Strengthened linear formulations for the part families with precedence constraints problem

Lídia Lampreia Lourenço*

Faculdade de Ciências e Tecnologia, New University of Lisbon

Centro de Investigação Operacional, Faculdade de Ciências, University of Lisbon

email:lll@fct.unl.pt fax: 351 212 948 391

address: Faculdade de Ciências e Tecnologia - Quinta da Torre 2829-516 Caparica, Portugal

Margarida Vaz Pato*

Instituto Superior de Economia e Gestão, Technical University of Lisbon

Centro de Investigação Operacional, Faculdade de Ciências, University of Lisbon

email:mpato@iseg.utl.pt

address: Instituto Superior de Economia e Gestão - Rua do Quelhas, 6, 1200-781 Lisboa, Portugal

Abstract

The part families with precedence constraints problem (PFP) arises in industry, when flexible manufacturing systems are designed within a group technology approach. The aim of this problem is to group N parts into K families by imposing capacity constraints, concerning both the number of parts and processing times, besides precedence constraints in the building of families.

Mixed binary linear programming formulations for the PFP are presented. In endeavoring to strengthen the linear relaxations for the formulations, and hence generating better lower bounds for the optimal value of PFP, some valid inequalities based on the properties of the problem.

The lower bounds obtained by the strengthened linear relaxations significantly improved through the very weak, frequently null, bounds resulting from the original linear relaxation. Moreover, one may conclude that these models can be a useful methodology to enforce the performance of branch-and-bound for this very important problem in flexible manufacturing systems.

Keywords: flexible manufacturing systems; part families problem; precedence constraints; mixed binary linear formulations; valid inequalities.

* Research partially supported by POCTI/ISFL 152-1.

1 Introduction

The part families with precedence constraints problem arises in industrial plants that operate with flexible manufacturing systems (Stecke [21], Ng [15]). A flexible manufacturing system is designed to manufacture a wide range of products ordered, denoted by parts, each produced in small quantities. In the manufacturing of each part various types of operations are required, executed by a tool machine, following a specific sequence. For this reason, one must know the N parts (or N lots of parts) ordered and for each one, let i be the part, what tools are used to produce it, its execution time p_i, considered independent of the machine, and which parts have production priority over it. In this organization of the production process, one assumes the formation of K $(2 \leq K \leq N)$ disjoint families of parts, with the purpose of setting up families with similar manufacturing features. The part families are processed in one or more machines whose tools are stored in magazine tools. The machines may be different or otherwise, but their tool magazines always have clearly defined capacities. These specific features lead to capacity restrictions in the formation of families. One also accepts precedence restrictions in the attribution of parts to the families, insofar as, in the production of certain parts, other parts are incorporated which should already have been produced beforehand.

The optimization criterion that directs the grouping of the parts into families consists in minimizing the sum of dissimilarities among parts to be found in the same family. The dissimilarity between two parts i and j, represented by d_{ij}, results from the ratio between the number of different tools and the total number of tools involved in the production of these parts.

In short, the problem of part families with precedence constraints (PFP) requires the grouping of N parts into K or less disjoints families, subject to capacity constraints as to the number of parts and the processing time, as well as the constraints that impose precedence relationships in the formation of families, minimizing the total dissimilarity among parts within the same family.

It should be noted that the optimal solution of a PFP instance may have a number of non empty families less than K due to efect of the capacity and precedence constraints. However, should these constraints be redundant then there is at least one optimal solution formed by K non empty families (Lourenço [9]).

Part of the PFP characterization was inspired by a problem due to Kusiak [8] for which this author presented a p-median formulation. Precedence constraints were mentioned in [4] by Finke and Kusiak and in [14] by Moon, Lee and Seo. Viswanathan [23] refers to the capacity constraints imposed by the number of parts to attribute to a machine or cell of machines, noting that this parameter is indicated by the production design analyst. The PFP was also motivated by the problem proposed by Gunther et al. [6], in which the objective is to assign to the same work station operations that use common tools.

Now one exemplifies a feasible solution for the PFP instance with $N = 10$ parts, $K = 4$ families, processing times: $p_1 = 6$, $p_2 = 6$, $p_3 = 6$, $p_4 = 3$, $p_5 = 10$, $p_6 = 10$, $p_7 = 10$, $p_8 = 10$, $p_9 = 10$, $p_{10} = 10$; capacities of each family relative to the number of parts and the processing time: $M_1 = 4$, $M_2 = 2$, $M_3 = 7$, $M_4 = 2$, $T_1 = 20$, $T_2 = 5$, $T_3 = 70$, $T_4 = 4$; direct precedences among the parts (figure 1.1): $1 \prec 2$, $1 \prec 3$, $2 \prec 4$, $3 \prec 4$, $1 \prec 5$, $1 \prec 6$, $1 \prec 7$, $1 \prec 8$, $1 \prec 9$, $8 \prec 9$, $1 \prec 10$.

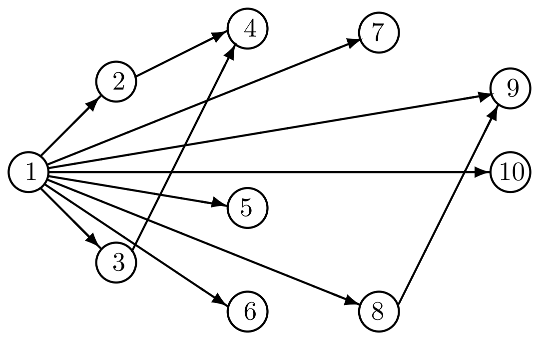

Figure 1.1: Precedence network.

A feasible solution for this problem is the following grouping: $F_1 = \{1, 2, 3\}$, $F_2 = \{4\}$, $F_3 = \{5, 6, 7, 8, 9, 10\}$ and $F_4 = \{\}$, represented in figure 1.2.

