On divisors of pseudovarieties generated by some classes of full transformation semigroups

Vítor H. Fernandes*

Departamento de Matemática
Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa
Monte da Caparica
2829-516 Caparica
Portugal
E-mail: *vhf@fct.unl.pt*

Abstract

In this paper we present a division theorem for the pseudovariety of semigroups OD [OR] generated by all semigroups of order-preserving or order-reversing [orientation-preserving or orientation-reversing] full transformations on a finite chain.

2000 Mathematics Subject Classification: 20M20, 20M07.

Keywords: transformations, order-preserving, orientation-preserving, order-reversing, orientation-reversing, pseudovarieties.

1 Introduction and preliminaries

In the 1987 "Szeged International Semigroup Colloquium" J.-E. Pin asked for an *effective* description of the pseudovariety (i.e. an algorithm to decide whether or not a finite semigroup belongs to the pseudovariety) of semigroups O generated by all semigroups of order-preserving full transformations on a finite chain. This problem only had essential progresses after 1995. First, Higgins [23] proved that O is self-dual and does not contain all $\mathcal{R}$-trivial semigroups (and so O is properly contained in A, the pseudovariety of all finite aperiodic semigroups), although every finite band belongs to O. Next, Vernitskii and Volkov [28] generalised Higgins's result by showing that every finite semigroup whose idempotents form an ideal is in O and in [11] the author proved that the pseudovariety of semigroups

* The author was (partially) supported by FCT and FEDER whithin the project POCTI-ISFL-1-143 of the Centro de Álgebra da Universidade de Lisboa.

POI generated by all semigroups of injective order-preserving partial transformations on a finite chain is a (proper) subpseudovariety of O. On the other hand, Almeida and Volkov [1] showed that the interval $[\mathsf{O},\mathsf{A}]$ of the lattice of all pseudovarieties of semigroups has the cardinality of the continuum and Repnitskiĭ and Volkov [27] proved that O is not finitely based. In fact, moreover, Repnitskiĭ and Volkov proved in [27] that any pseudovariety of semigroups V such that $\mathsf{POI} \subseteq \mathsf{V} \subseteq \mathsf{O} \vee \mathsf{R} \vee \mathsf{L}$, where R and L are the pseudovarieties of semigroups of all $\mathcal{R}$-trivial semigroups and of all $\mathcal{L}$-trivial semigroups, respectively, is not finitely based. Another contribution to the resolution of Pin's problem was given by the author [17] who showed that O contains all semidirect products of a chain (considered as a semilattice) by a semigroup of injective order-preserving partial transformations on a finite chain. Nevertheless, Pin's question is still unanswered.

The pseudovariety OP generated by all semigroups of orientation-preserving full transformations on a finite chain was studied by Catarino and Higgins in [6]. They showed that OP is self-dual and contains the join of O with the pseudovariety of all finite commutative monoids. Moreover, Catarino and Higgins also proved in [6] that the interval between these two pseudovarieties contains a chain of pseudovarieties isomorphic to the chain of real numbers. A division theorem for OP was presented by the author in [16]. He proved that the pseudovariety POPI generated by all semigroups of injective orientation-preserving partial transformations on a finite chain is a (proper) subpseudovariety of OP.

Semigroups of order-preserving transformations have long been considered in the literature. In 1962, Aĭzenštat [2] and Popova [26] exhibited presentations for $\mathcal{O}_n$, the monoid of all order-preserving full transformations on a chain with n elements, and for $\mathcal{PO}_n$, the monoid of all order-preserving partial transformations on a chain with n elements. Some years later, in 1971, Howie [24] studied some combinatorial and algebraic properties of $\mathcal{O}_n$ and, in 1992, Gomes and Howie [22] revisited the monoids $\mathcal{O}_n$ and $\mathcal{PO}_n$. More recently, the injective counterpart of $\mathcal{O}_n$, i.e. the monoid $\mathcal{POI}_n$ of all injective members of $\mathcal{PO}_n$, has been object of study by the author in several papers [11, 12, 14, 15, 17] and also by Cowan and Reilly [8].

On the other hand, the notion of an orientation-preserving transformation was introduced by McAlister in [25] and, independently, by Catarino and Higgins in [5]. The monoid $\mathcal{OP}_n$, of all orientation-preserving full transformations on a chain with n elements, was also considered by Catarino in [4] and by Arthur and Ruškuc in [3]. The injective counterpart of $\mathcal{OP}_n$, i.e. the monoid $\mathcal{POPI}_n$ of all injective orientation-preserving partial transformations on a chain with n elements, was studied by the first author in [13, 16].

