let V be a n-dimensional vector space over $\mathbb{C}$ and let $(e_1, \ldots, e_n)$ be a basis of V. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ $(\lambda_s > 0)$ be a partition of m and $\chi = (\lambda_1, \ldots, \lambda_s, 1)$ the partition of $m+1$ obtained from λ by adding one part equal to 1. The irreducible complex characters of S_m correspond canonically in a one to one way to the partitions of m. So, we identify λ with the corresponding irreducible complex character of S_m and χ with the corresponding irreducible complex character of S_{m+1}.

We denote by $\otimes^m V$ the mth tensor power of V. If $\sigma \in S_m$, then $\mathcal{P}(\sigma)$ is the unique linear operator on $\otimes^m V$ satisfying

$$\mathcal{P}(\sigma)(x_1 \otimes \cdots \otimes x_m) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}$$

for all $x_1, \ldots, x_m \in V$.

We define the *symmetrizer* associated with λ as the linear operator,

$$T_\lambda := \frac{\lambda(\mathrm{id})}{m!} \sum_{\sigma \in S_m} \lambda(\sigma)\mathcal{P}(\sigma).$$

The range of T_λ is called *symmetry class of tensors* associated with λ and is denoted by V_λ. The image by T_λ of the decomposable tensor $x_1 \otimes \cdots \otimes x_m$, where $x_1, \ldots, x_m \in V$, is called *decomposable symmetrized tensor* or *decomposable tensor* of V_λ and is denoted by

$$x_1 * \cdots * x_m := T_\lambda(x_1 \otimes \cdots \otimes x_m).$$

Let $z \in V_\lambda$. A family $(x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ that satisfies

$$\sum_{i=1}^{k} x_{i1} \otimes \cdots \otimes x_{im} \in T_\lambda^{-1}(\{z\})$$

is called *pre-image family* of z in V. Let $\mathcal{X} = (x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ be a pre-image family of $z \in V_\lambda$. We call the pair

$$(\mathcal{X}, z) = ((x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}, z)$$

a *presentation* of z. By abuse of language *presentation* of z is the expression

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

where $(x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ is a pre-image family of z. The tensor

$$z_{\mathcal{X}}^{\otimes} = \sum_{i=1}^{k} x_{i1} \otimes \cdots \otimes x_{im}$$

is a *root of the presentation* $(\mathcal{X}, z)$. The vectors x_{ij} are the *vectors of the presentation* and the dimension of the subspace of V

$$\langle x_{ij} : i = 1, \ldots, k, j = 1, \ldots, m \rangle$$

is called the *dimension of the presentation.*

If λ is the alternating character ε, V_λ is denoted by $\wedge^m V$, the well known *m-Grassmann space*, or the *m-th exterior power* of V and the decomposable symmetrized tensors $T_\varepsilon(x_1 \otimes \cdots \otimes x_m)$ are the *decomposable tensors of Grassmann* denoted by

$$x_1 \wedge \cdots \wedge x_m.$$

It is well known that the tensors of the form $x_1 \wedge \cdots \wedge x_m$, with $x_1, \ldots, x_m \in V$, are an algebraic variety of $\mathcal{A}^{n^m}(\otimes^m V)$. This algebraic variety is the affine cone of a projective variety whose defining polynomials are the quadratic Plücker polynomials.

We define *reach* of a nonzero tensor of V_λ, the smallest (by inclusion) subspace W of V such that $z \in W_\lambda$. We define also *annihilator* of a nonzero tensor z of V_λ as the subspace of the reach of z whose elements v satisfy

$$T_\chi(\sum_{i=1}^{k} x_{i1} \otimes \cdots \otimes x_{im} \otimes v) = 0$$

whenever

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

is a presentation of z with vectors in the reach of z. The concepts of reach and annihilator of a tensor of V_λ have a crucial role in this approach. The elementary properties of these concepts and the relations with the critical elements of V_λ are established. When λ is the alternating character the annihilator of z is the subspace of V whose elements $v \in V$ satisfy

$$z \wedge v = 0.$$

We prove that a Grassmann tensor is critical if and only if it is Grassmann decomposable. This observation allow us to conclude that the first of the main theorems of this paper generalizes the well known result (see [7]):

Theorem 1.1 *Let z be a nonzero vector in $\wedge^m V$. Then z is decomposable (in $\wedge^m V$) if and only if there exists a linearly independent set of m vectors $u_1, \ldots, u_m$ such that*

$$z \wedge u_i = 0, \quad i = 1, \ldots, m.$$

Following the strategy presented by M.Marcus in [7] we construct a family of Plücker polynomials to the set of critical decomposable tensors of V_λ.

2 Combinatorial tour

Let X be a finite set, we denote by $\Gamma_{m,X}$ the set of all mappings from $\{1, \ldots, m\}$ into X. When $X = \{1, \ldots, n\}$, we use the notation $\Gamma_{m,n}$ ($\Gamma^0_{m,n}$) to the set of the mappings from $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$ (respectively $\{1, \ldots, m\}$ into $\{0, \ldots, n\}$. We will call *multiplicity partition* of $\alpha \in \Gamma_{m,X}$ the partition of m obtained by rearranging in decreasing order the components of the family of nonnegative integers $(|\alpha^{-1}(x)|)_{x \in X}$. We denote the multiplicity partition of α by $M(\alpha)$.

Let $\omega \in \Gamma_{m,n}$ we denote by ω_i the element of $\Gamma_{m-1,n}$

$$\omega_i := (\omega(1), \ldots, \omega(i-1), \omega(i+1), \ldots, \omega(m)), \; i = 1, \ldots, m.$$

If $\nu \in \Gamma_{m-1,n}$, $t \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ we will denote by $\nu \overset{t}{\hookleftarrow} j$ the element of $\Gamma_{m,n}$ defined by

$$\nu \overset{t}{\hookleftarrow} j := (\nu(1), \ldots, \nu(t-1), j, \nu(t), \ldots, \nu(m-1)), \; \text{if} \quad t = 1, \ldots, m-1,$$

and

$$\nu \overset{m}{\hookleftarrow} j := (\nu(1), \ldots, \nu(m-1), j).$$

If $\alpha \in \Gamma_{m,n}$ we denote by $\hat{\alpha}$ the element of $\Gamma^0_{m+1,n}$,

$$\hat{\alpha} = (\alpha(1), \dots, \alpha(m), 0).$$

The subset of the increasing functions of $\Gamma_{m,n}$ is denoted by $G_{m,n}$.

We define an action

$$(\sigma, \alpha) \to \alpha\sigma^{-1}$$

of S_m (respectively S_{m+1}) on $\Gamma_{m,n}$ (respectively on $(\Gamma^0_{m+1,n})$). If $\alpha \in \Gamma_{m,n}$ the orbit of α is denoted by $\mathcal{O}_\alpha$. If α and β belongs to the same orbit we will say $\alpha \equiv \beta(\mathrm{mod} S_m)$. Observe that $G_{m,n}$ is the system of distinct representatives of the orbits of this action, choosing in each orbit $\mathcal{O}_\alpha$ the smallest element by the lexicographic order. We denote by H_α the stabilizer of α.

Lemma 2.1 *Let α and β be elements of $\Gamma_{m,n}$. Then $\alpha \equiv \beta(\mathrm{mod} S_m)$ if and only if*

$$|\alpha^{-1}(i)| = |\beta^{-1}(i)|, \quad i = 1, ..., n.$$

Proposition 2.1 *Let α and β be elements of $\Gamma_{m,n}$. Then $\alpha \equiv \beta(\mathrm{mod}\ S_m)$ if and only if $\hat{\alpha} \equiv \hat{\beta}(\mathrm{mod}\ S_{m+1})$.*

Let $(e_1, \dots, e_n)$ be a basis of V and $\alpha \in \Gamma_{m,n}$. We denote by $e^\otimes_\alpha$ the element of $\otimes^m V$

$$e^\otimes_\alpha := e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(m)}.$$

In the same way e^*_α is the element of V_λ

$$e^*_\alpha := e_{\alpha(1)} * \cdots * e_{\alpha(m)}.$$

Denote by Ω_λ (or just by Ω) the subset of $\Gamma_{m,n}$,

$$\Omega := \{\alpha \in \Gamma_{m,n} : e^*_\alpha \neq 0\}.$$

By the definitions it is easy to conclude that

$$V_\lambda \subseteq \langle e^\otimes_\alpha : \alpha \in \Omega\rangle. \tag{1}$$

So, if $z = \sum_{\alpha\in\Omega} c_\alpha e^\otimes_\alpha \in V_\lambda$ we define *support* of z and denote by $\mathrm{supp}(z)$ the subset of Ω

$$\mathrm{supp}(z) := \{\alpha \in \Omega : c_\alpha \neq 0\}.$$

Let m be a positive integer and $\lambda = (\lambda_1, \dots, \lambda_t)$ be a partition of m. We identify λ with an m-tuple of nonnegative integers by adding, if necessary, a list of zeros, i.e.

$$\lambda = (\lambda_1, \dots, \lambda_t) \equiv (\lambda_1, \dots, \lambda_t, 0, \dots, 0).$$

If λ is a partition of m, then $\lambda' = (\lambda'_1, \dots, \lambda'_{\lambda_1})$ defined by

$$\lambda'_k = |\{j \in \{1, \dots, t\} : \lambda_j \geq k\}|, \quad k = 1, \dots, \lambda_1,$$

is also a partition of m called the *conjugate partition* of λ.

