Graphs and minimum rank of matrices

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Abstract

For a given connected (undirected) graph G, the minimum rank of $G = (V(G), E(G))$ is defined to be the smallest possible rank over all hermitian matrices A whose (i, j)th entry is non-zero whenever $i \neq j$ and $\{i, j\}$ is an edge in G ($\{i, j\} \in E(G)$). For each vertex x in G ($x \in E(G)$), $\Gamma(x)$ is the set of all neighbors of x. Let R be the equivalence relation on $V(G)$ such that

$$\forall_{x,y \in V(G)} \quad xRy \Leftrightarrow \Gamma(x) = \Gamma(y).$$

Our aim is define connected graphs $G = (V(G), E(G))$ such that the minimum rank of G is equal to the number of equivalence classes for the relation R on $V(G)$.

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1 Introduction

Let $G = (V(G), E(G))$ be an undirected connected graph on n vertices. With G we associate a matrix $A = [a_{ij}]$ such that for $i \neq j$, $a_{ij} = 0$ if, and only if, $\{x_i, x_j\} \notin E(G)$, and a set $S(G)$ of all hermitian matrices that we can associate with this graph, i. e.

$$\mathcal{S}(G) = \{A = [a_{ij}] \quad hermitian : a_{ij} \neq 0 \text{ whenever } i \neq j \text{ and } \{x_i, x_j\} \in E(G)\}.$$

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With G we consider:

$M(G)$ =the maximum multiplicity occurring for an eigenvalue of an $A \in \mathcal{S}(G)$;

$P(G)$ = the minimum number of vertex disjoint paths, occurring as induced subgraphs of G that cover all the vertices of G;

$m(G) = n - min_{A \in S(G)} rank(A)$.

Several authors have been interested on multiplicity of eigenvalues of matrices whose graph is a tree, e.g. [4], [5].

When G is a tree (a connected graph without cycles) we denote G by T. Johnson and Leal Duarte [3] proved that, if T is a tree, we have

$$P(T) = m(T) = M(T).$$

Let $x \in V(G)$. We denote by $\Gamma(x)$ the set of all neighbors of x in G, i.e., $\Gamma(x) = \{z \in V(G) : \{x, z\} \in E(G)\}$. If we consider the equivalence relation R on $V(G)$ such that

$$\forall_{x,y \in V(G)} \quad xRy \Leftrightarrow \Gamma(x) = \Gamma(y),$$

we obtain the result:

$$n - M(G) \leq \left| \frac{V(G)}{R} \right|,$$

where $\left|\frac{V(G)}{R}\right|$ is the number of equivalence classes for the relation R.

Let $X_1, \dots, X_p$ be the equivalence classes for the relation R on $V(G)$. In Section 2 we define the equivalence classes graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ of G where $V(\mathcal{G}) = \{X_1, \dots, X_p\}$ and $\{X_i, X_j\} \in E(\mathcal{G})$ if, and only if, there are $x \in X_i$ and $y \in X_j$ such that $\{x, y\}$ is an edge in G. In section 3 we will study the previous inequality for all graphs whose equivalence graph is a path.

In section 4 we define the "star tree", a tree $T = (V(T), E(T))$ such that if $x \in V(T)$ and the degree of x is greater than or equal to two, $d_T(x) \geq 2$, than there exist at least two neighbors of x with degree one. Finally in this section we prove that these trees are the only trees that verify

$$n - M(G) = \left| \frac{V(G)}{R} \right|.$$

2 The equivalence relation R

Let $G = (V(G), E(G))$ be a graph on n vertices. All graphs discussed in this paper are connected and undirected.

Firstly, in this section, we are going to see some properties of the relation R on $V(G)$. Next we are going to construct thr equivalence classes graph of G. The main result of this section is the Proposition 2.3. This Proposition enables us to know graphs that verify

$$min_{A \in \mathcal{S}(G)} \quad rank(A) = \left| \frac{V(G)}{R} \right|.$$

The first result that we can prove is the following:

Proposition 2.1 *Let $G = (V(G), E(G))$ be a graph, then*

$$min_{A \in \mathcal{S}(G)} \quad rank(A) \leq \left| \frac{V(G)}{R} \right|.$$

Proof Let $X_1, X_2, ..., X_p$ be the distinct equivalence classes for the relation R. Then $X_1 \dot{\cup} X_2 \dot{\cup} \ldots \dot{\cup} X_p = V(G)$. Let A be the adjacency matrix of G considering the ordering $(X_1, \ldots, X_p)$, i. e., first we consider the vertices of X_1, after we consider the vertices of X_2 and so one. It is easy to prove that $A \in \mathcal{S}(G)$. Since the submatrices of A corresponding to the rows $\sum_{l=0}^{i} |X_l| + 1, \ldots, \sum_{l=0}^{i+1} |X_l|$, for $i = 0, \ldots, p-1$, where $|X_0| = 0$, are matrices of rank one, then $rank(A) \leq p = \left|\frac{V(G)}{R}\right|$. Thus

$$min_{A \in \mathcal{S}(G)} rank(A) \leq \left| \frac{V(G)}{R} \right|.$$

□

Proposition 2.2 *Let $G = (V(G), E(G))$ be a graph. If X_1 is an equivalence class for the relation R on $V(G)$, then the subgraph of G induced by the vertices of X_1 is isomorphic to $N_{|X_1|}$, the null graph with $|X_1|$ vertices.*