Recently, the author together with Gomes and Jesus [18] exhibited presentations for the monoids $\mathcal{PODI}_n$ of all injective order-preserving or order-reversing partial transformations on a chain with n elements, and for the monoid $\mathcal{PORI}_n$ of all injective orientation-preserving or orientation-reversing partial transformations on a chain with n elements. The same authors in [19] also gave presentations for the monoid $\mathcal{OD}_n$ of all order-preserving or order-reversing full transformations on a chain with n element; for the monoid $\mathcal{POD}_n$ of all order-preserving or order-reversing partial transformations on a chain with n elements; for the monoid $\mathcal{POP}_n$ of all orientation-preserving partial transformations on a chain with n elements; and for the monoid $\mathcal{POR}_n$ of all orientation-preserving or orientation-reversing partial transformations on a chain with n elements. The lattice of the congruences of some of these monoids were studied in [20, 21] by the same authors.

Together with Delgado the author [9, 10] have computed the abelian kernels of the monoids $\mathcal{POI}_n$, $\mathcal{POPI}_n$, $\mathcal{PODI}_n$ and $\mathcal{PORI}_n$. More recently, the same authors together with Cordeiro determined all relative abelian kernels of these four monoids [7].

Next, we will introduce or precise some definitions.

Denote by $\mathcal{PT}_n$ [$\mathcal{T}_n$] the monoid of all partial [full] transformations of a set with n elements, say $X_n = \{1, 2, \ldots, n\}$, and by $\mathcal{I}_n$ the symmetric inverse monoid, i.e. the submonoid of $\mathcal{PT}_n$ of all injective (partial) transformations of X_n.

From now on, we consider X_n as a chain with the usual order: $X_n = \{1 < 2 < \cdots < n\}$.

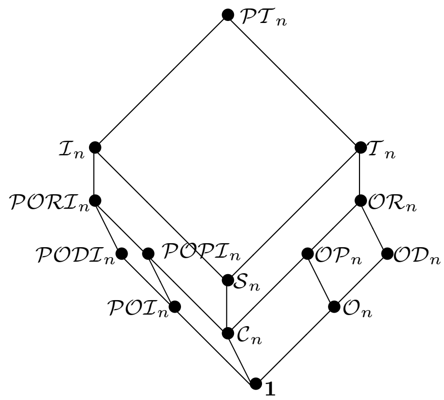
We say that a transformation s in $\mathcal{PT}_n$ is *order-preserving* [*order-reversing*] if, for all $x, y \in \mathrm{Dom}(s)$, $x \leq y$ implies $xs \leq ys$ [$xs \geq ys$]. Clearly, the product of two order-preserving transformations or of two order-reversing transformations is order-preserving and the product of an order-preserving transformation by an order-reversing transformation is order-reversing.

Denote by $\mathcal{O}_n$ [$\mathcal{POI}_n$] the submonoid of $\mathcal{T}_n$ [$\mathcal{I}_n$] whose elements are order-preserving and by $\mathcal{OD}_n$ [$\mathcal{PODI}_n$] the submonoid of $\mathcal{T}_n$ [$\mathcal{I}_n$] whose elements are either order-preserving or order-reversing.

Next, let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t ($t \geq 0$) elements from the chain X_n. We say that a is *cyclic* [*anti-cyclic*] if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ [$a_i < a_{i+1}$], where a_{t+1} denotes a_1. Let $s \in \mathcal{PT}_n$ and suppose that $\mathrm{Dom}(s) = \{a_1, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < \cdots < a_t$. We say that s is an *orientation-preserving* [*orientation-reversing*] transformation if the sequence of its images $(a_1 s, \ldots, a_t s)$ is cyclic [anti-cyclic]. It is also clear that the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving and the product of an orientation-preserving transformation by an orientation-reversing transformation is orientation-reversing.

Denote by $\mathcal{OP}_n$ [$\mathcal{POPI}_n$] the submonoid of $\mathcal{T}_n$ [$\mathcal{I}_n$] whose elements are orientation-preserving and by $\mathcal{OR}_n$ [$\mathcal{PORI}_n$] the submonoid of $\mathcal{T}_n$ [$\mathcal{I}_n$] whose elements are either orientation-preserving or orientation-reversing.

