Bootstrap Methods for Dependent Data. Application to Extremal Index Estimation

D. Prata Gomes[1⋆], M. Manuela Neves[2] and J. Tiago Mexia[1⋆⋆]

[1] Departamento de Matemtica, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Portugal
e-mail address: dsrp@fct.unl.pt, jtm@fct.unl.pt

[2] Departamento de Matemtica, Instituto Superior de Agronomia, Universidade Técnica de Lisboa, Portugal
e-mail address: manela@isa.utl.pt

1 Introduction

The main objective of statistics of extremes is the estimation of parameters of rare events. The introduction of a new parameter, the *extremal index* enables a straightforward extension of the classic results for the independent case to stationary processes. Extremal index estimators proposed in the literature are strongly dependent on the high level u_n. Our objective is to use and to compare block bootstrap procedures to estimate the *extremal index*.

2 The Extremal Index

Let $\{X_n\}_{n\geq 1}$ be a stationary process where random variables have distribution function F. Assume that for every $\tau > 0$ there exists a sequence $\{u_n(\tau)\}$ of constants such that,

$$n(1 - F(u_n(\tau))) \to \tau \quad \text{as} \quad n \to \infty. \tag{1}$$

The process $\{X_n\}$ will have extremal index θ, Leadbetter (1974) [7], if with $M_n = \max(X_i : i = 1, \cdots, n)$,

$$P\{M_n \leq u_n(\tau)\} \to e^{-\theta\tau} \quad \text{as} \quad n \to \infty. \tag{2}$$

Moreover, under some conditions the possible limit distribution for the maxima from the stationary process is the same as in the independent case.

⋆ Support for this author was provided by Centro de Matemtica e Aplicações da Universidade Nova de Lisboa and by FCT

⋆⋆ Support for this author was provided by Centro de Matemtica e Aplicações da Universidade Nova de Lisboa

We now have the important result,

Theorem 2.1:(Leadbetter (1983) [8]) If, with $\{\widehat{X}_n\}_{n\geq 1}$ a sequence of i.i.d. random variables with distribution function F and $\widehat{M}_n = \max(\widehat{X}_i : i = 1, \cdots, n)$, there exist sequences $\{a_n > 0\}$ and $\{b_n\}$ of constants such that

$$P\left\{\frac{\widehat{M}_n - b_n}{a_n} \leq x\right\} \to G\left(x\right) \quad \text{as} \quad n \to \infty$$

for a non-degenerate distribution function G, then, under $D(u_n)$ condition with $u_n = a_n x + b_n$, for each x such that $G(x) > 0$,

$$P\left\{\frac{M_n - b_n}{a_n} \leq x\right\} \to G^{\theta}\left(x\right) \quad \text{as} \quad n \to \infty,$$

whenever $\{X_n\}_{n\geq 1}$ is a stationary process whose random variables also have distribution F. $\theta \in [0,1]$ is the extremal index.

We now point out that the extremal index also rules the clustering of exceedances for increasing thresholds. Thus, if $\theta = 1$, exceedances occur singly at the limit, while, if $\theta < 1$, they tend to cluster at the limit. Thus, Leadbetter (1983) [8] interprets θ as the reciprocal mean cluster size. The identification of approximately independent clusters of high level exceedances is then a key issue for the estimation of θ. With $N_n(u_n)$ and $C_n(u_n)$ the number of the level u_n exceedances and of their clusters in a size n sample, respectively, we get the estimator:

$$\widehat{\theta}_n = \frac{C_n(u_n)}{N_n(u_n)}. \tag{3}$$

One of the most popular estimators of the extremal index is based on this characterization. As a first attempt to identify clusters, $C_n(u_n)$ is measured by the number of up-crossings of a high threshold u_n. This gives, Nandagopalan (1990) [10] and Gomes(1990) [3], the up-crossing estimator,

$$\widehat{\theta}_n^{UC} := \frac{\sum_{i=1}^{n-1} I(X_i \leq u_n < X_{i+1})}{\sum_{i=1}^{n} I(X_i > u_n)}. \tag{4}$$

Nandagopalan (1990) [10] derived, under general conditions, the weak consistency and the asymptotic distribution of the estimator $\widehat{\theta}_n^{UC}$ of θ. This estimator is also easy to compute and does not require any knowledge of clustering characteristics of the process.

