# A non-trivial explicit solution for the 14-velocity Cabannes kinetic model

Filipe Oliveira* and Ana Jacinta Soares†

* Centro de Matemática e Aplicações, FCT – Universidade Nova de Lisboa, Portugal
† Centro de Matemática, Universidade do Minho, Braga, Portugal

**Abstract**

We study the shock wave problem for the Cabannes 14-velocity model of the Boltzmann equation in one space dimension ($x$-axis) and obtain a non-trivial explicit solution which asymptotically connects two particular Maxwellian states. We prove that such a solution exists provided that a suitable condition on the differential elastic cross sections hold.

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## 1 Introduction

In kinetic theory of gases, Discrete Velocity Models (DVMs) of the Boltzmann Equation (BE) describe the time-space evolution of a gas whose particles can only attain a finite number of selected velocities and are subjected to a binary collision mechanism. The discretization of the velocity space allows to replace the integral collision operator of the BE by a finite sum over all admissible velocities. The resulting kinetic equations constitute an hyperbolic system of semilinear partial differential equations for the unknown number densities linked to the selected velocities. Due to their mathematical simplicity, after the pioneer work by Broadwell [1], DVMs have been used by several authors in the study of some relevant thermodynamical problems as shock wave propagation, Couette and Rayleigh flows, sound waves, etc. (see Ref.[2] for a detailed and systematic treatment of general DVMs and fluid dynamical applications for particular models).

*fso@fct.unl.pt
†ajsoares@math.uminho.pt

Certainly one of the more interesting DVM is known in literature as the Cabannes 14-velocity model, proposed by Cabannes in Ref.[3]. The admissible velocities are defined joining the centre of a cube at the origin of the velocity space to its vertices and to the centres of its faces. The inclusion of two different velocity moduli probably constitutes the novelty of this model since collisions among particles with different moduli of velocities can also occur and the temperature of the gas becomes an independent macroscopic variable. The Cabannes' model is simple enough to be mathematically tractable and, at the same time, contains enough physics to derive interesting hydrodynamic equations as the Euler or Navier-Stokes equations for the fluid density, mean velocity and total energy. The problem of shock wave propagation can then be treated by means of the Cabannes model using the Euler or Navier-Stokes equations as well as the kinetic model Boltzmann equations.

Within discrete kinetic theory, the problem of shock wave propagation has been studied by several authors. We quote for example Ref. [1], related to the Broadwell model, and Ref. [4] which gives a systematic treatment of the general problem and present some results for a particular model with six coplanar velocities. Moreover, paper [5] gives an exhaustive analysis on the existence of shock profile solutions to the discrete BE in unbounded domains by means of Euler and Navier-Stokes equations as well as model Boltzmann equations. Some recent contributions [7], [8] investigate the existence and stability of stationary wave solutions to the discrete Boltzmann equation in the half-space, considering, in particular, reflective boundary conditions.

In this paper, we consider the one space dimensional version of the Cabannes model and derive an explicit expression for the shock profile solution of the kinetic equations, traveling with constant velocity and connecting two limiting Maxwellian equilibrium states. The Euler equations are integrated across the shock and the resulting Rankine-Hugoniot conditions are derived. The solution is assured when a suitable condition on the differential elastic cross sections is verified. A pertinent choice of the Maxwellian parameters defining the state behind the shock is crucially used to obtain the above said exact solution.

In fact, the general equations can not be solved exactly, but this choice allows to introduce a particular change of variables and an explicit solution comes out. Accordingly, it seems natural that the content of the present paper can be used to develop numerical simulations and establish comparisons with approximate solutions. To our knowledge, explicit travelling wave solutions of the Cabannes 14-velocity model did not exist so far in the literature.

The paper is organized as follows: After this introduction, in section 2, we briefly recall the kinetic and the corresponding conservation equations of the Cabannes 14-velocity model reduced to the case of one space dimension. In section 3, steady profile solutions of plane shock wave are characterized after integrating the conservation equations between asymptotic states.