The following diagram, with respect to the inclusion relation, clarifies the relationship between these various monoids:



(**1** denotes the trivial monoid, $\mathcal{C}_n$ the cyclic group of order n and $\mathcal{S}_n$ the symmetric group on X_n).

Recall that a pseudovariety of semigroups is a class of finite semigroups closed under formation of finite direct products, subsemigroups and homomorphic images.

Let S and T be two semigroups. A *division of semigroups* $\tau : S \longrightarrow T$ is a relation from S into T (i.e. a function from S into the power set of T) such that:

- For all $s \in S$, $(s)\tau \neq \emptyset$, i.e. s is *totally defined*;
- For all $s_1, s_2 \in S$, $(s_1)\tau(s_2)\tau \subseteq (s_1 s_2)\tau$, i.e. s is a *relation of semigroups*; and
- For all $s_1, s_2 \in S$, $(s_1)\tau \cap (s_2)\tau \neq \emptyset \implies s_1 = s_2$, i.e. s is *injective*.

We say that S *divides* T if there exists a division of semigroups $\tau : S \longrightarrow T$.

Notice that, given a family $\mathcal{X}$ of finite semigroups, it is easy to show that the pseudovariety of semigroups generated by $\mathcal{X}$ is the class of all semigroups that divide a finite direct product of members of $\mathcal{X}$.

With the above notation, we have that:

- O is the pseudovariety of semigroups generated by $\{\mathcal{O}_n \mid n \in \mathbb{N}\}$;
- POI is the pseudovariety of semigroups generated by $\{\mathcal{POI}_n \mid n \in \mathbb{N}\}$;
- OP is the pseudovariety of semigroups generated by $\{\mathcal{OP}_n \mid n \in \mathbb{N}\}$; and
- POPI is the pseudovariety of semigroups generated by $\{\mathcal{POPI}_n \mid n \in \mathbb{N}\}$.

Next, also define:

- OD as the pseudovariety of semigroups generated by $\{\mathcal{OD}_n \mid n \in \mathbb{N}\}$;
- PODI as the pseudovariety of semigroups generated by $\{\mathcal{PODI}_n \mid n \in \mathbb{N}\}$;
- OR as the pseudovariety of semigroups generated by $\{\mathcal{OR}_n \mid n \in \mathbb{N}\}$; and
- PORI as the pseudovariety of semigroups generated by $\{\mathcal{PORI}_n \mid n \in \mathbb{N}\}$.

Now, we can state the main results of this paper, which are the analogues of the result presented by the author in [11] (and in [16]).

Theorem 1 *Every semigroup of injective order-preserving or order-reversing partial transformations on a finite chain belongs to* OD.

Theorem 2 *Every semigroup of injective orientation-preserving or orientation-reversing partial transformations on a finite chain belongs to* OR.

2 The proofs

Let X be a finite set and let Y be a subset of X. Denote by $\mathcal{PT}(X)$ [$\mathcal{PT}(Y)$] the monoid of all partial transformations of X [Y]. Let S be a subsemigroup of $\mathcal{PT}(Y)$ and let T be a subsemigroup of $\mathcal{PT}(X)$. Define a relation $\tau : S \longrightarrow T$ by

$$(s)\tau = \{t \in T \mid Yt^{-1} \subseteq Y \text{ and } t|_{Yt^{-1}} = s\},$$

for all $s \in S$. Notice that, Yt^{-1} denotes the set $\{x \in \mathrm{Dom}(t) \mid (x)t \in Y\}$ and $t|_{Yt^{-1}}$ the restriction of the map t to the set Yt^{-1}. Hence, we have:

Proposition 2.1 *With the foregoing, $\tau : S \longrightarrow T$ is an injective relation of semigroups. Moreover, if τ is completely defined then S divides T.*

Proof. First, notice that τ is clearly an injective relation. Indeed, given $s_1, s_2 \in S$ such that $(s_1)\tau \cap (s_2)\tau \neq \emptyset$, we can take $t \in (s_1)\tau \cap (s_2)\tau$ and so, in particular, we have $s_1 = t|_{Yt^{-1}} = s_2$.