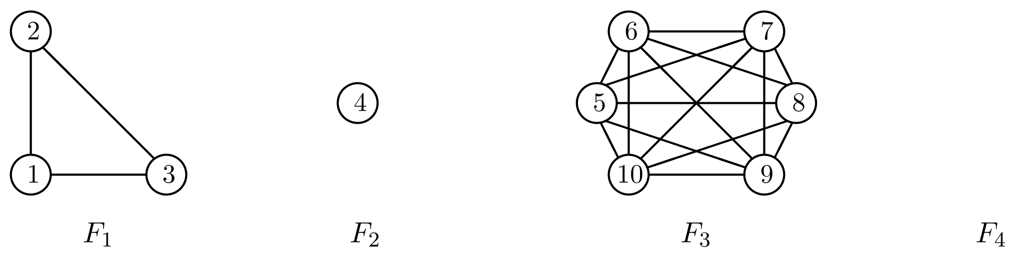

Figure 1.2: A feasible solution for the above instance.

This is a very difficult problem. In fact, it was proved to be NP-hard (Lourenço [9]) because the clustering problem is a particular case of it. It was heuristically solved in Lourenço and Pato [10] and formulations where studied in Lourenço [9]. In order to evaluate the quality of such solutions and also to improve the performance of branch and bound procedures applied to some of these formulations, lower bounds were developed and are presented in this paper.

In section 2 one presents a mixed binary linear formulation with aggregated variables for the PFP, denoted by BF1, followed, in section 3, by some results concerning the linear relaxation of BF1. In section 4 a preprocessing of the solution is carried out by fixing the value of some of the variables. Valid inequalities were developed to strengthen the above related mixed binary linear formulation and one presented in section 5. Section 6 contains a mixed binary linear formulation with disaggregated variables for the PFP. The proofs for the results presented in this paper may be consulted in Lourenço [9]. Finally, section 7 describes the computational experiment and section 8 is devoted to the conclusions.

2 Mixed binary linear formulation - BF1

The PFP may be formulated as a mixed binary linear programming problem with aggregated variables (Lourenço and Pato [11]) with the following indexes, parameters and variables:

i, j - part indexes

k - family index

N - number of parts ($N \in \mathbb{N}$)

K - maximum number of families ($K \in \mathbb{N}, 2 \leq K \leq N$)

M_k - maximum number of parts for family k ($M_k \in \mathbb{N}, M_k \leq N$)

p_i - production time of part i ($p_i \in \mathbb{R}_0^+$)

T_k - maximum production time for family k ($T_k \in \mathbb{R}^+$)

g_{ij} - direct precedence relationships between parts i and j, $i < j$, ($g_{ij} \in \{0, 1\}$)

d_{ij} - dissimilarity between parts i and j, element of the symmetric matrix D which diagonal elements are zero ($d_{ij} \in \mathbb{R}_0^+$)

x_{ik} - binary variable which indicates whether part i is in family k (=1) or not (=0), ($i = 1, \ldots, N$; $k = 1, \ldots, K$);

y_{ij} - binary variable which indicates whether parts i and j are in the same family (=1) or not (=0), ($i < j$; $i = 1, \ldots, N-1$; $j = i+1, \ldots, N$).

The formulation, denoted by BF1, follows:

$$\text{Min} \quad \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} d_{ij} y_{ij} \tag{2.1}$$

$$\text{s. to} \quad y_{ij} \geq x_{ik} + x_{jk} - 1 \qquad i = 1, \ldots, N-1;\ j = i+1, \ldots, N;\ k = 1, \ldots, K \tag{2.2}$$

$$\sum_{k=1}^{K} x_{ik} = 1 \qquad i = 1, \ldots, N \tag{2.3}$$

$$\sum_{k=1}^{K} k x_{ik} \leq \sum_{k=1}^{K} k x_{jk} \qquad (i,j) : g_{ij} = 1 \tag{2.4}$$

$$\sum_{i=1}^{N} x_{ik} \leq M_k \qquad k = 1, \ldots, K \tag{2.5}$$

$$\sum_{i=1}^{N} p_i x_{ik} \leq T_k \qquad k = 1, \ldots, K \tag{2.6}$$

$$x_{ik} = 0, 1 \qquad i = 1, \ldots, N; k = 1, \ldots, K \tag{2.7}$$

$$0 \leq y_{ij} \leq 1 \qquad i = 1, \ldots, N-1;\ j = i+1, \ldots, N. \tag{2.8}$$

The objective function (2.1) represents the total dissimilarity, that is, the sum of the values of dissimilarity between pairs of parts placed in the same family. Constraints (2.2) force each variable y_{ij} in the optimal solution to take the value of 1 if i and j are in the same family. Should these parts be in different families in the optimal solution, the variable y_{ij} contributes with the value 0 to the objective function. The set of constraints (2.3) forces each part to belong to one and only one family. The precedence constraints (2.4), defined for the whole pair of parts (i, j) such that i directly precedes j, guarantee that part j cannot be placed in a family whose index is inferior than that of the family to which part i belongs. Capacity constraints (2.5) and (2.6) do not allow violation of the limits referring to the number of parts and the processing time of each family, respectively.

Note that this linear formulation for the problem of the part families with precedence constraints was used by Aronson and Klein [1] for a classification problem arising in information systems outlined to support software design.

3 Linear Relaxation

The linear relaxation of BF1, denoted by $\overline{\text{BF1}}$, is the problem resulting from elimination of the integrality of the variables x_{ik}, that is, by substituting $x_{ik} \in \{0, 1\}$ by $0 \leq x_{ik} \leq 1$, with $i = 1, \ldots, N; k = 1, \ldots, K$.

In some cases, an analysis of the various parameters of an instance of BF1 enables one to verify that the optimal value of its linear relaxation is null. The three results that follow contain suficient conditions for the optimal value of linear relaxation to be null.

The first refers to linear relaxation of the particular case of BF1 in which the capacity constraints are redundant. It also proves that the $\overline{\text{BF1}}$ has a null optimal value for instances with equal capacities or, in certain cases, with different capacities.

Result 1. The linear relaxation of any BF1 instance:

a) where the capacity constraints are redundant;

or

b) that is not infeasible and such that the maximum number of parts and the maximum processing time is equal for all the families;

or

c) that is not infeasible and such that $\sum_{k=1}^{K} \bar{x}_{ik} \geq 1$ $(i = 1, \ldots, N)$, with vector $\bar{x}$ defined as follows:

where $min_k = \min\{\frac{M_k}{N}, \frac{T_k}{\sum_{i=1}^{N} p_i}\}$

$$\bar{x}_{ik} = \begin{cases} min_k & \text{if} \quad min_k < 0.5 \\ & \\ 0.5 & \text{if} \quad min_k \geq 0.5 \end{cases} \qquad (i = 1, \ldots, N; k = 1, \ldots, K)$$

has null value.

