THE 2D EULER EQUATIONS AND THE STATISTICAL TRANSPORT EQUATIONS

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Abstract. We prove the existence of weak solutions for the forward and backward statistical transport equations associated with the 2D Euler equations. Such solutions can be interpreted, respectively, as a statistical Lagrangian and a statistical Eulerian description of the motion of the fluid.

1. Introduction

This article is concerned with the 2D Euler equations for the in-viscous incompressible fluid

$$\begin{cases} \dfrac{\partial v}{\partial t} = -(v \cdot \nabla)v - \nabla p \\ div\ v = 0 \end{cases} \tag{1.1}$$

subjected to periodic boundary conditions and given initial data

$$v(x,0) = v_0(x), \tag{1.2}$$

where $v(x,t) = (v_1(x_1,x_2,t), v_2(x_1,x_2,t))$ is the velocity field of the fluid, $p = p(x,t)$ is the pressure.

In the study of such problem two different approaches are possible:

A) (1.1) is considered as an usual P. D. E. with smooth or less smooth initial data;

B) (1.1) is considered as a statistical equation with very low regularity of the initial data.

Concerning A), the existence and uniqueness results of the classical weak solutions for the initial data with $rot\ v_0 \in L^\infty$ have been shown by the different approaches in [Y], [Ka], [C-S]. The description of less smooth fluctuation of the fluid, when we have a velocity discontinuity (mixing layers, jets), is related with the case when $rot\ v_0$ is a measure (delta function), but $v_0 \in L^2$. There is a wide literature on the subject of existence results; we refer to [K], [DiP-M], [D]. All the solutions of the Euler equations in the mentioned articles have the feature of *finite kinetic energy.*

However, many physical problems possess highly unstable structures, whose complete dynamics can not be described by a smooth model. To study such dynamics of the fluid, when we deal with the velocity field v_0 belonging to $H^{-s},\ s > 0$, the approach B) is more natural. Statistical solutions of (1.1) are defined almost everywhere with respect to some probability measure, which is associated to physical quantities, which are invariants of the system. The underlying state space of the solutions is an infinite dimensional space, where a suitable differential calculus has to be considered.

For our 2D case, in [A-H.K-M], [A-H.K-R.F], [B-F], [C-D.G] invariant measures of Gibbs type have been constructed. These Gibbs measures are determined by quantities such as the enstrophy, the energy and the re-normalized energy and it has been proved that such measures are infinitesimally invariant with respect to the Euler equation. The existence of global flows (weak statistical solutions), leaving the measures invariant, has been shown in [A-C], [Ci], where point-wise flows are carried by some families of probability measures. The flows take values in the support of the Gaussian measures with covariance given by the enstrophy, therefore such flows can be of *infinite kinetic energy.* Similar situations have arisen in Stochastic Analysis and, in particular, within the Malliavin's stochastic calculus of variations. In [Cr] (and also in [U-Z], [P]) flows on the classical Wiener space, associated with vector fields with low regularity, have been defined.

In our present article we develop the approach suggested in [A-C], [Ci] to study flows with $v_0 \in H^{-s},\ s > 0$. We introduce the concept of *generalized statistical forward and backward flows.* These are solutions of the corresponding *Transport Equations* defined on the invariant infinite Gibbs measure. This approach has many advantages, comparing with the previous results [A-C], [Ci]; using these statistical transport equations, one can try to develop methods, obtained for the usual PDE theory for solving important questions:

1) Uniqueness result;
2) Study of regularity of solutions;
3) Develop numerical methods for these statistical differential equations.

As to the more detailed structure of this paper, in Section 2 we formulate our problem, define the standard Gibbs measure μ_γ given by the enstrophy integral and present a very useful Lemma on the integrability of the operator $B(U)$ in $L^2_{\mu_\gamma}$, associated to the nonlinear part of the Euler equation (1.1). In Sections 4 and 5, we define the generalized forward and backward flow of (1.1), which are solutions of the transport equations, and show their existence. In Section 6 we describe how the generalized forward flow can be interpreted as the statistical Lagrangian viewpoint for the description of the motion of the fluid and the generalized backward flow as the corresponding statistical Euler viewpoint

2. Statement of the Problem

Let us return to the Euler equations (1.1), (1.2). Since $div\ v = 0$ and $div\ v_0 = 0$, there exist functions $U = U(x,t)$, $u = u(x)$, such that

$$v = \nabla^\perp U = (-\partial_{x_2} U, \partial_{x_1} U),$$
$$v_0 = \nabla^\perp u = (-\partial_{x_2} u, \partial_{x_1} u).$$

We can eliminate the pressure p in (1.1) by applying the differential operator $rot\ z = -\partial_{x_2} z_1 + \partial_{x_1} z_2$ to the first equation of (1.1) and obtain

$$\partial_t \Delta U = -\partial_{x_2} U \cdot \partial_{x_1} \Delta U + \partial_{x_1} U \cdot \partial_{x_2} \Delta U. \tag{2.1}$$

We consider solutions of (1.1), (1.2) on the 2-dimensional torus that we identify with $T^2 = [0, 2\pi] \times [0, 2\pi]$ subjected to periodic boundary conditions

$$\begin{aligned} &U(0, x_2, t) = U(2\pi, x_2, t), \\ &U(x_1, 0, t) = U(x_1, 2\pi, t), \quad \forall x = (x_1, x_2) \in T^2, \forall t \in [0, T]. \end{aligned} \tag{2.2}$$

Let us denote by $e_k(x) = \frac{1}{2\pi} e^{k \cdot x},\ \ k \in \mathbb{Z}^2$ the eigenfunctions for the operator $-\Delta$ with eigenvalues $k^2 = k_1^2 + k_2^2$, where $k \cdot x = k_1 x_1 + k_2 x_2$. They form a complete set of orthonormal functions in $L^2(T^2)$. We expand the solution $U(x,t)$ of (2.1) in the form of Fourier serie

$$U(x,t) = \sum_k U_k(t) e_k(x).$$

Since U is a real function and we can assume

$$\int_{T^2} U dx = 0$$

we have $U_{-k} = \overline{U}_k$ ($\overline{z}$ is the complex conjugate of z); then

$$U(x,t) = \sum_{k\in\mathbb{Z}^2_+} U_k(t) e_k(x), \tag{2.3}$$

where $\mathbb{Z}^2_+$ denotes the set $\{k \in \mathbb{Z}^2 :\ k_1 > 0,\ k_2 \in \mathbb{Z} \text{ or } k_1 = 0,\ k_2 > 0\}$. For the initial data $u(x)$ we have

$$u(x) = \sum_{k\in\mathbb{Z}^2_+} u_k e_k(x). \tag{2.4}$$

In the sequel by (2.3), (2.4), we can identify the functions U, u with infinite vector fields with Fourier coefficients

$$U = \big(U_k\big)_{k\in\mathbb{Z}^2_+} \text{ and } u = \big(u_k\big)_{k\in\mathbb{Z}^2_+},$$

where $k \in \mathbb{Z}^2_+$. We define

$$\mathbb{C}^\infty = \big\{u = \big(u_k\big)_{k\in\mathbb{Z}^2_+}\ :\ u_k \in \mathbb{C}\big\}.$$

