Near-exact distributions for the sphericity likelihood ratio test statistic

Filipe J. Marques [1] Carlos A. Coelho [1]

fjm@fct.unl.pt *cmac@fct.unl.pt*

Mathematics Department, Faculty of Sciences and Technology
The New University of Lisbon, Portugal

Abstract

In this paper three near-exact distributions are developed for the sphericity test statistic. The exact probability density function of this statistic is usually represented through the use of the Meijer G function, what renders the computation of quantiles impossible even for a moderately large number of variables. The main goal of this paper is thus twofold: to obtain near-exact distributions that i) lie closer to the exact distribution than the asymptotic distributions and also that ii) correspond to density and cumulative distribution functions pratical to use, allowing for an easy way to determine quantiles. On the way, also two asymptotic distributions that lie closer to the exact distribution than the existing ones were developed. Three measures are considered to evaluate the proximity between the exact and the asymptotic and near-exact distributions developed. As a reference we use the asymptotic distribution proposed by Box as well as some saddlepoint approximations developed by other authors.

Key words: asymptotic distributions, sphericity test, Generalized Near-Integer Gamma distribution, mixtures.

1 Introduction

Let $\underline{x}$ be a $p{\times}1$ vector of variables with a multivariate normal distribution $N_p(\underline{\mu}, \Sigma)$. We intend to test the hypotheses,

$$H_0 : \Sigma = \sigma^2 I_p \quad \text{versus} \quad H_1 : \Sigma \neq \sigma^2 I_p \,. \tag{1}$$

[1] This research was financially supported by the Portuguese Foundation for Science and Technology (FCT).

The likelihood ratio test statistic is defined, for a sample of size N, as (Mauchly, 1940; Anderson, 1958, Sec. 10.7)

$$\lambda = \frac{|A|^{\frac{1}{2}N}}{\left(tr\ \frac{1}{p}A\right)^{\frac{1}{2}pN}}\,, \tag{2}$$

where A is either the Maximum Likelihood Estimator of Σ, the sample variance-covariance matrix or yet the sample matrix of sums of squares and cross products of deviations about the mean.

The moments of $\Lambda = \lambda^{2/N}$, considering $n = N-1$, are

$$\begin{aligned} E(\Lambda^h) &= p^{hp}\frac{\Gamma\left(\frac{np}{2}\right)}{\Gamma\left(\frac{np}{2}+ph\right)}\frac{\Gamma_p\left(\frac{1}{2}n+h\right)}{\Gamma_p\left(\frac{1}{2}n\right)} \\ &= p^{hp}\frac{\Gamma\left(\frac{np}{2}\right)}{\Gamma\left(\frac{np}{2}+ph\right)}\prod_{j=1}^{p}\frac{\Gamma\left(\frac{n+1-j}{2}+h\right)}{\Gamma\left(\frac{n+1-j}{2}\right)}\,, \qquad \left(h > -\frac{n+1-p}{2}\right), \end{aligned} \tag{3}$$

where

$$\Gamma_p(t) = \pi^{\frac{p(p-1)}{4}}\prod_{j=1}^{p}\Gamma\left(t-\frac{1}{2}(j-1)\right)$$

is the multivariate Gamma function.

Since in (3) the Gamma functions are defined for any strictly complex h, we may write the characteristic function of $W = -\log\Lambda$ as

$$\begin{aligned} \phi_W(t) &= E(\mathrm{e}^{-\mathrm{i}tW}) = E\left(\mathrm{e}^{-\log\Lambda \mathrm{i}t}\right) = E(\Lambda^{-\mathrm{i}t}) \\ &= p^{-\mathrm{i}tp}\frac{\Gamma\left(\frac{np}{2}\right)}{\Gamma\left(\frac{np}{2}-\mathrm{i}tp\right)}\prod_{j=1}^{p}\frac{\Gamma\left(\frac{n+1-j}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{n+1-j}{2}\right)}\,. \end{aligned} \tag{4}$$