The following result states a necessary and sufficient condition for the optimal value of the $\overline{\text{BF1}}$ to be positive.

Result 2. The optimal value of the linear relaxation of a BF1 instance is positive if and only if in any feasible solution $(\bar{x}, \bar{y})$ of $\overline{\text{BF1}}$ $\exists i, j \in \{1, \ldots, N\}$, where $i < j$ and $\exists k \in \{1, \ldots, K\}$ such that $\bar{x}_{ik} + \bar{x}_{jk} > 1$ and $d_{ij} > 0$.

It follows, from the previous results, that the linear relaxation lower bounds will be weak and, in most practical cases, null.

4 Preprocessing

Taking advantage of the capacities of families and of the network of PFP precedences, one finds that, a priori, the value of certain variables in any feasible solution of the BF1 formulation can be determined.

Let $G = (V, E)$ the oriented network that represents the precedence relationships between the parts of PFP, where V is the set of vertices (parts) and E the set of arcs defined by the precedence matrix $[g_{ij}]$: vertices i and j define the initial and final extremities, respectively, of an arc if there exists a direct precedence relationship of i in relation to j ($i \prec j$), that is, $g_{ij} = 1$. If, in network G, there is a path of vertex i to j one may say that there is a transitive precedence relationship of i to j, represented by $i \prec\prec j$.

In any feasible solution of BF1 (x, y) the following rules can be applied in order to fix a priori the values of specific variables.

Rule 1. In any feasible solution of BF1 (x, y)

$$x_{ik} = 0 \qquad i = 1, \ldots, N;\ k = 1, \ldots, K\text{: } p_i > T_k \tag{4.1}$$

$$x_{ik} = 0 \ \text{ and } \ x_{ik'} = 1 \qquad i = 1, \ldots, N;\ k = \{1, \ldots, K\};\ \exists k' : p_i > T_k \text{ and } p_i < T_{k'} \tag{4.2}$$

$$y_{ij} = 0 \qquad i = 1, \ldots, N-1;\ j = i+1, \ldots, N\text{: } p_i + p_j > T^*. \tag{4.3}$$

Rule 2. Let i and j be a pair of parts such that $i \prec\prec j$. If $|Cam_{ij}| + 2 > M^*$ or if $(\sum_{l \in Cam_{ij}} p_l) + p_i + p_j > T^*$ then, in any feasible solution of BF1 (x, y)

$$y_{ij} = 0;\ x_{iK} = 0;\ x_{j1} = 0 \tag{4.4}$$

$$y_{ir} = 0 \qquad r : j \prec\prec r \tag{4.5}$$

$$y_{sj} = 0 \qquad s : s \prec\prec i \tag{4.6}$$

$$x_{ik} + x_{jk} \leq 1 \qquad k = 1, \ldots, K \tag{4.7}$$

where
$Cam_{ij} = \{l \in \{1, \ldots, N\} : i \prec\prec l \text{ and } l \prec\prec j\}$, $M^* = max\{M_1, M_2, \ldots, M_K\}$ and
$T^* = max\{T_1, T_2, \ldots, T_K\}$.

Aronson and Klein [1] developed a preprocessing result by imposing in constraint (2.3) a lower bound L_i and upper bound U_i on the index of the family in which each part can be placed. Next Rule 3 presents this issue for the BF1.

Rule 3. In any feasible solution of BF1 (x, y) one has

$$x_{ik} = 0 \qquad i = 1, \ldots, N;\ k = 1, \ldots, K\colon\ k < L_i \text{ or } k > U_i \tag{4.8}$$

where $U_i = \min\{u_i^1, u_i^2\}$ and $L_i = \min\{l_i^1, l_i^2\}$ with

$u_i^1 = \max\{g = K, \ldots, 1 : p_i + \sum_{j \in TSuc(i)} p_j - \sum_{k=g}^{K} T_k \leq 0\}$

$u_i^2 = \max\{g = K, \ldots, 1 : 1 + |TSuc(i)| - \sum_{k=g}^{K} M_k \leq 0\}$

$l_i^1 = \max\{g = 1, \ldots, K : p_i + \sum_{j \in TPred(i)} p_j - \sum_{k=g}^{K} T_k \leq 0\}$

$l_i^2 = \max\{g = 1, \ldots, K : 1 + |TPred(i)| - \sum_{k=g}^{K} M_k \leq 0\}$

and $TPred(i)$ and $TSuc(i)$ are, respectively, the set of transitive predecessors and transitive successors of part i.

5 Valid Inequalities

The particular characteristics of the BF1 formulation for the PFP, in particular, the different constraint systems associated with it, allow one to establish different types of valid inequalities. These constraints, once incorporated in this mixed binary linear formulation, can strengthen the respective linear relaxation lower bound.

The first valid inequalities presented are deduced from the grouping characteristics and from the dissimilarity function. There follows a study of the inequalities obtained from the precedence and from the capacity constraints.

5.1 Inequalities based on the grouping characteristics

Valid inequalities can be deduced because the PFP, regarded as an optimization problem on the network R, may be considered a partition of a complete graph in no more than K cliques with minimum total dissimilarity and subject to additional constraints. Here, the convex hull of the feasible region of BF1 is represented by $conv(F(\text{BF1}))$, where the feasible region is denoted by $F(\text{BF1})$.

Result 3. The following constraint is a valid inequality for $conv(F(\text{BF1}))$:

$$l_{min} \leq \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} y_{ij} \tag{5.1.1}$$

where

$$l_{min} = \begin{cases} \lfloor \frac{N}{K} \rfloor (N - \frac{K}{2}(\lfloor \frac{N}{K} \rfloor + 1)) & \text{if} \quad \lfloor \frac{N}{K} \rfloor \geq 2 \\ N - K & \text{if} \quad \lfloor \frac{N}{K} \rfloor < 2. \end{cases} \tag{5.1.2}$$

There follows another type of constraint, which also only concerns variables y_{ij}.