Let $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\nu = (\nu_1, \dots, \nu_m)$ be partitions of m. We say that λ majorizes ν, and denote $\lambda \succeq \nu$, if

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \nu_i, \quad k = 1, \dots, m.$$

3 Auxiliary results

Let W be a subspace of V and $(e_1, \dots, e_n)$ be a basis of W. Let $e_0 \notin W$ and denote by U the subspace of V, $U = W + \langle e_0 \rangle$. Then $(e_0, \dots, e_n)$ is a basis of U and

$$(e^{\otimes}_{\beta} : \beta \in \Gamma^0_{m+1,n})$$

is a basis of $\otimes^{m+1} U$. Therefore

$$\otimes^{m+1} U = \langle e^{\otimes}_{\beta} : \beta \in \Gamma^0_{m+1,n},\ |\beta^{-1}(\{0\})| = 1,\ \beta(m+1) = 0 \rangle$$

$$\oplus \langle e^{\otimes}_{\beta} : \beta \in \Gamma^0_{m+1,n},\ |\beta^{-1}(\{0\})| = 1,\ \beta(m+1) \neq 0 \rangle$$

$$\oplus \langle e^{\otimes}_{\beta} : \beta \in \Gamma^0_{m+1,n},\ |\beta^{-1}(\{0\})| \neq 1 \rangle$$

$$= \langle e^{\otimes}_{\hat{\alpha}} : \alpha \in \Gamma_{m,n} \rangle \oplus \langle e^{\otimes}_{\beta} : \beta \in \Gamma^0_{m+1,n},\ |\beta^{-1}(\{0\})| = 1,\ \beta(m+1) \neq 0 \rangle$$

$$\oplus\langle e^{\otimes}_{\beta} : \beta \in \Gamma^0_{m+1,n},\ |\beta^{-1}(\{0\})| \neq 1\rangle. \qquad (2)$$

Let $(x_1, \ldots, x_m)$ be a family of nonzero vectors of V and $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of m. A *μ-coloring* of $(x_1, \ldots, x_m)$ or *coloring of shape* μ, is a decomposition of $(x_1, \ldots, x_m)$ in linearly independent subfamilies,

$$(x_1, \ldots, x_m) = (x_i)_{i\in\Delta_1} \dot{\cup} \ldots \dot{\cup} (x_i)_{i\in\Delta_k}$$

where $(\Delta_1, \ldots, \Delta_k)$ is a set partition of $\{1, \ldots, m\}$ and $|\Delta_i| = \mu_i, \quad i = 1, \ldots, k$. We say that the family $(x_1, \ldots, x_m)$ is *μ-colorable* if there exists a coloring of $(x_1, \ldots, x_m)$ of shape μ.

In [1] was proved that in the majorization order, the set of the shapes of the colorings of $(x_1, \ldots, x_m)$ has a maximum. This maximum partition is the *rank partition* of $(x_1, \ldots, x_m)$ and is denoted by

$$\rho(x_1, \ldots, x_m).$$

In [4] Gamas proved the following result that we present here with the formulation referred to [1]:

Proposition 3.2 *Let λ be an irreducible character of S_m. Let $(x_1, \ldots, x_m)$ be a family of nonzero vectors of V. Then $T_\lambda(x_1 \otimes \cdots \otimes x_m) \neq 0$ if and only if*

$$\rho(x_1, \ldots, x_m) \succeq \lambda'.$$

Remark

1. By the proposition, if z is a nonzero decomposable tensor of V_λ the dimension of the presentations of z is greater or equal to λ'_1.
2. The proposition is a generalization of the following result previously established by R.Merris [8].

Proposition 3.3 *Let λ be an irreducible character of S_m. Let $(e_1, \ldots, e_n)$ be a basis of V. If $\alpha \in \Gamma_{m,n}$ then $T_\lambda(e^{\otimes}_\alpha) \neq 0$ if and only if $\lambda \succeq M(\alpha)$.*

The next proposition is another formulation of the Gamas Theorem presented for the first time in [1].

Proposition 3.4 *Let λ be an irreducible character of S_m. Let $(x_1, \ldots, x_m)$ be a family of nonzero vectors of V. Then $T_\lambda(x_1 \otimes \cdots \otimes x_m) \neq 0$ if and only if the family $(x_1, \ldots, x_m)$ is λ'-colorable.*

The relation between the principal result of this article and the classical results of the Grassmann spaces depends to the following theorem:

Theorem 3.2 *Let $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_m)$ be families of linearly independent vectors of V. Then*

$$\langle x_1 \wedge \cdots \wedge x_m \rangle = \langle y_1 \wedge \cdots \wedge y_m \rangle$$

if and only if

$$\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_m \rangle.$$

4 Pre-image families and reach of a tensor of V_λ

Let

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

be a presentation of z. If, for all subset $L \subseteq \{1, \ldots, k\}$, we have

$$\sum_{l \in L} x_{l1} * \cdots * x_{lm} \neq 0 \tag{3}$$

we say that $z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$ $((x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}})$ is a *simple presentation* of z (a *simple pre-image family* of z).

From now on we assume that all the presentations (pre-image families) considered are simple.

Definition 4.1 Let $0 \neq z \in V_\lambda$,

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

is a *critical presentation* of z and the family $((x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}})$ is a *critical pre-image family* of z if

$$\dim\langle x_{i1}, \ldots, x_{im}\rangle = \lambda'_1, \quad i = 1, \ldots, k.$$

Definition 4.2 Let $0 \neq z \in V_\lambda$. We say that z is *weakly decomposable* if exists a presentation of z

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

such that

$$\langle x_{i1}, \ldots, x_{im}\rangle = \langle x_{j1}, \ldots, x_{jm}\rangle, \quad i, j \in \{1, \ldots, k\}.$$

This presentation of z is called *weakly decomposable* and the corresponding pre-image family is also called *weakly decomposable* .

Definition 4.3 A nonzero vector of V_λ has k *rank* if it is a sum of k and not less than k decomposable symmetrized tensors of V_λ. If $z \in V_\lambda$ has k rank then the expression

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

is called a *rank presentation* of z and the family $((x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}})$ will be called a *rank pre-image family* of z.

In [5] M. H. Lim has proved the following result:

Lemma 4.2 *Let z be a nonzero tensor of V_λ. If*

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

is a rank presentation of z and

$$z = \sum_{j=1}^{q} y_{j1} * \cdots * y_{jm}$$

is another presentation of z, then

$$\sum_{i=1}^{k} \langle x_{id} : d = 1, \ldots, m \rangle \subseteq \sum_{j=1}^{q} \langle y_{jd} : d = 1, \ldots, m \rangle.$$

Proposition 4.5 *Let W and U be subspaces of V and $z \in V_\lambda$. If $z \in W_\lambda$ and $z \in U_\lambda$ then $z \in (W \cap U)_\lambda$.*

Proof

Let

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

be a rank presentation of z. Since $z \in U_\lambda$ there exists $u_{it} \in U$ with $i = 1, \ldots, s$, $t = 1, \ldots, m$, such that

$$z = \sum_{i=1}^{s} u_{i1} * \cdots * u_{im}.$$

In the same way there exists $w_{jr} \in W$ with $j = 1, \ldots, p$, $r = 1, \ldots, m$, such that

$$z = \sum_{j=1}^{p} w_{j1} * \cdots * w_{jm}.$$

Using the lemma 4.2 we obtain

$$\sum_{i=1}^{k}\langle x_{id} : d = 1, \ldots, m\rangle \subseteq \sum_{i=1}^{s}\langle u_{id} : d = 1, \ldots, m\rangle \subseteq U.$$

In the same way

$$\sum_{i=1}^{k}\langle x_{id} : d = 1, \ldots, m\rangle \subseteq \sum_{j=1}^{p}\langle w_{jd} : d = 1, \ldots, m\rangle \subseteq W.$$

From this two inclusions we conclude that

$$\sum_{i=1}^{k}\langle x_{id} : d = 1, \ldots, m\rangle \subseteq W \cap U.$$

Then $z \in (W \cap U)_\lambda$.

