The necessity of the star-shaped condition on Ding's version of the Poincaré-Birkhoff theorem.

Rogério Martins*

Dep. Matemática, FCT, Universidade Nova de Lisboa

Monte da Caparica, 2829-516 Caparica, Portugal. E-mail: roma@fct.unl.pt

Antonio J. Ureña†

Dep. Matemática Aplicada, Facultad de Ciencias, Universidad de Granada

Av. Fuentenueva, s.n., 18071, Granada, Spain. E-mail: ajurena@ugr.es

Abstract: W-Y. Ding proved a generalization of the Poincaré-Birkhoff fixed point Theorem not requiring that the boundary curves of the anular region to be circles and invariant. However the inner boundary is required to be star-shaped with respect to the origin. We construct a counterexample that shows that the star-shaped condition is necessary.

1 Introduction

The well known Poincaré-Birkhoff theorem states that any area-preserving homeomorphism T of the annulus $\{(x,y) \in \mathbb{R}^2 : 0 < R_1^2 \leq x^2 + y^2 \leq R_2^2\}$ to itself, keeping invariant both boundary circles, and rotating them in opposite directions, has at least two fixed points. Poincaré stated this theorem [11] and gave a proof for some particular cases. The theorem was subsequently proved in its full generality by Birkhoff [1, 2] (see also [3]). In order to make the theorem more suitable for applications, Ding [7] generalized it, allowing the boundary curves not to be circles or invariant. More precisely, he considered an annular region A bounded by an inner boundary C_1 and

*Supported by FCT, POCI/Mat/57258/2004.

†Supported by D.G.I. MTM2005-03483, Ministerio de Educación y Ciencia, Spain.

Mathematics Subject Classification: Primary 54H25.

an outer boundary C_2, two simple closed curves. We denote by D_1 and D_2 to the open, simply connected regions of the plane bounded respectively by C_1 and C_2, which are assumed to contain the origin. Consider the universal covering space $\mathcal{H} = \mathbb{R}\times]0,+\infty[= \{(\theta, r) \in \mathbb{R}^2 : r > 0\}$ of $\mathbb{R}^2\backslash\{(0,0)\}$, and the covering map $P : \mathcal{H} \to \mathbb{R}^2\backslash\{(0,0)\}$ provided by the polar coordinates

$$P(\theta, r) = (r\cos\theta, r\sin\theta)\,, \qquad\qquad (\theta, r) \in \mathcal{H}\,.$$

Given a continuous map $T : A \subset \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^2\backslash\{(0,0)\}$, a second continuous map $\tilde{T} : \tilde{A} = P^{-1}(A) \to \mathcal{H}$ is said to be *a lifting* of T if $P \circ \tilde{T} = T \circ P$.

Theorem 1.0 (Ding, [7])**.** *Let $T : A \subset \mathbb{R}^2 \to T(A) \subset \mathbb{R}^2\backslash\{(0,0)\}$ be an area-preserving homeomorphism. Suppose that:*

(i) the inner boundary C_1 is star-shaped about the origin;

(ii) T has a lifting $\tilde{T}$ given by

$$\tilde{T}(\theta, r) = (\theta + g(\theta, r), f(\theta, r))$$

such that $g(\theta, r) > 0$ on $\tilde{C}_1 = P^{-1}(C_1)$ and $g(\theta, r) < 0$ on $\tilde{C}_2 = P^{-1}(C_2)$;

(iii) there exists an area-preserving homeomorphism $T_0 : \bar{D}_2 \to \mathbb{R}^2$ which satisfies $T_0\big|_A = T$ and $(0,0) \in T_0(D_1)$.

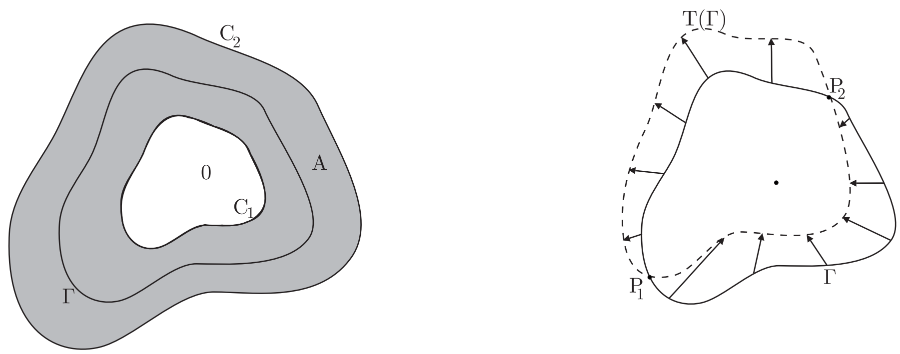
Then $\tilde{T}$ has at least two fixed points such that their images under P are two different fixed points of T.