Now, let $s_1, s_2 \in S$. We will prove that $(s_1)\tau(s_2)\tau \subseteq (s_1s_2)\tau$. If $(s_1)\tau = \emptyset$ or $(s_2)\tau = \emptyset$ then this inclusion is obvious. Thus, we can suppose that $(s_1)\tau \neq \emptyset$ and $(s_2)\tau \neq \emptyset$.

Let $t_1 \in (s_1)\tau$ and $t_2 \in (s_2)\tau$. Then $Yt_1^{-1} \subseteq Y$, $Yt_2^{-1} \subseteq Y$, $t_1|_{Yt_1^{-1}} = s_1$ and $t_2|_{Yt_2^{-1}} = s_2$. In order to prove that $t_1t_2 \in (s_1s_2)\tau$, we must show that $Y(t_1t_2)^{-1} \subseteq Y$ and $(t_1t_2)|_{Y(t_1t_2)^{-1}} = s_1s_2$.

Regarding the first condition, let $x \in Y(t_1t_2)^{-1}$. Then $(x)(t_1t_2) \in Y$, i.e. $((x)t_1)t_2 \in Y$, whence $(x)t_1 \in Yt_2^{-1} \subseteq Y$, i.e. $(x)t_1 \in Y$. It follows that $x \in Yt_1^{-1} \subseteq Y$ and so $x \in Y$. Hence, $Y(t_1t_2)^{-1} \subseteq Y$.

Next, we want to show that $(t_1t_2)|_{Y(t_1t_2)^{-1}} = s_1s_2$. We begin by proving that $(t_1t_2)|_{Y(t_1t_2)^{-1}}$ and s_1s_2 have the same domain. Let $x \in \mathrm{Dom}(s_1s_2)$. Then $x \in \mathrm{Dom}(s_1) = \mathrm{Dom}(t_1) \cap Yt_1^{-1}$ and $(x)s_1 \in \mathrm{Dom}(s_2) = \mathrm{Dom}(t_2) \cap Yt_2^{-1}$. Thus $(x)t_1 = (x)s_1$, whence $(x)t_1 \in \mathrm{Dom}(t_2) \cap Yt_2^{-1}$ and so $((x)t_1)t_2 \in Y$, i.e. $x \in Y(t_1t_2)^{-1}$. Hence $x \in \mathrm{Dom}(t_1t_2) \cap Y(t_1t_2)^{-1} = \mathrm{Dom}\big((t_1t_2)|_{Y(t_1t_2)^{-1}}\big)$ and so $\mathrm{Dom}(s_1s_2) \subseteq \mathrm{Dom}\big((t_1t_2)|_{Y(t_1t_2)^{-1}}\big)$. Conversely, let $x \in \mathrm{Dom}\big((t_1t_2)|_{Y(t_1t_2)^{-1}}\big)$. Then $x \in \mathrm{Dom}(t_1)$, $(x)t_1 \in \mathrm{Dom}(t_2)$ and $x \in Y(t_1t_2)^{-1}$, i.e. $(x)(t_1t_2) \in Y$. Hence $(x)t_1 \in Yt_2^{-1} \subseteq Y$ and so $(x)t_1 \in Y$, whence $x \in Yt_1^{-1}$. Then, we have $x \in \mathrm{Dom}(t_1) \cap Yt_1^{-1} = \mathrm{Dom}(s_1)$, from which it follows that $(x)t_1 = (x)s_1$, and so $(x)s_1 \in \mathrm{Dom}(t_2) \cap Yt_2^{-1} = \mathrm{Dom}(s_2)$. Thus $x \in \mathrm{Dom}(s_1s_2)$ and so $\mathrm{Dom}\big((t_1t_2)|_{Y(t_1t_2)^{-1}}\big) \subseteq \mathrm{Dom}(s_1s_2)$. Hence $\mathrm{Dom}(s_1s_2) = \mathrm{Dom}\big((t_1t_2)|_{Y(t_1t_2)^{-1}}\big)$. Finally, if $x \in \mathrm{Dom}(s_1s_2)$ then $x \in \mathrm{Dom}(s_1)$ and $(x)s_1 \in \mathrm{Dom}(s_2)$, whence $x \in \mathrm{Dom}(t_1)$ and $(x)t_1 = (x)s_1 \in \mathrm{Dom}(t_2)$ and so $(x)(t_1t_2) = ((x)t_1)t_2 = ((x)t_1)s_2 = ((x)s_1)s_2 = (x)(s_1s_2)$. Therefore $(t_1t_2)|_{Y(t_1t_2)^{-1}} = s_1s_2$, as required. $\square$