Substituting (2.3) in equation (2.1) we derive the following system

$$\begin{aligned} 2\pi k^2 \frac{d}{dt} U_k &= -\sum_{h+h'=k} (h^\perp \cdot h')(h')^2 U_h U_{h'} \\ &= \frac{1}{2} \sum_{h+h'=k} (h^\perp \cdot h')[h^2 - (h')^2] U_h U_{h'}, \end{aligned}$$

where $h^\perp = (-h_2, h_1)$. Hence if we denote the operator $B = B(U) : U \in \mathbb{C}^\infty \to \mathbb{C}^\infty$ by

$$B(U) = \big(B_k(U)\big)_{k\in\mathbb{Z}^2_+}$$

with coefficients $B_k = B_k(U)$ described by the equalities

$$\begin{aligned} B_k(U) &= \sum_{\substack{h\neq k \\ h,k\in\mathbb{Z}^2_+}} \alpha_{h,k} U_k U_{k-h}, \\ \alpha_{h,k} &= \frac{1}{2\pi}\Big[\frac{1}{k^2}(h^\perp\cdot k)(h\cdot k) - \frac{1}{2}(h^\perp \cdot k)\Big], \end{aligned} \tag{2.5}$$

the system (1.1), (1.2), (2.2) will be equivalent to

<u>Problem</u> *Find $U = U(t)$, the solution of the infinite dimensional system*

$$\frac{d}{dt} U(t) = B(U(t)) \tag{2.6}$$

with the initial conditions

$$U(0) = u. \tag{2.7}$$

Now we introduce the Sobolev spaces of order $\beta \in \mathbb{R}$ on the torus T^2,

$$\begin{aligned} H^\beta &= \{\varphi = \sum_k \varphi_k e_k : \sum_k k^{2\beta}|\varphi_k|^2 < +\infty,\ \varphi_{-k} = \overline{\varphi}_k\} \\ &\equiv \{\varphi = \big(\varphi_k\big)_{k\in\mathbb{Z}^2_+} \in \mathbb{C}^\infty : \sum_{k\in\mathbb{Z}^2_+} k^{2\beta}|\varphi_k|^2 < +\infty\}. \end{aligned} \tag{2.8}$$

The spaces H^β are complex Hilbert spaces with inner product and norm given by

$$< \varphi, \psi >_{H^\beta} = \sum_{k \in \mathbb{Z}^2_+} k^{2\beta} \varphi_k \bar{\psi}_k, \qquad \|\varphi\|^2_{H^\beta} =< \varphi, \varphi >_{H^\beta} .$$

The two dimensional Euler equation has an infinite number of invariants of motion, among which we mention the energy and the enstrophy, defined respectively by

$$E(t) = \frac{1}{2}\int_{T^2} v^2 dx = -\frac{1}{2}\int_{T^2} u\Delta u dx = \frac{1}{2}\sum_k k^2 u_k^2 = \frac{1}{2}\|u\|^2_{H^1},$$

$$S(t) = \frac{1}{2}\int_{T^2} (rot\ v)^2 dx = \frac{1}{2}\int_{T^2} (\Delta u)^2 dx = \frac{1}{2}\sum_k k^4 u_k^2 = \frac{1}{2}\|u\|^2_{H^2}. \tag{2.9}$$

Lemma 2.1. *Let $u(t)$ be a smooth solution of the Euler equation* (1.1)*,* (1.2)*,* (2.2)*. Then $E(t)$, $S(t)$ are constants of motion, i. e.,*

$$\begin{aligned} &E(t) = E(0), \\ &S(t) = S(0), \quad \forall t \in [0, T]. \end{aligned} \tag{2.10}$$

Proof. It is easy the check from (1.1), (1.2), (2.2) that

$$\frac{d}{dt}E(t) = 0 \text{ and } \frac{d}{dt}S(t) = 0.$$

□

We consider the Gaussian measures with covariance given by the enstrophy multiplied by a constant $\gamma > 0$, namely

$$\begin{aligned} d\mu_\gamma(\varphi) &= \prod_{k \in \mathbb{Z}^2_+} d\nu^k_\gamma(\varphi_k), \\ d\nu^k_\gamma(z) &= \frac{\gamma k^4}{2\pi}\exp(-\frac{1}{2}\gamma k^4 |z|^2) dx dy \end{aligned} \tag{2.11}$$

with $z = x + iy$. Taking into account that $\mu_\gamma(H^\beta) = 1$ for any $\beta < 1$, than, for simplification of notations, the integral will be written

$$\int \varphi(u) d\mu_\gamma(u) \ \text{ instead of } \ \int_{H^\beta} \varphi(u) d\mu_\gamma(u)$$

for arbitrary function $\varphi = \varphi(u) : \mathbb{C}^\infty \to \mathbb{C}$, which is measurable with respect to μ_γ. In the following we will assume that the value of $\beta < 1$ is given.

Definition 1. Let $\varphi = \varphi(u) : \mathbb{C}^\infty \to \mathbb{C}$ be arbitrary complex function and $p > 1$. We define

$$\begin{aligned} L^p_{\mu_\gamma}(H^\beta) = \{ &\varphi \text{ is measurable with respect to } \mu_\gamma : \\ &\|\varphi\|^p_{L^p_{\mu_\gamma}} = \int_{H^\beta} |\varphi(u)|^p d\mu_\gamma(u) < \infty \}. \end{aligned} \tag{2.12}$$

3. Useful previous results

In this paragraph we state some results that will be needed in next sections.

Let us denote $\mathbb{Z}^2_{+,n} = \{k \in \mathbb{Z}^2_+ : |k| \leq n\}$ and $d(n) = \#\mathbb{Z}^2_{+,n}$. We consider the finite dimensional approximations of $B_k(u)$ defined as

$$B^n_k(u) = \sum_{\substack{h \neq k \\ h,k \in \mathbb{Z}^2_{+,n}}} \alpha_{h,k} u_k u_{k-h}. \tag{3.1}$$

Lemma 3.1. *For any* $k \in \mathbb{Z}^2_+$ *we have*

$$B_k(u) \in L^p_{\mu_\gamma}(H^\beta), \quad \forall p > 1, \beta < 1$$

and

$$B(u) \in L^p_{\mu_\gamma}(H^\beta, H^\beta), \quad \forall p > 1, \beta < -1.$$

Proof. Let us give the idea of the proof for $p = 2$.

$$\int |B_k^n|^2 d\mu_\gamma \leq 8\gamma^{-2} \sum_{\substack{h \neq k \\ h \in \mathbb{Z}^2_{+,n}}} (k-h)^{-4} < \infty$$

Analogously we obtain, for $m < n$

$$\int |B_k^n - B_k^m|^2 d\mu_\gamma \leq 8\gamma^{-2} \sum_{\substack{h \neq k \\ |h| > m}} (k-h)^{-4} \to 0 \text{ if } m \to \infty.$$

This proves that

$$B_k^n \to B_k \text{ in } L^2_{\mu_\gamma}(H^\beta).$$

The functions $B_k(u)$ are measurable as the limit of continuous measurable functions in $L^2_{\mu_\gamma}(H^\beta)$. Following [A-C], [Ci], we have

$$B_k^n \to B_k \text{ in } L^p_{\mu_\gamma}(H^\beta), \quad p > 1. \tag{3.2}$$

The functional $B(u) = \sum_{k \in \mathbb{Z}^2_+} B_k(u) e_k$, defined on the space H^β, is integrable as a functional from H^β to H^β only if $\beta < -1$. More precisely,

$$\int \|B(u)\|^p_{H^\beta} d\mu_\gamma < \infty, \quad p > 1$$

for any $\beta < -1$.

□

Definition 2. An arbitrary complex function $f = f(u) : \mathbb{C}^\infty \to \mathbb{C}$ is a cylindrical function if, for some integer N

$$f = f(u) \equiv F(u_{\alpha_1}, \dots, u_{\alpha_{d(N)}})$$

where F is a $C^1_0(\mathbb{C}^{d(N)})$ - smooth function depending only on the components u_{α_i}, $\alpha_i \in \mathbb{Z}^2_{+,d(N)}$.

Definition 3. The operator $\delta_{\mu_\gamma} B : H^\beta \to \mathbb{C}$, which satisfies

$$\int B(u) \cdot \nabla f(u) d\mu_\gamma(u) = \int \delta_{\mu_\gamma} B(u) \cdot f(u) d\mu_\gamma(u)$$

for any cylindrical function f, is named as the divergence of the field $B(u)$ with respect to the measure μ_γ.