This theorem has been widely applied, see for instance [4, 5, 6, 8, 9, 12] and the references therein. All of its assumptions were shown to be necessary in [7], except condition (i). Quoting Ding [7]: 'The condition (i) of the theorem is crucial for our proof. However, we doubt of its necessity for the theorem'. Subsequently, there have been efforts [10] in the literature to remove this assumption from Theorem 1.0. The aim of this paper is to construct a counterexample that shows that this condition is necessary.

As before, we denote by C_1 and C_2 the inner and outer boundaries of the annular region A, which are assumed to be simple closed curves. They delimit, respectively, simply connected open regions $D_1 \subset D_2 \subset \mathbb{R}^2$ containing the origin.

Theorem 1.1. *It is possible to find closed curves C_1 and C_2 in the conditions above and an area-preserving $\mathcal{C}^\infty$ diffeomorphism $T : A \subset \mathbb{R}^2 \to T(A) \subset \mathbb{R}^2\backslash\{(0,0)\}$ verifying (ii) and (iii), which has not fixed points.*

First of all we would like to show in an intuitive basis why it is reasonable to think that the star-shaped condition is necessary. Consider an area-preserving homeomorphism $T : A \subset \mathbb{R}^2 \to T(A) \subset \mathbb{R}^2 \backslash \{(0,0)\}$ in the conditions of Ding's theorem. Denote by Γ to the set of points of A that are radially moved by T, i.e. $\Gamma = \{(x,y) \in A : g(\theta,r) = 0 \text{ if } (\theta,r) \in P^{-1}(x,y)\}$. Clearly all fixed points of T are in Γ. On the other hand, the twist condition *(ii)* implies that Γ should intersect every continuous curve starting from C_1 and ending on C_2. It seems reasonable to think that, in some cases, Γ will be a Jordan curve whose interior region contains the origin, as pictured in Fig.1 *a*).



a) b)

Figure 1:

Assume for a moment that Γ is a star-shaped Jordan curve. In this case, the inequality

$$f(\theta,r) > r \text{ for all } (\theta,r) \in P^{-1}(\Gamma)\,, \tag{1}$$

would imply Γ to be contained into the interior region of its image, which is not possible because T is area-preserving. Similarly, it cannot happen that $f(\theta,r) < r$ for all $(\theta,r) \in P^{-1}(\Gamma)$ and we get at least two points $P_1 = P(\theta_1,r_1)$ and $P_2 = P(\theta_2,r_2)$ in Γ with $f(\theta_i,r_i) = r_i$, or, what is the same, two fixed points of T as predicted by Theorem 1.0, see Fig. 1 *b*).

Observe that the last reasoning was only possible because we assumed that Γ is star-shaped. If Γ is not star-shaped about the origin, then (1) does not imply Γ to be contained inside the interior region of its image, and thus, it does not contradict the area preserving condition of $T : A \subset \mathbb{R}^2 \to \mathbb{R}^2$, see Fig. 2 a). On the other hand, since g does not vanish outside Γ, condition *(ii)* implies that $g(\theta,r) > 0$ for any (θ,r) in the interior region delimited by this curve, while $g(\theta,r) < 0$ in the exterior, meaning that the restriction of T