The proof of Theorem 1

Let us consider the chain $X = \{\bar{0} < 1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}\}$ (with $2n+1$ elements) and its subchain $Y = \{1 < 2 < \cdots < n\}$. Consider the semigroups $\mathcal{OD}_{2n+1}$ and $\mathcal{PODI}_n$ built over X and Y, respectively, and the relation $\tau : \mathcal{PODI}_n \longrightarrow \mathcal{OD}_{2n+1}$ defined by

$$(s)\tau = \{t \in \mathcal{OD}_{2n+1} \mid Yt^{-1} \subseteq Y \text{ and } t|_{Yt^{-1}} = s\},$$

for all $s \in \mathcal{PODI}_n$. We claim that τ is completely defined.

Indeed, for an element $s \in \mathcal{PODI}_n$ such that $\mathrm{Dom}(s) = \{i_1 < i_2 < \cdots < i_k\}$ $(1 \le k \le n)$, define $\overline{s} \in \mathcal{OD}_{2n+1}$ by

$$(x)\overline{s} = \begin{cases} \overline{0} & \text{if } \overline{0} \le x < i_1 \\ (i_p)s & \text{if } x = i_p, \text{ for some } 1 \le p \le k \\ \overline{(i_p)s} & \text{if } i_p < x < i_{p+1}, \text{ for some } 1 \le p \le k-1 \\ \overline{n} & \text{if } i_k < x \le \overline{n}\ , \end{cases}$$

if s is order-preserving, and by

$$(x)\overline{s} = \begin{cases} \overline{n} & \text{if } \overline{0} \le x < i_1 \\ (i_p)s & \text{if } x = i_p, \text{ for some } 1 \le p \le k \\ \overline{(i_{p+1})s} & \text{if } i_p < x < i_{p+1}, \text{ for some } 1 \le p \le k-1 \\ \overline{0} & \text{if } i_k < x \le \overline{n}\ , \end{cases}$$

if s is order-reversing and $k \ge 2$. If $s \in \mathcal{PODI}_n$ is the empty transformation, then define $\overline{s}$ as the constant transformation of $\mathcal{OD}_{2n+1}$ with image $\{\overline{0}\}$.

Examples 2.2 Let $n = 7$. Then:

- If $s = \begin{pmatrix} 2 & 4 & 5 \\ 1 & 2 & 7 \end{pmatrix}$ then $\overline{s} = \begin{pmatrix} \overline{0} & 1 & \overline{1} & \mathbf{2} & \overline{2} & 3 & \overline{3} & \mathbf{4} & \overline{4} & \mathbf{5} & \overline{5} & 6 & \overline{6} & 7 & \overline{7} \\ \overline{0} & \overline{0} & \overline{0} & \mathbf{1} & \overline{1} & \overline{1} & \overline{1} & \mathbf{2} & \overline{2} & \mathbf{7} & \overline{7} & \overline{7} & \overline{7} & \overline{7} & \overline{7} \end{pmatrix}$;
- If $s = \begin{pmatrix} 2 & 4 & 5 \\ 7 & 2 & 1 \end{pmatrix}$ then $\overline{s} = \begin{pmatrix} \overline{0} & 1 & \overline{1} & \mathbf{2} & \overline{2} & 3 & \overline{3} & \mathbf{4} & \overline{4} & \mathbf{5} & \overline{5} & 6 & \overline{6} & 7 & \overline{7} \\ \overline{7} & \overline{7} & \overline{7} & \mathbf{7} & \overline{2} & \overline{2} & \overline{2} & \mathbf{2} & \overline{1} & \mathbf{1} & \overline{0} & \overline{0} & \overline{0} & \overline{0} & \overline{0} \end{pmatrix}$.

It is clear that $\overline{s} \in (s)\tau$, for all $s \in \mathcal{PODI}_n$. Thus, by Proposition 2.1, we have:

Theorem 2.3 *The semigroup $\mathcal{PODI}_n$ divides $\mathcal{OD}_{2n+1}$.* $\square$

Now, as a corollary, we obtain (the following reformulation of) Theorem 1:

Corollary 2.4 $\mathsf{PODI} \subset \mathsf{OD}$. $\square$

Notice that, since PODI is generated by inverse semigroups, all elements of PODI have commuting idempotents. On the other hand, it is clear that, for instance, $\mathcal{OD}_2$ has non-commuting idempotents. Therefore, the inclusion $\mathsf{PODI} \subset \mathsf{OD}$ is strict.