Lemma 3.2. *For* $\mu_\gamma -$ *a. e.* u

$$\delta_{\mu_\gamma} B(u) = 0. \tag{3.3}$$

Proof. To prove this result we use the approximations B_k^n defined in (3.1). Since B_k^n does not depend on the component u_k, we have

$$\text{div} B^n = \sum_k \frac{\partial B_k^n}{\partial u_k} = 0.$$

Hence for any cylindrical function $f = f(u)$, using integration by parts and (2.11)

$$\int B^n(u)\nabla f(u) d\mu_\gamma(u) = -\int < B^n(u), u >_{H^2} f(u) d\mu_\gamma(u) = 0.$$

Since

$$< B^n(u), u >_{H^2} = \sum_{k\in\mathbb{Z}^2_{+,n}} k^4 B^n_k(u)\bar{u}_k = \int_{T^2} \text{rot } (U^n)\nabla(\Delta U^n)\cdot \Delta U^n dx = 0$$

for all n fixed. Here $U^n(x) = \sum_{k\in\mathbb{Z}^2_{+,n}} u_k e_k(x)$. Therefore

$$\delta_{\mu_\gamma} B^n = 0. \tag{3.4}$$

We deduce (3.3) from (3.2), noting that the definition of $\delta_{\mu_\gamma} B$ only involves integration against cylindrical functions. □

4. Euler equations and transport equations

Let us return to the **Problem**, written as

$$\frac{dU_k(t)}{dt} = B_k(U(t)), \quad \forall k \in \mathbb{Z}^2_+ \tag{4.1}$$

with the initial data

$$U_k(0) = u_k. \tag{4.2}$$

The solution of this system

$$U = U(t)$$

can be considered as a function of both the time parameter $t \in [0,T]$ and the initial data $u \in \mathbb{C}^\infty$, therefore in what follows we consider

$$U(t,u) = U(t)$$

for $(t,u) \in [0,T]\times\mathbb{C}^\infty$.

Let us assume, *just formally*, that $B = B(u)$ and $U = U(t,u)$ are C^1-differentiable functions; we can write the identity (flow property)

$$U(t+s,u) = U(t, U(s,u)), \quad \forall t,s \geq 0, \quad 0 \leq t+s \leq T. \tag{4.3}$$

Taking the derivative on the time variable s we obtain

$$\begin{aligned}\frac{\partial}{\partial t} U_k(t+s,u) &= \sum_l \frac{\partial}{\partial u_l} U_k(t, U(s,u)) \frac{\partial}{\partial s} U_l(s,u) \\ &= \sum_l \frac{\partial}{\partial u_l} U_k(t, U(s,u)) B_l(U(s,u))\end{aligned} \tag{4.4}$$

fo each $k \in \mathbb{Z}^2_+$. For $s = 0$ we deduce that the function $U = U(t,u)$ satisfies the linear transport equation

$$\frac{\partial}{\partial t} U_k(t,u) = B(u)\cdot\nabla U_k(t,u), \quad k \in \mathbb{Z}^2_+$$

with initial condition

$$U_k(0,u) = u_k.$$

Definition 4. A function $U = U(t,u) = \big(U_k(t,u)\big)_{k\in\mathbb{Z}^2_+}$ is called a *generalized forward flow* of the **Problem** if

$$U_k(t,u) \in W^{1,\infty}([0,T], L^p_{\mu_\gamma}(H^\beta)), \; p > 1, \;\; \forall k \in \mathbb{Z}^2_+$$

and the following identities hold

$$\begin{aligned}\int u_k \Phi(0) f(u) d\mu_\gamma(u) + \int_0^T \int U_k(t,u)\Phi'(t) f(u) d\mu_\gamma(u) dt \\ = \int_0^T \int U_k(t,u)\big(B(u)\cdot\nabla f(u)\big)\Phi(t) d\mu_\gamma(u) dt\end{aligned} \tag{4.5}$$

for any cylindrical function $f = f(u)$ and $\forall \Phi \in C^1([0,T])$, such that $\Phi(T) = 0$.

Theorem 4.1. *There exists a generalized forward flow $U(t,u)$ for* (4.1), (4.2)*, such that*

$$U_k(t,u) \in W^{1,\infty}([0,T], L^p_{\mu_\gamma}(H^\beta)) \tag{4.6}$$

and

$$\int |U_k(t,u)|^p d\mu_\gamma(u) \le \int |u_k|^p d\mu_\gamma(u) < \infty \tag{4.7}$$

for any $t \in [0,T]$ and $k \in \mathbb{Z}^2_+$.

Proof. From the theory of O. D. E. and the conservation of the energy (Lemma 2.1), there exists a unique solution of

$$\begin{cases} \dfrac{d}{dt} U^n_k(t) = B^n_k(U^n(t)), \quad \forall k \,:\, k \in \mathbb{Z}^2_{+,n}, \\ U^n(0) = u \in \mathbb{C}^{d(n)} \end{cases} \tag{4.8}$$

such that

$$U^n(t) = U^n(t,u) \in C^1\big([0,T]\times \mathbb{C}^{d(n)}\big). \tag{4.9}$$

Applying (4.3), (4.4) to each function $U^n_k(t,u)$, we deduce that any solution of (4.8) satisfies the system

$$\begin{cases} \dfrac{\partial}{\partial t} U^n_k(t,u) = \displaystyle\sum_{l\in\mathbb{Z}^2_{+,n}} B^n_l(u) \frac{\partial}{\partial u_l} U^n_k(t,u) \\ \qquad\qquad\quad = B^n(u)\cdot \nabla U^n_k(t,u), \\ U^n_k(0) = u_k. \end{cases} \tag{4.10}$$

Let us multiply the first equation in (4.10) by an arbitrary cylindrical function $f = f(u)$ and arbitrary $\Phi \in C^1\big([0,T]\big)$, such that $\Phi(T) = 0$. Taking into account that $\delta_{\mu_\gamma} B^n = 0$, we deduce

$$\begin{aligned}\int u_k \Phi(0) f(u) d\mu_\gamma(u) + \int_0^T \int U^n_k(t,u)\Phi'(t) f(u) d\mu_\gamma(u) dt \\ = \int_0^T \int U^n_k(t,u)\big(B^n(u)\cdot\nabla f(u)\big)\Phi(t) d\mu_\gamma(u) dt\end{aligned} \tag{4.11}$$

for any cylindrical function $f = f(u)$ and $\forall \Phi \in C^1([0,T])$, such that $\Phi(T) = 0$.

We also have

$$\int |U^n_k(t,u)|^p d\mu_\gamma(u) = \int |u_k|^p d\mu_\gamma(u) < \infty, \text{ (cf. [Ci])} \tag{4.12}$$

and

$$\int |\partial_t U^n_k(t,u)|^p d\mu_\gamma(u) \le \int |B_k(u)|^p d\mu_\gamma(u) < \infty.$$

Therefore there exists a subsequence of $\{U^n_k, B^n_l\}$ such that, when $n \to \infty$,

$$\begin{aligned} &U^n_k(t,u) \rightharpoonup U_k(t,u) \text{ (weakly) in } W^{1,\infty}\big([0,T], L^p_{\mu_\gamma}(H^\beta)\big) \\ &B^n_l(u) \to B_l(u) \text{ in } L^p_{\mu_\gamma}(H^\beta) \end{aligned} \tag{4.13}$$

for all k, l. This convergence allows to pass to the limit equation (4.11), written for $U^n_k(t,u)$, $B^n(u)$ and conclude that $U_k(t,u)$ satisfies equation (4.5) for every k. Inequality (4.7) follows from (4.12) and the weak convergence of $U^n_k(t,u)$. □

5. Transport equations and Liouville-type equations

As discussed in [F-M-R-T], in turbulent flow regimes, physical properties are universally recognized as randomly varying and characterized by suitable probability distribution functions. For instance, some turbulent processes, due to technical difficulties, can not be measured with good precision; measurements are therefore to be taken with some error estimates. This is why we speak about finding the solution in some distribution class of initial data. Mathematically this can be formulated as follows:

If the initial conditions are given according to a measure (distribution)

$$\nu_0 = \nu_0(u) \tag{5.1}$$

on the phase space $\mathbb{C}^\infty$, then the solution of the Euler equation (4.1) with initial distribution ν_0 at some later time t will be distributed according to another distribution

$$\nu_t = \nu_t(u). \tag{5.2}$$

How can we determine this time dependent distribution $\nu_t = \nu_t(u)$ with respect to the initial distribution $\nu_0 = \nu_0(u)$?