## 2 The model equations

The Cabannes 14-velocity model proposed in Ref.[3] assumes six velocities directed from the centre of a cube at the origin of the velocity space to the centre of each face and eight velocities directed from the centre of the cube to each vertex. The number densities associated to seleted velocities are represented by $(F_i)_{i\in\{1,\dots,14\}}$. In one space dimension ( $x$–axis), they reduce to five independent, $F_1$ and $F_2$ for particles moving with positive $x$–velocity directed to the face and vertices of the cube, $F_3$ and $F_4$ for particles moving in the opposite directions, and $F_5$ for those particles moving with velocities at right angle with the $x$-axis.

The kinetic equations, describing the space-time evolution of the number densities $F_i$, can be written in the form

$$\left\{\begin{array}{rcl} \partial_t F_1 + \partial_x F_1 &=& \sigma_1 q_1(F) + \sigma_2 q_2(F) \\ 4\partial_t F_2 + 4\partial_x F_2 &=& -\sigma_2 q_2(F) \\ \partial_t F_3 - \partial_x F_3 &=& \sigma_1 q_1(F) - \sigma_2 q_2(F) \\ 4\partial_t F_4 - 4\partial_x F_4 &=& \sigma_2 q_2(F) \\ 4\partial_t F_5 &=& -2\sigma_1 q_1(F) \end{array}\right. \tag{1}$$

where $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, $\sigma_1$ and $\sigma_2$ are positive constants depending on differential cross sections, collision frequencies and relative velocity of the colliding particles, and the nonlinear collision terms on the *r.h.s.* of Eqs.(1) are given by

$$q_1(F) := F_5^2 - F_1F_3 \ \text{ and } \ q_2(F) := F_2F_3 - F_1F_4. \tag{2}$$

Introducing the new variables $\rho, m, \tilde{m}, z, \tilde{\rho}$ defined by

$$\begin{aligned} \rho &= F_1 + 4F_2 + F_3 + 4F_4 \\ m &= F_1 + 4F_2 - F_3 - 4F_4 \\ \tilde{m} &= 4(F_2 - F_4) \\ z &= 4(F_2 + F_4) \\ \tilde{\rho} &= 4F_5 \end{aligned} \tag{3}$$

where $\rho+\tilde{\rho}$ is the total mass density of the gas, $m$ the total $x$–momentum component, $z$ the mass density of particles with velocities directed to the vertices of the cube, $\tilde{m}$ its $x$-flux and $\tilde{\rho}$ the mass density of particles with zero velocity along $x$-axis. From Eqs.(3) we obtain

$$\begin{array}{lll} F_1 = \frac{1}{2}(\rho + m + \tilde{m} - z) & F_2 = \frac{1}{8}(\tilde{m} + z) & F_3 = \frac{1}{2}(\rho - m - z + \tilde{m}) \\ F_4 = \frac{1}{8}(z - \tilde{m}) & F_5 = \frac{1}{4}\tilde{\rho} & \end{array} \tag{4}$$

and the system (1) can be re-written as

$$
\begin{aligned}
\partial_t(\rho + \tilde{\rho}) + \partial_x m &= 0 && (5a)\\
\partial_t m + \partial_x \rho &= 0 && (5b)\\
\partial_t z + \partial_x \tilde{m} &= 0 && (5c)\\
\partial_t \tilde{m} + \partial_x z &= -\frac{1}{4}\sigma_2 Q_2 && (5d)\\
\partial_t \tilde{\rho} &= -\frac{1}{8}\sigma_1 Q_1, && (5e)
\end{aligned}
$$

where

$$Q_1(\rho, m, z, \tilde{m}, \tilde{\rho}) = \tilde{\rho}^2 - 4(\rho - z)^2 + 4(m - \tilde{m})^2\,, \qquad Q_2(\rho, m, z, \tilde{m}, \tilde{\rho}) = \tilde{m}\rho - zm\,. \quad (6)$$

Equations ($5a$-$c$) express conservation of total mass density, total $x$-momentum and mass density of particles with velocities directed to the vertices of the cube.