We remark that, as $\overline{s}$ is order-preserving when s is order-preserving, by simply adapting the definition of τ to order-preserving transformations only, we recover the result presented by the author in [11]: $\mathsf{POI} \subset \mathsf{O}$.

The proof of Theorem 2

Now, we consider the chain $X = \{1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}\}$ (with $2n$ elements) and its subchain $Y = \{1 < 2 < \cdots < n\}$. Also, we consider the semigroups $\mathcal{OR}_{2n}$ and $\mathcal{PORI}_n$ built over X and Y, respectively, and the relation $\tau : \mathcal{PORI}_n \longrightarrow \mathcal{OR}_{2n}$ defined by

$$(s)\tau = \{t \in \mathcal{OR}_{2n} \mid Yt^{-1} \subseteq Y \ \text{ and } \ t|_{Yt^{-1}} = s\},$$

for all $s \in \mathcal{PORI}_n$. Again, we will prove that τ is completely defined.

Let $s \in \mathcal{PODI}_n$ be such that $\mathrm{Dom}(s) = \{i_1 < i_2 < \cdots < i_k\}$, with $1 \leq k \leq n$. If s is orientation-preserving, we define $\overline{s} \in \mathcal{OR}_{2n}$ by

$$(x)\overline{s} = \begin{cases} (i_p)s & \text{if } x = i_p, \text{ for some } 1 \leq p \leq k \\ \overline{(i_p)s} & \text{if } i_p < x < i_{p+1}, \text{ for some } 1 \leq p \leq k-1 \\ \overline{(i_k)s} & \text{if } \overline{1} \leq x < i_1 \text{ or } i_k < x \leq \overline{n}\ . \end{cases}$$

If s is orientation-reversing and $k \geq 3$, we define $\overline{s} \in \mathcal{OR}_{2n}$ by

$$(x)\overline{s} = \begin{cases} (i_p)s & \text{if } x = i_p, \text{ for some } 1 \leq p \leq k \\ \overline{(i_{p+1})s} & \text{if } i_p < x < i_{p+1}, \text{ for some } 1 \leq p \leq k-1 \\ \overline{(i_1)s} & \text{if } \overline{1} \leq x < i_1 \text{ or } i_k < x \leq \overline{n}\ . \end{cases}$$

Finally, if $s \in \mathcal{PORI}_n$ is the empty transformation, then define $\overline{s}$ as the constant transformation of $\mathcal{OR}_{2n}$ with image $\{\overline{1}\}$.

Examples 2.5 Let $n = 7$. Then:

- If $s = \begin{pmatrix} 2 & 4 & 5 \\ 1 & 2 & 7 \end{pmatrix}$ then $\overline{s} = \begin{pmatrix} 1 & \overline{1} & \mathbf{2} & \overline{2} & 3 & \overline{3} & \mathbf{4} & \overline{4} & \mathbf{5} & \overline{5} & 6 & \overline{6} & 7 & \overline{7} \\ \overline{7} & \overline{7} & \mathbf{1} & \overline{1} & \overline{1} & \overline{1} & \mathbf{2} & \overline{2} & \mathbf{7} & \overline{7} & \overline{7} & \overline{7} & \overline{7} & \overline{7} \end{pmatrix}$;

- If $s = \begin{pmatrix} 2 & 4 & 5 \\ 7 & 2 & 1 \end{pmatrix}$ then $\overline{s} = \begin{pmatrix} 1 & \overline{1} & \mathbf{2} & \overline{2} & 3 & \overline{3} & \mathbf{4} & \overline{4} & \mathbf{5} & \overline{5} & 6 & \overline{6} & 7 & \overline{7} \\ \overline{7} & \overline{7} & \mathbf{7} & \overline{2} & \overline{2} & \overline{2} & \mathbf{2} & \overline{1} & \mathbf{1} & \overline{7} & \overline{7} & \overline{7} & \overline{7} & \overline{7} \end{pmatrix}$;