Result 4. Considering three parts i, j e t, the following constraints, triangular inequalities, are valid inequalities for $conv(F(\text{BF1}))$:

$$y_{ij} \geq y_{ti} + y_{tj} - 1 \qquad t < i < j \tag{5.1.3}$$

$$y_{ij} \geq y_{it} + y_{tj} - 1 \qquad i < t < j \tag{5.1.4}$$

$$y_{ij} \geq y_{it} + y_{jt} - 1 \qquad i < j < t. \tag{5.1.5}$$

It should be mentioned that Grötschel and Wakabayashi [5] used the triangular inequalities as constraints to formule the partition problem of a network with N vertices into cliques. These inequalities are cited in the paper of Park, Lee and Park [17] as facets for the convex hull of the set of feasible solutions of a formulation for the maximum weight clique problem with a capacity constraint in the number of vertices.

5.2 Inequalities from precedence constraints

In the next result conditions are imposed which guarantee that, in the optimal solution, either the index family 1 or the index K family is not empty.

Result 5. Consider a PFP:

a) with at least one feasible solution, where the 1 index family is not empty, the following constraint is satisfied by at least one optimal solution of BF1:

$$\sum_{i \in Spred} x_{i1} \geq 1 \qquad (5.2.1)$$

b) with at least one feasible solution, where the K index family is not empty, the following constraint is satisfied by at least one optimal solution of BF1:

$$\sum_{i \in Ssuc} x_{iK} \geq 1. \qquad (5.2.2)$$

The Result 6 was developed by relating parts pairwise and considering the precedence constraints between parts

Result 6. Consider five parts i, j, l, t and u such that $i \prec\prec j \prec\prec t \prec\prec u$ and $i \prec\prec l \prec\prec t$, the following constraints are valid inequalities for $conv(F(\text{BF1}))$:

$$x_{j1} \leq x_{i1}; \qquad x_{iK} \leq x_{jK} \qquad (5.2.3)$$

$$(x_{ik} + x_{tk}) - x_{jk} \leq 1 \qquad k = 1, \ldots, K \qquad (5.2.4)$$

$$y_{ij} \geq y_{it}; \qquad y_{jt} \geq y_{it}; \qquad y_{it} \leq y_{jl} \qquad (5.2.5)$$

$$y_{ij} + y_{jt} \geq x_{ik} + x_{jk} + x_{tk} - 1 \qquad k = 1, \ldots, K \qquad (5.2.6)$$

$$y_{ij} + y_{jt} + y_{tu} \geq x_{ik} + x_{jk} + x_{tk} + x_{uk} - 1 \qquad k = 1, \ldots, K. \qquad (5.2.7)$$

5.3 Inequalities from capacities

The first two groups of inequalities presented in the Result 7 relate one part with all the others, through the variables y_{ij}. Note that these variables y_{ij} do not identify the family to which the parts are assigned, which is why one chooses the capacity of the family possessing with the greatest capacity. One knows that the lifting operation is performed on the coefficients of the variables of a constraint with a view to obtaining a valid inequality that dominates the inequality from which it resulted (Wolsey [24]). In this same result, the following group of inequalities involves the lifting of the coefficients of variables x_{ik} in the capacity constraints (2.6) of the formulation BF1. This type of inequalities was developed for the generalized assignment problem by Farias and Nemhauser [3].

Result 7. The following constraints are valid inequalities for $conv(F(\text{BF1}))$:

$$\sum_{i=1}^{j-1} y_{ij} + \sum_{i=j+1}^{N} y_{ji} \leq M^* - 1 \qquad j = 1, \ldots, N \qquad (5.3.1)$$

$$\sum_{i=1}^{j-1} p_i y_{ij} + \sum_{i=j+1}^{N} p_i y_{ji} \leq T^* - p_j \qquad j = 1, \ldots, N \qquad (5.3.2)$$

$$p_{i_1}x_{i_1 1} + (T_k - p_{i_2})x_{i_1 2} + \ldots + (T_k - p_{i_2})x_{i_1 k} + \ldots +$$
$$(T_k - p_{i_2})x_{i_1 K} + p_{i_2}x_{i_2 1} + (T_k - p_{i_1})x_{i_2 2} + \ldots + (T_k - p_{i_1})x_{i_2 k} +$$
$$+ \ldots + (T_k - p_{i_1})x_{i_2 K} \leq T_k \qquad i_1, i_2 = 1, \ldots, N\colon\ p_{i_1} + p_{i_2} \geq T_k \qquad (5.3.3)$$

with $M^* = max\{M_1, M_2, \ldots, M_K\}$ and $T^* = max\{T_1, T_2, \ldots, T_K\}$.

6 Mixed binary linear formulation - BF2

The introduction of yet another index in the variables y_{ij} disaggregated them, thus allowing one to inform what the family index is in which the two parts i and j are grouped. In this way, there appear the variables y_{ijk}, $\forall i < j$, $i, j \in \{1, \ldots, N\}$ $\forall k \in \{1, \ldots, K\}$ which characterize an extended formulation (Pulleyblank [18]) for the PFP. Though it has more variables and more constraints, this will permit the development of valid inequalities which dominate those obtained by substituting the y_{ij} by the sums of y_{ijk}.

The variables used in BF2 and the formulation itself are now defined:

y_{ijk} - binary variable which indicates whether parts i and j are in the same family k (=1) or not (=0) ($i = 1, \ldots, N-1$; $j = i+1, \ldots, N$; $k = 1 \ldots, K$);

$$\text{Min} \quad \sum_{k=1}^{K} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} d_{ij} y_{ijk} \qquad (6.1)$$

s. to

$$y_{ijk} \geq x_{ik} + x_{jk} - 1 \qquad i = 1, \ldots, N-1;\ j = i+1, \ldots, N;\ k = 1, \ldots, K \qquad (6.2)$$

constraints (2.3) to (2.7)

$$0 \leq y_{ijk} \leq 1 \qquad i = 1, \ldots, N-1;\ j = i+1, \ldots, N;\ k = 1, \ldots, K. \qquad (6.3)$$

It is proved that the optimal value of the linear relaxation of BF2 is equal to the optimum $\overline{\text{BF1}}$. Hence, in most cases, the linear relaxation bound from BF2 is expected to be null or of very poor quality, like the linear bound from BF1. Then, valid inequalities were studied in order to strengthen the optimal value of the linear relaxation of BF2.