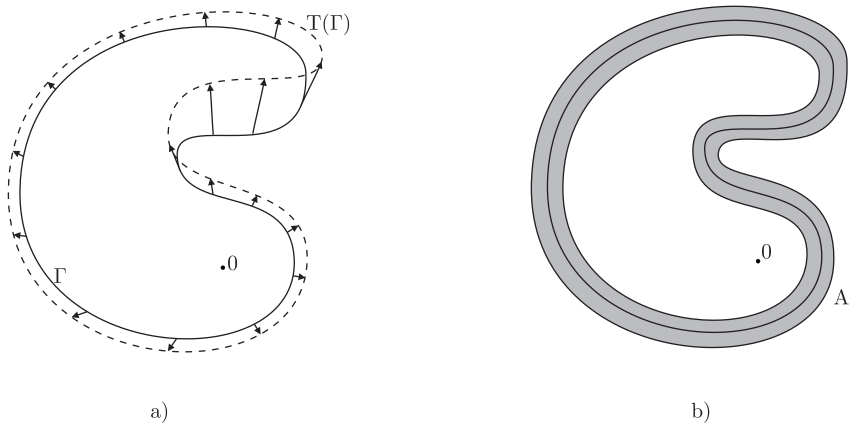


Figure 2:

to some small tubular neighborhood A of Γ in which $f(\theta, r) > r$ provides the example claimed by Theorem 1.1, see Fig. 2 b). We devote the remaining of the paper to make precise the details of this construction.

Before ending this introduction, we want to express our gratitude to Prof. Rafael Ortega, who read the first versions of this work, contributed to clarify it, put us in contact with each other, and encouraged us to write the paper.

2 Proof of Theorem 1.1

The lifting $\tilde{T}$ of the diffeomorphism T of the counterexample will be a time map of the flux of a suitable autonomous Hamiltonian system. The associated Hamiltonian will be defined on the universal covering space $\mathcal{H} = \mathbb{R}\times]0, +\infty[$ and 2π periodic in its first variable θ.

It will be convenient to establish some terminology. Given some subset $B \subset \mathbb{R}^2 \backslash \{(0,0)\}$, we shall denote by $\tilde{B}$ to its lifting to $\mathcal{H}$, i.e., $\tilde{B} = P^{-1}(B)$. It remains invariant after being translated by the vector $(2\pi, 0)$ i.e. $\tilde{B} + (2\pi, 0) = \tilde{B}$, and, throughout this paper, the subsets of $\mathcal{H}$ with this property will be simply called 2π-*periodic*. Observe that a set is 2π-periodic if and only if it coincides with the lifting of its image by P.

For instance, one can check that $\Gamma \subset \mathbb{R}^2 \backslash \{(0,0)\}$ is a closed curve for which the origin $(0,0)$ belongs to the bounded component of $\mathbb{R}^2 \backslash \Gamma$ if and only if $\tilde{\Gamma} = P^{-1}(\Gamma)$ is a connected and 2π-periodic curve in $\mathcal{H}$. This fact motivates us to consider this class of curves. Given some connected and 2π-periodic curve $\tilde{\Gamma} \subset \mathcal{H}$, the function $h : \mathcal{H} \to \mathbb{R}$ will be said to *change sign around* $\tilde{\Gamma}$ if there exists a 2π-periodic neighborhood $\mathcal{U} \subset \mathcal{H}$ of $\tilde{\Gamma}$ such that

$\mathcal{U}\backslash\tilde{\Gamma}$ has two connected components $\mathcal{U}_1$ and $\mathcal{U}_2$, and, further, either $h > 0$ on $\mathcal{U}_1$ and $h < 0$ on $\mathcal{U}_2$, or the reversed inequalities hold. Since $\tilde{\Gamma} = (\partial\mathcal{U}_1)\cap(\partial\mathcal{U}_2)$, h must be constantly zero on $\tilde{\Gamma}$.

Lemma 2.1. *There exist a $C^\infty(\mathcal{H})$ function $H : \mathcal{H} \to \mathbb{R}$ with*

$$H(\theta + 2\pi, r) = H(\theta, r)\,,\ \textit{for any}\ (\theta, r) \in \mathcal{H}\,, \tag{2}$$

and a connected, smooth, 2π-periodic curve $\tilde{\Gamma} \subset \mathcal{H}$ such that

- *(a) $\partial_r H$ changes sign around $\tilde{\Gamma}$.*
- *(b) $\partial_\theta H < 0$ on $\tilde{\Gamma}$.*

The construction of such a function H is postponed to Section 3. Let us rely now on this result to complete the proof of Theorem 1.1.