Definition 5. A distribution $\nu_t = \nu_t(u)$ is called a *generalized backward flow* of the Euler equation (4.1) with initial distribution ν_0, if $\nu_t(u)$ is a probability measure and satisfies the Liouville-type equations

$$\int f(u)d\nu_t(u) = \int f(u)d\nu_0(u) + \int_0^t \int B(u)\cdot\nabla f(u)d\nu_\tau(u)d\tau \tag{5.3}$$

for any cylindrical function $f = f(u)$.

5.1. **Existence for initial data absolutely continuous with respect to the measure μ_γ.** The class of all absolutely continuous probability measures with respect to the Gaussian measure of the form

$$v(u)d\mu_\gamma(u) \tag{5.4}$$

with $v \in L^q_{\mu_\gamma}(H^\beta)$, $q>1$ will be denoted as $\mathcal{M}^q_{\mu_\gamma}$.

Theorem 5.1. *For any initial probabilistic distribution $d\nu_0 = v_0(u)d\mu_\gamma(u) \in \mathcal{M}^q_{\mu_\gamma}$, $q>1$ there exists a generalized backward flow ν_t of the Euler equation (4.1) such that*

$$d\nu_t(u) = V(t,u)d\mu_\gamma(u) \text{ and } V(t,u) \in L^\infty\big([0,T], L^q_{\mu_\gamma}(H^\beta)\big)$$

that is, $\nu_t(u) \in \mathcal{M}^q_{\mu_\gamma}$, for a. e. $t\in[0,T]$.

Moreover the function $V = V(t,u)$ is a weak solution of the transport equation

$$\begin{cases} \partial_t V(t,u) + B(u)\cdot\nabla V(t,u) = 0, \\ V(0,u) = v_0(u). \end{cases} \tag{5.5}$$

Proof. Let us consider regular initial distributions

$$d\nu_0^n(u) = v_0^n(u)d\mu_\gamma(u),$$

where $v_0^n \in C^1(\mathbb{C}^{d(n)})$, $\int v_0^n(u)d\mu_\gamma(u) = \int v_0(u)d\mu_\gamma(u)$ and

$$v_0^n \to v_0 \;\text{ in } L^q_{\mu_\gamma}(H^\beta). \tag{5.6}$$

Here the function v_0^n can be taken as $\big(P_{\frac{1}{n^2}}v_0\big)(U_{\alpha_1},\ldots,U_{\alpha_{d(n)}})$, $\alpha_i \in \mathbb{Z}^2_{+,d(n)}$, where

$$\big(P_t f\big)(u) = \int f(\mathrm{e}^{-t}u + \sqrt{1-\mathrm{e}^{-2t}}y)d\mu_\gamma(y) \tag{5.7}$$

is the Ornstein-Uhlenbeck operator (see [U-Z]).

For any cylindrical function $f = f(u)$ we have

$$\frac{d}{dt}\int f\big(U^n(t,u)\big)v_0^n(u)d\mu_\gamma(u) = \int \nabla f\big(U^n(t,u)\big)\cdot\frac{d}{dt}U^n(t,u)v_0^n(u)d\mu_\gamma(u)$$
$$= \int \big(B^n\big(U^n(t,u)\big)\cdot\nabla f\big(U^n(t,u)\big)\big)v_0^n(u)d\mu_\gamma(u),$$

here

$$U^n(t,u)\in C^1\big([0,T]\times\mathbb{C}^{d(n)}\big)$$

is the solution of problem (4.8). Making a change of variables from u to $U = U^n(t,u)$ we obtain

$$\begin{aligned}&\frac{d}{dt}\int f(u)v_0^n\big(U^n(-t,u)\big)d\mu_\gamma(u)\\&\qquad=\int\big(B^n(u)\cdot\nabla f(u)\big)v_0^n(U^n(-t,u))d\mu_\gamma(u)\end{aligned}\tag{5.8}$$

where we have used (3.4). For the function

$$V^n(t,u) = v_0^n\big(U^n(-t,u)\big)\tag{5.9}$$

and according to (3.4), Lemma 2.1 we have

$$\begin{aligned}\int|V^n(t,u)|^q d\mu_\gamma(u) &= \int|v_0^n\big(U^n(-t,u)\big)|^q d\mu_\gamma(u)\\&=\int|v_0^n(u)|^q d\mu_\gamma(u) < C,\ \ \forall n.\end{aligned}\tag{5.10}$$

So there exists a subsequence of V^n such that

$$V^n(t,u)\rightharpoonup V(t,u)\ \text{(weakly) in } L^\infty\big([0,T],L^q_{\mu_\gamma}(H^\beta)\big).\tag{5.11}$$

Hence by (3.2), (5.11) and (5.8) the distribution

$$d\nu_t(u) = V(t,u)d\mu_\gamma(u)\tag{5.12}$$

satisfies (5.3). Since ν_0 is a probability measure, convergence (5.11) implies that ν_t is also a probability measure.

Let us now show that $V(t,U)$ can be obtained as a weak solution of the transport equation (5.5). To do it we consider the approximated system

$$\begin{cases}\partial_t V^n(t,u) + B^n(u)\cdot\nabla V^n(t,u) = 0,\\ V^n(0,u) = v_0^n(u).\end{cases}\tag{5.13}$$

This system has unique regular solution

$$V^n = V^n(t,u)\in C^1\big([0,T]\times\mathbb{C}^{d(n)}\big),$$

which satisfies the identity

$$\frac{d}{dt}V^n\big(U^n(t,u)\big) = 0,$$

i. e. $V^n(t,u)$ verifies (5.9). On the other hand $V^n(t,u)$ is a solution of the weak formulation for the problem (5.13), namely

$$\begin{aligned}&\int v_0^n(u)\Phi(0)f(u)d\mu_\gamma(u) + \int_0^T\int V^n(t,u)\Phi'(t)f(u)d\mu_\gamma(u)dt\\&\qquad=\int_0^T\int V^n(t,u)\big(B^n(u)\cdot\nabla f(u)\big)\Phi(t)d\mu_\gamma(u)dt\end{aligned}\tag{5.14}$$

for any fixed cylindrical function $f = f(u)$. From (3.2) and (5.11) we deduce that $V(t, U)$ is a weak solution of (5.5). □

Remark 1. The measure $\nu_t = \mu_\gamma$, $\forall t \geq 0$ is a particular generalized backward solution of the Euler Equation (4.1) with initial condition (5.1), a result which has been proved in paper [A-C].

In the following two paragraphs we study of the evolution of turbulent processes with the initial data, which are Dirac measures.

5.2. **Approximation of the Dirac measure.** Since we shall need the results of Lemma 3.1, in this subsection and in the subsection 5.3, we consider the space H^β with a fixed β, such that

$$\beta < -1. \tag{5.15}$$

Let δ_{z^0} be the Dirac measure concentrated at a given point $z^0 \in H^\beta$, that is for any set $A \subseteq H^\beta$, we have

(a) $\delta_{z^0}(A) = 1$, if $z^0 \in A$;
(b) $\delta_{z^0}(A) = 0$, if $z^0 \notin A$.