3 Blocks Bootstrapping

Bootstrap methodology was first introduced by Efron(1979) [2] in the context of i.i.d. data. Shing(1981) [14] showed the inadequacy of the classic bootstrap in the context of dependent data. His idea was to group the observations into blocks and carrying out the resampling at the block level. The motivation for this scheme is that the dependence structure of the underlying model is preserved within each block. Thus, if the block size is allowed to tend to infinity with the sample size, asymptotically correct inference can ensue. Several authors studied ways of blocking, Carlstein(1986) [1], K*ü*nsch(1989) [5], Liu and Singh(1992) [9], Politis and Romano(1992, 1994) [12],[13] and Lahiri(2003) [6].

In what follows we describe the principal blocking methods and discuss how to choose the block size and exceedance level.

3.1 Moving Block Bootstrap

This method was introduced by Künsch (1989) [5] and Liu and Singh (1982) [9]. The blocks are constituted by b contiguous observations so that they may be identified by the index of their first observation. From the population of $n_b = n-b+1$ blocks, a sample of size $[n/b]$ is taken with replacement. Since the underling model is stationary, the $[n/b]$ vectors constituted by the observations in the chosen blocks, may be considered as identically distributed. Moreover, if $(X_1^*, \dots, X_b^*)$ are the observations in a chosen block,

$$P\left\{\bigcap_{i=1}^{b}(X_i^* = X_{j-1+i})\right\} = n_b^{-1},\ \text{for } 1 \le j \le n_b, \tag{5}$$

since n_b^{-1} is the probability of observation with index j being chosen as first observation of a chosen block.

3.2 Non-overlapping Block Bootstrap

Carlstein (1986) [1] considered to extract, from the original sample divided into $n_b = [n/b]$ non-overlapping blocks of b contiguous observations, a sample of n_b blocks. The observations in the chosen blocks are rewrite into a sequence.

We point out that the vectors of observations contained in the chosen blocks are i.i.d.. Now, the first observations in the blocks have indexes $(j-1)b+1,\ j = 1, \dots, n_b$. Thus, if $(X_1^*, \dots, X_b^*)$ are the observations in a chosen block

$$P\left\{\bigcap_{i=1}^{b}(X_i^* = X_{(j-1)b+i})\right\} = n_b^{-1},\ \text{for } 1 \le j \le n_b. \tag{6}$$

3.3 The Circular Block Bootstrap

Both the previous methods assign less weights to the first and last observations in the sample than to those in the middle. To overcome this problem Politis and Romano (1992, 1994) [12] [13] put forward two resampling schemes:

circular block bootstrap and the stationary bootstrap. We will only consider the first one. The main idea is to wrap the sample around a circle. The original sample $\underline{X}_n = (X_1, X_2, \ldots, X_n)$ is replaced by $(X_1, X_2, \ldots, X_{n+b-1})$ with $X_{n+j} = X_j,\ j = 1, \ldots, b-1$. We thus will have $n_b = n$ blocks identified by the index of their first observation. From this population of blocks, a sample of size $[n/b]$ is taken with replacement. Each observation belongs to b blocks and once all blocks have the same probability of being chosen, the weights are the same for all observations. A problem arises in applying these methods: the choice of the block size, b.