■

Definition 4.4 We call *reach* of z, and denote by $W(z)$, the intersection of the subspaces W of V such that $z \in W_\lambda$.

By the definition, $W(z)$ is the smallest subspace, by inclusion, that contains a pre-image family of z.

Definition 4.5 Let $0 \neq z \in V_\lambda$. We say that z is *critical* if $\dim W(z) = \lambda'_1$.

Proposition 4.6 *Let $0 \neq z$ be a critical tensor of V_λ. Then all the presentations of z with vectors in $W(z)$ are simultaneously critical and weakly decomposable.*

Proof

Let

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

be a presentation of z with vectors in $W(z)$. Since z is critical we have

$$\dim\langle x_{i1}, \ldots, x_{im}\rangle \leq \dim W(z) = \lambda'_1.$$

By proposition 3.2 (we assume $z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$ simple) we conclude that

$$\dim\langle x_{i1}, \ldots, x_{im}\rangle \geq \lambda'_1.$$

So, $\dim\langle x_{i1}, \ldots, x_{im}\rangle = \dim W(z) = \lambda'_1$ and then $\langle x_{i1}, \ldots, x_{im}\rangle = W(z)$ for all $i = 1, \ldots, k$.

$\blacksquare$

Theorem 4.3 *Let $z \in V_\lambda$ and let*

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

be a rank presentation of z. Then

$$W(z) = \langle x_{ij} : i = 1, \ldots, k, j = 1, \ldots, m\rangle.$$

Proof

By definition of reach we conclude that

$$W(z) \subseteq \langle x_{ij} : i = 1, \ldots, k, j = 1, \ldots, m\rangle.$$

Since $z \in W(z)_\lambda$, there exists $y_{ij} \in W(z)$, $i = 1, \ldots, l$, $j = 1, \ldots, m$, such that $z = \sum_{i=1}^{l} y_{i1} * \cdots * y_{im}$ is a presentation of z. Then, by lemma 4.2, we have

$$W(z) \subseteq \sum_{i=1}^{k}\langle x_{ij} : j = 1, \ldots, m\rangle \subseteq \sum_{i=1}^{l}\langle y_{ij} : j = 1, \ldots, m\rangle \subseteq W(z).$$

$\blacksquare$

Corollary 1 *If $V_\lambda = \wedge^m V$ then $z \in \wedge^m V$ is critical if and only if it is decomposable.*

Proof

We observe first that the partition corresponding to ε is (1^m). Then, all nonzero decomposable tensors $x_1 \wedge \cdots \wedge x_m$ are critical, since they satisfy

$$\dim\langle x_1, \ldots, x_m\rangle = \varepsilon_1' = m.$$

By the previous theorem, proposition 4.6 and theorem 3.2 it is easy to conclude that if z is critical, then z is a decomposable element of $\wedge^m V$.

■

Lemma 4.3 *Let $0 \neq z \in V_\lambda$ and let*

$$z = \sum_{i=1}^{l} u_{i1} * \cdots * u_{im} \tag{4}$$

be a weakly decomposable presentation of z. Then there exists a weakly decomposable presentation of z with vectors in $W(z)$ and dimension less or equal to the dimension of the presentation (4).

Proof

Let $(x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ be a rank pre-image family of z. Let $(u_{ij})_{\substack{i=1,\ldots,l \\ j=1,\ldots,m}}$ be the weakly decomposable pre-image family of z. Then

$$\sum_{i=1}^{k} x_{i1} * \cdots * x_{im} = z = \sum_{i=1}^{l} u_{i1} * \cdots * u_{im}. \tag{5}$$

Let P be a projection of V over $W(z)$. Then, by theorem 4.3, $P(x_{ij}) = x_{ij}$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, m$. The images by $\otimes^m P = P \otimes \cdots \otimes P$ in the both sides of the equality (5) are

$$\begin{aligned}(\otimes^m P)(z) &= (\otimes^m P)(\sum_{i=1}^{k} x_{i1} * \cdots * x_{im}) \\ &= \sum_{i=1}^{k} P(x_{i1}) * \cdots * P(x_{im}) \\ &= \sum_{i=1}^{k} x_{i1} * \cdots * x_{im} \\ &= z \\ &= \sum_{i=1}^{l} P(u_{i1}) * \cdots * P(u_{im}).\end{aligned}$$

Suppose, without loss of generality, that $s \leq l$ is a positive integer and

$$z = \sum_{i=1}^{s} P(u_{i1}) * \cdots * P(u_{im})$$

is simple. But,

$$P(\langle u_{i1}, \ldots, u_{im} \rangle) = \langle P(u_{i1}), \ldots, P(u_{im}) \rangle$$

so, we conclude from $(u_{ij})_{\substack{i=1,\ldots,l \\ j=1,\ldots,m}}$ beeing weakly decomposable, that $(P(u_{ij}))_{\substack{i=1,\ldots,s \\ j=1,\ldots,m}}$ is a pre-image family of z weakly decomposable with elements in $W(z)$ and the dimension of the presentation is less or equal to the dimension of the presentation (4).

■

Proposition 4.7 *Let $0 \neq z \in V_\lambda$. The tensor z is critical if and only if admits a presentation simultaneously critical and weakly decomposable. Moreover, if z is critical, a presentation of z is critical and weakly decomposable if and only if the vectors are in $W(z)$.*

Proof

If $z \neq 0$ have a critical and weakly decomposable presentation, by the previous lemma there exists a presentation of z,

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im},$$

weakly decomposable with elements in $W(z)$ and dimension less or equal to λ'_1. Then

$$W(z) \subseteq \sum_{i=1}^{k} \langle x_{ij} : j = 1, \ldots, m \rangle = \langle x_{11}, \cdots, x_{1m} \rangle \subseteq W(z).$$

So, as $\dim\langle x_{11}, \cdots, x_{1m} \rangle = \lambda'_1$, z is critical.
Conversly, if z is critical, by proposition 4.6, the presentations of z with vectors in $W(z)$ are critical and weakly decomposable.

Finally, we know by proposition 4.6, that if z is critical all the presentations of z with vectors in $W(z)$ are critical and weakly decomposable. Conversly, if

$$z = \sum_{i=1}^{k} y_{i1} * \cdots * y_{im}$$

is critical and weakly decomposable, we have

$$W(z) \subseteq \sum_{i=1}^{k} \langle y_{ij} : j = 1, \ldots, m \rangle = \langle y_{s1}, \ldots, y_{sm} \rangle$$

for all $s \in \{1, \ldots, k\}$. Then, by an argument of dimension, we have

$$W(z) = \langle y_{s1}, \ldots, y_{sm} \rangle, \quad s = 1, \ldots, k.$$

■

Definition 4.6 Let $\mathcal{U} = (x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ be a pre-image family of a nonzero z of V_λ. We call *annihilator* of $\mathcal{U}$ (or the presentation $z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$), and denote by $\mathrm{Ann}^{\mathcal{U}}(z)$ the subspace of V

$$\mathrm{Ann}^{\mathcal{U}}(z) := \{v \in V : T_\chi(z_{\mathcal{U}}^{\otimes} \otimes v) = 0\}.$$

We are now prepared to prove the following theorem:

Theorem 4.4 *Let V be a vector space over $\mathbb{C}$, $\lambda = (\lambda_1, \ldots, \lambda_s) \in Irr(S_m)$ with $\lambda_s > 0$ and $\chi = (\lambda_1, \ldots, \lambda_s, 1) \in Irr(S_{m+1})$. Let z be a nonzero and critical element of V_λ. Then all the pre-image family of z,*

$$\mathcal{U} = (x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$$

with elements in $W(z)$, satisfy

$$W(z) = Ann^{\mathcal{U}}(z).$$

Proof

Observe first that $W(z) = \langle x_{i1}, \ldots, x_{im} \rangle$, $i = 1, \ldots, k$. So, if $v \in W(z)$, we have

$$(\rho(x_{i1}, \ldots, x_{im}, v))_1 = (\rho(x_{i1}, \ldots, x_{im}))_1 = \lambda_1', \quad i = 1, \ldots, k.$$

Then,

$$(\rho(x_{i1}, \ldots, x_{im}, v))_1 < \chi_1' = \lambda_1' + 1, \quad i = 1, \ldots, k.$$

So,

$$\rho(x_{i1}, \ldots, x_{im}, v) \not\trianglerighteq \chi'.$$

Then, by proposition 3.2, we have

$$T_\chi(x_{i1} \otimes \cdots \otimes x_{im} \otimes v) = 0, \quad i = 1, \ldots, k.$$