For $m \in I\!N$ (where $I\!N$ is the set of positive integers) we can use the expression

$$\Gamma(mz) = (2\pi)^{-\frac{m-1}{2}}\, m^{mz-\frac{1}{2}}\prod_{k=1}^{m}\Gamma\left(z+\frac{k-1}{m}\right)$$

and write $\phi_W(t)$ as

$$\begin{aligned} \phi_W(t) &= \prod_{j=1}^{p}\frac{\Gamma\left(\frac{n}{2}+\frac{j-1}{p}\right)}{\Gamma\left(\frac{n}{2}+\frac{j-1}{p}-\mathrm{i}t\right)}\frac{\Gamma\left(\frac{n+1-j}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{n+1-j}{2}\right)} \\ &= \prod_{j=1}^{p}\frac{\Gamma\left(\frac{n+1}{2}-\frac{j}{2}+\frac{j-1}{p}+\frac{j-1}{2}\right)}{\Gamma\left(\frac{n+1}{2}-\frac{j}{2}\right)}\frac{\Gamma\left(\frac{n+1}{2}-\frac{j}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{n+1}{2}-\frac{j}{2}+\frac{j-1}{p}+\frac{j-1}{2}-\mathrm{i}t\right)}\,. \end{aligned} \tag{5}$$

It we will be based on (5) that we will obtain the expressions for the near-exact approximations.

2 Asymptotic approximations

In order to compare and evaluate the quality of the near-exact approximations, we will use the asymptotic approximation of Box (1949) and Anderson (1958, Sec. 10.7) and two other asymptotic approximations that we develop, which lie closer to the exact distribution, as it will be shown in section 5 through the use of the measures described in section 4.

Box (1949) and Anderson (1958, Sec. 10.7) give, under the null hypothesis in (1), an asymptotic distribution for Λ, under the form

$$P(-n\rho\log\Lambda \leq z) = P(\chi^2_f \leq z) + \omega_2\left(P(\chi^2_{f+4} \leq z) - P(\chi^2_f \leq z)\right) + O\left(n^{-3}\right) \tag{6}$$

where

$$\omega_2 = \frac{(p+2)(p-1)(p-2)(2p^3+6p^2+3^p+2)}{288p^2n^2p^2}\,, \quad f = \frac{1}{2}p(p+1)-1$$

and

$$\rho = 1 - \frac{2p^2+p+2}{6pn}\,.$$

From (6) we may write $\phi_{Box}(t)$ as an aproximation for $\phi_W(t)$, where $W = -\log\Lambda$, under the form

$$\begin{aligned}\phi_{Box}(t) &= (1-\omega_2)\left(\tfrac{1}{2}\right)^{\frac{f}{2}}\left(\tfrac{1}{2}-\mathrm{i}\tfrac{t}{n\rho}\right)^{-\frac{f}{2}} + \omega_2\left(\tfrac{1}{2}\right)^{\frac{f}{2}+2}\left(\tfrac{1}{2}-\mathrm{i}\tfrac{t}{n\rho}\right)^{-\frac{f}{2}-2} \\ &= (1-\omega_2)\left(\tfrac{n\rho}{2}\right)^{\frac{f}{2}}\left(\tfrac{n\rho}{2}-\mathrm{i}t\right)^{-\frac{f}{2}} + \omega_2\left(\tfrac{n\rho}{2}\right)^{\frac{f}{2}+2}\left(\tfrac{n\rho}{2}-\mathrm{i}t\right)^{-\frac{f}{2}-2}\,.\end{aligned} \tag{7}$$

Considering that the random variable X has a Gamma distribution with shape parameter $r > 0$ and rate parameter $\lambda > 0$, that is,

$$X \sim \Gamma(r, \lambda)$$

if its probability density and characteristic functions are respectively given by

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)}\, e^{-\lambda x}\, x^{r-1}\,, \qquad \phi_X(t) = \lambda^r (\lambda - \mathrm{i}t)^{-r}\,,$$

we may thus say that the characteristic function in (7) is the characteristic function of a mixture of two Gamma distributions, both with rate parameter $n\rho/2$ and one of them with weight $1-\omega_2$ and shape parameter $f/2$ and the other with weight ω_2 and shape parameter $f/2+2$.

In order to obtain other asymptotic distributions we propose two other mixtures of Gamma distributions, which match the first four or six exact moments; the mixture of two Gamma distributions, both with the same rate parameter (matching the first four exact moments), with characteristic function

$$\phi_{M2G}(t) = \sum_{j=1}^{2} p_j\, \lambda^{r_j} (\lambda - \mathrm{i}t)^{-r_j}\,, \tag{8}$$

where $p_2 = 1 - p_1$ with p_j, r_j, $\lambda > 0$, and the mixture of three Gamma distributions, all with the same rate parameter (matching the first six exact moments), with characteristic function

$$\phi_{M3G}(t) = \sum_{j=1}^{3} p_j^*\, \mu^{r_j^*} (\mu - \mathrm{i}t)^{-r_j^*}\,, \tag{9}$$

where $p_3^* = 1 - p_1^* - p_2^*$, with p_j^*, r_j^*, $\mu > 0$.