A local Maxwellian state to Eqs.($5a$-$e$) is characterized by equilibrium functions $(a, b, c, d, e)$ such that $Q_i(a, b, c, d, e) = 0$, for i=1,2. Therefore

$$e^2 = 4(a - c)^2 - 4(b - d)^2\,, \qquad ad = bc. \quad (7)$$

# 3 Shock profile solutions

We are interested in steady shock profile solutions to Eqs.($5a$—$e$) of the form

$$(\rho(x,t), m(x,t), z(x,t), \tilde{m}(x,t), \tilde{\rho}(x,t)) = (A(\xi), B(\xi), C(\xi), D(\xi), E(\xi))\,, \quad (8)$$

where

$$\xi = x - vt\,, \quad (9)$$

traveling with constant velocity $v$ in the direction of the $x$-axis and connecting, asymptoticaly in space, two Maxwellian states $(a, b, c, d, e)$ and $(\alpha, \beta, \gamma, \delta, \epsilon)$, that is

$$
\begin{aligned}
\lim_{\xi\to-\infty}(A(\xi), B(\xi), C(\xi), D(\xi), E(\xi)) &= (a, b, c, d, e)\\
\lim_{\xi\to+\infty}(A(\xi), B(\xi), C(\xi), D(\xi), E(\xi)) &= (\alpha, \beta, \gamma, \delta, \epsilon)
\end{aligned} \quad (10)
$$

Solutions of type (8-9) with limiting conditions (10) correspond to state variables such that, far from the shock at $x = vt$, are close to their local equilibrium.

Now we write system ($5a$-$e$) in terms of the new variable $\xi$. Partial derivatives transform to

$$\frac{\partial}{\partial t} = -v\frac{d}{d\xi}\,, \qquad \frac{\partial}{\partial x} = \frac{d}{d\xi} \quad (11)$$

and ($5a$-$e$) becomes the system of ordinary differential equations:

$$\begin{aligned}
-v(A+E)'+B' &= 0 && (12a)\\
-vB'+A' &= 0 && (12b)\\
-vC'+D' &= 0 && (12c)\\
-vD'+C' &= -\sigma_2(AD-BC)/4 && (12d)\\
-vE' &= -\sigma_1\left(E^2-4(A-C)^2+4(B-D)^2\right)/8 && (12e)
\end{aligned}$$

where the primes indicate total derivative with respect to $\xi$.

The two Maxwellian states introduced in (10) are not independent and need to be expressed one in terms of the other through the so called Rankine-Hugoniot conditions. At this end we integrate the conservation equations ($12a$-$c$) with limiting conditions (10), resulting

$$\begin{aligned}
-v(a+e)+b &= -v(\alpha+\epsilon)+\beta && (13a)\\
-vb+a &= -v\beta+\alpha && (13b)\\
-vc+d &= -v\gamma+\delta && (13c)
\end{aligned}$$

which constitute the Rankine-Hugoniot conditions and express a relation between the properties of the gas in the limiting Maxwellian states.

If we assume a Maxwellian state behind the shock of type $(a,b,c,d,e)=(a,b,a,b,0)$, we get the solutions

$$\begin{aligned}
\alpha &= \frac{(a-vb)(2v^2-\sqrt{3v^2+1})}{(1-v^2)\sqrt{3v^2+1}}, & \beta &= \frac{v(a-vb)(2-\sqrt{3v^2+1})}{(1-v^2)\sqrt{3v^2+1}},\\
\gamma &= \frac{(va-b)(-\sqrt{3v^2+1}+2v^2)}{2v(v^2-1)}, & \delta &= \frac{(va-b)(-\sqrt{3v^2+1}+2))}{2(v^2-1)}, \qquad (14)\\
\epsilon &= \frac{\sqrt{3v^2+1}(va-b)+2v|a-vb|}{v\sqrt{3v^2+1}}.
\end{aligned}$$

Again, integration of the conservation laws ($12a$-$c$), now from $-\infty$ to $\xi$, leads to

$$\begin{aligned}
A(\xi) &= a+v(B(\xi)-b) && (15a)\\
D(\xi) &= b+v(C(\xi)-a) && (15b)\\
E(\xi) &= \frac{1-v^2}{v}(B(\xi)-b) && (15c)
\end{aligned}$$