Proof Let H be the subgraph of G induced by the vertices of X_1. Suppose that there are two vertices $x, y \in X_1$ such that $\{x, y\}$ is an edge of H. Then $y \in \Gamma(x)$ and $x \in \Gamma(y)$. Since G is an undirected graph we have $x \notin \Gamma(x)$ and $y \notin \Gamma(y)$. Thus $\Gamma(x) \neq \Gamma(y)$. But this is impossible because $x, y \in X_1$ and $\Gamma(x) = \Gamma(y)$. So, for all vertices $x, y \in X_1$, we have $\{x, y\}$ isn't an edge of H i.e., $H \cong N_{|X_1|}$. □

Let $G = (V(G), E(G))$ be a graph and $X_1, \ldots, X_p$ be the equivalence classes for the relation R on $V(G)$. We define the equivalence classes graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ of G where $V(\mathcal{G}) = \{X_1, \ldots, X_p\}$ and $\{X_i, X_j\} \in E(\mathcal{G})$ if, and only if, there are $x \in X_i$ and $y \in X_j$ such that $\{x, y\}$ is an edge in G.

In a similar way of that we have defined $\mathcal{S}(G)$ for the graph G , we can also define the set, $\mathcal{S}(\mathcal{G})$, of all hermitian matrices $B = [b_{ij}]$ that verify:

For $i \neq j$ we have $b_{ij} \neq 0$, if, and only if, $\{X_i, X_j\} \in E(\mathcal{G})$ and for $i = j$, $b_{ii} = 0$ whenever $|X_i| > 1$.

Remark that $|\frac{V(G)}{R}| = |V(\mathcal{G})|$. So, $min_{B \in \mathcal{S}(\mathcal{G})} rank(B) \leq \left|\frac{V(G)}{R}\right|$.

Now we can establish a relation between the rank of the matrices of $\mathcal{S}(G)$ and of the matrices of $\mathcal{S}(\mathcal{G})$.

Proposition 2.3 *Let $G = (V(G), E(G))$ be a graph and $\mathcal{G}$ the equivalence classes graph of G. If $min_{A \in \mathcal{S}(G)} rank(A) = \left|\frac{V(G)}{R}\right|$ then $min_{B \in \mathcal{S}(\mathcal{G})} rank(B) = \left|\frac{V(G)}{R}\right|$ i.e., all $B \in \mathcal{S}(\mathcal{G})$ are non-singular.*

Proof Let $X_1, \ldots, X_p$ be the equivalence classes for R. Suppose that $min_{B \in \mathcal{S}(\mathcal{G})} rank(B) = k < \left|\frac{V(G)}{R}\right|$. Let $C \in \mathcal{S}(\mathcal{G})$, considering the ordering $(X_1, \ldots, X_p)$, such that $rankC = k$. Consider $A = [a_{ij}]$ the matrix of G, considering the ordering first the vertices of X_1, after the vertices of X_2, ..., and last the vertices of X_p, and such that if $x_i \in X_r$, $x_j \in X_l$ then $a_{ij} = c_{rl}$. It is easy to prove that $A \in \mathcal{S}(G)$ and the rows of A corresponding to vertices in the same equivalent class X_i, are equal.

Consequently, $rank(A) = rank(B) = k < \left|\frac{V(G)}{R}\right|$ which is impossible. Thus

$$min_{B \in \mathcal{S}(\mathcal{G})} rank(B) = \left|\frac{V(G)}{R}\right|.$$

□

3 Equivalence classes graph

In this section we are going to search graphs $G = (V(G), E(G))$ such that

$$min_{A \in \mathcal{S}(G)} \ \ rank(A) = \left|\frac{V(G)}{R}\right|.$$

Using the Proposition 2.3 we know that if $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is the equivalence classes graph of G and there exists $B \in \mathcal{S}(\mathcal{G})$ singular, then $min_{A \in \mathcal{S}(G)} \ \ rank(A) < \left|\frac{V(G)}{R}\right|$.

The main result of this section is Theorem 3.6 where we describe all the graphs G whose equivalence classes graph is a path and verify the condition

$$min_{A \in \mathcal{S}(G)} \ \ rank(A) = \left|\frac{V(G)}{R}\right|.$$

For this we have to define:

Definition 3.1 *Let $G = (V(G), E(G))$ be a graph and $x \in V(G)$. We call a **branch incident with x** to a path of G which first vertex is adjacent to x,(x doesn't belong to this path), all the path vertices, except the last one, are vertices of degree two in G and the last one is a vertex of degree one in G.*

Observation: Remember that the length of a path is the number of edges of the path.

If $x, \ y \in V(G)$ we denote by $d(x, y)$ the minimum length of the paths between the vertices x and y and we denote by $d_G(x)$ the degree of x in G.