The parameters in (8) and (9) are respectively obtained by solving the systems of equations

$$\mathrm{i}^h \sum_{j=1}^{2} p_j\, \frac{\Gamma(r_j+h)}{\Gamma(r_j)}\, \lambda^{-h} = \left.\frac{\partial^h \phi_{M2G}(t)}{\partial t^h}\right|_{t=0} = \left.\frac{\partial^h \phi_W(t)}{\partial t^h}\right|_{t=0} \qquad (h = 1, \ldots, 4)$$

and

$$\mathrm{i}^h \sum_{j=1}^{3} p_j^*\, \frac{\Gamma(r_j^*+h)}{\Gamma(r_j^*)}\, \mu^{-h} = \left.\frac{\partial^h \phi_{M3G}(t)}{\partial t^h}\right|_{t=0} = \left.\frac{\partial^h \phi_W(t)}{\partial t^h}\right|_{t=0} \qquad (h = 1, \ldots, 6)$$

for those parameters.

3 Near-exact distributions

Theorem 1 *The characteristic function of* $W = -\log \Lambda = -\log\left(\lambda^{\frac{2}{N}}\right)$*, where* λ *is the statistic in* (2)*, may be written as*

$$\phi_W(t) = \underbrace{\left[\prod_{j=2}^{p} a_j^{s_{j,p}} \left(a_j - \mathrm{i}t\right)^{-s_{j,p}}\right]}_{\phi_{W_1}(t)} \times \underbrace{\prod_{j=1}^{p} \frac{\Gamma(a_j + b_j^* + c_j)}{\Gamma(a_j + b_j^*)} \frac{\Gamma(a_j + b_j^* - \mathrm{i}t)}{\Gamma(a_j + b_j^* + c_j - \mathrm{i}t)}}_{\phi_{W_2}(t)} = \phi_{W_1}(t) \times \phi_{W_2}(t)\,, \tag{10}$$

where, for $j = 1, \ldots, p$*,*

$$a_j = \frac{n+1}{2} - \frac{j}{2}\,, \qquad b_j^* = \begin{cases} \frac{j-1}{2} & j \text{ odd} \\ \frac{j-2}{2} & j - 1 < \frac{p}{2} \text{ and } j \text{ even} \\ \frac{j}{2} & j - 1 \geq \frac{p}{2} \text{ and } j \text{ even}\,, \end{cases} \tag{11}$$

$$c_j = \begin{cases} \frac{j-1}{p} & j \text{ odd} \\ \frac{j-1}{p} + \frac{1}{2} & j - 1 < \frac{p}{2} \text{ and } j \text{ even} \\ \frac{j-1}{p} - \frac{1}{2} & j - 1 \geq \frac{p}{2} \text{ and } j \text{ even}\,, \end{cases} \tag{12}$$

and

$$s_{j,p} = \begin{cases} \left\lfloor \frac{p-j}{2} + 1 \right\rfloor & j = 3, \ldots p \\ \left\lfloor \frac{p}{4} + \frac{1}{2} \right\rfloor & j = 2 \text{ and } p \text{ even} \\ \left\lfloor \frac{p}{4} \right\rfloor & j = 2 \text{ and } p \text{ odd}\,. \end{cases} \tag{13}$$

Proof: For a_j given by (11),

$$b_j = \frac{j-1}{p} + \frac{j-1}{2} \qquad \text{and} \qquad b_j^* = \lfloor b_j \rfloor \tag{14}$$

we may write (5) as

$$\phi_W(t) = \prod_{j=1}^{p} \frac{\Gamma(a_j+b_j)}{\Gamma(a_j)} \frac{\Gamma(a_j-\mathrm{i}t)}{\Gamma(a_j+b_j-\mathrm{i}t)}$$
$$= \prod_{j=1}^{p} \frac{\Gamma(a_j+b_j^*)}{\Gamma(a_j)} \frac{\Gamma(a_j-\mathrm{i}t)}{\Gamma(a_j+b_j^*-\mathrm{i}t)} \frac{\Gamma(a_j+b_j)}{\Gamma(a_j+b_j^*)} \frac{\Gamma(a_j+b_j^*-\mathrm{i}t)}{\Gamma(a_j+b_j-\mathrm{i}t)}\,.$$