Inserting conditions ($15a$-$c$) into Eqs.($12d$-$e$), we obtain the following system of ordinary differential equations

$$\begin{cases}
(1-v^2)\widetilde{C}' = \dfrac{\sigma_2}{4}\left((1-v^2)\widetilde{B}\widetilde{C}-(vb-a)\widetilde{B}-(va-b)\widetilde{C}\right)\\[2ex]
\widetilde{B}' = \dfrac{\sigma_1}{8}\left(\dfrac{1+3v^2}{v^2}\widetilde{B}^2-4\widetilde{C}^2\right),
\end{cases} \qquad (16)$$

where $\widetilde{B} = \widetilde{B}(\xi) := B(\xi) - b$ and $\widetilde{C} = \widetilde{C}(\xi) := C(\xi) - a$.

In the particular case that the Maxwellian state $(a,b,c,d,e) = (a,b,a,b,0)$ ahead the shock verifies $a = vb$, the system (16) has an exact solution. In fact, the two limiting Maxwellian states can be written as

$$(a,b,c,d,e) = b(v,1,v,1,0)\,, \tag{17}$$

and

$$(\alpha,\beta,\gamma,\delta,\epsilon) = b\left(0\,,\,0\,,\,\frac{2v^2-\sqrt{3v^2+1}}{2v}\,,\,\frac{2-\sqrt{3v^2+1}}{2}\,,\,\frac{v^2-1}{v}\right) \tag{18}$$

as it follows from Eqs.(14).

From Eqs.(17) and (18), system (16) transform to

$$\begin{cases} \widetilde{C}' = \dfrac{\sigma_2}{4}\left(\widetilde{B}\widetilde{C} + b\widetilde{C}\right) \\[2ex] \widetilde{B}' = \dfrac{\sigma_1}{8}\left(\dfrac{1+3v^2}{v^2}\widetilde{B}^2 - 4\widetilde{C}^2\right) \end{cases}$$

and, if in addition, we consider

$$\frac{1}{2}\left(\frac{\sigma_1}{8}(3+\frac{1}{v^2})\right) = \frac{\sigma_2}{4} := \theta\,, \tag{19}$$

system (16) finally transforms to

$$\begin{cases} \widetilde{C}' = \theta\widetilde{B}\widetilde{C} + b\theta\widetilde{C} \\[2ex] \widetilde{B}' = 2\theta\widetilde{B}^2 - \dfrac{\sigma_1}{2}\widetilde{C}^2 \end{cases} \tag{20}$$

System (20) can be solved explicitly. In fact, if we set $\mathcal{F}(\xi) = \tilde{B}(\xi)$ and $\mathcal{G}(\xi) = \tilde{C}(\xi)e^{-b\theta\xi}$, it becomes

$$\begin{cases} \mathcal{G}' = \theta\mathcal{G}\mathcal{F} \\[2ex] \mathcal{F}' = 2\theta\mathcal{F}^2 - \dfrac{\sigma_1}{2}\mathcal{G}^2\mathrm{e}^{2\theta b\xi} \end{cases} \tag{21}$$

Assuming now that $\mathcal{G}$ does not vanish, we get $\mathcal{F} = \dfrac{1}{\theta}\dfrac{\mathcal{G}'}{\mathcal{G}}$ and from (21) it results

$$\frac{1}{\theta}\left(\frac{\mathcal{G}'}{\mathcal{G}}\right)' = \frac{2}{\theta}\left(\frac{\mathcal{G}'}{\mathcal{G}}\right)^2 - \frac{\sigma_1}{2}\mathcal{G}^2\mathrm{e}^{2\theta b\xi}\,. \tag{22}$$

Finally, putting $\mathcal{A}(\xi) = \dfrac{1}{\mathcal{G}^2(\xi)}$ , (22) states exacly that $\mathcal{A}''(\xi) = \sigma_1 \theta \mathrm{e}^{2\theta b \xi}$ , and integrating twice Eqs.(22), it results

$$\mathcal{A} = \frac{\sigma_1}{4b^2\theta} e^{2\theta b \xi} + c_1(\xi - c_2) \, .$$