3.4 Block Size choice

Let $\widehat{\theta}^*(n, b)$ be the bootstrap block estimator of θ obtained from the sample of size n with block length b. Following Hall et al. (1995) [4] we take a sequence $\{s_n\}_{n\geq 1}$ of sub-sample sizes such that,

$$s_n^{-1} + n^{-1}s_n = o(1) \text{ as } n \to \infty. \tag{7}$$

With $\widehat{\theta}_i^*(s_n, h)$ the block bootstrap estimator derived from the i-th chosen sub-sample, $i = 1, \ldots, n - s_n + 1$, with block length $h < b$, we get the estimator

$$\widehat{MSE}^*(s_n, h) = (n - s_n + 1)^{-1} \sum_{i=1}^{n-s_n+1} [\widehat{\theta}_i^*(s_n, h) - \widehat{\theta}^*(n, b)]^2, \tag{8}$$

of the mean squared error of $\widehat{\theta}(s_n, h)$. In (8), b will be the block pilot size. Let

$$\widehat{h}_{opt,s_n} = \arg\min\{\widehat{MSE}^*(s_n, h)\}$$

in a suitable range of block sizes, h. Rescaling it, Hall et al. (1995) [4] propose the estimate $\widehat{b}_{opt,n}$ given by

$$\widehat{b}_{opt,n} = \widehat{h}_{opt,s_n}[n/s_n]^{\frac{1}{C+2}}. \tag{9}$$

where $C = 1$ for estimation of bias or variance, $C = 2$ for one-sided distribution function and $C = 3$ for two-sided distribution function. Hall et al. (1995) [4] suggest iterating this algorithm, taking in the next iteration as pilot size the result obtained in the previous one.

3.5 Exceedance Level

After having chosen the block size we must choose the exceedance level. The level u_n for which (1) holds depends on the distribution so, as usual, we will consider an upper order statistic, i.e., $u_n := X_{k:n}$, then, the choice is made over k. To do this we:

- consider an auxiliary threshold k_{aux} from a suitable range (small values) of thresholds and compute $\widehat{\theta}_n^{UC}(k_{aux})$;
- divide the sample into blocks of size $\widehat{b}_{opt,n}$; resample B times from the sample and for $1 \leq k \leq n-1$, compute the bootstrap estimator $\widehat{\theta}_n^{UC*}(k)$;
- select the threshold which minimize bootstrap estimate of mean squared error of $\widehat{\theta}^*(k)$ over the previous set of thresholds, i.e,

$$\widehat{MSE}^*(k) = B^{-1}\sum_{j=1}^{B}[\widehat{\theta}_n^{UC*}(k) - \widehat{\theta}_n^{UC}(k_{j-1})]^2, \text{ where } k_0 = k_{aux},$$

  replacing in second iteration the k_{aux} by the threshold selected in the first iteration.
- The process stop in $k_0(k_{aux}) = k_j$ when either $k_j = k_{j-1}$ or

$$E\left\{[\widehat{\theta}_n^{UC*}(k_{j+1}) - \widehat{\theta}_n^{UC}(k_j)]^2\right\} > E\left\{[\widehat{\theta}_n^{UC*}(k_j) - \widehat{\theta}_n^{UC}(k_{j-1})]^2\right\}.$$

4 Simulation study

Here we consider some of the simulation study for the block bootstrap methods presented above. This simulation study is carried out for three different models:

Model A - *Max autoregressive process*

Let $\{Y_n\}_{n\geq 1}$ be independent unit Fréchet random variables, $\theta \in (0,1]$ and

$$X_1 = Y_1/\theta \text{ and } X_n = \max\{(1-\theta)X_{n-1}, Y_n\}, \text{ for } n \geq 2.$$

Then $\{X_n\}$ is a strictly stationary process with *extremal index* θ.

Model B - *Markovian max-autoregressive sequence*

Let X_0 be a random variable with distribution function $H_0(x)$ and $\{Y_n\}$ an independent and identically distributed sequence, independent of X_0, with common distribution function $F(x)$. We consider $H_0(x) = \exp(-x^{-\alpha}/\beta^{-\alpha-1})$ and $F(x)$ the Fréchet distribution function with parameter α. The stationary process is defined by

$$X_n = \beta \max(X_{n-1}, Y_n), \ n \geq 1, \ 0 < \beta < 1.$$

and has *extremal index* $\theta = 1 - \beta^\alpha$.

Model C - *Max two-dependent sequence*

Let $\{Y_n\}_{n\geq 1}$ be independent unit exponential random variables and let

$$X_n = \max(Y_{n-1}, Y_n), \ n \geq 1.$$

$\{X_n\}$ is a stationary process with *extremal index* $\theta = 1/2$.