After multiplying $H = H(\theta, r)$ by some cut-off function of r with compact support contained in $]0, \infty[$, is not restrictive to assume that H vanishes outside the horizontal band $\mathbb{R}\times]1/M, M[$ for some $M > 0$. In this case, the support of H is compact on the cylinder $(\mathbb{R}/2\pi\mathbb{Z})\times]0, \infty[$ and the solutions of the Hamiltonian system

$$\left\{ \begin{array}{l} \theta' = \partial_r H(\theta, r) \\ r' = -\partial_\theta H(\theta, r) \end{array} \right. \tag{3}$$

are defined, and remain in $\mathcal{H}$, for all time. Given $\epsilon > 0$ we consider the associated time map $\tilde{T}_\epsilon : \mathcal{H} \to \mathcal{H}$, defined by

$$\tilde{T}_\epsilon(\theta, r) = (\phi_1(\epsilon; \theta, r), \phi_2(\epsilon; \theta, r))\,,$$

where $(\phi_1(\cdot; \theta, r), \phi_2(\cdot; \theta, r))$ denotes the solution of (3) starting from the point (θ, r) at time $t = 0$. We recall that, as a consequence of Liouville's theorem, these mappings are area-preserving. They are indeed $\mathcal{C}^\infty$ diffeomorphisms which verify $\tilde{T}_\epsilon(\theta + 2\pi, r) = \tilde{T}_\epsilon(\theta, r) + (2\pi, 0)$ for any $(\theta, r) \in \mathcal{H}$ and coincide with the identity outside the horizontal band $\mathbb{R}\times]1/M, M[$. Consequently, they can be seen as liftings of area-preserving diffeomorphisms $T_\epsilon : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^2\backslash\{(0,0)\}$ coinciding with the identity on some neighborhood of the origin and on some neighborhood of infinity.

Since, by assumption, $\partial_r H$ changes sign around $\tilde{\Gamma}$, we can find a 2π-periodic open neighborhood $\mathcal{U} \subset \mathcal{H}$ of $\tilde{\Gamma}$ such that $\mathcal{U}\backslash\tilde{\Gamma}$ has two connected components $\mathcal{U}_1$ and $\mathcal{U}_2$ with $\partial_r H > 0$ on $\mathcal{U}_1$ and $\partial_r H < 0$ on $\mathcal{U}_2$. After possibly replacing H by $-H$, we may further assume that $\mathcal{U}_1$ is the lower connected component of $\mathcal{U}\backslash\tilde{\Gamma}$ and that $\mathcal{U}_2$ is the upper one. Elementary arguments based in the use of tubular neighborhoods of $\tilde{\Gamma}$ show that it is possible to find 2π-periodic and connected curves $\tilde{C}_i \subset \mathcal{U}_i$ close enough to $\tilde{\Gamma}$ so that $\partial_\theta H < 0$ on the closure of the open region $\tilde{A}$ of $\mathcal{H}$ bounded by these curves. They are destined to be the liftings in $\mathcal{H}$ of the closed curves C_i and the annular region A which Theorem 1.1 refers to. Fix now some $\epsilon_* > 0$ small enough so that

(c) $\partial_t \phi_1(t;\theta,r) = \partial_r H(\phi_1(t;\theta,r),\phi_2(t;\theta,r)) > 0$ for any $(\theta,r) \in \tilde{C}_1$ and any $t \in [0,\epsilon_*]$.

(d) $\partial_t \phi_1(t;\theta,r) = \partial_r H(\phi_1(t;\theta,r),\phi_2(t;\theta,r)) < 0$ for any $(\theta,r) \in \tilde{C}_2$ and any $t \in [0,\epsilon_*]$.

(e) $\partial_t \phi_2(t;\theta,r) = -\partial_\theta H(\phi_1(t;\theta,r),\phi_2(t;\theta,r)) > 0$ if (θ,r) belong to the closure of $\tilde{A}$ and $t \in [0,\epsilon_*]$.