Therefore, Eqs.(20) has the following exact solutions

$$\widetilde{B}(\xi) = \pm \frac{b(c_1 b + 2e^{2\theta b \xi}\sigma_1)}{2(\mathrm{e}^{2b\theta\xi}\sigma_1 + \theta b^2 c_1(\xi - c_2))} \; , \; \widetilde{C}(\xi) = \mp \frac{2b\theta^{\frac{1}{2}} e^{\theta b \xi}}{\sqrt{\mathrm{e}^{2b\theta\xi}\sigma_1 + \theta b^2 c_1(\xi - c_2)}} .$$

where $c_1, c_2 \in \mathbb{R}$ . Taking into account the Maxwellian states assigned in (10) as limiting conditions, the signs must be correctly chosen, and we finally obtain

$$B(\xi) = b - \frac{b(c_1 b + 2e^{2\theta b \xi}\sigma_1)}{2(e^{2b\theta\xi}\sigma_1 + \theta b^2 c_1(\xi - c_2))} \qquad (23a)$$

$$C(\xi) = vb - \frac{2b\theta^{\frac{1}{2}} e^{b\theta\xi}}{\sqrt{\mathrm{e}^{2b\theta\xi}\sigma_1 + \theta b^2 c_1(\xi - c_2)}} \qquad (23b)$$

**Remark 1**
As mentionned in the introduction, the solution (23*a*-*b*) exists when the following condition holds true

$$\forall \xi \in \mathbb{R} \, , \quad \mathrm{e}^{2b\theta\xi}\sigma_1 + \theta b^2 c_1(\xi - c_2) > 0 \, . \qquad (24)$$

Constraint (24) can be assured by taking, for example, $c_2 = 0$, $c_1 < 0$ and $\dfrac{b|c_1|}{\sigma_1} < 2e$ .

**Remark 2**
The expressions for $A$, $D$ and $E$ come directly from Eqs.(15*a*-*e*).

**Remark 3**
From expressions (23*a*-*b*) it is easy to verify that, as it is expected, the following limiting conditions hold true

$$\lim_{\xi \to -\infty} B(\xi) = b \, , \qquad \lim_{\xi \to -\infty} C(\xi) = vb = a \, ,$$

$$\lim_{\xi \to +\infty} B(\xi) = 0 = \beta \, , \qquad \lim_{\xi \to +\infty} C(\xi) = b\frac{2v^2 - \sqrt{3v^2 + 1}}{2v} = \gamma \, .$$

From the exact solution obtained from (23*a*-*b*) together with (15*a*-*e*), we can compute, for example, the total mass density $\rho + \tilde{\rho}$ through definitions (3).

In Figure 1 we have drawn two shock profiles for the gas density versus $\xi$ . (with $v = 1$, $c = -1$, $b = 1$, $\sigma_1 = 12$ in the left frame and with $v = 1$, $c = -3$, $b = 1$, $\sigma_1 = 12$ in the right frame). Each profile corresponds to a continuous solution to the model Boltzmann equation interpolating the corresponding limit Maxellian states and showing a finite and nonzero wave thickness.

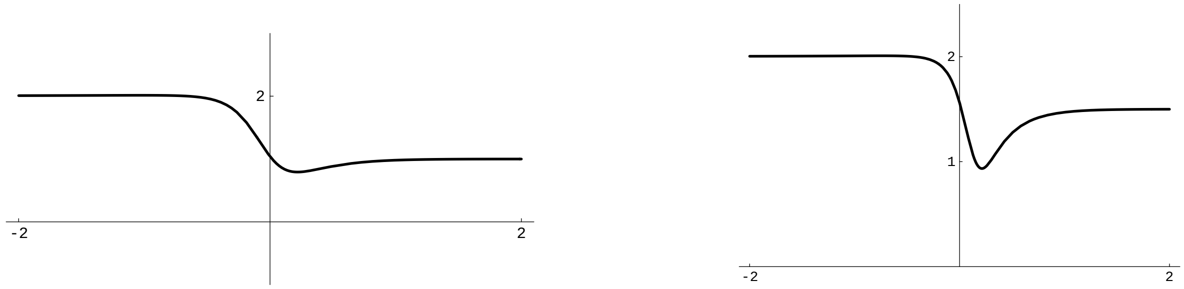


Figure 1: Exact shock profile solutions for the gas density.

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