Proposition 3.2 *Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a graph such that all $B \in \mathcal{S}(\mathcal{G})$ are invertible, and let $x \in E(\mathcal{G})$ be a vertex such that $d_{\mathcal{G}}(x) \geq 2$. Then there exists, at most, one branch incident with x of even length.*

Proof Suppose that there are two branches incident with x of even length which are $y-z$ and $w-t$. Let B be the adjacency matrix of $\mathcal{G}$. Then $B \in \mathcal{S}(\mathcal{G})$. The submatrix B' of B whose rows and columns correspond to the vertex x, the vertices of the branch $y-z$ and the vertices of the branch $w-t$ is the matrix

$$B' = \begin{array}{c} x \\ y \\ \\ \vdots \\ \\ z \\ w \\ \\ \vdots \\ \\ t \end{array} \begin{bmatrix} 0\,1\,0 \cdots 0\,0\,1\,0 \cdots 0\,0 \\ 1\,0\,1 \cdots 0\,0\,0\,0 \cdots 0\,0 \\ 0\,1\,0 \cdots 0\,0\,0\,0 \cdots 0\,0 \\ \vdots\;\vdots\;\vdots\;\ddots\;\vdots\;\vdots\;\vdots\;\vdots\;\ddots\;\vdots\;\vdots \\ 0\,0\,0 \cdots 0\,1\,0\,0 \cdots 0\,0 \\ 0\,0\,0 \cdots 1\,0\,0\,0 \cdots 0\,0 \\ 1\,0\,0 \cdots 0\,0\,0\,1 \cdots 0\,0 \\ 0\,0\,0 \cdots 0\,0\,1\,0 \cdots 0\,0 \\ \vdots\;\vdots\;\vdots\;\ddots\;\vdots\;\vdots\;\vdots\;\vdots\;\ddots\;\vdots\;\vdots \\ 0\,0\,0 \cdots 0\,0\,0\,0 \cdots 0\,1 \\ 0\,0\,0 \cdots 0\,0\,0\,0 \cdots 1\,0 \end{bmatrix}$$

If r is the length of the branch $y-z$, p is the length of the branch $w-t$ and R_i denote the i-row of B', then B' is a matrix with $r+1+p+1+1 = r+p+3$ rows. We can see easily that

$$\sum_{k=1}^{\frac{r+2}{2}} (-1)^{k+1} R_{2k} + \sum_{l=1}^{\frac{p+2}{2}} (-1)^{l} R_{r+1+2l} = 0.$$

So, B' is singular and B either. That is impossible. Then $\mathcal{G}$ has, at most, an even length branch incident with x. □

Corollary 3.3 *If $\mathcal{G}$ is a path such that all $B \in \mathcal{S}(\mathcal{G})$ are nonsingular, then $\mathcal{G}$ isn't an even length path. This is equivalent to say that there aren't any even length path $\mathcal{G}$ such that all $B \in \mathcal{S}(\mathcal{G})$ are nonsingular.*

Proof Suppose that $\mathcal{G}$ is an even length path. If $\mathcal{G}$ is the null graph, N_1, then the matrix $B = [0] \in \mathcal{S}(\mathcal{G})$ and is nonsingular. So $\mathcal{G}$ is a non null even length path. If $X_1, X_2, \ldots, X_p$ is a non null even length path (p is odd and $p \geq 3$) then $d_{\mathcal{G}}(X_2) = 2$ and X_1 and $X_3 - X_p$ are two branches incident with X_2 of even length. By Proposition 3.2 we have a contradiction. □

Now, we can ask: "All graphs $G = (V(G), E(G))$ such that the equivalence classes graph $\mathcal{G}$ of G is an odd length path verify

$$min_{A \in \mathcal{S}(G)} rank(A) = \left| \frac{V(G)}{R} \right| ?"$$

The answer is no.

Example 3.4 *The graph $G \cong K_2$ is a graph such that $\mathcal{G} \cong G \cong K_2$. Therefore $\mathcal{G}$ is an odd length path, however*

$$A = \begin{bmatrix} 1\,1 \\ 1\,1 \end{bmatrix} \in \mathcal{S}(G)$$

and $rank(A) = 1 < 2 = \left|\frac{V(G)}{R}\right|$.

Lemma 3.5 *Let G be a graph such that $\mathcal{G}$ is a path of length $p-1$. Then*

$$min_{A\in\mathcal{S}(G)} rank(A) \geq p-1.$$

Proof Let $C \in \mathcal{S}(G)$ such that $rank(C) = min_{A\in\mathcal{S}(G)} rank(A)$.

If we reorder the rows and the columns of C in such way that the matrix C_1 that we obtain is the matrix C when we consider the ordering $x_{11}, \cdots, x_{1r_1}, x_{21}, \cdots, x_{2r_2}, \cdots, x_{p1}, \cdots, x_{pr_p}$ where $X_i = \{x_{i1}, \cdots, x_{ir_i}\}$ for $i = 1, \cdots, p$ are the equivalence classes for the relation R. The submatrix C_2 that corresponds to the rows $1, r_1+1, r_1+r_2+1, r_1+r_2+r_3+1, \cdots, r_1 + r_2 + r_3 + \cdots + r_p + 1$ of C is

$$C_2 = \begin{bmatrix} b_1 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ c_2 & b_2 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & c_3 & b_3 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_p & b_p \end{bmatrix}$$

where $b_i = [\, d_i \ 0 \ \cdots \ 0 \,] \in \mathcal{M}_{1\times r_i}, d_i \in \mathbb{C},\ i = 1, \ldots, p$
$a_i = [\, a_{i1} \ \cdots \ a_{ir_{i+1}} \,] \in \mathcal{M}_{1\times r_{i+1}}, a_{ij} \in \mathbb{C}\setminus\{0\},\ i = 1, \ldots, p-1$
$c_i = [\, c_{i1} \ \cdots \ c_{ir_{i-1}} \,] \in \mathcal{M}_{1\times r_{i-1}}, c_{ij} \in \mathbb{C}\setminus\{0\},\ i = 2, \ldots, p.$

Easily, we can see that the rows $2, \ldots, p$ of C_2 are linearly independent. So $rank(C_2) \geq p-1$. Consequently, $rank(C) \geq p-1$. □

Theorem 3.6 *Let $G = (V(G), E(G))$ be a graph such that $\mathcal{G}$ is an odd length path $X_1, \ldots, X_p$. Then*

$$min_{A\in\mathcal{S}(G)} rank(A) = \left|\frac{V(G)}{R}\right| = p,$$

if, and only if, there aren't two distinct vertices in $\mathcal{G}$ X_i and X_j such that $i < j$ and $|X_i| = |X_j| = 1$, $d(X_i, X_j)$ is odd and $d(X_1, X_i)$ is even.