Therefore,

$$T_\chi(\sum_{i=1}^{k} x_{i1} \otimes \cdots \otimes x_{im} \otimes v) = 0.$$

Conversly, we will show that if $v \notin W(z)$ then

$$T_\chi(z_{\mathcal{U}}^{\otimes} \otimes v) \neq 0.$$

In order to prove this result we start by introducing terminology, notation and some results about the symmetric group. We will denote by $S_m^{'}$ the subgroup of S_{m+1}

$$S_m^{'} = \{\sigma \in S_{m+1} : \sigma(m+1) = m+1\}.$$

Consider in S_{m+1} the permutations $\tau_0 = id$, $\tau_i = (m+1\ i)$ for $i = 1, \ldots, m$. Then,

$$S_{m+1} = S_m^{'} \dot{\cup} S_m^{'} \tau_1 \dot{\cup} \ldots \dot{\cup} S_m^{'} \tau_m$$

is a right coset decomposition of $S_m^{'}$ in S_{m+1} . Then, we have

$$\begin{aligned} T_\chi &= \tfrac{\chi(\mathrm{id})}{(m+1)!} \sum_{\sigma \in S_{m+1}} \chi(\sigma)\mathcal{P}(\sigma) \\ &= \tfrac{\chi(\mathrm{id})}{(m+1)!} \sum_{i=0}^{m} \sum_{\sigma \in S'_m} \chi(\sigma\tau_i)\mathcal{P}(\sigma\tau_i) \\ &= \tfrac{\chi(\mathrm{id})}{(m+1)!} [\sum_{\sigma \in S'_m} \chi(\sigma)\mathcal{P}(\sigma) + \sum_{i=1}^{m} \sum_{\sigma \in S'_m} \chi(\sigma\tau_i)\mathcal{P}(\sigma)\mathcal{P}(\tau_i)] \\ &= T_{\chi_{|S'_m}} + \tfrac{\chi(\mathrm{id})}{(m+1)!} \sum_{i=1}^{m} \sum_{\sigma \in S'_m} \chi(\sigma\tau_i)\mathcal{P}(\sigma)\mathcal{P}(\tau_i). \end{aligned} \tag{6}$$

By the "Branching Theorem", λ is a constituint of $\chi_{|S'_m}$. Then, there exists irreducibles characters of S_m, $\lambda = \lambda^{(1)}, \ldots, \lambda^{(l)}$, such that

$$\chi_{|S'_m} = \lambda + \lambda^{(2)} + \ldots + \lambda^{(l)}.$$

Therefore, we can express $T_{\chi_{|S'_m}}$ as a sum of the pairwise orthogonal projections

$$T_{\chi_{|S'_m}} = T_\lambda + T_{\lambda^{(2)}} + \ldots + T_{\lambda^{(l)}}. \tag{7}$$

Let $(e_1, \ldots, e_{\lambda'_1})$ be a basis of $W(z)$. Then $(e_0 = v, e_1, \ldots, e_{\lambda'_1})$ is a linearly independent family.
As $z_{\mathcal{U}}^{\otimes}$ is the root of the presentation $(\mathcal{U}, z)$,

$$z_{\mathcal{U}}^{\otimes} = \sum_{i=1}^{k} x_{i1} \otimes \cdots \otimes x_{im}.$$

then,

$$T_\chi(z_{\mathcal{U}}^{\otimes} \otimes v) = \underbrace{T_{\chi_{|S'_m}}(z_{\mathcal{U}}^{\otimes} \otimes v)}_{\mathcal{A}}$$

$$+\underbrace{\frac{\chi(\mathrm{id})}{(m+1)!}\sum_{i=1}^{m}\sum_{\sigma\in S'_m}\chi(\sigma\tau_i)\mathcal{P}(\sigma)\mathcal{P}(\tau_i)(z_{\mathcal{U}}^{\otimes}\otimes v)}_{\mathcal{B}}.$$

Our purpose is to prove that $T_\chi(z_{\mathcal{U}}^{\otimes}\otimes v)$ is not equal to zero. We compute separately parts $\mathcal{A}$ and $\mathcal{B}$. Bearing in mind that $(e_1,\ldots,e_{\lambda'_1})$ is a basis of $W(z)$, we have

$$z_{\mathcal{U}}^{\otimes}=\sum_{\alpha\in\Gamma_{m,\lambda'_1}} c_\alpha e_\alpha^{\otimes}. \tag{8}$$

Part $\mathcal{B}$

$$\begin{aligned}
&\tfrac{\chi(\mathrm{id})}{(m+1)!}\sum_{i=1}^{m}\sum_{\sigma\in S'_m}\chi(\sigma\tau_i)\mathcal{P}(\sigma)\mathcal{P}(\tau_i)(z_{\mathcal{U}}^{\otimes}\otimes v)\\
=\;&\tfrac{\chi(\mathrm{id})}{(m+1)!}\sum_{i=1}^{m}\sum_{\sigma\in S'_m}\chi(\sigma\tau_i)\mathcal{P}(\sigma)\mathcal{P}(\tau_i)(\sum_{\alpha\in\Gamma_{m,\lambda'_1}} c_\alpha e_\alpha^{\otimes}\otimes e_0)\\
=\;&\tfrac{\chi(\mathrm{id})}{(m+1)!}\sum_{i=1}^{m}\sum_{\sigma\in S'_m}\chi(\sigma\tau_i)\mathcal{P}(\sigma)(\sum_{\alpha\in\Gamma_{m,\lambda'_1}} c_\alpha e_{\alpha_i\overset{i}{\hookleftarrow}0}^{\otimes}\otimes e_{\alpha(i)})
\end{aligned}$$

and as $\alpha(i)\neq 0$, $i=1,\ldots,m$,

$$\subseteq\langle e_\beta^{\otimes}:\beta\in\Gamma^0_{m+1,\lambda'_1},\ |\beta^{-1}(\{0\})|=1,\ \beta(m+1)\neq 0\rangle.$$

Part $\mathcal{A}$ According to (8) we have

$$\begin{aligned}
T_{\chi_{|S'_m}}(z_{\mathcal{U}}^{\otimes}\otimes v) &= T_{\chi_{|S'_m}}(\sum_{\alpha\in\Gamma_{m,\lambda'_1}} c_\alpha e_\alpha^{\otimes}\otimes e_0)\\
&= T_{\chi_{|S'_m}}(\sum_{\alpha\in\Gamma_{m,\lambda'_1}} c_\alpha e_{\hat\alpha}^{\otimes}).
\end{aligned}$$

Since $\sigma\in S'_m$, we have

$$\begin{aligned}
\mathcal{P}(\sigma)(e_{\hat\alpha}^{\otimes}) &= \mathcal{P}(\sigma)(e_\alpha^{\otimes}\otimes e_0)\\
&= e_{\alpha\sigma}^{\otimes}\otimes e_0\\
&= e_{\widehat{\alpha\sigma}}^{\otimes}
\end{aligned}$$

and we conclude that

$$T_{\chi_{|S'_m}}(z_{\mathcal{U}}^{\otimes} \otimes v) \subseteq \langle e_{\hat{\alpha}}^{\otimes} : \alpha \in \Gamma_{m+1,\lambda'_1} \rangle.$$

So, according to (2), if we show that part $\mathcal{A}$ is not equal to zero, we conclude that $T_\chi(z_{\mathcal{U}}^{\otimes} \otimes v) \neq 0$. But, by (7),

$$T_{\chi_{|S'_m}}(\otimes^{m+1}V) = (T_\lambda(\otimes^m V) \otimes V) \oplus (T_{\lambda^{(2)}}(\otimes^m V) \otimes V) \oplus \cdots \oplus (T_{\lambda^{(l)}}(\otimes^m V) \otimes V).$$

But the component of $T_{\chi_{|S'_m}}(z_{\mathcal{U}}^{\otimes} \otimes v)$ to $T_\lambda(\otimes^m V) \otimes V$ is

$$T_\lambda(z_{\mathcal{U}}^{\otimes}) \otimes v = z \otimes v$$

not equal to zero because z and v are nonzero.

■

Remark Using the arguments of the second part of the proof of the last theorem we can conclude that if $\mathcal{U} = (x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ is a pre-image family of z with elements in $W(z)$ then

$$\mathrm{Ann}^{\mathcal{U}}(z) \subseteq W(z). \tag{9}$$

Definition 4.7 Let z be a nonzero element of V_λ. We call *annihilator* of z and denote by $\mathrm{Ann}(z)$ the set of the elements $v \in V$ such that

$$T_\chi(z_{\mathcal{U}}^{\otimes} \otimes v) = 0$$

for all pre-image family $\mathcal{U} = (x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ of z with elements in $W(z)$.