Finally, define

$$\tilde{T}(\theta,r) = \tilde{T}_{\epsilon_*}(\theta,r) = (\phi_1(\epsilon_*;\theta,r),\phi_2(\epsilon_*;\theta,r)) = (\theta + g(\theta,r), f(\theta,r)),$$

and observe that

$$g(\theta,r) = \phi_1(\epsilon_*;\theta,r) - \phi_1(0;\theta,r) = \int_0^{\epsilon_*} \partial_t \phi_1(t;\theta,r)\,dt\,,$$

which, by virtue of *(c)* and *(d)*, is positive if $(\theta,r) \in \tilde{C}_1$ and negative if $(\theta,r) \in \tilde{C}_2$, meaning that $\tilde{T}$ rotates both boundary curves in opposite directions. Finally, condition *(e)* implies

$$f(\theta,r) - r = \phi_2(\epsilon_*;\theta,r) - \phi_2(0;\theta,r) = \int_0^{\epsilon_*} \partial_t \phi_2(t;\theta,r)\,dt > 0\,,$$

for any (θ,r) in the closure of $\tilde{A}$, and we deduce that neither $\tilde{T}$ has fixed points on this set nor the diffeomorphism $T : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^2\backslash\{(0,0)\}$ from which it is the lifting, has fixed points on the closure of the annular region $A = P(\tilde{A})$.

3 Constructing the Hamiltonian

We devote this Section to construct the function $H : \mathcal{H} \to \mathbb{R}$ and the curve $\tilde{\Gamma} \subset \mathcal{H}$ whose existence was stated in Lemma 2.1. Observe that $\tilde{\Gamma}$ cannot be the graph of a function of θ, since *(a)* and *(b)* would imply H to be strictly decreasing along $\tilde{\Gamma}$, contradicting its periodicity in the variable θ. What is the same, the projection $\Gamma = P(\tilde{\Gamma})$ of this curve on $\mathbb{R}^2\backslash\{(0,0)\}$ cannot be star-shaped with respect to the origin.

We consider the following 2π-periodic subsets of the covering space $\mathcal{H}$ (see Fig. 3):

$$L_- := \Big(\{-2\pi/3\} \times]0,2] \cup [-2\pi/3, \pi/3] \times \{2\} \Big) + (2\pi, 0)\mathbb{Z}\,,$$
$$L_+ := \{(-t, 1/r) : (t,r) \in L_-\}\ .$$

Observe that they do not disconnect $\mathcal{H}$ i.e. the open set $W := \mathcal{H} \backslash (L_- \cup L_+)$

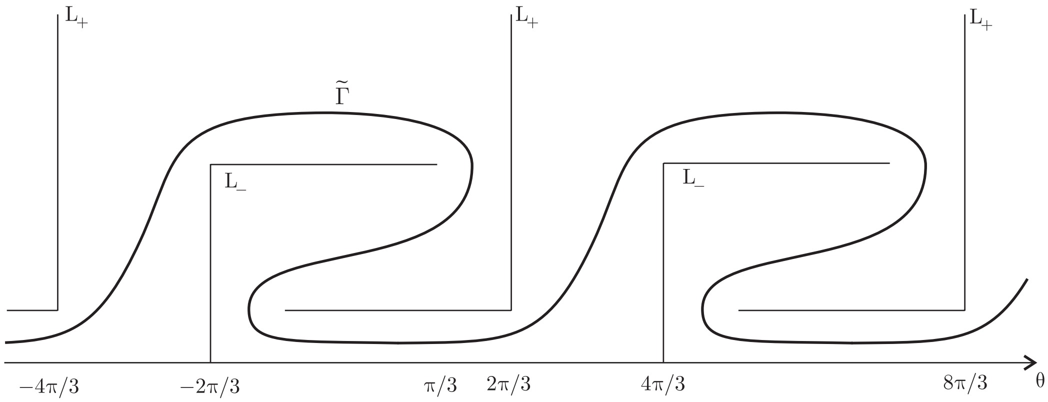


Figure 3:

is still connected. Consequently, it is arcwise connected i.e. each two points of W may be joined by means of a curve. Lemma 3.1 states the existence of a smooth curves $\tilde{\Gamma}$ travelling through W (see Fig. 3):

Lemma 3.1. *There exists a $\mathcal{C}^\infty$, connected and simple curve $\tilde{\Gamma} \subset W$ which is 2π-periodic.*