The valid inequalities for the set of feasible solutions of BF1 which involve only variables x_{ik} can be directly included in BF2. As for the valid inequalities developed for BF1 which involve variables y_{ij}, they require an adaptation to BF2, which may consist in substituting y_{ij} by $\sum_{k=1}^{K} y_{ijk}$ or simply substituting variables y_{ij} by y_{ijk} or else, in other deductions performed with the purpose of benefiting the optimum of the linear relaxation of this extended formulation BF2.

Note that, in BF2 one may deduce constraints by making use of the exact values of the capacity limits as regards the number of parts and processing time for each family k, instead of the maximum of all the capacities, as in the case of BF1.

Result 8. The following constraints are valid inequalities for $conv(F(\text{BF2}))$:

$$\sum_{i=1}^{j-1} y_{ijk} + \sum_{i=j+1}^{N} y_{jik} \leq (M_k - 1)x_{jk} \qquad j = 1, \ldots, N;\ k = 1, \ldots, K \tag{6.4}$$

$$\sum_{i=1}^{j-1} p_i y_{ijk} + \sum_{i=j+1}^{N} p_i y_{jik} \leq (T_k - p_j)x_{jk} \qquad j = 1, \ldots, N;\ k = 1, \ldots, K \tag{6.5}$$

$$y_{ijk} \leq x_{ik} \qquad i = 1, \ldots, N-1;\ j = i+1, \ldots, N;\ k = 1, \ldots, K \tag{6.6}$$

$$y_{ijk} \leq x_{jk} \qquad i = 1, \ldots, N-1;\ j = i+1, \ldots, N;\ k = 1, \ldots, K \tag{6.7}$$

$$\sum_{i=1}^{N} x_{ik} - \sum_{\substack{i=1 \\ i<j}}^{j-1} y_{ijk} - \sum_{\substack{i=j+1 \\ i>j}}^{N} y_{jik} \leq M_k + (1 - M_k)x_{jk} \qquad j = 1, \ldots, N;\ k = 1, \ldots, K \tag{6.8}$$

$$\sum_{i=1}^{N} p_i x_{ik} - \sum_{\substack{i=1 \\ i<j}}^{j-1} p_i y_{ijk} - \sum_{\substack{i=j+1 \\ i>j}}^{N} p_i y_{jik} \leq T_k + (p_j - T_k)x_{jk} \qquad j = 1, \ldots, N;\ k = 1, \ldots, K. \tag{6.9}$$

Inequalities similar to (6.4)-(6.5) were deduced for the maximal clique problem with a capacity constraint in the number of vertices (Park, Lee and Park [17], Macambira and Souza [13], Hunting, Faile and Kern [7]). These define facets for the polyhedron of the maximal clique problem with a capacity constraint if and only if $b \leq N - 1$, where b is the capacity of the clique and N the number of vertices of the complete network. It should be noted that the inequalities (6.4)-(6.5) are an adaptation of the (5.3.3) – (5.3.4) developed for BF1 but these dominate the inequalities deduced from the mere introduction of the index k, that is, substitution of y_{ij} by y_{ijk}.

The last four types of inequalities presented are specifically valid only for F(BF2): in BF2 the variables x_{ik} and y_{ijk} are defined for a given family and for this reason it is possible to establish a relationship between the two types of variables.

Finally, one should mention that the formulation BF2 strengthened by the valid inequalities (6.4) $-$ (6.9) coincides with the mixed binary linear formulation resulting from application of the hierarchy linear reformulation technique due to Sherali and Adams [20] to a quadratic formulation for the PFP given in Lourenço [9].

7 Computational Experiments

Although the formation of part families within flexible manufacturing systems amounts to a problem whose application is real, and is often mentioned in the literature, it has not been object of publication in the literature, as far as the authors' knowledge is concerned. In view of the difficulty in ascertaining what the dimensions of the instances are and what are the appropriate values for the parameters for a real application of the problem, the computational experiment is undertaken with instances that claim to represent different situations.

Two sets, each consisting of 25 instances each, were built partly from the literature concerning the assembly line balancing problem in industry (Scholl [19] and [16]). This option was made in view of the fact that data is available on the number of tasks (parts) to be assigned and on the number of work stations (families), the network of precedences between the tasks and the execution times for the tasks. The data referring to the capacities of families and the dissimilarities between parts were randomly generated, using the random function of the Pascal programming language (Lourenço [9]). In the first set, A - instances I_1 to I_{25} -, each instance has equal capacities for all the families, as regards the number of parts, and also the processing time. In the second set, B - instances I_{26} até I_{50} -, each instance has different capacities for the various families. The capacities were generated and then adapted to give rise to feasible problems.

In Table 7.1 the data is displayed. This table also indicates the order strength value, that is, the ratio between the total number of direct and transitive precedences and $\frac{N(N-1)}{2}$, the maximum number of direct precedences. This value tells us if the network of precedence constraints is sparse

($0 \leq orderstrength \leq 0.5$) or dense ($0.5 < orderstrength \leq 1$).