Remark Let $0 \neq z = T_\lambda(x_1 \otimes \cdots \otimes x_m)$ a decomposable and critical tensor of V_λ. If $\mathcal{U} = (u_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ is a pre-image family of z with elements in $W(z)$, by theorem 4.3 and the previous theorem, we have

$$\mathrm{Ann}^{\mathcal{U}}(z) = \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) = W(z),$$

and then $\mathrm{Ann}(z) = W(z)$.

5 Decomposable tensors

Proposition 5.8 *Let* $0 \neq z = T_\lambda(x_1 \otimes \cdots \otimes x_m) \in V_\lambda$. *If*

$$\{y_{11}, \ldots, y_{1\lambda'_1}\}\dot{\cup}\ldots\dot{\cup}\{y_{\lambda_1 1}, \ldots, y_{\lambda_1\lambda'_{\lambda_1}}\}$$

is a λ'*-coloring of* $(x_1, \ldots, x_m)$*, then*

$$\mathrm{Ann}^{(x_1,\ldots,x_m)}(z) \subseteq \langle y_{11}, \ldots, y_{1\lambda'_1}\rangle.$$

Proof

Let

$$\{y_{11}, \ldots, y_{1\lambda'_1}\}\dot{\cup}\ldots\dot{\cup}\{y_{\lambda_1 1}, \ldots, y_{\lambda_1\lambda'_{\lambda_1}}\}$$

be a λ'-coloring of $(x_1, \ldots, x_m)$. If $x \notin \langle y_{11}, \ldots, y_{1\lambda'_1}\rangle$ then $(x, y_{11}, \ldots, y_{1\lambda'_1})$ is linearly independent and so

$$\{x, y_{11}, \ldots, y_{1\lambda'_1}\}\dot{\cup}\{y_{21}, \ldots, y_{2\lambda'_2}\}\dot{\cup}\ldots\dot{\cup}\{y_{\lambda_1 1}, \ldots, y_{\lambda_1\lambda'_{\lambda_1}}\}$$

is a χ'-coloring of $(x_1, \ldots, x_m, x)$. So, by proposition 3.4, we conclude that

$$T_\chi(x_1 \otimes \cdots \otimes x_m \otimes x) \neq 0$$

and then $x \notin \mathrm{Ann}^{(x_1,\ldots,x_m)}(z)$. So,

$$\mathrm{Ann}^{(x_1,\ldots,x_m)}(z) \subseteq \langle y_{11}, \ldots, y_{1\lambda'_1}\rangle.$$

■

Corollary 1 *If* $0 \neq z = T_\lambda(x_1 \otimes \cdots \otimes x_m)$ *is a decomposable tensor of* V_λ *we have*

$$\dim \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) \leq \lambda'_1.$$

Theorem 5.5 *Let* $0 \neq z = T_\lambda(x_1 \otimes \cdots \otimes x_m)$ *be a decomposable tensor of* V_λ*, then* z *is critical if and only if*

$$\dim \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) = \lambda'_1.$$

Proof

If z is critical and decomposable, by theorem 4.4, we have that

$$W(z) = \mathrm{Ann}^{(x_1,\dots,x_m)}(z)$$

So, $\dim \mathrm{Ann}^{(x_1,\dots,x_m)}(z) = \lambda'_1$.
For the converse condition we need the following:

Fact *If* $(x_1, \dots, x_m)$ *is* λ'*-colorable and* $\dim\langle x_1, \dots, x_m\rangle > \lambda'_1$ *then there exists two* λ'*-colorings*

$$\{x_{11}, \dots, x_{1\lambda'_1}\}\dot\cup\{x_{21}, \dots, x_{2\lambda'_2}\}\dot\cup \dots \dot\cup\{x_{\lambda_1 1}, \dots, x_{\lambda_1\lambda'_{\lambda_1}}\}$$

and

$$\{y_{11}, \dots, y_{1\lambda'_1}\}\dot\cup\{y_{21}, \dots, y_{2\lambda'_2}\}\dot\cup \dots \dot\cup\{y_{\lambda_1 1}, \dots, y_{\lambda_1\lambda'_{\lambda_1}}\}$$

such that

$$\langle x_{11}, \dots, x_{1\lambda'_1}\rangle \neq \langle y_{11}, \dots, y_{1\lambda'_1}\rangle.$$

Proof
Let

$$\{x_{11}, \dots, x_{1\lambda'_1}\}\dot\cup\{x_{21}, \dots, x_{2\lambda'_2}\}\dot\cup \dots \dot\cup\{x_{\lambda_1 1}, \dots, x_{\lambda_1\lambda'_{\lambda_1}}\}$$

be a λ'-coloring of $(x_1, \dots, x_m)$. By hypothesis, $\dim\langle x_1, \dots, x_m\rangle > \lambda'_1$, so, there exists $i \in \{2, \dots, \lambda_1\}$ and $k \in \{1, \dots, \lambda'_i\}$ such that

$$x_{ik} \notin \langle x_{11}, \dots, x_{1\lambda'_1}\rangle.$$

Also

$$\langle x_{11}, \dots, x_{1\lambda'_1}\rangle \not\subseteq \langle x_{i1}, \dots, x_{i\lambda'_i}\rangle.$$

In fact if $\langle x_{11}, \dots, x_{1\lambda'_1}\rangle \subseteq \langle x_{i1}, \dots, x_{i\lambda'_i}\rangle$ then

$$\dim\langle x_{11}, \dots, x_{1\lambda'_1}\rangle \leq \dim\langle x_{i1}, \dots, x_{i\lambda'_i}\rangle$$

so $\lambda'_1 \leq \lambda'_i$ which implies $\lambda'_1 = \lambda'_i$ and so $x_{ik} \in \langle x_{11}, \dots, x_{1\lambda'_1}\rangle$.
Contradiction.

We can conclude that exists $j \in \{1, \ldots, \lambda'_1\}$ such that

$$x_{1j} \notin \langle x_{i1}, \ldots, x_{i\lambda'_i} \rangle.$$

So $(x_{11}, \ldots, x_{1j-1}, x_{ik}, x_{1j+1}, \ldots, x_{1\lambda'_1})$ and $(x_{i1}, \ldots, x_{ik-1}, x_{1j}, x_{ik+1}, \ldots, x_{i\lambda'_i})$ are linearly independent families.
Consequently

$$\{x_{11}, \ldots, x_{1\lambda'_1}\} \dot{\cup} \{x_{21}, \ldots, x_{2\lambda'_2}\} \dot{\cup} \ldots \dot{\cup} \{x_{\lambda_1 1}, \ldots, x_{\lambda_1 \lambda'_{\lambda_1}}\}$$

and

$$\{x_{11}, \ldots, x_{1j-1}, x_{ik}, x_{1j+1}, \ldots, x_{1\lambda'_1}\} \dot{\cup} \{x_{21}, \ldots, x_{2\lambda'_2}\} \dot{\cup} \ldots$$

$$\ldots \dot{\cup} \{x_{i1}, \ldots, x_{ik-1}, x_{1j}, x_{ik+1}, \ldots, x_{i\lambda'_i}\} \dot{\cup} \ldots \dot{\cup} \{x_{\lambda_1 1}, \ldots, x_{\lambda_1 \lambda'_{\lambda_1}}\}$$

are two λ'-colorings of $(x_1, \ldots, x_m)$ satisfying the referred conditions. □

Suppose that z is not critical. Then, by the theorem 4.3 and $z \neq 0$, we have $\dim\langle x_1, \ldots, x_m \rangle = \dim W(z) > \lambda'_1$. According now to the proved fact and the proposition 5.8 we conclude that

$$\mathrm{Ann}^{(x_1, \ldots, x_m)}(z) \subseteq \langle x_{11}, \ldots, x_{1\lambda'_1} \rangle \cap \langle y_{11}, \ldots, y_{1\lambda'_1} \rangle$$

wich leads

$$\dim \mathrm{Ann}^{(x_1, \ldots, x_m)}(z) \leq \dim(\langle x_{11}, \ldots, x_{1\lambda'_1} \rangle \cap \langle y_{11}, \ldots, y_{1\lambda'_1} \rangle) < \lambda'_1.$$

■

Next proposition gives us a necessary and sufficient condition for the criticality of decomposable tensors.

Corollary 1 *Let $0 \neq z = T_\lambda(x_1 \otimes \cdots \otimes x_m)$ be a decomposable tensor of V_λ. Then z is critical if and only if there exists a linearly independent family $(v_1, \ldots, v_{\lambda'_1})$ with elements in $\mathrm{Ann}^{(x_1, \ldots, x_m)}(z)$.*

Proof

Suppose that z is critical. By theorems 4.4 and 4.3 we conclude that

$$\mathrm{Ann}(z) = W(z).$$

Consequently $\dim \mathrm{Ann}(z) = \lambda'_1$, so there exists λ'_1 vectors in the conditions of the statement.