But then, using, for integer b_j^*,

$$\Gamma(\alpha+b_j^*) = \Gamma(\alpha) \prod_{h=0}^{b_j^*-1} (\alpha+h)\,,$$

we may write

$$\phi_W(t) = \left[\prod_{j=1}^{p} \prod_{h=0}^{b_j^*-1} (a_j+h)(a_j+h-\mathrm{i}t)^{-1}\right] \times \prod_{j=1}^{p} \frac{\Gamma(a_j+b_j)}{\Gamma(a_j+b_j^*)} \frac{\Gamma(a_j+b_j^*-\mathrm{i}t)}{\Gamma(a_j+b_j-\mathrm{i}t)}\,,$$

or, taking $c_j = b_j - b_j^*$,

$$\phi_W(t) = \underbrace{\left[\prod_{j=1}^{p} \prod_{h=0}^{b_j^*-1} (a_j+h)(a_j+h-\mathrm{i}t)^{-1}\right]}_{\phi_{W_1}(t)} \times \underbrace{\prod_{j=1}^{p} \frac{\Gamma(a_j+b_j^*+c_j)}{\Gamma(a_j+b_j^*)} \frac{\Gamma(a_j+b_j^*-\mathrm{i}t)}{\Gamma(a_j+b_j^*+c_j-\mathrm{i}t)}}_{\phi_{W_2}(t)} \qquad (15)$$
$$= \phi_{W_1}(t) \times \phi_{W_2}(t)$$

where, given the definiton of b_j in (14), b_j^* and c_j ($j = 1, \ldots, p$) are respectively given by (11) and (12).

In (15), $\phi_{W_1}(t)$ is the characteristic function of the sum of $\sum_{j=1}^{p} b_j^*$ independent exponencial random variables with parameters $a_j + h$ ($j = 1, \ldots, p$; $h = 0, \ldots, b_j^* - 1$) and $\phi_{W_2}(t)$ is the characteristic function of the sum of p independent random variables whose exponential has a Beta distribution with parameters $a_j + b_j^*$ and c_j ($j = 1, \ldots, p$).

Observing with attention expression (15) we may see that

$$a_j = a_{j+2k} + k \quad \text{for} \quad k \le b_{j+2k}^* - 1\ , \quad k \in I\!N\,,$$

or in a more general form

$$a_j + n_0 = a_{j+2k} + k + n_0$$

for $n_0 \leq b_j^* - 1$ and $n_0 + k \leq b_{j+2k}^* - 1$, with $k \in I\!N$, $n_0 \in I\!N_0$. Using this relation we may write $\phi_{W_1}(t)$ under the form

$$\phi_{W_1}(t) = \prod_{j=2}^{p} a_j^{s_{j,p}} (a_j - \mathrm{i}t)^{-s_{j,p}} \tag{16}$$

where $s_{j,p}$ $(j = 1, \ldots, p)$, are given by (13). See Appendix A for further details on the proof of this latter result. $\square$

Further ahead we will take advantage of the fact that $\phi_{W_2}(t)$ is the characteristic function of the sum of p independent Logbeta random variables by using characteristic functions of Gamma distributions or mixtures of these distributions to approximate $\phi_{W_2}(t)$.

Our goal is to obtain a near-exact approximation (Coelho, 2004) to $\phi_W(t)$, by keeping $\phi_{W_1}(t)$ unchanged and replacing $\phi_{W_2}(t)$ by another characteristic function, having in mind that the final characteristic function has to correspond to a known and manageable cumulative distribution function. We intend to make the computation of near-exact quantiles as accurate as possible by using for $\phi_{W_2}(t)$ the best possible approximation, while keeping the above context unchanged and the computation of quantiles feasible.

As we show in Theorem 2 ahead, the near-exact distributions obtained for W will be either a Generalized Near-Integer Gamma distribution (GNIG) or mixtures of two or three of these distributions.