The construction of such a curve is elementary and will not be needed in our subsequent arguments, so that it will be skipped. Choose now some smooth parametrization $\gamma : \mathbb{R} \to \tilde{\Gamma} \subset \mathcal{H}$ with $\gamma' \neq 0$ on $\mathbb{R}$ and $\gamma(t+2\pi) = \gamma(t) + (2\pi, 0)$ for any $t \in \mathbb{R}$. It can be seen as chart from the circumference $\mathbb{R}/2\pi\mathbb{Z}$ into the projection of $\tilde{\Gamma}$ on the cylinder $(\mathbb{R}/2\pi\mathbb{Z}) \times]0, +\infty[$. Now, the compactness of $\mathbb{R}/2\pi\mathbb{Z}$ makes it possible to construct uniform tubular neighborhoods for this chart. This means that, if $\epsilon > 0$ is small enough, the function $\psi : \mathbb{R} \times]-\epsilon, \epsilon[\to \mathbb{R}^2$ defined by

$$\psi(t,x) := \gamma(t) + x \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma'(t)\,,$$

is a $\mathcal{C}^\infty$ diffeomorphism into its image, which, after possibly replacing ϵ by some smaller number, we may assume contained in W. We define

$$\mathcal{U} = \psi(\mathbb{R}\times]-\epsilon/2, \epsilon/2[)\,, \tag{4}$$

which is a 2π-periodic and open subset of $\mathcal{H}$. Moreover, $\bar{\mathcal{U}} \subset \mathcal{H}$. Observe that $\tilde{\Gamma} = \gamma(\mathbb{R}) = \{\psi(t,0) : t \in \mathbb{R}\}$ is connected and $\mathcal{U}\backslash\tilde{\Gamma}$ has two connected components,

$$\mathcal{U}_1 := \{\psi(t,x) : t \in \mathbb{R}, x \in]-\epsilon/2, 0[\}\,, \quad \mathcal{U}_2 := \{\psi(t,x) : t \in \mathbb{R}, x \in]0, \epsilon/2[\}\,.$$

We shall start by showing 'half' of Lemma 3:

Lemma 3.2. *There exists a $\mathcal{C}^\infty$ function $H_1 : \mathcal{H} \to \mathbb{R}$ such that*

1. *$H_1(\theta + 2\pi, r) = H_1(\theta, r)$ for any $(\theta, r) \in \mathcal{H}$.*
2. *$\partial_r H_1 > 0$ on $\mathcal{U}_1$ and $\partial_r H_1 < 0$ on $\mathcal{U}_2$.*

Proof. Choose some $\mathcal{C}^\infty$, cut-off function $m : \mathbb{R} \to \mathbb{R}$ with

$$m \equiv 1 \text{ in } [-1/2, 1/2],\ m \equiv 0 \text{ on some neighborhood of } \mathbb{R}\backslash]-1, 1[\,, \tag{5}$$

and define $h_1 : \mathbb{R} \times [0, +\infty[\to \mathbb{R}$ by

$$h_1(\psi(t,x)) := -x\, m(x/\epsilon) \text{ if } (t,x) \in \mathbb{R}\times]-\epsilon, \epsilon[\,,$$

$$h_1(\theta, r) := 0 \text{ if } (\theta, r) \notin \psi(\mathbb{R}\times]-\epsilon, \epsilon[)\,.$$

Observe that h_1 is a $\mathcal{C}^\infty$ function on the upper half plane $\mathbb{R} \times [0, +\infty[$ which is 2π-periodic in its first variable and verifies $h_1 > 0$ on $\mathcal{U}_1$ and $h_1 < 0$ on $\mathcal{U}_2$. Now, it suffices to define

$$H_1(\theta, r) := \int_0^r h_1(\theta, s)\, ds\,, \qquad (\theta, r) \in \mathcal{H}\,.$$

□

Observe that no horizontal straight line $\mathbb{R} \times \{r\}$ is contained in W. For this reason, it is possible to find functions $H_2 : W \to \mathbb{R}$ which combine the periodicity with respect to θ with the positivity of the partial derivative $\partial_\theta H$. We show it below:

Lemma 3.3. *There exists a $\mathcal{C}^\infty$ function $H_2 : W \to \mathbb{R}$ such that*

1. *$H_2(\theta + 2\pi, r) = H_2(\theta, r)$ for any $(\theta, r) \in W$.*

2. $\partial_\theta H_2 \geq 1$ *on* W.

3. $\partial_r H_2 \equiv 0$ *on* W.

Proof. We start by defining H_2 on the set

$$S_+ :=]-4\pi/3, -2\pi/3[\times]0, +\infty[\cup[-2\pi/3, 2\pi/3[\times]2, +\infty[\cup\{-4\pi/3\}\times]0, 1/2[$$

by the rule $H_2(\theta, r) := \theta$, see Fig. 4 *a*).