Proof

Necessity Suppose there exist two vertices X_i and X_j in $V(\mathcal{G})$ such that $|X_i| = |X_j| = 1$, $i < j$, $d(X_i, X_j)$ is odd and $d(X_1, X_i)$ is even.

Consider the matrix

$$B = \begin{array}{c} X_1 \\ \\ \vdots \\ \\ X_i \\ \\ \vdots \\ \\ X_j \\ \\ \vdots \\ \\ X_p \end{array} \begin{bmatrix} 0\,1\ldots 0\,0\,0\ldots 0\,0\,0\ldots 0\,0 \\ 1\,0\ldots 0\,0\,0\ldots 0\,0\,0\ldots 0\,0 \\ \vdots\,\vdots\,\ddots\,\vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots \\ 0\,0\ldots 0\,1\,0\ldots 0\,0\,0\ldots 0\,0 \\ 0\,0\ldots 1\,1\,1\ldots 0\,0\,0\ldots 0\,0 \\ 0\,0\ldots 0\,1\,0\ldots 0\,0\,0\ldots 0\,0 \\ \vdots\,\vdots\,\ddots\,\vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots \\ 0\,0\ldots 0\,0\,0\ldots 0\,1\,0\ldots 0\,0 \\ 0\,0\ldots 0\,0\,0\ldots 1\,1\,1\ldots 0\,0 \\ 0\,0\ldots 0\,0\,0\ldots 0\,1\,0\ldots 0\,0 \\ \vdots\,\vdots\,\ddots\,\vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots\,\vdots\,\ddots\,\vdots\,\vdots \\ 0\,0\ldots 0\,0\,0\ldots 0\,0\,0\ldots 0\,1 \\ 0\,0\ldots 0\,0\,0\ldots 0\,0\,0\ldots 1\,0 \end{bmatrix}$$

Then $B \in \mathcal{S}(\mathcal{G})$. Since $d(X_1, X_p)$, $d(X_i, X_j)$ are odd, $d(X_1, X_i)$ is even and $d(X_1, X_p) = d(X_1, X_i) + d(X_i, X_j) + d(X_j, X_p)$, then $d(X_j, X_p)$ is even. Consequently, p is even, i is odd and j is even. Let R_l be the l-row of B. Then

$$R_1 = \sum_{k=1}^{\frac{i-1}{2}} (-1)^{k+1} R_{2k+1} + \sum_{u=1}^{\frac{j-i-1}{2}} (-1)^{u+\frac{i+1}{2}} (R_{i+2u-1} + R_{i+2u}) + \sum_{t=1}^{\frac{p-j+2}{2}} (-1)^{t+\frac{j}{2}} R_{2k+j-2}.$$

So B is not invertible and by the Proposition 2.3 we have $min_{A\in\mathcal{S}(G)} rank(A) < \left|\frac{V(G)}{R}\right| = p$, which is impossible.

Consequently, there aren't two distinct vertices of $\mathcal{G}$ X_i and X_j such that $i < j$ and $|X_i| = |X_j| = 1$, $d(X_i, X_j)$ is odd and $d(X_1, X_i)$ is even.

Sufficiency

Suppose that there is a graph $G = (V(G), E(G))$ such that $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ is a path of odd length $X_1, \cdots, X_p$ where does not exist two vertices X_i and X_j in $\mathcal{G}$ with $i < j$ and $|X_i| = |X_j| = 1$, $d(X_i, X_j)$ is odd and $d(X_1, X_i)$ is even but $min_{A\in\mathcal{S}(G)} rank(A) < \left|\frac{V(G)}{R}\right| = p$.

Let $C \in \mathcal{S}(G)$ such that $rank(C) = min_{A\in\mathcal{S}(G)} rank(A) < p$.

Using the Lemma 3.5, $rank(C) \geq p - 1$. But by hypothesis, $rank(C) < p$, then $rank(C) = p - 1$. Since $\mathcal{G}$ is a path of length $p - 1$ it is easy to prove that

$$min_{B\in\mathcal{S}(G)} rank(B) \geq p - 1.$$

If we reorder the rows and the columns of C in such way that the matrix C_1 that we obtain is the matrix C when we consider the ordering $x_{11}, \cdots, x_{1r_1}, x_{21}, \cdots, x_{2r_2}, \cdots, x_{p1}, \cdots, x_{pr_p}$ where $X_i = \{x_{i1}, \cdots, x_{ir_i}\}$ for $i = 1, \cdots, p$ are the equivalence classes for the relation R. The submatrix C_2 that corresponds to the rows $1, r_1 + 1, r_1 + r_2 + 1, r_1 + r_2 + r_3 + 1, \cdots, r_1 + r_2 + r_3 + \cdots + r_p + 1$ of C is