Fig 1. shows a simple path of $n = 1000$ sample size for the estimator $\widehat{\theta}_n^{UC}$. For models A and B we used $\theta = 0.1,\ 0.5$ and 0.9. For level k we considered up to 20% of the sample size.

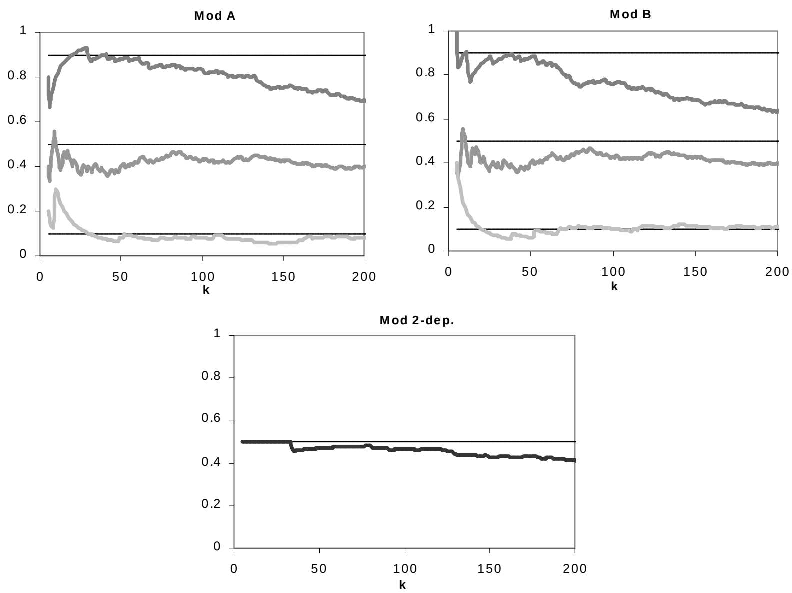

Fig. 1. One sample path

For illustration, two different subsample sizes, ($s_n = 200, 400$), were considered for each model. For several block pilot size ($b = 10, 20, 25, 40, 50, 100$) the optimal h ($h < b$) was calculated using 500 Monte -Carlo replicates. We will present some results for the Moving Block Bootstrap and the Non-overlapping Block Bootstrap. Tables 1., 2. and 3. give the optimal block size $\widehat{b}_{opt,n}$ for the estimation of θ computed by (9) (we present the divisor of n nearest to $\widehat{b}_{opt,n} = \left[(\widehat{h}_{opt,s_n} [n/s_n]^{\frac{1}{3}}) \right]$). It seems that $\widehat{b}_{opt,n}$ depends on s_n.

	Model A					
	$\theta = \mathbf{0.1}$		$\theta = \mathbf{0.5}$		$\theta = \mathbf{0.9}$	
	$\mathbf{s_n = 200}$	$\mathbf{s_n = 400}$	$\mathbf{s_n = 200}$	$\mathbf{s_n = 400}$	$\mathbf{s_n = 200}$	$\mathbf{s_n = 400}$
Non-overlapping Bootstrap	20	20	10	10	10	20
Moving Bootstrap	25	25	10	10	10	10

Table 1. Optimal block sizes for some values of θ – model A.

	Model B					
	$\theta = \mathbf{0.1}$		$\theta = \mathbf{0.5}$		$\theta = \mathbf{0.9}$	
	$\mathbf{s_n = 200}$	$\mathbf{s_n = 400}$	$\mathbf{s_n = 200}$	$\mathbf{s_n = 400}$	$\mathbf{s_n = 200}$	$\mathbf{s_n = 400}$
Non-overlapping Bootstrap	10	20	10	10	10	25
Moving Bootstrap	10	25	10	10	10	25

Table 2. Optimal block sizes for some values of θ - model B.