- If $s = \begin{pmatrix} 2 & 4 & 5 \\ 7 & 1 & 2 \end{pmatrix}$ then $\overline{s} = \begin{pmatrix} 1 & \overline{1} & \mathbf{2} & \overline{2} & 3 & \overline{3} & \mathbf{4} & \overline{4} & \mathbf{5} & \overline{5} & 6 & \overline{6} & 7 & \overline{7} \\ \overline{2} & \overline{2} & \mathbf{7} & \overline{7} & \overline{7} & \overline{7} & \mathbf{1} & \overline{1} & \mathbf{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} \end{pmatrix}$;

- If $s = \begin{pmatrix} 2 & 4 & 5 \\ 2 & 1 & 7 \end{pmatrix}$ then $\overline{s} = \begin{pmatrix} 1 & \overline{1} & \mathbf{2} & \overline{2} & 3 & \overline{3} & \mathbf{4} & \overline{4} & \mathbf{5} & \overline{5} & 6 & \overline{6} & 7 & \overline{7} \\ \overline{2} & \overline{2} & \mathbf{2} & \overline{1} & \overline{1} & \overline{1} & \mathbf{1} & \overline{7} & \mathbf{7} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} \end{pmatrix}$;

- If $s = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 6 & 1 \end{pmatrix}$ then $\overline{s} = \begin{pmatrix} \mathbf{1} & \overline{1} & 2 & \overline{2} & 3 & \overline{3} & \mathbf{4} & \overline{4} & \mathbf{5} & \overline{5} & 6 & \overline{6} & 7 & \overline{7} \\ \mathbf{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \mathbf{6} & \overline{6} & \mathbf{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} \end{pmatrix}$;

- If $s = \begin{pmatrix} 1 & 4 & 5 \\ 1 & 6 & 3 \end{pmatrix}$ then $\overline{s} = \begin{pmatrix} \mathbf{1} & \overline{1} & 2 & \overline{2} & 3 & \overline{3} & \mathbf{4} & \overline{4} & \mathbf{5} & \overline{5} & 6 & \overline{6} & 7 & \overline{7} \\ \mathbf{1} & \overline{6} & \overline{6} & \overline{6} & \overline{6} & \overline{6} & \mathbf{6} & \overline{3} & \mathbf{3} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} \end{pmatrix}$.

It is a routine matter to show that, for all $s \in \mathcal{PORI}_n$, in fact, $\overline{s} \in \mathcal{OR}_{2n}$. On the other hand, clearly $\overline{s} \in (s)\tau$, for all $s \in \mathcal{PORI}_n$. Now, applying Proposition 2.1, we also have:

Theorem 2.6 *The semigroup* $\mathcal{PORI}_n$ *divides* $\mathcal{OR}_{2n}$. □

Consequently, we deduce Theorem 2 (reformulated as):

Corollary 2.7 $\mathsf{PORI} \subset \mathsf{OR}$. $\square$

Notice that, likewise for the order case, the inclusion $\mathsf{PORI} \subset \mathsf{OR}$ is also strict.

Observe also that, as $\overline{s}$ is orientation-preserving when s is orientation-preserving, again by simply adapting the definition of τ to just orientation-preserving transformations, we obtain that $\mathsf{POPI} \subset \mathsf{OP}$ [16].

References

[1] Almeida, J., and M.V. Volkov, *The gap between partial and full,* Int. J. Algebra Comput. Vol. 8, No. 3 (1998), 399-430.

[2] A.Ya. Aĭzenštat, *The defining relations of the endomorphism semigroup of a finite linearly ordered set,* Sibirsk. Mat. **3** (1962), 161-169 (Russian).

[3] R.E. Arthur and N. Ruškuc, *Presentations for two extensions of the monoid of order-preserving mappings on a finite chain,* Southeast Asian Bull. Math. **24** (2000), 1-7.

[4] P.M. Catarino, Monoids of orientation-preserving transformations of a finite chain and their presentations, *Semigroups and Applications,* eds. J.M. Howie and N. Ruškuc, World Scientific, (1998), 39-46.

[5] P.M. Catarino and P.M. Higgins, *The monoid of orientation-preserving mappings on a chain,* Semigroup Forum **58** (1999), 190-206.

[6] P.M. Catarino and P.M. Higgins, *The pseudovariety generated by all semigroups of orientation-preserving transformations on a finite cycle,* Internat. J. Algebra Comput.**12** (2002), no. 3, 387-405.