$$C_2 = \begin{bmatrix} b_1 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ c_2 & b_2 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & c_3 & b_3 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_p & b_p \end{bmatrix}$$

where $b_i = [\, d_i \ 0 \ \cdots \ 0 \,] \in \mathcal{M}_{1\times r_i}, d_i \in \mathbb{C},\ i = 1, \ldots, p$
$a_i = [\, a_{i1} \ \cdots \ a_{ir_{i+1}} \,] \in \mathcal{M}_{1\times r_{i+1}}, a_{ij} \in \mathbb{C} \setminus \{0\},\ i = 1, \ldots, p-1$
$c_i = [\, c_{i1} \ \cdots \ c_{ir_{i-1}} \,] \in \mathcal{M}_{1\times r_{i-1}}, c_{ij} \in \mathbb{C} \setminus \{0\},\ i = 2, \ldots, p.$

So, $rank(C_2) = p-1$, i.e., if R_l is the l-row of C_2 then there exist $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$, not all zeros , such that

$$R_1 = \alpha_2 R_2 + \ldots + \alpha_p R_p.$$

Now we have to consider several cases:

Case 1: $|X_1| = r_1 = 1$.

If $r_p = 1$ as though $d(X_1, X_p)$ is odd and $d(X_1, X_1)$ is even, we have a contradiction. So, $|X_p| = r_p > 1$. For the same reason, all classes of even indices verify $|X_2| > 1, \ldots, |X_{p-2}| > 1$.

If we have $d_p \neq 0$, as we have $a_{p-1,r_p} \neq 0$, and if we look to the elements that are immediately above of the principal diagonal elements of C_2,we can conclude that $rank(C_2) = p$, which is impossible. So $d_p = 0$ and $b_p = 0 \in \mathcal{M}_{1\times r_p}$.

Since $R_1 = \alpha_2 R_2 + \ldots + \alpha_p R_p$, $a_{p-1} \neq 0$ and $b_p = 0$ then $p > 2$, $\alpha_{p-1} = 0$,

$$R_1 = \alpha_2 R_2 + \ldots + \alpha_{p-2} R_{p-2} + \alpha_p R_p$$

and $\alpha_{p-2} a_{p-2} + \alpha_p c_p = 0$.

If we consider

$$C_3 = \begin{bmatrix} b_1 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ c_2 & b_2 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & c_3 & b_3 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_{p-2} & b_{p-2} \end{bmatrix}$$

and R'_l denotes the l-row of C_3, we will have

$$R'_1 = \alpha_2 R'_2 + \ldots + \alpha_{p-2} R'_{p-2}.$$

For the reason that we have previous mentioned $d_{p-2} = 0$ and also $b_{p-2} = 0$. Therefore, $p > 4$ and $\alpha_{p-3} = 0$.

In a same way we will conclude that $b_2 = 0$, $b_4 = 0, \ldots b_{p-2} = 0, b_p = 0$ and $\alpha_3 = \ldots = \alpha_{p-1} = 0$. So

$$R_1 = \alpha_2 R_2 + \alpha_4 R_4 + \ldots + \alpha_p R_p$$

which is impossible.

Case 2: $|X_1| = r_1 > 1$

In this case, $d_1 = 0$, (if it isn't, as we have $c_2 r_1 \neq 0$ then the equality $R_1 = \alpha_2 R_2 + \ldots + \alpha_p R_p$ will be impossible). So $b_1 = 0$ and $\alpha_2 = 0$. Consequently, $p > 2$ and $a_1 = \alpha_3 c_3$. If we consider

$$C_4 = \begin{bmatrix} b_3 & a_3 & 0 & 0 & \cdots & 0 & 0 \\ c_4 & b_4 & a_4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_p & b_p \end{bmatrix}$$

and R'_l denotes the l-row of C_4, we will have

$$R'_1 = \alpha_4 R'_2 + \ldots + \alpha_p R'_{p-2}.$$

If $|X_3| = r_3 = 1$ (pay attention that $d(X_1, X_3) = 2$ is even then $d(X_3, X_p)$ is odd) and if we think analogously to the prior form (case 1) we conclude that is impossible. So, $|X_3| = r_3 > 1$ and using a similar argument used before (case 2), we conclude that $b_3 = 0$ and $\alpha_4 = 0$. If we repeat this process we obtain an absurd. Then,

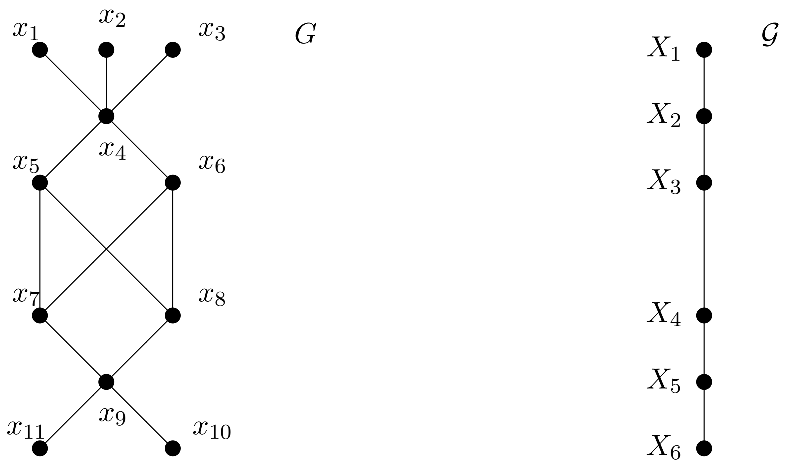
$$min_{A \in \mathcal{S}(G)} rank(A) \geq p.$$