Table 7.1: Data regarding the test instances

instances / parameters	Mertens $I_1;I_{26}$	Bowman $I_2;I_{27}$	Jaeschke $I_3;I_{28}$	Jackson $I_4;\ I_{29}$	Mansoor $I_5;I_{30}$
N	7	8	9	11	11
K	2	5	3	4	3
no. of direct precedence relationships	6	8	11	13	11
part's processing time	1-6	3-17	1-6	1-7	2-45
maximum no. of parts per family	5;4	5-7;2	3-8;3	6-9;3	6-8;4
maximum time per family	14-20;17	23-47;17	15-17;13	15-21;12	125-149;62
dissimilarity between two parts	0.1-1.0	0.1-1.0	0.1-1.0	0.1-1.0	0.1-1.0
order strength	0.52	0.75	0.83	0.58	0.60
instances / parameters	Mitchell $I_6;I_{31}$	Roszieg $I_7;I_{32}$	Heskia $I_8;I_{33}$	Buxey $I_9;I_{34}$	Sawyer $I_{10};\ I_{35}$
N	21	25	28	29	30
K	5	6	5	7	7
no. of direct precedence relationships	27	32	39	36	32
part's processing time	1-9	1-13	1-108	19-21	1-25
maximum no. of parts per family	4-10;5	5-7;5	6-10;6	5-10;5	6-8;5
maximum time per family	29-44;21	27-42;21	393-590;205	59-83;47	53-84;47
dissimilarity between two parts	0.1-1.0	0.1-1.0	0.1-1.0	0.1-1.0	0.1-1.0
order strength	0.70	0.70	0.22	0.51	0.44
instances / parameters	Lutz1 $I_{11};I_{36}$	Gunther $I_{12};I_{37}$	Kilbridge $I_{13};I_{38}$	Hahn $I_{14};\ I_{39}$	Warnecke $I_{15};\ I_{40}$
N	32	35	45	53	58
K	7	8	8	8	10
no. of direct precedence relationships	38	45	62	82	70
part's processing time	100-1400	1-30	3-29	40-1775	7-52
maximum no. of parts per family	6-9;5	7-11;5	4-12;7	9-12;7	10-12;8
maximum time per family	3585-4644;2096	89-138;54	115-158;69	3604-5153;1907	111-249;155
dissimilarity between two parts	0.1-1.0	0.1-1.0	0.1-1.0	0.1-1.0	0.1-1.0
order strength	0.83	0.59	0.45	0.84	0.59
instances / parameters	Tonge $I_{16};I_{41}$	Wee-Mag $I_{17};I_{42}$	Arc83 $I_{18};I_{43}$	Lutz2 $I_{19};I_{44}$	Lutz3 $I_{20};\ I_{45}$
N	70	75	83	89	89
K	23	30	21	15	24
no. of direct precedence relationships	86	87	113	118	118
part's processing time	1-152	5-27	233-2881	1-10	1-74
maximum no. of parts per family	5-10;4	5-10;4	6-10;5	4-12;8	5-10;5
maximum time per family	324-457;162	55-102;72	7886-10713;4068	21-49;33	152-213;76
dissimilarity between two parts	0.1-1.0	0.1-1.0	0.1-1.0	0.1-1.0	0.1-1.0
order strength	0.59	0.23	0.59	0.77	0.77

instances parameters	Mukherje I_{21};I_{46}	Arc111 I_{22};I_{47}	Barthold1 I_{23};I_{48}	Barthold2 I_{24};I_{49}	Scholl I_{25};I_{50}
N	94	111	148	148	297
K	19	27	14	40	50
no. of direct precedence relationships	181	176	175	175	300
part's processing time	8-123	10-5200	7-811	5-383	9-1386
maximum no. of parts per family	8-11;6	6-11;7	11-19;11	10-19;4	11-19;11
maximum time per family	354-549;549	7702-10172;19413	789-1156;1016	171-268;176	2808-4136;9545
dissimilarity between two parts	0.1-1.0	0.1-1.0	0.1-1.0	0.1-1.0	0.1-1.0
order strength	0.45	0.40	0.26	0.26	0.04

Besides these instances, four other instances sets were tested (Lourenço [12]), three of which were randomly generated for a grouping problem in the information systems. The fourth concerns the assembly line balancing problem (Tonge [22]). The values for the parameters of these instances were taken from Aronson and Klein [1]. However, the authors opted against presenting the results of these tests here as the instances had parameters that were very similar to some of the instances belonging to the sets A and B characterized above, besides the fact that the computational results were similar.

The computational experiment was performed on a PC with a Pentium 4 processor, with 2,53 GHZ and 512 Mb of RAM memory. To solve the continuous or mixed binary linear models, one used CPLEX software version 8.1 [2], in which all the parameters assumed pre-defined values, with the exception of the time limit parameter used in the search for the optimum solution.

The lower bounds obtained from the linear relaxations $\overline{BF1}$ and $\overline{BF2}$ for all the instances, except the I_{26}, are null. The optimal value of both linear relaxations for this instance is 0.31, while the minimum total dissimilarity equals 3.9. One may conclude from this experiment that this linear lower bound for the PFP is very weak indeed. Hence,So the strengthened formulations were tested with a view to improving the lower bounds.

As to the effect of the order strength of the instances on the computational results, one found that the greater the order strength the greater the proportion of variables with a fixed value through the preprocessing phase, application of the Rules (1), (2), and (3).

Table 7.2 displays data, results and execution times referring to each instance given in column (1). Column (2) lists the number of parts and column (3) the number of families. In column (4) there is information on the optimal value obtained by application of the CPLEX software to the formulation BF1. As only for the small instances does the CPLEX reach the optimum, in the remaining instances one points out the upper bound with ** and the lower bound with *. Column (5) records the results obtained from strengthening formulation BF1 with preprocessing and all valid inequalities developed - Rules (1)-(3) and Results (3)-(7)-, denoted by BF1cut, and in column (6) the results of the respective linear relaxation, $\overline{\text{BF1cut}}$ appear. Column (7) was built from BF2 and columns (8) and (9) contain, respectively, the figures obtained from BF2 strengthened by all the valid inequalities developed (Results (3)-(7) adapted to BF2 and Result (8), BF2cut, and the results from the respective linear relaxation, $\overline{\text{BF2cut}}$.