	Model C	
	$\mathbf{s_n = 200}$	$\mathbf{s_n = 400}$
Non-overlapping Bootstrap	100	100
Moving Bootstrap	100	100

Table 3. Optimal block sizes for some values of θ – model C

Model A - Non-overlapping Bootstrap

	$\theta = 0.1$			$\theta = 0.5$			$\theta = 0.9$		
k_{aux}	k_{opt}	est_{boot}	n_{iter}	k_{opt}	est_{boot}	n_{iter}	k_{opt}	est_{boot}	n_{iter}
10	890	0.0281	7	766	0.1723	21	140	0.7714	4
11	890	0.0281	6	766	0.1723	21	193	0.7047	7
12	890	0.0281	6	766	0.1723	21	33	0.9091	2
13	890	0.0281	6	766	0.1723	21	26	0.9231	3
14	890	0.0281	6	766	0.1723	21	26	0.9231	3
15	885	0.0282	6	766	0.1723	22	26	0.9231	2
16	885	0.0282	6	766	0.1723	22	26	0.9231	2
17	896	0.0268	6	766	0.1723	22	26	0.9231	2
18	900	0.0256	6	766	0.1723	23	26	0.9231	2
19	890	0.0281	5	766	0.1723	23	26	0.9231	2
20	890	0.0281	5	766	0.1723	23	140	0.7714	4
21	878	0.0273	5	766	0.1723	24	193	0.7047	7
22	878	0.0273	5	766	0.1723	24	193	0.7047	7
23	878	0.0273	5	766	0.1723	23	33	0.9091	2
24	878	0.0273	5	766	0.1723	23	33	0.9091	2
25	878	0.0273	5	766	0.1723	23	33	0.9091	2
26	900	0.0256	6	766	0.1723	22	26	0.9231	3
27	885	0.0282	5	766	0.1723	22	26	0.9231	3
28	895	0.0279	5	766	0.1723	22	26	0.9231	3
29	885	0.0282	5	766	0.1723	22	26	0.9231	2
30	890	0.0281	5	766	0.1723	22	26	0.9231	2

Table 4. Values of k_{opt} corresponding to k_{aux}, bootstrap estimate and number of iterations, for some values of θ.

Using the value for the block size given in tables 1.,2. and 3. (if it is not the same for $s_n = 200$ and $s_n = 400$, we choose one of them), we considered a set of values for k_{aux}, ($k_{aux} = 10, 11, ..., 30$) to use in the iterative procedure described above. Tables 4., 5. 6., 7. and 8. show the results of the simulation procedure.

Model A - Moving Bootstrap

	$\theta = 0.1$			$\theta = 0.5$			$\theta = 0.9$		
k_{aux}	k_{opt}	est_{boot}	n_{iter}	k_{opt}	est_{boot}	n_{iter}	k_{opt}	est_{boot}	n_{iter}
10	659	0.0349	6	126	0.4524	5	359	0.6017	5
11	687	0.0378	7	126	0.4524	4	359	0.6017	5
12	659	0.0349	6	135	0.4519	3	359	0.6017	5
13	653	0.0352	5	126	0.4524	4	359	0.6017	5
14	687	0.0378	5	126	0.4524	4	359	0.6017	5
15	653	0.0352	6	126	0.4524	5	359	0.6017	5
16	653	0.0352	7	126	0.4524	6	359	0.6017	5
17	653	0.0352	7	126	0.4524	4	359	0.6017	6
18	687	0.0378	7	126	0.4524	4	359	0.6017	6
19	687	0.0378	7	126	0.4524	6	359	0.6017	6
20	687	0.0378	7	109	0.4862	3	359	0.6017	5
21	653	0.0352	7	109	0.4862	3	359	0.6017	5
22	687	0.0378	7	126	0.4524	7	359	0.6017	5
23	687	0.0378	7	109	0.4862	3	359	0.6017	5
24	687	0.0378	7	126	0.4524	4	359	0.6017	5
25	687	0.0378	7	126	0.4524	4	359	0.6017	5
26	659	0.0349	6	126	0.4524	4	359	0.6017	5
27	659	0.0349	6	126	0.4524	4	359	0.6017	5
28	617	0.0373	6	126	0.4524	5	359	0.6017	5
29	659	0.0349	6	126	0.4524	4	359	0.6017	5
30	859	0.0349	6	126	0.4524	5	359	0.6017	5