Since the adjacency matrix of G is a matrix of rank p, then

$$min_{A \in \mathcal{S}(G)} rank(A) = p = \left| \frac{V(G)}{R} \right|.$$

□

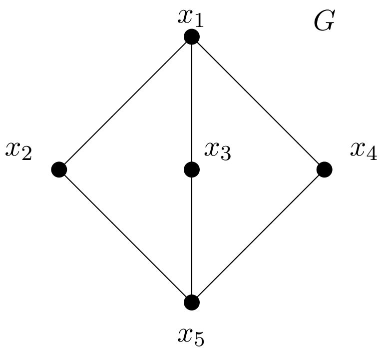
Example 3.7 *Let $G = (V(G), E(G))$ be the graph*

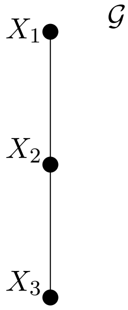


The equivalence classes for the relation R on $V(G)$, vertices in $\mathcal{G}$, are $X_1 = \{x_1, x_2, x_3\}$, $X_2 = \{x_4\}$, $X_3 = \{x_5, x_6\}$, $X_4 = \{x_7, x_8\}$, $X_5 = \{x_9\}$ and $X_6 = \{x_{10}, x_{11}\}$.

We have $|X_2| = |X_5| = 1$, $d(X_1, X_2)$ is odd and $d(X_2, X_5)$ is odd. There aren't another classes with cardinal one, then G verifies the previous Theorem conditions, so $min_{A \in \mathcal{S}(G)} rank(A) = 6$.

Example 3.8 *If $G = (V(G), E(G))$ is the graph*

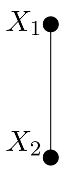




As we can see, $\mathcal{G}$ is an even length path whose vertices are $X_1 = \{x_1\}$, $X_2 = \{x_2, x_3, x_4\}$ and $X_3 = \{x_5\}$, so by Corollary 3.3 $min_{A\in\mathcal{S}(G)}rank(A) < 3$. But, by Lemma 3.5, $\min_{A\in\mathcal{S}(G)} rank(A) \geq 2$ then

$$min_{A\in\mathcal{S}(G)}rank(A) = 2.$$

Example 3.9 *If $G = (V(G), E(G)) = (X_1 \cup X_2, E(G))$ is a complete bipartite graph different from $K_{1,1}$, $\mathcal{G}$ is the graph*



where $|X_1| > 1$ or $|X_2| > 1$, thus, by the previous Theorem we have

$$min_{A\in\mathcal{S}(G)}rank(A) = 2.$$

Corollary 3.10 *Let $G = (V(G), E(G))$ be a graph such that $\mathcal{G}$ is the path $X_1, \ldots, X_p$. Then $min_{A\in\mathcal{S}(G)}rank(A) = p - 1$ if, and only if, $\mathcal{G}$ is an even length path or $\mathcal{G}$ is an odd length path and there are two distinct vertices in $\mathcal{G}$ X_i and X_j such that $i < j$ and $|X_i| = |X_j| = 1$, $d(X_i, X_j)$ is odd and $d(X_1, X_i)$ is even.*

Proof

Necessity

By Proposition 2.1, $min_{A\in\mathcal{S}(G)}rank(A) \leq p$. If $min_{A\in\mathcal{S}(G)}rank(A) = p - 1$ and $\mathcal{G}$ is a path then by Theorem 3.6, $\mathcal{G}$ is an even length path or $\mathcal{G}$ is an odd length path and there are two distinct vertices in $\mathcal{G}$, X_i and X_j such that $i < j$ and $|X_i| = |X_j| = 1$, $d(X_i, X_j)$ is odd and $d(X_1, X_i)$ is even.

Sufficiency

If $\mathcal{G}$ is an even length path or $\mathcal{G}$ is an odd length path and there are two distinct vertices in $\mathcal{G}$, X_i and X_j such that $i < j$ and $|X_i| = |X_j| = 1$, $d(X_i, X_j)$ is odd and $d(X_1, X_i)$ is even then using the Lemma 3.5, $min_{A \in \mathcal{S}(G)} rank(A) \geq p - 1$. By the Corollary 3.3 and Theorem 3.6, $min_{A \in \mathcal{S}(G)} rank(A) < p$. Thus, $min_{A \in \mathcal{S}(G)} rank(A) = p - 1$.

□

Example 3.11 *Consider G the path P_4*

Then we have $G \cong \mathcal{G}$. We also see that $|X_3| = 1$, $|X_4| = 1$, $d(X_3, X_4)$ is odd and $d(X_1, X_3)$ is even. Then, for the previous Corollary,

$$min_{A \in \mathcal{S}(G)} rank(A) = 3 < 4.$$

4 Star Trees

Let $T = (V(T), E(T))$ be a tree. We can ask what kind of trees $T = (V(T), E(T))$ verify the condition $min_{A \in \mathcal{S}(G)} rank(A) = \left| \frac{V(T)}{R} \right|$. In this section we are going to describe all trees that verify the condition.