Conversly, let $(v_1, \dots, v_{\lambda'_1})$ be a family of linearly independent vectors in $\mathrm{Ann}^{(x_1,\dots,x_m)}(z)$. Corollary 1 of proposition 5.8 gives us

$$\dim \mathrm{Ann}^{(x_1,\dots,x_m)}(z) \leq \lambda'_1.$$

So, we can conclude that $\dim \mathrm{Ann}^{(x_1,\dots,x_m)}(z) = \lambda'_1$. Then, by the theorem 5.5, we have that z is critical.

■

6 Plücker polynomials

The main purpose of this section is to construct a family of polynomials characterizing the criticality of a decomposable tensor of V_λ. The idea behind this construction is to use corollary 1 to theorem 5.5 to extend the argument referred by M. Marcus in [7].

We start with some basic computations. Recall we are fixing a basis $(e_1, \dots, e_n)$ of V. Consider a tensor $z \in \otimes^m V$,

$$z = \sum_{\alpha \in \Gamma_{m,n}} a_\alpha e_\alpha^\otimes.$$

Let $\nu \in \Gamma_{m-1,n}$ and $t \in \{1, \dots, m\}$, we denote by $u_{t,\nu}^{(z)}$ or briefly by $u_{t,\nu}$ the vector of V,

$$u_{t,\nu} := \sum_{j=1}^{n} a_{\nu \overset{t}{\hookleftarrow} j} e_j.$$

Let $\gamma \in \Gamma_{m+1,n}$. Let $\pi_1^{(\gamma)}, \dots, \pi_{s_\gamma}^{(\gamma)}$ be a system of representatives of the right cosets of H_γ in S_{m+1}, i. e. ,

$$S_{m+1} := H_\gamma \pi_1^{(\gamma)} \dot\cup \cdots \dot\cup H_\gamma \pi_{s_\gamma}^{(\gamma)}. \tag{10}$$

For $i \in \{1, \ldots, s_\gamma\}$ we denote the mapping $\gamma\pi_i^{(\gamma)}$ by $\gamma^{(i)}$.

It can be easily seen that $(e^{\otimes}_{\gamma^{(i)}})_{i=1,\ldots,s_\gamma}$ is a basis of the orbital subspace associated to γ, i.e. ,

$$\langle e^{\otimes}_{\gamma\sigma} : \sigma \in S_{m+1}\rangle = \langle e^{\otimes}_{\gamma^{(1)}}, \ldots, e^{\otimes}_{\gamma^{(s_\gamma)}}\rangle.$$

Therefore, if $l \in \{1, \ldots, s_\gamma\}$, we have

$$\begin{aligned} T_\chi(e^{\otimes}_{\gamma^{(l)}}) &= \frac{\chi(\mathrm{id})}{(m+1)!}\sum_{k=1}^{s_\gamma}(\sum_{\tau\in H_\gamma}\chi(\pi_k^{-1}\tau\pi_l))e^{\otimes}_{\gamma^{(k)}} \\ &= \sum_{k=1}^{s_\gamma}\mathfrak{c}_{\gamma,k,l}e^{\otimes}_{\gamma^{(k)}} \end{aligned} \tag{11}$$

where $\mathfrak{c}_{\gamma,k,l}$ denotes

$$\mathfrak{c}_{\gamma,k,l} := \frac{\chi(\mathrm{id})}{(m+1)!}\sum_{\tau\in H_\gamma}\chi(\pi_k^{-1}\tau\pi_l).$$

Definition 6.8 Let $\gamma \in \Gamma_{m+1,n}$, $\nu \in \Gamma_{m-1,n}$, $t \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, s_\gamma\}$. The polynomial of $\mathbb{C}[X_\alpha : \alpha \in \Gamma_{m,n}]$

$$f_{\gamma,\nu,t,k}(X_\alpha : \alpha \in \Gamma_{m,n}) := \sum_{l=1}^{s_\gamma}\mathfrak{c}_{\gamma,k,l}X_{\nu \overset{t}{\hookleftarrow} \gamma^{(l)}(m+1)}X_{\gamma^{(l)}_{m+1}}$$

is called λ-*Plücker polynomial* associated with (γ, ν, t, k).

We denote by η the element of $\Gamma_{m,n}$, $\eta = (1, \ldots, m)$. If $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ and $\alpha \in \Gamma_{m,n}$, we denote by $A[\eta|\alpha]$ the $m \times m$ matrix whose jth column is the column $\alpha(j)$ of A, $j = 1, \ldots, m$; i.e., the (i, j) entry of $A[\eta|\alpha]$ is $a_{i,\alpha(j)}$, $i, j = 1, \ldots, m$.

If $B = (b_{ij}) \in \mathbb{C}^{m\times m}$, we denote by $d_\lambda(B)$ the value of the generalized matrix function d_λ on B,

$$d_\lambda(B) := \sum_{\sigma\in S_m}\lambda(\sigma)\prod_{t=1}^{m}b_{t,\sigma(t)} \quad .$$

The Hadamard function on B will be denoted by $\mathfrak{h}(B)$, i.e., $\mathfrak{h}(B) := \prod_{i=1}^{m} b_{ii}$.

Next result is technical and prepares the computations needed for the main results of this section.

Lemma 6.4 *Let z be an element of $\otimes^m V$,*

$$z = \sum_{\alpha\in\Gamma_{m,n}} a_\alpha e_\alpha^\otimes.$$

Let u be a vector of V, $u = \sum_{j=1}^{n} c_j e_j$. The following equality holds:

$$T_\chi(z\otimes u) = \sum_{\gamma\in G_{m+1,n}} \sum_{k=1}^{s_\gamma} \left(\sum_{l=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} c_{\gamma^{(l)}(m+1)} a_{\gamma^{(l)}_{m+1}} \right) e^\otimes_{\gamma^{(k)}}.$$

Proof

By the assumptions of the theorem we have

$$\begin{aligned} T_\chi(z\otimes u) &= T_\chi\left((\sum_{\alpha\in\Gamma_{m,n}} a_\alpha e_\alpha^\otimes) \otimes (\sum_{j=1}^{n} c_j e_j) \right) \\ &= \sum_{j=1}^{n} \sum_{\alpha\in\Gamma_{m,n}} c_j a_\alpha T_\chi(e_\alpha^\otimes \otimes e_j). \end{aligned}$$

Therefore, since $\Gamma_{m+1,n} = \Gamma_{m,n} \times \{1,\ldots,n\}$, we get,

$$T_\chi(z\otimes u) = \sum_{\gamma\in\Gamma_{m+1,n}} c_{\gamma(m+1)} a_{\gamma_{m+1}} T_\chi(e_\gamma^\otimes).$$

As $G_{m+1,n}$ is a system of distinct representatives of the orbits for the action of S_{m+1} on $\Gamma_{m+1,n}$ and due to (10) and (11) we obtain, from the previous equalities,

$$T_\chi(z\otimes u) = \sum_{\gamma\in G_{m+1,n}} \sum_{l=1}^{s_\gamma} c_{\gamma^{(l)}(m+1)} a_{\gamma^{(l)}_{m+1}} T_\chi(e^\otimes_{\gamma^{(l)}})$$

$$
\begin{aligned}
&= \sum_{\gamma\in G_{m+1,n}} \sum_{l=1}^{s_\gamma} c_{\gamma^{(l)}(m+1)} a_{\gamma^{(l)}_{m+1}} \sum_{k=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} e^{\otimes}_{\gamma^{(k)}} \\
&= \sum_{\gamma\in G_{m+1,n}} \sum_{k=1}^{s_\gamma} \left(\sum_{l=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} c_{\gamma^{(l)}(m+1)} a_{\gamma^{(l)}_{m+1}} \right) e^{\otimes}_{\gamma^{(k)}}.
\end{aligned}
$$

■

Lemma 6.4 can be restated in view of definition of λ-Plücker polynomial as follows:

Corollary 2 *If $\nu \in \Gamma_{m-1,n}$ and $t \in \{1, \ldots, m\}$, we have the following equality*

$$
T_\chi(z \otimes u_{t,\nu}) = \sum_{\gamma\in G_{m+1,n}} \sum_{k=1}^{s_\gamma} f_{\gamma,\nu,t,k}(a_\alpha : \alpha \in \Gamma_{m,n}) e^{\otimes}_{\gamma^{(k)}}.
$$