The density and distribution functions for the GNIG is given by Coelho (2004). Let

$$Z = Z_1 + Z_2$$

where $Z_2 \sim \Gamma(r, \lambda)$, with $\lambda > 0$ and r a positive non-integer and

$$Z_1 = \sum_{i=1}^{g} X_i \,, \qquad \text{with} \qquad X_i \sim \Gamma(r_i, \lambda_i)\,,$$

where $r_1, \ldots, r_g$ are positive integers and $\lambda_1, \ldots, \lambda_g > 0$ are all different. Furthermore, let Z_1 and Z_2 be independent and $\lambda \neq \lambda_i$ $(i = 1 \ldots, g)$. Then the distribution of Z is a GNIG distribution of depth $g + 1$. We will denote this by

$$Z \sim GNIG(r_1, \ldots, r_g, r;\, \lambda_1, \ldots, \lambda_g, \lambda)\,.$$

The probability density function of Z is given by

$$f_Z(z|r_1,\ldots,r_g,r;\,\lambda_1,\ldots,\lambda_g,\lambda)=$$
$$K\lambda^r\sum_{j=1}^{g}\mathrm{e}^{-\lambda_j z}\sum_{k=1}^{r_j}\left\{c_{j,k}\frac{\Gamma(k)}{\Gamma(k+r)}z^{k+r-1}{}_1F_1(r,k+r,-(\lambda-\lambda_j)z)\right\},\quad (z>0) \tag{17}$$

and the cumulative distribution function by

$$F_Z(z|r_1,\ldots,r_g,r;\,\lambda_1,\ldots,\lambda_g,\lambda)=\lambda^r\frac{z^r}{\Gamma(r+1)}{}_1F_1(r,r+1,-\lambda z)$$
$$-K\lambda^r\sum_{j=1}^{g}\mathrm{e}^{-\lambda_j z}\sum_{k=1}^{r_j}c^*_{j,k}\sum_{i=0}^{k-1}\frac{z^{r+i}\lambda_j^i}{\Gamma(r+1+i)}{}_1F_1(r,r+1+i,-(\lambda-\lambda_j)z)\quad (z>0) \tag{18}$$

where

$$K=\prod_{j=1}^{g}\lambda_j^{r_j}\qquad\text{and}\qquad c^*_{j,k}=\frac{c_{j,k}}{\lambda_j^k}\Gamma(k)$$

with c_{jk} given by (11) through (13) in Coelho (1998). In the above expressions ${}_1F_1(a,b;z)$ is the Kummer confluent hypergeometric function (Abramowitz and Stegun, 1974). Such functions have usually very good convergence properties and are nowadays handled by a number of software packages.

Theorem 2 *Using for $\phi_{W_2}(t)$ in (10), the following approximations*

- $\lambda^r(\lambda-\mathrm{i}t)^{-r}$ *with $r,\lambda>0$ (matching the first two moments);*
- $\sum_{k=1}^{2}\theta_k\,\mu^{r_k}(\mu-\mathrm{i}t)^{-r_k}$, *where $\theta_2=1-\theta_1$ with θ_k, r_k, $\mu>0$ (matching the first four moments);*
- $\sum_{k=1}^{3}\theta^*_k\,\nu^{r^*_k}(\nu-\mathrm{i}t)^{-r^*_k}$, *where $\theta^*_3=1-\theta^*_1-\theta^*_2$ with θ^*_k, r^*_k, $\nu>0$ (matching the first six moments);*

we obtain as near-exact distributions for W, respectively, a GNIG distribution of depth $p+1$ with distribution function (using the notation as in (18))

$$F(w|s_{1,p},\ldots,s_{p,p},r;a_1,\ldots,a_p,\lambda) \tag{19}$$

or a mixture of two GNIG distributions of depth $p+1$, with distribution func-

tion

$$\sum_{k=1}^{2} \theta_k \, F(w|s_{1,p},\ldots,s_{p,p},r_k;a_1,\ldots,a_p,\mu) \tag{20}$$

or a mixture of three GNIG distributions of depth $p+1$, with distribution function

$$\sum_{k=1}^{3} \theta_k^* \, F(w|s_{1,p},\ldots,s_{p,p},r_k^*;a_1,\ldots,a_p,\nu)\,, \tag{21}$$

with $s_{j,p}$ and a_j $(j=1,\ldots,p)$ given respectively by (13) *and* (11)*, and where*

$$\lambda = \frac{m_1}{m_2-m_1^2} \qquad \text{and} \qquad r = \frac{m_1^2}{m_2-m_1^2} \tag{22}$$

with

$$m_h = \mathrm{i}^h \, \frac{\partial^h}{\partial t^h} \, \phi_{W_2}(t)\,, \qquad h=1,2\,,$$