Table 5. Values of k_{opt} corresponding to k_{aux}, bootstrap estimate and number of iterations, for some values of θ

Model B - Non-overlapping Bootstrap

	$\theta = 0.1$			$\theta = 0.5$			$\theta = 0.9$		
k_{aux}	k_{opt}	est_{boot}	n_{iter}	k_{opt}	est_{boot}	n_{iter}	k_{opt}	est_{boot}	n_{iter}
10	799	0.0375	3	33	0.5455	2	10	0.9	3
11	796	0.0389	3	27	0.6296	2	10	0.9	3
12	799	0.0375	3	22	0.7273	2	10	0.9	2
13	809	0.0358	3	27	0.6296	2	13	0.7692	3
14	809	0.0358	3	27	0.6296	2	13	0.7692	2
15	816	0.0355	3	22	0.7273	2	13	0.7692	2
16	816	0.0355	3	20	0.7	2	13	0.7692	2
17	809	0.0358	3	27	0.6296	3	10	0.9	2
18	822	0.0353	3	27	0.6296	3	10	0.9	2
19	796	0.0389	2	27	0.6296	3	10	0.9	2
20	799	0.0375	2	27	0.6296	3	10	0.9	2
21	809	0.0358	2	33	0.5455	3	9	0.8889	2
22	796	0.0389	3	33	0.5455	3	9	0.8889	2
23	799	0.0375	3	27	0.6296	3	9	0.8889	2
24	799	0.0375	3	27	0.6296	3	9	0.8889	2
25	799	0.0375	3	27	0.6296	3	9	0.8889	2
26	809	0.0358	3	22	0.7273	3	9	0.8889	3
27	822	0.0353	4	20	0.7	2	10	0.9	2
28	799	0.0375	3	21	0.7143	2	9	0.8889	2
29	816	0.0355	4	20	0.7	2	9	0.8889	2
30	809	0.0358	4	22	0.7273	2	9	0.8889	2

Table 6. Values of k_{opt} corresponding to k_{aux}, bootstrap estimate and number of iterations, for some values of θ

Model B - Moving Bootstrap

	$\theta = 0.1$			$\theta = 0.5$			$\theta = 0.9$		
k_{aux}	k_{opt}	est_{boot}	n_{iter}	k_{opt}	est_{boot}	n_{iter}	k_{opt}	est_{boot}	n_{iter}
10	900	0.0222	10	899	0.0779	38	54	0.8519	3
11	900	0.0222	10	607	0.2356	12	54	0.8519	3
12	900	0.0222	10	607	0.2356	12	404	0.5297	21
13	900	0.0222	9	607	0.2356	12	404	0.5297	18
14	900	0.0222	9	607	0.2356	12	404	0.5297	19
15	900	0.0222	9	607	0.2356	13	404	0.5297	19
16	900	0.0222	9	607	0.2356	13	404	0.5297	20
17	900	0.0222	8	607	0.2356	13	404	0.5297	20
18	900	0.0222	8	607	0.2356	14	404	0.5297	21
19	900	0.0222	10	607	0.2356	14	65	0.8462	2
20	900	0.0222	10	607	0.2356	15	404	0.5297	23
21	900	0.0222	9	607	0.2356	15	60	0.85	2
22	900	0.0222	9	607	0.2356	15	59	0.8475	2
23	900	0.0222	9	607	0.2356	15	54	0.8519	2
24	900	0.0222	9	607	0.2356	14	54	0.8519	2
25	900	0.0222	9	607	0.2356	14	46	0.8696	2
26	900	0.0222	8	607	0.2356	14	46	0.8696	2
27	900	0.0222	8	607	0.2356	13	404	0.5297	23
28	900	0.0222	8	607	0.2356	13	60	0.85	2
29	900	0.0222	8	607	0.2356	13	60	0.85	2
30	900	0.0222	9	607	0.2356	13	59	0.8475	2