Definition 6.9 Let γ be an element of $\Gamma_{m+1,n}$. Let ν be an element of $\Gamma_{m-1,n}$, and t and k positive integers respectively in $\{1, \ldots, m\}$ and $\{1, \ldots, s_\gamma\}$. We denote by $F_{\gamma,\nu,t,k}$ the polynomial of $\mathbb{C}[X_\alpha : \alpha \in \Gamma_{m,n}]$,

$$
F_{\gamma,\nu,t,k} := \sum_{\sigma\in S_m} \lambda(\sigma) f_{\gamma,\xi_{\nu,\sigma,t,\gamma^{(l)}(m+1)},\sigma^{-1}(t),k} ,
$$

where $\xi_{\nu,\sigma,t,j} := [(\nu \overset{t}{\hookleftarrow} j)\sigma]_{\sigma^{-1}(t)}$.
We denote by D_α the polynomial of $\mathbb{C}[X_\alpha : \alpha \in \Gamma_{m,n}]$,

$$
D_\alpha(X_\beta : \beta \in \Gamma_{m,n}) := \sum_{\sigma\in S_m} \lambda(\sigma) X_{\alpha\sigma} .
$$

Proposition 6.9 *Let $\gamma \in \Gamma_{m+1,n}$, $\nu \in \Gamma_{m-1,n}$, $t \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, s_\gamma\}$. Then, we have*

$$
F_{\gamma,\nu,t,k} = \sum_{l=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} D_{\nu \overset{t}{\hookleftarrow} \gamma^{(l)}(m+1)} X_{\gamma^{(l)}_{m+1}}.
$$

Proof

By definitions, we have

$$
\begin{aligned}
F_{\gamma,\nu,t,k} &= \sum_{\sigma\in S_m}\lambda(\sigma)f_{\gamma,[(\nu\overset{t}{\hookleftarrow}\gamma^{(l)}(m+1))\sigma]_{\sigma^{-1}(t)},\sigma^{-1}(t),k} \\
&= \sum_{\sigma\in S_m}\lambda(\sigma)\sum_{l=1}^{s_\gamma}\mathfrak{c}_{\gamma,k,l}X_{[(\nu\overset{t}{\hookleftarrow}\gamma^{(l)}(m+1))\sigma]_{\sigma^{-1}(t)}\overset{\sigma^{-1}(t)}{\hookleftarrow}\gamma^{(l)}(m+1)}X_{\gamma^{(l)}_{m+1}} \\
&= \sum_{l=1}^{s_\gamma}\mathfrak{c}_{\gamma,k,l}(\sum_{\sigma\in S_m}\lambda(\sigma)X_{(\nu\overset{t}{\hookleftarrow}\gamma^{(l)}(m+1))\sigma})X_{\gamma^{(l)}_{m+1}} \\
&= \sum_{l=1}^{s_\gamma}\mathfrak{c}_{\gamma,k,l}D_{\nu\overset{t}{\hookleftarrow}\gamma^{(l)}(m+1)}X_{\gamma^{(l)}_{m+1}} \quad .
\end{aligned}
$$

■

Next lemma makes the connection between the polynomials D_α and the generalized matrix functions as it was done in [2].

Lemma 6.5 *Let $A=(a_{ij})\in\mathbb{C}^{m\times n}$ and*

$$x_i=\sum_{j=1}^{n}a_{ij}e_j \quad i=i,...,m.$$

Let z be the decomposable tensor

$$z=x_1\otimes\cdots\otimes x_m=\sum_{\beta\in\Gamma_{m,n}}a_\beta e^{\otimes}_\beta.$$

Then the following equality holds

$$D_\alpha(a_\beta:\beta\in\Gamma_{m,n})=d_\lambda(A[\eta|\alpha]) \;.$$

Proof

Since $a_\beta=\mathfrak{h}(A[\eta|\beta])$, $\forall\beta\in\Gamma_{m,n}$, then

$$
\begin{aligned}
D_\alpha(a_\beta : \beta \in \Gamma_{m,n}) &= \sum_{\sigma \in S_m} \lambda(\sigma)\mathfrak{h}(A[\eta|\alpha\sigma]) \\
&= \sum_{\sigma \in S_m} \lambda(\sigma) \prod_{t=1}^{m} a_{t,\alpha\sigma(t)} \\
&= d_\lambda(A[\eta|\alpha]) \ .
\end{aligned}
$$

■

A special linearly independent family of vectors is constructed in the following proposition.

Proposition 6.10 *Let* $A = (a_{ij}) \in \mathbb{C}^{m\times n}$,

$$x_i = \sum_{j=1}^{n} a_{ij} e_j \quad i = i, ..., m.$$

Assume that $z^* = T_\lambda(x_1 \otimes \cdots \otimes x_m) \neq 0$. *Let* $\omega \in \operatorname{supp}(z^*)$ *such that* $M(\omega)$ *is maximal for the majorization order of* $\{M(\alpha) : \alpha \in \operatorname{supp}(z^*)\}$. *Let* $\omega(\{1, \dots, m\}) = \{p_1, \dots, p_l\}$, $(|\omega^{-1}(p_1)| \geq \cdots \geq |\omega^{-1}(p_l)|)$ *and* $r_i = \min \omega^{-1}(p_i)$, $\quad i = 1, \dots, l$. *Then*

$$v_i := u^{(z^*)}_{r_i,\omega_{r_i}} = \sum_{j=1}^{n} \frac{\lambda(\mathrm{id})}{m!} d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} j])e_j \ , \quad i = 1, \dots, l$$

is a linearly independent family.

Proof

We begin by proving the following

Fact
If $j < i$, *then* $M(\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j) \gneqq M(\omega)$.

Proof
If $j < i$, we have,

$$M(\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j) = (|\omega^{-1}(p_1)|, \dots, |\omega^{-1}(p_{j-1})|, |\omega^{-1}(p_j)| + 1, \dots, |\omega^{-1}(p_l)|) \ .$$

Therefore,

$$M(\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j) \gneqq M(\omega) \ .$$

□

Then, we have

$$\begin{aligned} v_i &= \sum_{j=1}^{n} \frac{\lambda(\mathrm{id})}{m!} d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} j])e_j \\ &= \frac{\lambda(\mathrm{id})}{m!}(d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} p_1])e_{p_1} + \cdots + d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} p_l])e_{p_l} + \\ &\qquad + \sum_{j \notin \{p_1,\ldots,p_l\}} d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} j])e_j). \end{aligned}$$

Since for $j < i$, we have $M(\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j) \gneqq M(\omega)$, we can conclude that $\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j \notin \mathrm{supp}(z^*)$ if $j < i$. Then

$$\begin{aligned} v_i &= \frac{\lambda(\mathrm{id})}{m!}(d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} p_i])e_{p_i} + \cdots + d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} p_l])e_{p_l} + \\ &\qquad + \sum_{j \notin \{p_1,\ldots,p_l\}} d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} j])e_j). \end{aligned}$$

But, by definition, $\omega_{r_i} \overset{r_i}{\hookleftarrow} p_i = \omega$, so we have that $(v_1, \ldots, v_l)$ is linearly independent.

■

Lemma 6.6 *Let ν be an element of $\Gamma_{m-1,n}$ and $t \in \{1, \ldots, m\}$.*

Let $A = (a_{ij}) \in \mathbb{C}^{m \times n}$ and

$$x_i = \sum_{j=1}^{n} a_{ij} e_j \quad i = i, ..., m.$$

Then, if

$$z = x_1 \otimes \cdots \otimes x_m = \sum_{\alpha \in \Gamma_{m,n}} a_\alpha e_\alpha^{\otimes},$$

we have

$$u_{t,\nu} \in \langle x_t \rangle \ .$$

Proof

It is well known [6] that the coeficient of $x_1 \otimes \cdots \otimes x_m$ in $e^{\otimes}_{\alpha}$ is the value of the Hadamard function on the matrix $A[\eta|\alpha]$, i.e.,

$$x_1 \otimes \cdots \otimes x_m = \sum_{\alpha \in \Gamma_{m,n}} \mathfrak{h}(A[\eta|\alpha]) e^{\otimes}_{\alpha}.$$

Therefore taking $k = a_{1,\nu(1)} \dots a_{t-1,\nu(t-1)} a_{t+1,\nu(t)} \dots a_{m,\nu(m-1)}$, we have

$$\begin{aligned}
u_{t,\nu} &= \sum_{j=1}^{n} \mathfrak{h}(A[\eta|\nu \overset{t}{\hookleftarrow} j]) e_j \\
&= \sum_{j=1}^{n} \left(\prod_{r=1}^{t-1} a_{r,\nu(r)} \right) a_{tj} \left(\prod_{r=t+1}^{m} a_{r,\nu(r-1)} \right) e_j \\
&= \left(\prod_{r=1}^{t-1} a_{r,\nu(r)} \prod_{r=t+1}^{m} a_{r,\nu(r-1)} \right) \sum_{j=1}^{n} a_{tj} e_j \\
&= k \sum_{j=1}^{n} a_{tj} e_j \\
&= k x_t \ .
\end{aligned}$$