Table 7.2: Results of the computacional experiment

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
instances	N	K	BF1 optimal or lower*,upper** (time)	BF1cut optimal or lower*,upper** (time)	$\overline{\text{BF1cut}}$ lower (time)	BF2 optimal or lower*,upper** (time)	BF2cut optimal or lower*,upper** (time)	$\overline{\text{BF2cut}}$ lower (time)
I_1	7	2	3.9 (0.0 s)	3.9 (0.1 s)	3.9 (0.0 s)	3.9 (0.0 s)	3.9 (0.0 s)	3.9 (0.0 s)
I_2	8	5	1.7 (0.0 s)	1.7 (0.0 s)	1.7 (0.0 s)	1.7 (0.1 s)	1.7 (0.0 s)	1.7 (0.1 s)
I_3	9	3	5.5 (0.0 s)	5.5 (0.0 s)	5.5 (0.0 s)	5.5 (0.0 s)	5.5 (0.0 s)	5.5 (0.0 s)
I_4	11	4	3.3 (0.3 s)	3.3 (0.0 s)	3.3 (0.0 s)	3.3 (0.3 s)	3.3 (0.1 s)	3.3 (0.0 s)
I_5	11	3	5.0 (34.1 s)	5.0 (0.5 s)	4.0 (0.0 s)	5.0 (0.1 s)	5.0 (0.3 s)	4.2 (0.1 s)
I_6	21	5	17.0 (34.1 s)	17.0 (4.9 s)	14.9 (0.3 s)	17.0 (17.4 s)	17.0 (5.8 s)	16.2 (0.9 s)
I_7	25	6	18.4 (1.6 h)	18.4 (51.0 s)	14.6 (1.2 s)	12.5*,18.4** (47.1 s)	18.4 (396.2 s)	15.1 (4.5 s)
I_8	28	5	22.9 (7.5 h)	19.6*,24.5** (10.0 h)	16.1 (3.1 s)	11.3*,24.6** (1.0 h)	16.5*,24.6* (1.0 h)	16.2 (52.2 s)
I_9	29	7	16.0*,18.0** (10.0 h)	14.8*,18.6** (10.0 h)	11.7 (3.0 s)	10.1*,19.5** (1.0 h)	9.4*,19.5** (1.0 h)	11.9 (60.0 s)
I_{10}	30	7	11.4*,20.5** (10.0 h)	16.1*,22.0** (10.0 h)	11.5 (1.7 s)	5.8*,21.9** (1.0 h)	11.4*,20.0** (1.0 h)	12.4 (56.6 s)
I_{11}	32	7	21.3*,28.5** (10.0 h)	27.5,-** (10.0 h)	23.2 (3.7 s)	5.8*,- (1.0 h)	27.5 (123.9 s)	24.3 (436.0 s)
I_{12}	35	8	8.7*,24.3** (10.0 h)	16.6*,27.0** (10.0 h)	14.2 (18.5 s)	22.6*,26.4** (1.0 h)	8.6*,- (1.0 h)	14.2 (453.0 s)
I_{13}	45	8	3.8*,48.3** (10.0 h)	26.3*,- (10.0 h)	23.9 (10.7 s)	11.6*,55.1** (1.0 h)	19.1*,- (1.0 h)	24.7 (0.8 h)

I_{14}	53	8	4.8*,- (10.0 h)	65.8*,78.6** (10.0 h)	60.0 (557.6 s)	0.3*,- (1.0 h)	44.9*,83.2** (1.0 h)	62.7 (1234.5 s)
I_{15}	53	10	-,- (10.0 h)	37.8*,- (10.0 h)	35.9 (175.7 s)	3.7*,- (1.0 h)	-,- (1.0 h)	36.7 (3.8 h)
I_{26}	7	2	3.9 (0.0 s)	3.9 (0.1 s)	3.9 (0.0 s)	3.9 (0.0 s)	3.9 (0.0 s)	3.9 (0.0 s)
I_{27}	8	5	1.3 (0.0 s)	1.3 (0.1 s)	0.7 (0.0 s)	1.3 (0.0 s)	1.3 (0.0 s)	0.7 (0.0 s)
I_{28}	9	3	5.5 (0.1 s)	5.5 (0.0 s)	5.5 (0.0 s)	5.5 (0.1 s)	5.5 (0.0 s)	5.5 (0.0 s)
I_{29}	11	4	3.3 (0.3 s)	3.3 (0.1 s)	2.9 (0.0 s)	3.3 (0.3 s)	3.3 (1.0 s)	3.1 (0.1 s)
I_{30}	11	3	4.4 (0.1 s)	4.4 (0.5 s)	3.9 (0.1 s)	4.4 (0.1 s)	4.4 (1.0 s)	3.9 (0.1 s)
I_{31}	21	5	15.4 (33.1 s)	15.4 (96.3 s)	13.7 (1.7 s)	15.4 (17.7 s)	15.4 (119.0 s)	13.8 (6.3 s)
I_{32}	25	6	17.4 (0.6 h)	17.4 (0.9 h)	13.7 (3.2 s)	17.4 (1854.5 s)	14.4*,18.6** (1.0 h)	13.8 (5.5 s)
I_{33}	28	5	13.6*,23.3** (10.0 h)	20.5*,22.2** (10.0 h)	15.9 (27.6 s)	12.9*,24.2** (1.0 h)	16.0*,27.8** (1.0 h)	15.9 (47.6 s)
I_{34}	29	7	15.9*,17.8** (10.0 h)	13.8*,18.5** (10.0 h)	11.3 (4.7 s)	9.9*,18.9** (1.0 h)	8.6*,19.8** (1.0 h)	11.4 (136.0 s)
I_{35}	30	7	14.8*,18.4** (10.0 h)	13.2*,19.4** (10.0 h)	10.3 (4.2 s)	8.7*,19.9** (1.0 h)	8.6*,22.6** (1.0 h)	10.4 (98.0 s)
I_{36}	32	7	17.1*,24.1** (10.0 h)	23.7 (7.7 h)	20.0 (18.0 s)	22.4*,23.7** (1.0 h)	11.1*,31.1** (1.0 h)	20.0 (109.4 s)
I_{37}	35	8	11.1*,25.0** (10.0 h)	14.9*,30.1** (10.0 h)	13.7 (45.4 s)	24.5*,48.4** (1.0 h)	8.1*,29.7** (1.0 h)	13.7 (418.4 s)
I_{38}	45	8	6.1*,46.1** (10.0 h)	24.2*,66.0** (10.0 h)	22.6 (102.6 s)	1.9*,48.0** (1.0 h)	15.2*,- (1.0 h)	22.7 (1487.7 s)
I_{39}	53	8	4.9*,- (10.0 h)	57.0*,118.1** (10.0 h)	56.5 (1054.0 s)	0.6*,- (1.0 h)	26.5*,- (1.0 h)	56.8 (7.1 h)
I_{40}	53	10	-,- (10.0 h)	33.3*,80.5** (10.0 h)	31.9 (467.0 s)	22.1*,- (1.0 h)	24.4*,- (1.0 h)	33.6 (4.1 h)

Firstly, one began by testing the formulations without and with strengthening, using the branch-and-bound algorithm of the CPLEX software and verified that the strengthened formulations (BF1cut and BF2cut in columns (6) and (9) of table 7.2), for the medium size instances, generated bounds for the optimum of PFP that are significantly better than those given by the corresponding basic formulations BF1 and BF2 (columns (5) and (8)). However, for the larger instances it is the non-strengthened formulation that manages to obtain better bounds for the same computational time. This is due to the excess of variables and constraints of the formulations with strengthening which require an additional computational effort.