Table 7. Values of k_{opt} corresponding to k_{aux}, bootstrap estimate and number of iterations, for some values of θ

Model C

	NOB			MB		
k_{aux}	k_{opt}	est_{boot}	n_{iter}	k_{opt}	est_{boot}	n_{iter}
10	10	0.5	1	21	0.5	2
11	10	0.5	2	21	0.5	2
12	10	0.5	2	21	0.5	2
13	10	0.5	2	21	0.5	2
14	10	0.5	2	21	0.5	2
15	10	0.5	2	21	0.5	2
16	10	0.5	2	21	0.5	2
17	10	0.5	2	21	0.5	2
18	10	0.5	2	21	0.5	2
19	10	0.5	2	21	0.5	2
20	10	0.5	2	21	0.5	2
21	10	0.5	2	21	0.5	1
22	10	0.5	2	21	0.5	2
23	10	0.5	2	21	0.5	2
24	10	0.5	2	21	0.5	2
25	10	0.5	2	21	0.5	2
26	10	0.5	2	21	0.5	2
27	10	0.5	2	21	0.5	2
28	10	0.5	2	21	0.5	2
29	10	0.5	2	21	0.5	2
30	10	0.5	2	21	0.5	2

Table 8. Values of the k_{opt} corresponding to k_{aux}, bootstrap estimate and number of iterations for Non-overlapping Block Bootstrap (NOB) and Moving Block Bootstrap (MB)

It seems that there is still a lot of work that needs to be done in order to choose the optimal block size for resampling (it seems to depend on the subsampling sample although, as much as we know, nothing is considered in the literature). The iterative procedure used for choosing k, converges quickly although for some values of θ we have not obtained pretty good values for the estimates. Once again it seems that it depends on the block size. Only as an example, here we considered the same block size for different values of θ within each model.

References

1. Carlstein, E. (1986). The use of subseries values for estimating the variance of a general statistic from a stationary sequence. *The Ann. of Stat.* 14, 1171-1179.
2. Efron, B. (1979). Bootstrap methods: another look at the Jackknife. *Ann. Stat.* 7, 1-26.
3. Gomes, M.I. (1990). Statistical inference in an extremal markovian model.*COMPSTAT*, 257-262.
4. Hall, P., Horowitz, J. L. and Jing, B.-Y. (1996). On blocking rules for the bootstrap with dependent data. *Biometrika* 50, 561-574.
5. Künsch, H. (1989). The jackknife and the bootstrap for general stationary observations. *Ann. Math.*, 17, 1217-1241.
6. Lahiri, S.N. (2003). *Resampling Methods for Dependent data.* Springer Verlag, Inc.
7. Leadbetter, M. R. (1974). On extreme values in stationary sequences. *Z. Wahr. Ver. Geb.* , 28, 289-303.
8. Leadbetter, M. R. (1983). Extremes and local dependence in stationary sequences. *Z. Wahr. Ver. Geb.* 65, 291-306.
9. Liu, R. Y. and Singh, K. (1992). Moving blocks jackknife and bootstrap capture weak dependence, in R. Lepage and L. Billard, eds, *Exploring the Limits of Bootstrap*, Wiley, New York, pp. 225-248.
10. Nandagopalan, S. (1990). *Multivariate Extremes and Estimation of the Extremal Index.* Ph.D. Thesis. Techn. Report 315, Center for Stochastic Processes, Univ. North-Caroline.
11. Oliveira, O. A. (2002). *In Extremis.* Ph.D. Thesis DEIO. Faculdade de Ciências da Universidade de Lisboa.
12. Politis, D. N. and Romano, J. P. (1992). A circular block resampling procedure for stationary data, in R. Lepage and L. Billard, eds, *Exploring the Limits of Bootstrap*, Wiley, New York, pp. 263-270.
13. Politis, D. N. and Romano, J. P. (1994). The Sationary Bootstrap, *J. of Amer. Stat. Assoc.*, 89, 1303-1313.
14. Singh, K. (1981). On the asymptotic accuracy of the Efron's bootstrap. *Ann. of Stat.*, 9, 345-362.