θ_1, μ, r_1 and r_2 are obtained from the numerical solution of the system of equations

$$\sum_{k=1}^{2} \theta_k \, \frac{\Gamma(r_k+h)}{\Gamma(r_k)} \, \mu^{-h} \; = \; \mathrm{i}^{-h} \frac{\partial^h}{\partial t^h} \, \phi_{W_2}(t) \qquad (h=1,\ldots,4) \tag{23}$$

with $\theta_2 = 1-\theta_1$, for those parameters, and θ_1^*, θ_2^*, ν, r_1^*, r_2^* and r_3^* are obtained from the numerical solution of the system of equations

$$\sum_{k=1}^{3} \theta_j^* \, \frac{\Gamma(r_k^*+h)}{\Gamma(r_k^*)} \, \nu^{-h} \; = \; \mathrm{i}^{-h} \frac{\partial^h}{\partial t^h} \, \phi_{W_2}(t) \qquad (h=1,\ldots,6) \tag{24}$$

with $\theta_3^* = 1-\theta_1^*-\theta_2^*$, for those parameters.

Proof: If in the characteristic function of W in (10) we replace $\phi_{W_2}(t)$ by $\lambda^r(\lambda-\mathrm{i}t)^{-r}$ we obtain

$$\phi_W(t) \approx \underbrace{\left[\prod_{j=1}^{p} a_j^{s_{j,p}} \, (a_j-\mathrm{i}t)^{-s_{j,p}}\right]}_{\phi_{W_1}(t)} \, \lambda^r(\lambda-\mathrm{i}t)^{-r}$$

that is the characteristic function of the sum of $p+1$ independent Gamma random variables, p of which with integer shape parameters $s_{j,p}$ and rate parameters a_j, and a further Gamma random variable with rate parameter r and shape parameter λ, that is, the characteristic function of the GNIG distribution of depth $p+1$ with distribution function given in (19). The parameters r and λ are taken in such a way that the two first moments of this approximation are the same as the two first exact moments of W, that is, in such a way that

$$\frac{\partial^h}{\partial t^h}\lambda^r(\lambda-\mathrm{i}t)^{-r} = \frac{\partial^h}{\partial t^h}\,\phi_{W_2}(t)\,, \qquad h=1,\ldots,2\,,$$

what gives rise to the definition of r and λ in (22).

If in the characteristic function of W in (10) we replace $\phi_{W_2}(t)$ by $\sum_{k=1}^{2}\theta_k\,\mu^{r_k}(\mu-\mathrm{i}t)^{-r_k}$ we obtain

$$\begin{aligned}\phi_W(t) &\approx \underbrace{\left[\prod_{j=1}^{p} a_j^{s_{j,p}}\,(a_j-\mathrm{i}t)^{-s_{j,p}}\right]}_{\phi_{W_1}(t)}\;\sum_{k=1}^{2}\theta_k\,\mu^{r_k}(\mu-\mathrm{i}t)^{-r_k}\\ &= \sum_{k=1}^{2}\theta_k\left[\prod_{j=1}^{p} a_j^{s_{j,p}}\,(a_j-\mathrm{i}t)^{-s_{j,p}}\right]\,\mu^{r_k}(\mu-\mathrm{i}t)^{-r_k}\end{aligned}$$

that is the characteristic function of the mixture of two GNIG distributions of depth $p+1$ with density function given in (20). The parameters θ_1, μ, r_1 and r_2 are defined in such a way that the first four moments of this approximation match the first four exact moments of W, that is, in such a way that

$$\frac{\partial^h}{\partial t^h}\sum_{k=1}^{2}\theta_k\,\mu^{r_k}(\mu-\mathrm{i}t)^{-r_k} = \frac{\partial^h}{\partial t^h}\,\phi_{W_2}(t)\,, \qquad h=1,\ldots,4\,,$$

giving rise to the evaluation of these parameters as the numerical solution of the system of equations in (23).