Now let us consider the Gaussian measure μ_γ for a fixed constant $\gamma > 0$. The main objective of this subsection is to construct an approximation of the Dirac measure δ_{z^0} with respect of the measure μ_γ.

Let $\varepsilon > 0$ be a fixed real. If we take an arbitrary $z^0 \in H^\beta$, we define the set

$$B_{\varepsilon_k}(z_k^0) = \{z_k \in \mathbb{C} : |z_k - z_k^0| \leqslant \varepsilon_k\}$$

with $\varepsilon_k = \dfrac{\varepsilon}{k^{3/2}}$ and z_k^0 being the k-th coordinate of z^0. Then for the characteristic function of the set $B_{\varepsilon_k}(z_k^0)$

$$\chi_{z_k^0}^{\varepsilon_k}(z_k) = \begin{cases} 1, \text{ if } z_k \in B_{\varepsilon_k}(z_k^0), \\ 0, \text{ if } z_k \notin B_{\varepsilon_k}(z_k^0), \end{cases} \tag{5.16}$$

we have that the integral

$$g_{k,\varepsilon}(z_k^0) := \int_{\mathbb{C}} \chi_{z_k^0}^{\varepsilon_k}(z_k) d\mu_\gamma^k(z_k) \tag{5.17}$$

is equal to

$$\frac{\gamma k^4}{2\pi} \int_{B_{\varepsilon_k}(0)} e^{-\frac{\gamma k^4}{2}|z_k + z_k^0|^2} dz_k.$$

Let us compute the integral of $g_{k,\varepsilon}(z_k^0)$ on $\mathbb{C}$ with respect to the measure $\mu_\gamma^k(z_k^0)$

$$S_k := \int_{\mathbb{C}} g_{k,\varepsilon}(z_k^0) d\mu_\gamma^k(z_k^0) = \left[\frac{\gamma k^4}{2\pi}\right]^2 \int_{B_{\varepsilon_k}(0)} I_k dz_k \tag{5.18}$$

with

$$I_k = \int_{\mathbb{C}} e^{-\frac{\gamma k^4}{2}(|z_k + z_k^0|^2 + |z_k^0|^2)} dz_k^0.$$

Since

$$|z_k + z_k^0|^2 + |z_k^0|^2 = \left|\sqrt{2} z_k^0 + \frac{1}{\sqrt{2}} z_k\right|^2 + \frac{1}{2}|z_k|^2,$$

introducing the new variable

$$\overline{z}_k^0 = \sqrt{2} z_k^0 + \frac{1}{\sqrt{2}} z_k,$$

we easily find that

$$I_k = \frac{\pi}{\gamma k^4} e^{-\frac{\gamma k^4}{4}|z_k|^2}.$$

Therefore by (5.18) we obtain

$$0 < S_k = 1 - e^{-|k|A} < 1 \tag{5.19}$$

with $A = \frac{\gamma \varepsilon^2}{4}$.

Let us now define three sequences

$$\begin{aligned} L_n : &= \prod_{j=1}^{n} (1 - e^{-jA})^{2j}, \\ R_j : &= \prod_{\substack{k \in \mathbb{Z}_2^+ \\ |k|=j}} S_k \equiv \prod_{\substack{k \in \mathbb{Z}_2^+ \\ |k|=j}} (1 - e^{-jA}), \quad j = 1, 2, ..., n, \\ T_n : &= \prod_{j=1}^{n} R_j, \quad \forall n = 1, 2, ... \end{aligned} \tag{5.20}$$

By (5.19), (5.20) and $\#\{|k| = j\} \leqslant 2j$, we have that $\{L_n\}, \{T_n\}$ are monotone decreasing sequences, such that

$$0 < L_n \leq T_n < 1, \quad \forall n = 1, 2, ... \tag{5.21}$$

Using the comparison criteria of convergence of series and the convergence of the serie $\sum_{j=1}^{\infty} 2j \cdot e^{-jA}$ we see that the monotone increasing sequence

$$0 < -ln(L_n) = \sum_{j=1}^{n} 2j \; ln(1 - e^{-jA})$$

is bounded above, hence by (5.21)

$$0 < \lim_{n \to \infty} L_n \leqslant T_\infty := \lim_{n \to \infty} T_n < 1$$

or

$$0 < T_\infty < 1. \tag{5.22}$$

Let us consider the sequence $\{\chi_{z^0}^{(n,\varepsilon)}(z)\}_{n=1}^{\infty}$ of the functions defined as

$$\chi_{z^0}^{(n,\varepsilon)}(z) := \prod_{\substack{k \in \mathbb{Z}_2^+ \\ |k| \leq n}} \chi_{z_k^0}^{\varepsilon_k}(z_k), \forall (z^0, z) \in H^\beta \times H^\beta \tag{5.23}$$

that is monotone decreasing in n and bounded

$$0 \leqslant \chi_{z^0}^{(n,\varepsilon)}(z) \leqslant 1, \quad \forall (z^0, z) \in H^\beta \times H^\beta. \tag{5.24}$$

It is clear that $\forall (z^0, z) \in H^\beta \times H^\beta$

$$\chi_{z^0}^{(n,\varepsilon)}(z) \to \chi_{z^0}^{\varepsilon}(z) := \prod_{k \in \mathbb{Z}_2^+} \chi_{z_k^0}^{\varepsilon_k}(z_k), \quad \text{when } n \to +\infty.$$

Let us introduce the following functions

$$f^{(n,\varepsilon)}(z^0) = \int \chi_{z^0}^{(n,\varepsilon)}(z) \; d\mu_\gamma(z), \;\; f^\varepsilon(z^0) = \int \chi_{z^0}^{\varepsilon}(z) \; d\mu_\gamma(z).$$

By Lebesgue's theorem of dominated convergence and (5.17), (5.18), (5.20), (5.22), we have

$$\lim_{n\to\infty} f^{(n,\varepsilon)}(z^0) = f^{\varepsilon}(z^0), \ \forall z^0 \in H^{\beta}$$

and

$$\int\int \chi^{\varepsilon}_{z^0}(z)\, d\mu_\gamma\,(z)\, d\mu_\gamma(z^0) =$$

$$= \lim_{n\to\infty}\int \Big[\int \chi^{(n,\varepsilon)}_{z^0}(z)\, d\mu_\gamma\,(z)\Big]\, d\mu_\gamma(z^0) =$$

$$= \lim_{n\to\infty} T_n = T_\infty \in (0,1).$$

Fubini's theorem implies

$$0 < f^{\varepsilon}(z^0) \quad \text{for} \quad \mu_\gamma-\text{ a.e. } z^0 \text{ and } \int f^{\varepsilon}(z^0)\, d\mu_\gamma(z^0) \in (0,1).$$

Therefore the functions

$$\delta^{(n,\varepsilon)}_{z^0}(z) := \frac{1}{f^{(n,\varepsilon)}(z^0)}\cdot \chi^{(n,\varepsilon)}_{z^0}(z), \ \ \delta^{\varepsilon}_{z^0}(z) := \frac{1}{f^{\varepsilon}(z^0)}\cdot \chi^{\varepsilon}_{z^0}(z) \tag{5.25}$$

are well defined and

$$\delta^{(n,\varepsilon)}_{z^0}(z) \to \delta^{\varepsilon}_{z^0}(z), \text{ when } n\to\infty \tag{5.26}$$

for μ_γ–a.e. z^0 and $\forall z \in H^\beta$.