■

Lemma 6.7 *Let $A = (a_{ij}) \in \mathbb{C}^{m \times n}$. Let $\alpha \in \Gamma_{m,n}$, such that $\alpha \notin \Omega$. Then*

$$d_\lambda(A[\eta|\alpha]) = 0\ .$$

Proof

Let $\theta_1, \dots, \theta_{s_\alpha}$ be a system of distinct representatives of the left coset decomposition of H_α in S_m. Then, we have

$$e^*_\alpha = \frac{\lambda(\mathrm{id})}{m!} \sum_{\sigma \in S_m} \lambda(\sigma) e^{\otimes}_{\alpha\sigma^{-1}} = \frac{\lambda(\mathrm{id})}{m!} \sum_{j=1}^{s_\alpha} (\sum_{\tau \in H_\alpha} \lambda(\theta_j \tau)) e^{\otimes}_{\alpha\theta_j^{-1}} \ .$$

Then, since $\alpha \notin \Omega$

$$\sum_{\tau\in H_\alpha} \lambda(\theta_j\tau) = 0\ , \quad j = 1,\dots,s_\alpha.$$

So, as λ is a character of S_m, $\lambda(\sigma) = \lambda(\sigma^{-1})$,

$$\begin{aligned} d_\lambda(A[\eta|\alpha]) &= \sum_{\sigma\in S_m} \lambda(\sigma)\prod_{t=1}^{m} a_{t,\alpha\sigma(t)} \\ &= \sum_{\sigma\in S_m} \lambda(\sigma^{-1})\prod_{t=1}^{m} a_{t,\alpha\sigma^{-1}(t)} \\ &= \sum_{\sigma\in S_m} \lambda(\sigma)\prod_{t=1}^{m} a_{t,\alpha\sigma^{-1}(t)} \\ &= \sum_{j=1}^{s_\alpha}(\sum_{\tau\in H_\alpha} \lambda(\theta_j\tau))\prod_{t=1}^{m} a_{t,\alpha\theta_j^{-1}(t)} \\ &= 0 \end{aligned}$$

■

Theorem 6.6 *Let* $0 \neq z^* = T_\lambda(x_1 \otimes \cdots \otimes x_m)$ *and*

$$z = x_1 \otimes \cdots \otimes x_m = \sum_{\alpha\in\Gamma_{m,n}} a_\alpha e_\alpha^\otimes.$$

Then z^* *is critical if and only if* $(a_\alpha : \alpha \in \Gamma_{m,n})$ *is a zero of the* λ*-Plücker polynomials associated with* (γ, ν, t, k)*, when* $\gamma \in G_{m+1,n}$*,* $\nu \in \Gamma_{m-1,n}$*,* $t \in \{1,\dots,m\}$ *and* $k \in \{1,\dots,s_\gamma\}$*.*

Proof

Let $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ such that

$$x_i = \sum_{j=1}^{n} a_{ij}e_j \quad i = i, ..., m.$$

Let

$$z = x_1 \otimes \cdots \otimes x_m = \sum_{\beta\in\Gamma_{m,n}} a_\beta e_\beta^\otimes,$$

such that

$$z^* = T_\lambda(x_1 \otimes \cdots \otimes x_m) = \sum_{\alpha\in\Omega_\lambda} \frac{\lambda(\mathrm{id})}{m!} d_\lambda(A[\eta|\alpha]) e^{\otimes}_{\alpha} \neq 0 \ .$$

Let $\omega \in \mathrm{supp(z^*)}$ such that $M(\omega)$ is maximal for the majorization order of

$$\{M(\alpha) : \alpha \in \mathrm{supp}(z^*)\}.$$

From lemma 6.7, we conclude that $|\{\omega(1), \ldots, \omega(m)\}| \geq \lambda'_1$.
Let

$$u^{(z^*)}_{t,\nu} = \sum_{j=1}^{n} \frac{\lambda(\mathrm{id})}{m!} d_\lambda(A[\eta|\nu \overset{t}{\hookleftarrow} j]) e_j \ .$$

According to lemma 6.4, lemma 6.5 and proposition 6.9 we have

$$\begin{aligned} T_\chi(z \otimes u^{(z^*)}_{t,\nu}) &= \sum_{\gamma\in G_{m+1,n}} \sum_{k=1}^{s_\gamma}\sum_{l=1}^{s_\gamma} \frac{\lambda(\mathrm{id})}{m!} \mathfrak{c}_{\gamma,k,l} d_\lambda(A[\eta|\nu \overset{t}{\hookleftarrow} \gamma^{(l)}(m+1)]) a_{\gamma^{(l)}_{m+1}} e^{\otimes}_{\gamma^{(k)}} \\ &= \sum_{\gamma\in G_{m+1,n}} \sum_{k=1}^{s_\gamma}\sum_{l=1}^{s_\gamma} \frac{\lambda(\mathrm{id})}{m!} \mathfrak{c}_{\gamma,k,l} D_{\nu \overset{t}{\hookleftarrow} \gamma^{(l)}(m+1)}(a_\alpha : \alpha \in \Gamma_{m,n}) a_{\gamma^{(l)}_{m+1}} e^{\otimes}_{\gamma^{(k)}} \\ &= \frac{\lambda(\mathrm{id})}{m!} \sum_{\gamma\in G_{m+1,n}} \sum_{k=1}^{s_\gamma} F_{\gamma,\nu,t,k}(a_\alpha : \alpha \in \Gamma_{m,n}) e^{\otimes}_{\gamma^{(k)}} \\ &= \frac{\lambda(\mathrm{id})}{m!} \sum_{\gamma\in G_{m+1,n}} \sum_{k=1}^{s_\gamma} \Big(\sum_{\sigma\in S_m} \lambda(\sigma) f_{\gamma,\xi_{\nu,\sigma,t,\gamma^{(l)}(m+1)},\sigma^{-1}(t),k}(a_\alpha : \alpha \in \Gamma_{m,n})\Big) e^{\otimes}_{\gamma^{(k)}} \\ &= 0 \ . \end{aligned} \tag{12}$$

But, by proposition 6.10, $(v_1, \ldots, v_{\lambda'_1})$ is a linearly independent family of vectors, and by (12) the vectors belongs to

$$\mathrm{Ann}^{(x_1,\ldots,x_m)}(z^*) \ .$$

Then, by corollary 1 to theorem 5.5, z^* is critical.

Conversely, assume that z^* is critical.
Let $t \in \{1, \ldots, m\}$, $\nu \in \Gamma_{m-1,n}$. Then, according to corollary 2 to lemma 6.4, we have

$$T_\chi(z \otimes u_{t,\nu}) = \sum_{\gamma \in G_{m+1,n}} \sum_{k=1}^{s_\gamma} f_{\gamma,\nu,t,k}(a_\alpha : \alpha \in \Gamma_{m,n}) e^{\otimes}_{\gamma(k)}.$$

But, by lemma 6.6 and theorem 4.3, for all t and all ν, $u_{t,\nu} \in \langle x_t \rangle \subseteq W(z^*)$. Consequently, by theorem 4.4, since z^* is critical, for all t and all ν,

$$u_{t,\nu} \in \mathrm{Ann}^{(x_1,\dots,x_m)}(z^*).$$

Then,

$$T_\chi(z \otimes u_{t,\nu}) = 0 \quad .$$

So, we have that $(a_\alpha : \alpha \in \Gamma_{m,n})$ is a root of $f_{\gamma,\nu,t,k}$, for all $\gamma \in G_{m+1,n}$, $\nu \in \Gamma_{m-1,n}$, $t \in \{1, \dots, m\}$, $k \in \{1, \dots, s_\gamma\}$.

■

References

[1] J. A. Dias da Silva, On the μ-colorings of a matroid, *Linear and Multilinear Algebra* 27 (1990) 25-32 .

[2] J. A. Dias da Silva, A note on preservers of decomposability, *Linear Algebra Appl.* 186 (1993) 215-225.

[3] J. A. Dias da Silva, Colorings and equality of tensors, *Linear Algebra Appl.* 342 (2002) 79-91.

[4] C. Gamas, Conditions for a symmetrized decomposable tensor to be zero, *Linear Algebra Appl.* 108 (1988) 83-119.

[5] M. H. Lim, Rank k vectors in symmetry classes of tensors, *Canad. Math. Bull.* Vol. 19(1) (1976).

[6] M. Marcus,*Finite Dimensional Multilinear Algebra I*, Marcel Dekker, New York, 1973.

[7] M. Marcus,*Finite Dimensional Multilinear Algebra II*, Marcel Dekker, New York, 1975.

[8] R. Merris, Nonzero decomposable symmetrized tensors, *Linear Algebra Appl.* 17 (1977) 287-292.

[9] R. Merris, *Multilinear Algebra*, Gordon and Breach Science Publishers, Amsterdam, 1997.