We define the following subsets of $V(T)$:

1) $V_2(T) = \{x \in V(T) : d_T(x) \geq 2\}$ whose subsets are
   - 1.1) $T_0 = \{x \in V_2(T) : \forall y \in \Gamma(x), d_T(y) \neq 1\}$
   - 1.2) $T_1 = \{x \in V_2(T) : \exists^1 y \in \Gamma(x)$ such that $d_T(y) = 1\}$
   - 1.3) $T_2 = V_2(T) \setminus (T_0 \cup T_1)$
2) $V_1(T) = \{z \in V(T) : d_T(z) = 1\}$ whose subsets are
   - 2.1) $S_1 = \{z \in V_1(T) : \Gamma(z) \subseteq T_1\}$
   - 2.2) $S_2 = \{z \in V_1(T) : \Gamma(z) \subseteq T_2\}$
   - 2.3) $S_0 = V_1(T) \setminus (S_1 \cup S_2)$

Observation: It is easy to see that

1) $V(T) = V_1(T) \cup V_2(T) = S_1 \cup S_2 \cup S_0 \cup T_0 \cup T_1 \cup T_2$.

2) $|S_1| = |T_1|$.

Definition 4.1 *A* **Star Tree** *is a tree $T = (V(T), E(T))$ such that $V_2(T) \neq \emptyset$ and $T_0 = T_1 = \emptyset$.*

Observation 1: If $T = (V(T), E(T))$ is a tree, there are three types of equivalence classes for the relation R on $V(T)$

1$-$ Singular classes whose element is a vertex of $V_2(T)$

2$-$ Singular classes whose element is a vertex of $V_1(T)$ and it neighbor is a vertex of $(T_1 \cup S_0)$.

3$-$ Classes with more than one element where each element is a vertex of $V_1(T)$ and all are neighbors of the same vertex of $V_2(T)$. The number of these classes is $|T_2|$.

Observation 2: If T is a star tree there are not equivalence classes of type 2 and $|T_2| = |V_2(T)|$.

Lemma 4.2 *Let $T = (V(T), E(T))$ be a star tree. Then*

$$P(T) = |V_1(T)| - |V_2(T)|.$$

Proof We will prove the result by induction in the number of elements of $V_2(T)$.

If $|V_2(T)| = 1$ then T is a star.

In this case, if we suppose that $|V(T)| = k$, we can easily see that $P(T) = (k-3)+1 = (k-1)-1 = |V_1(T)| - |V_2(T)|$. Thus the result holds for $|V_2(T)| = 1$.

Suppose that the result is true for all star trees with $|V_2(T)| = t \geq 1$.

Let T be a star tree with $|V_2(T)| = t+1$.

Let z be a vertex of $V_2(T)$ such that $|\Gamma(z) \cap V_2(T)| = 1$ (this vertex exists because T doesn't have cycles). Let T' be the subgraph induced by the vertices $V(T) \setminus (\{z\} \cup (\Gamma(z) \cap S_2))$ and T'' be the subgraph induced by the vertices of $(\{z\} \cup (\Gamma(z) \cap S_2))$. It is easy to see that T' is a star tree, T'' is a star and $|V_2(T')| = t$. By induction hypothesis, $P(T') = |V_1(T')| - |V_2(T')|$. Consequently, $P(T) \leq P(T') + P(T'') = |V_1(T')| - |V_2(T')| + |V_1(T'')| - 1 = |V_1(T)| - |V_2(T)|$.

Since T'' is a star with more than two vertices then $P(T') \leq P(T) - (|\Gamma(z) \cap V_1(T)| - 1)$. If $P(T) < |V_1(T)| - |V_2(T)|$ then

$$P(T^{'}) < |V_1(T)| - |V_2(T)| - (|\Gamma(z) \cap V_1(T)| - 1) = |V_1(T')| - |V_2(T')|.$$

But by the induction hypothesis this is equal to $P(T^{'})$. So we have a contradiction and consequently, $P(T) = |V_1(T)| - |V_2(T)|$.

$\square$

Lemma 4.3 *Let $T = (V(T), E(T))$ be a tree, $T_2 = \{x_1, \ldots, x_{|T_2|}\} \neq \emptyset$ and $\{y_i, z_i\} \subseteq (\Gamma(x_i) \cap S_2)$, $|\{y_i, z_i\}| = 2$, $i \in \{1, 2, \ldots, |T_2|\}$. There exists a path tree of T that realizes $P(T)$, $D_1, \ldots, D_{|T_2|}, D_{|T_2|+1}, \ldots, D_{P(T)}$, such that D_i is the path y_i, x_i, z_i for $i \in \{1, 2, \ldots, |T_2|\}$.*

Proof Let $E_1, \ldots, E_{P(T)}$, be a path tree of T that realizes $P(T)$ and suppose that $x_1 \in E_1$. We can verify that $d_{E_1}(x_1) = 2$. If $d_{E_1}(x_1) \leq 1$, since $x_1 \in T_2$ there is $y_1 \in \Gamma(x_1)$ such that $d_T(y_1) = 1$, that implies an existence of j, $2 \leq j \leq P(T)$ such that E_j is the singular path with the only vertex y_1.

But, in these conditions, $E_1 \cup E_j, E_2, \ldots, E_{j-1}, E_{j+1}, \ldots, E_{P(T)}$, is steel a path tree of T with $P(T) - 1$ paths, which is impossible. So, $d_{E_1}(x_1) = 2$. Let u, v vertices of E_1 adjacent to x_1. If $u \notin S_2$ (adjacent to x_1 with degree one in T), then, since $x_1 \in T_2$, there are $y_1, z_1 \in (\Gamma(x_1) \cap S_2)$ with $|\{y_1, z_1\}| = 2$.