Choose next some $\mathcal{C}^\infty$ function $q : [-2\pi/3, 2\pi/3] \to \mathbb{R}$ with $q' \geq 1$ and

$$q(\theta) = \begin{cases} \theta - 2\pi & \text{if } -2\pi/3 \leq \theta \leq -\pi/3\,, \\ \theta & \text{if } \pi/3 \leq \theta \leq 2\pi/3\,, \end{cases}$$

(see Fig. 4 *b*)), and define H_2 on $S_0 :=]-2\pi/3, 2\pi/3[\times]1/2, 2[\cup]\pi/3, 2\pi/3[\times\{2\}$ by $H_2(\theta, r) := q(\theta)$.

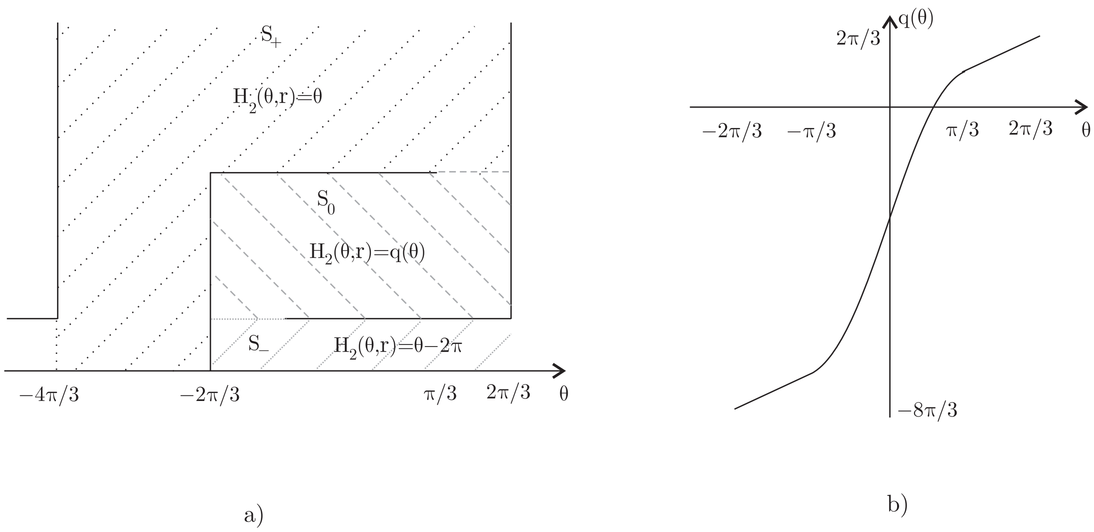


Figure 4:

Finally, let H_2 be defined on

$$S_- :=]-2\pi/3, 2\pi/3[\times]0, 1/2[\cup]-2\pi/3, -\pi/3[\times\{1/2\}$$

by the rule $H_2(\theta, r) := \theta - 2\pi$.

Observe that $S_+ \cup S_0 \cup S_- = W \cap \big([-4\pi/3, 2\pi/3[\times]0, +\infty[\big)$ and H_2 can be uniquely extended to $\mathcal{H}$ in such a way that *1.* holds. Moreover, this extension is $\mathcal{C}^\infty$ and it satisfies *2.* and *3.* because these conditions do hold in S_+, S_0, and S_-. $\square$

Of course, Lemma 3.3 does not hold if the set W is replaced by $\mathcal{H}$. However, multiplying H_2 by a convenient cut-off function, we may get a similar result on $\mathcal{H}$ if *2.* and *3.* are required only locally near $\tilde{\Gamma}$. In the next corollary $\mathcal{U}$ denotes the tubular neighborhood of $\tilde{\Gamma}$ constructed in (4):

Corollary 3.4. *There exists a $\mathcal{C}^\infty$ function $H_3 : \mathcal{H} \to \mathbb{R}$ such that*

1. $H_3(\theta + 2\pi, r) = H_3(\theta, r)$, *for any* $(\theta, r) \in \mathcal{H}$.
2. $\partial_\theta H_3 \geq 1$ *on* $\mathcal{U}$.
3. $\partial_r H_3 \equiv 0$ *on* $\mathcal{U}$.