Moreover we have

1) $\int \delta^{\varepsilon}_{z^0}(z)\, d\mu_\gamma\,(z) = 1$ for μ_γ–a.e. z^0;
2) For μ_γ–a.e. z^0, the function $\delta^{\varepsilon}_{z^0}(z)$ has a compact support with respect to the weak topology in H^β:

$$\text{supp}(\delta^{\varepsilon}_{z^0}) = \left\{ z\in H^\beta : |z_k - z^0_k| \leqslant \frac{\varepsilon}{k^{1+1/2}}, \ \forall k \in \mathbb{Z}^+_2 \right\}$$

$$\subset B_{\varepsilon D}(z^0) = \left\{ z \in H^\beta : ||z-z^0||^2_{H^\beta} \leqslant \right.$$

$$\left. \leqslant \sum_{k\in\mathbb{Z}^+_2} |z_k - z^0_k|^2 \leqslant \varepsilon^2 \sum_{k\in\mathbb{Z}^+_2} \frac{1}{k^3} =: \varepsilon^2 D^2 \right\}.$$

By properties 1), 2) and Prokhorov's theorem [G-S], there exists a subsequence of the measures $\delta^{\varepsilon}_{z^0}(z)\, d\mu_\gamma(z)$ converging weakly to a measure $dm_{z^0}(z)$,

$$\delta^{\varepsilon}_{z^0}(z)\, d\mu_\gamma(z) \rightharpoonup dm_{z^0}(z) \quad \text{weakly} \ \text{ for } \mu_\gamma \text{ -a.e. } z^0,$$

when $\varepsilon \to 0$. By 1), 2), we have that

a) $\int 1\, dm_{z^0}\,(z) = 1 \quad \text{for} \quad \mu_\gamma$ -a.e. z^0;
b) For any cylindrical function $\varphi = \varphi(z)$, such that $z^0 \notin \text{supp}(\varphi)$, we deduce

$$\int \varphi(z)\, dm_{z^0}(z) = 0.$$

Therefore m_{z^0} coincides with the Dirac measure δ^{z^0}, that is, for $\varepsilon \to 0$

$$\delta^{\varepsilon}_{z^0}(z)\, d\mu_\gamma(z) \rightharpoonup d\delta_{z^0}(z) \ \text{ weakly for } \ \mu_\gamma\text{-a.e. } z^0. \tag{5.27}$$

In what follows we use the approximations $\delta^{(n,\varepsilon)}_{z^0}$, $\delta^{\varepsilon}_{z^0}$ to construct the measure valued solutions of problem (5.1)-(5.2), where the initial data is the Dirac measure.

5.3. **Existence for Dirac measure initial data.** In this paragraph we will assume that good measurements of turbulent process can be made. In that case the evolution of system should be described by a generalized solution associated with initial Dirac measure.

Let us assume that the initial distribution (5.1) is the Dirac measure, concentrated at a given point

$$u^0 \in \mathrm{supp}(\mu_\gamma). \tag{5.28}$$

In the sequel we prove the following theorem.

Theorem 5.2. *For μ_γ-a.e. u^0 there exists a generalized backward flow*

$$\nu_t = \nu_t(u),\ u \in H^\beta \text{ for } \beta < -1$$

of (4.1), satisfying equality (5.3) with the initial distribution (5.1), such that

$$\nu_0 = \delta_{u^0}.$$

Proof. The proof will be done in three steps. In the first step we construct approximated solutions of our problem, satisfying the Liouville-type equations (5.3). The second step will be devoted to show that this set of approximated solutions is pre-compact in a corresponding space of measures. In the last step we pass to the limit integral equations (5.3).

1st step. Let us note that the approximation $\delta_{u^0}^{(n,\varepsilon)}$ of the Dirac measure δ_{u^0} satisfies

$$\int |\delta_{u^0}^{(n,\varepsilon)}(u)|^2 d\mu_\gamma(u) = \frac{1}{f^{(n,\varepsilon)}(u^0)} < \infty. \tag{5.29}$$

Then for the regular initial distribution $\delta_{u^0}^{(n,\varepsilon)}$, by Theorem 5.1, the function

$$v^{(n,\varepsilon)}(t,u) = \delta_{u^0}^{(n,\varepsilon)}(U^n(-t,u)), \tag{5.30}$$

where $U^n = U^n(t,u)$ is the solution of problem (4.8), fulfills the identity

$$\Phi(0)\cdot \int f(u)\ \delta_{u^0}^{(n,\varepsilon)}(u)\,d\mu_\gamma(u) + \int_0^T \Phi'(t)\left[\int f(u)\,v^{(n,\varepsilon)}(t,u)\,d\mu_\gamma(u)\right]dt$$
$$= \int_0^T \Phi(t)\left[\int (B^n(u)\cdot\nabla f(u))\,v^{(n,\varepsilon)}(t,u)\,d\mu_\gamma(u)\right]dt \tag{5.31}$$

for any fixed cylindrical function $f = f(u)$ and any $\Phi(t) \in C^1([0,T])$, such that $\Phi(T) = 0$.

2nd step. We show that this identity converges when $n \to \infty$ and $\varepsilon \to 0$. Namely we prove that the set of measures

$$d\nu_t^{(n,\varepsilon)}(u) := v^{(n,\varepsilon)}(t,u)\,d\mu_\gamma(u) \tag{5.32}$$

is relatively compact for the weak topology of measures on H^β.

Let $C([0,T],H^\beta)$ be the space of continuous functions, depending on the parameter $t \in [0,T]$, with values in H^β. We introduce the measure

$$\nu^{(n,\varepsilon)}(\Gamma) := \int_{S_\Gamma} \delta_{u^0}^{(n,\varepsilon)}(u)\,d\mu_\gamma(u), \tag{5.33}$$

defined for any set $\Gamma \subset C([0,T],H^\beta)$. Here

$$S_\Gamma := \{u \in H^\beta : U^n(\cdot,u) \in \Gamma\}. \tag{5.34}$$

For any integrable functional $F : C([0,T],H^\beta) \to R$, we have

$$\int_{H^\beta} F(U^n(\cdot,u))\delta_{u^0}^{(n,\varepsilon)}(u)d\mu_\gamma(u) = \int_{C([0,T],H^\beta)} F(y)d\nu^{(n,\varepsilon)}(y). \tag{5.35}$$

For a fixed time moment $t \in [0,T]$ and a given function $f : C(H^\beta) \to \mathbb{R}$, we define the functional $F : C([0,T], H^\beta) \to \mathbb{R}$ by

$$F(y) := f(y(t)) \quad \text{for any } y \in C([0,T], H^\beta).$$

Then, in this particular case, the right side of the identity (5.35) is equal to

$$\begin{aligned}\int f(U^n(t,u))\delta_{u^0}^{(n,\varepsilon)}(u)d\mu_\gamma(u) &= \int f(u)\delta_{u^0}^{(n,\varepsilon)}\big(U^n(-t,u)\big)d\mu_\gamma(u) \\ &= \int f(u)d\nu_t^{(n,\varepsilon)}(u).\end{aligned} \tag{5.36}$$

Hence by (5.35) and (5.36), the relative compactness of the set of measures $\{d\nu^{(n,\varepsilon)}(y)\}$ in the sense of weak convergence of measures on $C([0,T], H^\beta)$ implies the relative compactness of $\{d\nu_t^{(n,\varepsilon)}(u)\}$ in the sense of weak convergence of measures on H^β for a.e. fixed $t \in [0,T]$.

Then, in the sequel, the objective is to show that the set of measures $\{d\nu^{(n,\varepsilon)}(y)\}$ is relatively compact in the sense of weak convergence of measures on $C([0,T], H^\beta)$. Let us denote by $C_c(H^\beta)$ the space of continuous functions on H^β with a compact support (in weak topology) on H^β. We introduce the operator

$$V^{(n,\varepsilon)}(\Psi,\Gamma) := \int \Psi(u^0)\left[\int_{S_\Gamma} \delta_{u^0}^{(n,\varepsilon)}(u)\, d\mu_\gamma(u)\right] d\mu_\gamma(u^0), \tag{5.37}$$

defined for any function $\Psi \in C_c(H^\beta)$ and any set $\Gamma \subset C([0,T], H^\beta)$.