Moreover, in Table 7.2 one may see that the optima given by the linear relaxations of the strengthened formulations (columns (8) and (10)) significantly improve the initial null values (which do not appear in the table). Additionally, one finds that the $\overline{\text{BF2cut}}$ produces lower bounds that are appreciably better than those obtained from $\overline{\text{BF1cut}}$, and this difference between the lower bounds is slightly more marked and frequent in the case of instances with equal capacity for the families (for instance (I_6)). Note that the strengthening includes in BF2cut inequalities which have no correspondent in BF1cut, hence forcing a best lower bound from $\overline{\text{BF2cut}}$.

In the computational experiments it was found that, of the various valid inequalities studied, the one that has greatest importance in the improvement of the lower bounds is valid inequality (5.1.1). This inequality is also beneficial, in relation to the remaining ones, in that it can be adapted to any problem of grouping elements, with or without constraints.

8 Conclusions

An analysis of the PFP and of its properties lead to perform a preprocessing of the solution, by fixing the value of a large number of variables. In the experimentation undertaken, this preprocessing benefits the application of the CPLEX software to the strengthened formulations BF1 and BF2, specially, in the case of the higher dimension instances.

Several valid inequalities were also developed to strength the formulations. For the smaller instances one may conclude that the strengthened formulations enable the ILP algorithm of CPLEX to perform better in determining the optimum, as they require less time. For the medium-sized instances these formulations found bounds (upper and lower) for the optimum in relatively short time. Finally, for the larger instances, the BF1 with strengthening obtains better lower bounds than the BF2, which is not appropriate due to it has an excessive consumption of computational resources.

The experiments indicate that, by using more computational resources and, possibly, with a more discerning study of the constraints to include for strengthened, the disaggregated formulation, duly strengthened with valid inequalities will be the basis of a solution methodology for this important, though difficult problem that may be used within the flexible manufacturing systems.

References

[1] Aronson, J. E. and Klein, G. (1989). "A clustering algorithm for computer assisted process organization". *Decision Sciences*, 20, pp. 730-745.

[2] CPLEX Manual, (Version 8.2). *Using the* $CPLEX^R$ *callable library* and $CPLEX^R$ *Mixed Integer Library.* $CPLEX^R$ Division, ILOG, INC., Incline Village, Nevada, 2000.

[3] Farias, I. R. and Nemhauser, G. L. (2001). "A family of inequalities for the generalized assignment polytope ". *Operations Research Letters*, 29, pp. 49-55.

[4] Finke, G. and Kusiak, A. (1985). "Network approach to modelling of flexible manufacturing modules and cells". *R.A.I.R.O. APII*, 19(4), pp. 359-370.

[5] Grötschel, M. and Wakabayashi, Y. (1989). "A cutting plane algorithm for a clustering problem". *Mathematical Programming*, 45, pp. 59-96.

[6] Gunther, R. E., Johnson, G. D. and Peterson, R. S. (1983). "Currently practiced formulations for the assembly line balance problem". *Journal of Operations Management*, 3(4), pp. 209-221.

[7] Hunting, M., Faigle, U. and Kern, W. (2001). "A Lagrangian relaxation approach to the edge-weighted clique problem". *European Journal of Operational Research*, 131, pp. 119-131.

[8] Kusiak, A. (1985). "The part families problem in flexible manufacturing systems". *Annals of Operations Research*, 3, pp. 279-300.

[9] Lourenço, L. L. (2004). *Classificação com Objectivo Globalizante e Restrições.* PhD thesis in Operations Research, Faculdade de Ciências e Tecnologia, New University of Lisbon.

[10] Lourenço, L. L. and Pato, M. V. (2004). "An improved genetic heuristic to support the design of flexible manufacturing systems". *Computers & Industrial Engineering*, 46, pp. 141-157.

[11] Lourenço, L. L. and Pato, M. V. (2001). "A binary linear programming formulation for the part families with precedence constraints problem". Proceedings of the IIE Conference. Atlanta, publication of the Institute of Industrial Engineering, production CD-ROM EP Innovations.

[12] Lourenço, L. L. (1995). *Contributos da Optimização Discreta para a Análise Classificatória.* Aplicação de Heurísticas Genéticas a uma Classificação com Precedências. Master thesis, Instituto Superior de Economia e Gestão, Techical University of Lisbon.

[13] Macambira, E. M. and Souza, C. C. (2000). "The edge-weighted clique problem: valid inequalities, facets and polyhedral computations". *European Journal of Operational Research*, 123, pp. 346-371.

[14] Moon, C., Lee, M., Seo, Y. and Lee, Y.H. (2002). "Integrated machine tool selection and operation sequenting with capacity and precedence constraints using genetic algorithm". *Computers & Industrial Engineering*, 43(3), pp. 605-621.

[15] Ng, M. (2000). "Clustering methods for printed circuit board insertion problems". *Journal of the Operational Research Society*, 51(10), pp. 1205-1211.

[16] OR-Library, http://www.ms.ic.ac.uk//info.html.

[17] Park, K., Lee and Park, S. (1996). "An extended formulation approach to the edge-weighted maximal clique problem". *European Journal of Operational Research*, 95, pp. 671-682.

[18] Pulleyblank, P. (1989). *Polyhedral Combinatorics*. Handbooks in OR & MS, vol. 1 Elsevier Science Publishers B. V., North-Holland.

[19] Scholl, A. (1995). *Balancing and Sequencing of Assembly Lines*. Physica-Verlag, Heidelberg.

[20] Sherali, H. D. and Adams, W. P. (1999). *A Reformulation-Linearization Technique for Solving Discrete and Continuous Nonconvex Problems*. Kluwer Academic Publishers, Dordrecht.

[21] Stecke, K. E. (1986). "A hierarchical approach to solving machine grouping and loading problem of flexible manufacturing systems". *European Journal of Operational Research*, 24, pp. 369-378.

[22] Tonge, F. (1965). "Assembly line balancing using probabilistic combinations of heuristics". *Management Science*, 11(7), pp. 727-735.

[23] Viswanathan, S. (1995). "Configuring cellular manufacturing systems: a quadratic integer programming formulation and a simple interchange heuristic". *International Journal of Production Research*, 33(2), pp. 361-376.

[24] Wolsey, L. A. (1998). *Integer Programming*. John Wiley and Sons, New York.