[7] E. Cordeiro, M. Delgado and V.H. Fernandes, *Relative abelian kernels of some classes of transformation monoids,* Submitted.

[8] D.F. Cowan and N.R. Reilly, *Partial cross-sections of symmetric inverse semigroups,* Int. J. Algebra Comput. **5** (1995) 259-287.

[9] M. Delgado and V.H. Fernandes, *Abelian kernels of some monoids of injective partial transformations and an application*, Semigroup Forum **61** (2000) 435-452.

[10] M. Delgado and V.H. Fernandes, *Abelian kernels of monoids of order-preserving maps and of some of its extensions.* Semigroup Forum **68** (2004), 335-356

[11] V.H. Fernandes, *Semigroups of order-preserving mappings on a finite chain: a new class of divisors,* Semigroup Forum **54** (1997), 230-236.

[12] V.H. Fernandes, *Normally ordered inverse semigroups,* Semigroup Forum **58** (1998) 418-433.

[13] V.H. Fernandes, *The monoid of all injective orientation preserving partial transformations on a finite chain,* Comm. Algebra **28** (2000), 3401-3426.

[14] V.H. Fernandes, *The monoid of all injective order preserving partial transformations on a finite chain,* Semigroup Forum **62** (2001), 178-204.

[15] V.H. Fernandes, Presentations for some monoids of partial transformations on a finite chain: a survey, *Semigroups, Algorithms, Automata and Languages,* eds. Gracinda M. S. Gomes & Jean-Éric Pin & Pedro V. Silva, World Scientific, (2002), 363-378.

[16] V.H. Fernandes, *A division theorem for the pseudovariety generated by semigroups of orientation preserving transformations on a finite chain,* Comm. Algebra **29** (2001) 451-456.

[17] V.H. Fernandes, *Semigroups of order-preserving mappings on a finite chain: another class of divisors,* Izvestiya VUZ. Matematika **3** (478) (2002) 51-59 (Russian).

[18] V.H. Fernandes, G.M.S. Gomes and M.M. Jesus, *Presentations for some monoids of injective partial transformations on a finite chain,* Southeast Asian Bull. Math. **28** (2004), 903-918.

[19] V.H. Fernandes, G.M.S. Gomes and M.M. Jesus, *Presentations for some monoids of partial transformations on a finite chain,* Comm. Algebra **33** (2005), 587-604.

[20] V.H. Fernandes, G.M.S. Gomes and M.M. Jesus, *Congruences on monoids of order-preserving or order-reversing transformations on a finite chain,* Glasgow Mathematical Journal **47** (2005), 413-424.

[21] V.H. Fernandes, G.M.S. Gomes and M.M. Jesus, *Congruences on monoids of transformations preserving the orientation on a finite chain,* Submitted.

[22] G.M.S. Gomes and J.M. Howie, *On the ranks of certain semigroups of order-preserving transformations,* Semigroup Forum **45** (1992), 272-282.

[23] P.M. Higgins, *Divisors of semigroups of order-preserving mappings on a finite chain,* Int. J. Algebra Comput. 5 (1995), 725-742.

[24] J.M. Howie, *Product of idempotents in certain semigroups of transformations,* Proc. Edinburgh Math. Soc. **17** (1971) 223-236.

[25] D. McAlister, *Semigroups generated by a group and an idempotent,* Comm. Algebra **26** (1998), 515-547.

[26] L.M. Popova, *The defining relations of the semigroup of partial endomorphisms of a finite linearly ordered set,* Leningradskij gosudarstvennyj pedagogicheskij institut imeni A. I. Gerzena, Uchenye Zapiski **238** (1962) 78-88 (Russian).

[27] V.B. Repnitskiĭ and M.V. Volkov, *The finite basis problem for the pseudovariety* O, Proc. R. Soc. Edinb., Sect. A, Math. 128, No.3 (1998), 661-669.

[28] A. Vernitskii and M.V. Volkov, *A proof and generalisation of Higgins' division theorem for semigroups of order-preserving mappings,* Izv.vuzov. Matematika, (1995), No.1, 38-44.

Author's second address:
Centro de Álgebra da Universidade de Lisboa
Av. Prof. Gama Pinto, 2
1649-003 Lisboa
Portugal

Version of December 7, 2006