We fix an arbitrary function $\Psi \in C_c(H^\beta)$, put

$$A := ||\Psi||_{C_c(H^\beta)}$$

and choose a sufficiently large real $R > 0$, such that

$$\text{supp }(\Psi) \subset B_R = \{u^0 \in H^\beta : ||u^0||_{H^\beta} < R\}.$$

Let us verify that the set of measures

$$V^n(\Psi,\cdot) := V^{(n,\varepsilon_n)}(\Psi,\cdot), \quad n = 1,2,3,..., \tag{5.38}$$

where the exact value of ε_n is defined below in (5.43), satisfies the following conditions

a) $\lim\limits_{\rho\to+\infty} \sup\limits_n V^n(\Psi, ||y(0)||_{H^\beta} > \rho) = 0;$

b) For any fixed $\rho > 0$, we have that

$$\lim_{\delta\to 0} \sup_n V^n(\Psi, \sup_{\substack{0\leqslant t<t_1\leqslant T\\ t_1-t\leqslant\delta}} ||y(t) - y(t_1)||_{H^\beta} \geqslant \rho) = 0.$$

By Prokhorov's criteria ([Mal], Theorem 2.6) these two conditions guarantee that the set of measures $\{V^n(\Psi,\cdot)\}$ is tight (see, for instance, [Mal], Theorems 4.2 and 4.3), therefore this set is relatively compact in the sense of weak convergence of measures on $C([0,T], H^\beta)$.

Let us start from the proof of condition a). We have

$$V^{(n,\varepsilon)}(\Psi, ||y(0)||_{H^\beta} > \rho) \leqslant A \cdot \int_{B_R}\left[\int \chi_{S_\Gamma}(u)\cdot \delta_{u^0}^{(n,\varepsilon)}(u)\, d\mu_\gamma(u)\right] d\mu_\gamma(u^0).$$

Since

$$\begin{aligned} S_\Gamma &= \{u \in H^\beta : U^n(\cdot,u) \in \Gamma = \{y(\cdot) \in C([0,T], H^\beta) : ||y(0)||_{H^\beta} > \rho\}\} \\ &= \{u \in H^\beta : U^n(0,u) = u \text{ and } \rho < ||u||_{H^\beta}\},\end{aligned}$$

using the definition (5.25), we deduce

$$V^{(n,\varepsilon)}(\Psi, ||y(0)||_{H^\beta} > \rho) \leqslant \frac{A}{\rho} \int_{B_R} \left[\int ||u||_{H^\beta} \cdot \delta_{u^0}^{(n,\varepsilon)}(u) d\mu_\gamma(u) \right] d\mu_\gamma(u^0).$$

For any $u \in H^\beta$, $u^0 \in B_R$, which satisfy $||u - u^0||_{H^\beta} < \varepsilon \cdot D$, we have $||u||_{H^\beta} < ||u^0||_{H^\beta} + \varepsilon \cdot D < C$, hence

$$V^{(n,\varepsilon)}(\Psi, ||y(0)||_{H^\beta} > \rho) \leqslant \frac{A \cdot C}{\rho},$$

that implies condition a).

Now we show that the set of measures $\{V^n(\Psi, \cdot)\}$ satisfy condition b). We have

$$V^{(n,\varepsilon)} \left(\Psi, \sup_{\substack{0 \leqslant t < t_1 \leqslant T \\ t_1 - t \leqslant \delta}} ||y(t) - y(t_1)||_{H^\beta} \geqslant \rho \right) \tag{5.39}$$

$$\leqslant A \cdot \int_{B_R} \left[\int_S \delta_{u^0}^{(n,\varepsilon)}(u)\, d\mu_\gamma(u) \right] d\mu_\gamma(u^0) = I,$$

where

$$S = \left\{ u \in H^\beta : \sup_{\substack{0 \leqslant t < t_1 \leqslant T \\ t_1 - t \leqslant \delta}} ||U^n(t,u) - U^n(t_1,u)||_{H^\beta} \geqslant \rho \right\}.$$

From (4.8)

$$||U^n(t_1,u) - U^n(t,u)||_{H^\beta} \leqslant \int_t^{t_1} ||B^n(U^n(s,u))||_{H^\beta} ds$$

for any $0 \leqslant t < t_1 \leqslant T$. Hence, using Lemma 3.1 and (3.2),

$$\begin{aligned} I &\leqslant \frac{A}{\rho} \cdot \int_t^{t_1} \left\{ \int_{B_R} \int ||B^n(U^n(s,u))||_{H^\beta} \delta_{u^0}^{(n,\varepsilon)}(u) d\mu_\gamma(u) d\mu_\gamma(u^0) \right\} dt \\ &\leqslant \frac{A}{\rho} \cdot \int_t^{t_1} \left\{ \int_{B_R} ||B^n(U^n(s,u^0))||_{H^\beta} d\mu_\gamma(u^0) + \right. \end{aligned}$$

$$\left. + \int_{B_R} \left[\int ||B^n(U^n(s,u)) - B^n(U^n(s,u^0))||_{H^\beta} \delta_{u^0}^{(n,\varepsilon)}(u) d\mu_\gamma(u) \right] d\mu_\gamma(u) \right\} dt. \tag{5.40}$$

From (4.9), in the bounded domain $M := \{(t, u_0, u) \in [0,T] \times \mathbb{C}^\infty \times \mathbb{C}^\infty : u^0 \in B_R \text{ and } ||u - u^0||_{H^\beta} \leqslant \varepsilon \cdot D\}$ the functions $U^n(t,u), U^n(t,u_0)$ are bounded. By (3.1), there exists a constant C_n, depending only on the parameter n and satisfying the inequality

$$||B^n(U^n(s,u)) - B^n(U^n(s,u^0))||_{H^\beta} \leqslant C_n ||U^n(s,u) - U^n(s,u^0)||_{H^\beta}, \tag{5.41}$$

Let us note that $C_n \to \infty$, when $n \to \infty$. By (4.8), (5.41), the function

$$z(t) := ||U^n(s,u) - U^n(s,u^0)||_{H^\beta}$$

satisfies the Gronwall type inequality $\frac{dz(t)}{dt} \leqslant C_n \cdot z(t)$ with the initial condition $z(0) = ||u - u^0||_{H^\beta}$, that implies

$$z(t) \leqslant z(0) \cdot \exp(C_n T) \quad \text{for any } t \in [0,T]. \tag{5.42}$$

In the following considerations, we assume that the parameter ε depends on n and is equal to

$$\varepsilon_n := C_n^{-1} \cdot \exp(-C_n T). \tag{5.43}$$

Considering estimates (5.40)-(5.43) and Lemma 3.1, we deduce the inequality

$$V^n\left(\Psi, \sup_{\substack{0\leqslant t<t_1\leqslant T\\ t_1-t\leqslant\delta}} ||y(t)-y(t_1)||_{H^\beta}\geqslant\rho\right) \quad\leqslant\quad \frac{\delta\cdot A}{\rho}\cdot C,$$

with the constant C, which is independent on n. Taking $\delta\to 0$, we conclude that the measures $V^n(\Psi,\cdot)$ fulfill condition (b).

Therefore, for any fixed function $\Psi\in C_c(H^\beta)$, there exist a measure $V(\Psi,\cdot)$ on $C([0,T],H^\beta)$ and a subsequence $n=n(\Psi)\to\infty$, such that

$$V^{n(\Psi)}(\Psi,\Gamma) \underset{n(\Psi)\to\infty}{\rightarrow} V(\Psi,\Gamma) \tag{5.44}$$

for all $\Gamma\subset C([0,T],H^\beta)$.

Let us show that such subsequence can be choosen independently of $\Psi\in C_c(H^\beta)$.

1) Let $\mathcal{P}:=\{\Psi_i\in C_c(H^\beta):\ i=1,2,3,...\}$ be a dense set in $C_c(H^\beta)$.