Suppose that $y_1 \neq v$ (otherwise we can do the same with z_1). There is $2 \leq r \leq P(T)$ such that E_r is the path y_1. If $E_1 \backslash x_1 = G_1 \cup G_2$ then $\{G_1 \cup \{x_1\} \cup \{y_1\}\}, G_2, E_2, \ldots, E_{r-1}, E_{r+1}, \ldots, E_{P(T)}$ is a path tree of T with $P(T)$ paths. If we think analogously with v we conclude that there is $F_1, F_2, \ldots, F_{P(T)}$ a path tree of T such that F_1 is the path y_1, x_1, z_1. If we do the same with each element of T_2, we obtain the result.

□

Lemma 4.4 *Let $T = (V(T), E(T))$ be a tree such that $V_2(T) = \emptyset$ or $T_2 \subset V_2(T) \neq \emptyset$, then*

$$P(T) > |S_2| - |T_2|.$$

Proof If $T_2 = \emptyset$ then $S_2 = \emptyset$ and trivially we have $P(T) > |S_2| - |T_2| = 0$. Suppose that $T_2 = \left\{x_1, \ldots, x_{|T_2|}\right\} \neq \emptyset$. Using the Lemma 4.3 there is $D_1, \ldots, D_{|T_2|}, D_{|T_2|+1}, \ldots, D_{P(T)}$, a path tree of T such that, for $i = 1, \ldots, |T_2|$, D_i is the path y_i, x_i, z_i, with $y_i, z_i \in (S_2 \cap \Gamma(x_i))$ and $|\{y_i, z_i\}| = 2$.

Let $M = S_2 \backslash \{y_1, \ldots, y_{|T_2|}, z_1, \ldots, z_{|T_2|}\}$. It's easy to see that each $t \in M$ is a path of the previous tree path of T, $D_1, \ldots, D_{P(T)}$. So, we can suppose that $D_{|T_2|+1}, \ldots, D_{|T_2|+|M|}$ are these paths.

Since $\emptyset \neq T_2 \subset V_2(T)$, by hypothesis, then $T_0 \neq \emptyset$ or $T_1 \neq \emptyset$. Thus

$$|T_2| + |M| < P(T).$$

But, $|M| = |S_2| - 2|T_2|$ which implies that

$$P(T) > |S_2| - |T_2|.$$

□

Theorem 4.5 *Let $T = (V(T), E(T))$ be a tree. Then $min_{A \in \mathcal{S}(T)} \quad rank(A) = \left|\frac{V(T)}{R}\right|$ if, and only if, T is a star tree.*

Proof

Necessity

If $min_{A \in \mathcal{S}(T)} \quad rank(A) = \left|\frac{V(T)}{R}\right|$ then, by [3], we know that

$$min_{A \in \mathcal{S}(T)} \quad rank(A) = n - P(T) = \left|\frac{V(T)}{R}\right|.$$

If T is not a star tree then $T_2 \subset V_2(T)$ or $V_2(T) = \emptyset$ and $P(T) > |S_2| - |T_2|$.
But, using the observation 1 we have

$$\left|\frac{V(T)}{R}\right| = |T_0| + |T_1| + 2|T_2| + |S_1| + |S_0|,$$

$$n = |T_0| + |T_1| + |T_2| + |S_1| + |S_2| + |S_0|$$

and

$$P(T) = n - min_{A \in \mathcal{S}(T)} \quad rank(A) = |S_2| - |T_2|.$$

By the Lemma 4.4, if $T_2 \subset V_2(T)$ or $V_2(T) = \emptyset$ then $P(T) > |S_2| - |T_2|$. But we have $P(T) = |S_2| - |T_2|$ then $T_2 = V_2(T) \neq \emptyset$. So T is a star tree.

Sufficiency

Let T be a star tree with n vertices. By [3], we know that $min_{A \in \mathcal{S}(G)} rank(A) = n - P(T)$. Using the Lemma 4.2 we have

$$min_{A \in \mathcal{S}(G)} rank(A) = n - |V_1(T)| + |V_2(T)|.$$

Using the observation 2,

$$\left|\frac{V(T)}{R}\right| = |V_2(T)| + |V_2(T)|$$

and

$$n = |V_2(T)| + |V_1(T)|.$$

Thus

$$min_{A \in \mathcal{S}(G)} rank(A) = |V_2(T)| + |V_1(T)| - |V_1(T)| + |V_2(T)| = \left|\frac{V(T)}{R}\right|.$$

□

References

[1] C.D. Godsil, Algebraic Combinatorics, *Chapman and Hall*, **New York and London** (1993).

[2] R.A. Horn and C.R. Johnson, Matrix Analysis, *Cambridge Univ. Press*, **New York** (1985).

[3] C.R. Johnson and A.Leal Duarte, The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, *Linear and Multilinear Algebra* **46** (1999), 139-144.

[4] S. Parter, On the eigenvalues and eigenvectors of a class of matrices, *J. Soc. Indust. Appl. Math* **8** (1960), 376-388.

[5] G. Wiener, Spectral multiplicity and splitting results for a class of qualitative matrices, *Linear Algebra Appl.* **61** (1984), 15-29.