Proof. Let the $\mathcal{C}^\infty$ function $m : \mathbb{R} \to \mathbb{R}$ be chosen as in (5), and consider the $\mathcal{C}^\infty$ mapping $\rho : \mathcal{H} \to \mathbb{R}$ defined by

$$\rho(\psi(t, x)) = m(x/\epsilon) \text{ if } (t, x) \in \mathbb{R}\times]-\epsilon, \epsilon[\,,$$

$$\rho(\theta, r) = 0 \text{ if } (\theta, r) \notin \psi(\mathbb{R}\times]-\epsilon, \epsilon[)\,.$$

Now, observe that $H_3(\theta, r) := H_2(\theta, r)\rho(\theta, r)$ is a $\mathcal{C}^\infty$ function on $\mathcal{H}$ which coincides with H_2 on $\mathcal{U}$. Conditions *1.*, *2.* and *3.* follow. □

We are now ready to prove Lemma 2.1. Together with the discussion carried out in Section 2, this will complete the proof of Theorem 1.1:

Proof of Lemma 2.1. Let the function $H_1 : \mathcal{H} \to \mathbb{R}$ be given by Lemma 3.2. When seen on the cylinder $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$, the curve $\tilde{\Gamma}$ becomes compact, so that $\partial_\theta H_1$ is bounded there. Accordingly, we can find a constant $k > 0$ such that $\partial_\theta H_1 < k$ on $\tilde{\Gamma}$. Let the function $H_3 : \mathcal{H} \to \mathbb{R}$ be as given by Corollary 3.4 and define $H := H_1 - kH_3$. Then, $\partial_\theta H = \partial_\theta H_1 - k\partial_\theta H_3 < 0$ on $\tilde{\Gamma}$, while $\partial_r H = \partial_r H_1$ changes sign around this curve. The result follows. □

References

[1] G.D. Birkhoff, *Proof of Poincaré's last geometric theorem*, Trans. Am. Math. Soc., **18** (1913), 14-22. (Collected Mathematical Papers, Vol. 1, pp. 673-681. Dover, New York, 1968).

[2] G.D. Birkhoff, *An extension of the Poincaré's last geometric theorem*, Acta Math., **47** (1925), 297-311. (Collected Mathematical Papers, Vol. 2, pp. 252-266. Dover, New York, 1968).

[3] M. Brown, W.D. Neumann, *Proof of the Poincaré-Birkhoff fixed point theorem*, Michigan Math.J., **24** (1977), 21-31.

[4] M. Del Pino, R. Manásevich, A. Murua, *On the number of 2π periodic solutions for $u''+g(u)=s(1+h(t))$ using the Poincaré- Birkhoff theorem*, J. Diff. Eqns., **95** (1992), 240-258.

[5] T. Ding, *An infinite class of periodic solutions of periodically perturbed Duffing equations at resonance.*, Proc. Am. Math. Soc., **86** (1982), 47-54.

[6] T. Ding, R. Iannacci, F. Zanolin, *Existence and multiplicity results for periodic solutions of semilinear Duffing equations*, J. Diff. Eqns.,**105** (1993), 364-409.

[7] W-Y. Ding, *A generalization of the Poincaré-Birkhoff theorem*, Proc. Am. Math. Soc., **88** (1983), 341-346.

[8] A. Fonda, R. Manásevich, F. Zanolin, *Subharmonic solutions for some second-order differential equations with singularities*, SIAM J. Math. Anal., **24** (1993), 1294-1311.

[9] F. Hausrath, R. Manásevich, *Periodic solutions of a periodically perturbed Lotka-Volterra equation using the Poincaré-Birkhoff theorem*, J. Math. Anal. Appl., **157** (1991), 1-9.

[10] Y. Li, Z.H. Lin, *A constructive proof of the Poincaré-Birkhoff theorem*, Trans. Amer. Math. Soc. 347 (1995), no. 6, 2111-2126.

[11] H. Poincaré, *Sur un théoréme de geométrie*, Rend. Circ. Mat. Palermo, **33** (1912), 375-391.

[12] C. Rebelo, *A note on the Poincaré-Birkhoff fixed point theorem and periodic solutions of planar systems*, Nonlinear Analysis, TMA, **29** (1997), 291-311.