2) Since the set $\mathcal{P}$ is countable, we can select a subsequence of $n\to\infty$, that for simplicity of notations we still denote by n, such that for any $\Psi\in\mathcal{P}$, we have

$$V^n(\Psi,\Gamma) \underset{n\to\infty}{\rightarrow} V(\Psi,\Gamma) \tag{5.45}$$

for all $\Gamma\subset C([0,T],H^\beta)$.

3) Let us now fix an arbitrary $\Psi\in C_c(H^\beta)$. There exists a subsequence $n=n(\Psi)\to\infty$ of the sequence $\{n:\ n=1,2,3,...\}$, constructed in 2), and a measure $V(\Psi,\cdot)$, satisfying (5.44). In fact, we can verify that the sequence $\{V^n(\Psi,\cdot)\}_{n=1}^\infty$ itself is convergent. To do it, it is enough to show that $\{V^n(\Psi,\cdot)\}_{n=1}^\infty$ is a Cauchy sequence. Since there exists a sequence $\Psi_{i_k}\in\mathcal{P},\ i_k\to\infty$, such that

$$||\Psi_{i_k}-\Psi||_{C(H^\beta)} \underset{i_k\to\infty}{\rightarrow} 0,$$

then by (5.37), the convergence (5.45) and the inequality

$$|V^l(\Psi)-V^j(\Psi)|\leqslant|V^l(\Psi)-V^l(\Psi_{i_k})|+$$
$$+|V^l(\Psi_{i_k})-V^j(\Psi_{i_k})|+|V^j(\Psi_{i_k})-V^j(\Psi)|,$$

we deduce that $\{V^n(\Psi,\cdot)\}_{n=1}^\infty$ is the Cauchy sequence. Hence the sequence $\{n:\ n=1,2,3,...\}$, constructed in 2), satisfies the convergence (5.45) for any $\Psi\in C_c(H^\beta)$.

Let us note that for any fixed $\Gamma\subset C([0,T],H^\beta)$, the operator $V^n(\Psi,\Gamma)$ as the function of the parameter $\Psi\in C_c(H^\beta)$ is linear, then it is easy to verify that the operator $V(\Psi,\Gamma)$ is also linear in Ψ. By Kakutani-Riesz Representation theorem, if we denote by $\mathcal{B}$ the set of all Borel sets in H^β, there exist a σ-algebra Σ on the space H^β, that contains $\mathcal{B}$, and a unique positive measure μ_Γ on Σ, which represent the operator V as

$$V(\Psi,\Gamma)=\int\Psi(u^0)\ d\mu_\Gamma(u^0),\ \ \Psi\in C_c(H^\beta) \tag{5.46}$$

for any fixed $\Gamma\subset C([0,T],H^\beta)$.

Let us consider the constructed measure μ_Γ and the measure μ_γ, which are two measures on the topological space $(H^\beta,\mathcal{B})$. By the Radon-Nikodym theorem (see, for instance, [U-Z]) there exists an integrable positive real function $\Lambda_\Gamma(u^0)$ on the space $(H^\beta,\ \mathcal{B},\ \mu_\gamma)$ and a set $\mathcal{N}$ of zero measure on the space $(H^\beta,\ \mu_\gamma)$, such that for each set $A\in\mathcal{B}$ we have

$$\mu_\Gamma(A)=\int_A\Lambda_\Gamma(u^0)\ d\mu_\gamma(u^0)+\mu_\Gamma(A\cap\mathcal{N})$$

for any fixed $\Gamma\subset C([0,T],H^\beta)$. Moreover for any set $A\in\mathcal{B}$ such that $\mu_\gamma(A)=0$, using (5.37), (5.45) and (5.46), we can easily show that $\mu_\Gamma(A)=0$. Therefore by the particular case of the Radon-Nikodym theorem ([U-Z], page 3) the set $\mathcal{N}$ is empty, which implies

$$\mu_\Gamma(A) = \int_A \Lambda_\Gamma(u^0)\ d\mu_\gamma(u^0)$$

and

$$V(\Psi,\Gamma) = \int \Psi(u^0)\,\Lambda_\Gamma(u^0)\ d\mu_\gamma(u^0),$$

for any function $\Psi \in C_c(H^\beta)$ and any set $\Gamma \subset C([0,T],H^\beta)$. Considering (5.37) and (5.45), we obtain for μ_γ- a.e. u_0

$$\nu^{(n,\varepsilon_n)}(\Gamma) = \int_{S_\Gamma} \delta_{u^0}^{(n,\varepsilon_n)}(u)\, d\mu_\gamma(u) \underset{n\to\infty}{\to} \Lambda_{u^0}(\Gamma) := \Lambda_\Gamma(u^0)$$

for any set $\Gamma \subset C([0,T],H^\beta)$ with the set S_Γ, defined by (5.34). The function $\Lambda_{u^0}(\cdot)$ is a probability measure on $C([0,T],H^\beta)$, therefore, taking into account (5.32), (5.35), (5.36), for μ_γ-a.e. u^0 we get that there exists a probability measure

$$d\nu_t^{u^0} = d\nu_t^{u^0}(u), \quad \text{for each } t\in[0,T]$$

on H^β, such that

$$\nu_t^{(n,\varepsilon_n)}(\Gamma) \underset{n\to\infty}{\to} \nu_t^{u^0}(\Gamma) \tag{5.47}$$

for any set $\Gamma \subset C(H^\beta)$.

3d step. With the help of (3.2), (5.25)-(5.27) and (5.47), passing to the limit on $n \to \infty$ in equality (5.31) written for $\varepsilon = \varepsilon_n$, we obtain that the measures $\{\nu_t^{u^0},\ t\in[0,T]\}$ satisfy the equality

$$\Phi(0)\cdot f(u^0) + \int_0^T \Phi'(t)\left[\int f(u) d\nu_t^{u^0}(u)\right] dt =$$

$$= \int_0^T \Phi(t)\left[\int (B(u)\cdot\nabla f(u))\ d\nu_t^{u^0}(u)\right] dt$$

for any fixed cylindrical function $f = f(u)$ and any $\Phi(t) \in C^1([0,T])$, such that $\Phi(T) = 0$.

Hence we have shown that for μ_γ-a.e. u^0 the set of the probability measures $\{\nu_t^{u^0},\ t\in[0,T]\}$ is the generalized backward flow of the Euler equation (4.1) with the initial distribution $\nu_0 = \delta_{u^0}$. □

6. Conclusion

From (4.7) we have

$$\int \|U(t,u)\|^2_{H^\beta} d\mu_\gamma < \infty, \quad t\in[0,T],\ \beta<1.$$

Hence for μ_γ - a. e. fixed u, using (2.8), $U(t,u)$ is a function of (t,x)

$$U(t,x) = \sum_{k\in\mathbb{Z}^2_+} U_k(t,u)e_k(x) \in L^\infty\big([0,T],H^\beta\big)$$

satisfying the periodic conditions (2.2) and the initial data (2.4): $U(0,x) = u(x)$. The function $U(t,x)$ may not satisfy Euler equation (2.1), because the Fourier coefficients may not correspond to a solution of the infinite dimensional equation (2.6).

One possible approach to this open problem is the method of the re-normalized solutions [DiP-L].

As is well known, there are two different ways of expressing the behaviour of the fluid: the Lagrangian and the Eulerian point of view. Their difference lies in the choice of coordinates to describe flow phenomena.

In the Lagrangian description, the fluid is viewed as a collection of fluid particles (elements) that are freely translating, rotating, and deforming. To obtain a full description of the flow we need to identify the initial position of the elements.

In our case, the relationship

$$U = U(t, u)$$

is the statistical Lagrangian description of the fluid.

In the Eulerian description, an observed point of the physical space remains unchanged by time t. Quantities (velocities, temperature, pressure, etc.) are measured at different instances of t.

Hence, our quantities

$$\mu_t = \mu_t(U), \;\; V = V(t, U)$$

are determined as functions of Euler parameters: the time t and the observed point U.

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