If in the characteristic function of W in (10) we replace $\phi_{W_2}(t)$ by

$\sum_{k=1}^{3} \theta_k^* \, \nu^{r_k^*} (\nu - \mathrm{i}t)^{-r_k^*}$ we obtain

$$\phi_W(t) \approx \underbrace{\left[\prod_{j=1}^{p} a_j^{s_{j,p}} \, (a_j - \mathrm{i}t)^{-s_{j,p}}\right]}_{\phi_{W_1}(t)} \; \sum_{k=1}^{3} \theta_k^* \, \nu^{r_k^*} (\nu - \mathrm{i}t)^{-r_k^*}$$

$$= \sum_{k=1}^{3} \theta_k^* \left[\prod_{j=1}^{p} a_j^{s_{j,p}} \, (a_j - \mathrm{i}t)^{-s_{j,p}}\right] \; \nu^{r_k^*} (\nu - \mathrm{i}t)^{-r_k^*}$$

that is the characteristic function of the mixture of three GNIG distributions of depth $p+1$ with density function given in (21). The parameters θ_1^*, θ_2^*, ν, r_1^*, r_2^* and r_3^* are defined in such a way that the first six moments of this approximation match the first six exact moments of W, that is, in such a way that

$$\frac{\partial^h}{\partial t^h} \sum_{k=1}^{3} \theta_k^* \, \nu^{r_k^*} (\nu - \mathrm{i}t)^{-r_k^*} = \frac{\partial^h}{\partial t^h} \, \phi_{W_2}(t) \, , \qquad h = 1, \ldots, 6 \, ,$$

what gives rise to the evaluation of these parameters as the numerical solution of the system of equations in (24). $\square$

All of the approximations considered were determined for the random variable $W = -\log \Lambda$ so, having in mind the final goal of evaluating quantiles we must note that

$$F_\Lambda(z) = 1 - F_W(-\log z)$$

where $F_\Lambda(z)$ is the cumulative distribution function of Λ and $F_W(z)$ is the cumulative distribution function of W, so that

$$\Lambda_{1-\alpha} = e^{-W_\alpha} \, ,$$

where $\Lambda_{1-\alpha}$ is the $(1-\alpha)$-quantile of Λ and W_α is the α-quantile of W $(0 < \alpha < 1)$.

4 Measures of proximity between distributions and characteristic functions

In order to evaluate the quality of the approximations developed we use three measures. The first two are measures of proximity between characteristic functions which also can be used as a reference of the proximity of the distributions. The third one is a measure of proximity between distributions based on their moments.

Let Y be a random variable defined on S with distribution function $F_Y(y)$ and characteristic function $\phi_Y(t)$. Let us consider also $\phi(t)$ and $F(t)$ the characteristic function and the distribution function that are supposed to be approximatios of $\phi_Y(t)$ and $F_Y(y)$. The first measure is

$$\Delta_1 = \int_{-\infty}^{\infty} |\phi_Y(t) - \phi(t)|\, \mathrm{d}t$$

and the second one is

$$\Delta_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\phi_Y(t) - \phi(t)}{t} \right| \mathrm{d}t\,,$$

based on the Berry-Esseen upper bound on $|F_Y(y) - F(y)|$ (Berry, 1941; Esseen, 1945; Loève, 1977, Chap. VI, Sec. 21; Hwang, 1998) which may, for any $b > 1/(2\pi)$ and any $T > 0$, be written as

$$\max_{y \in S} |F_Y(y) - F(y)| \le b \int_{-T}^{T} \left| \frac{\Phi_Y(t) - \Phi(t)}{t} \right| dt + C(b)\, \frac{M}{T} \qquad (25)$$

where $M = max_{y \in S} f(y)$ and $C(b)$ is a positive constant that only depends of b. But if in (25) above we take $T \to \infty$ then we will have Δ_2, since then we may take $b = 1/(2\pi)$.

These measures were used by Grilo & Coelho (2005) to study the application of near-exact approximations to the distribution of the product of independent Beta random variables.

The third measure of proximity,

$$\Delta_3 = 2 \sum_{i=1}^{12} \frac{\left| M_Y^{(i)}(0) - \widetilde{M}_Y^{(i)}(0) \right|}{(i+1)!}\,,$$

where $M_Y^{(i)}(0)$ and $\widetilde{M}_Y^{(i)}(0)$ are respectively the exact and approximate moments of order i of Y, was used by Alberto & Coelho (2005) in the study of the quality of several asymptotic and near-exact approximations for the distribution of the Wilks Lambda statistic.

Alberto & Coelho (2005) choose as upper limit in the summation 12, because it seems to work well with most distributions, therefore we use the same criterion. This measure is particularly useful when the moments are the only available information about the exact distribution, in the sense that the density function and the distribution function expressions are not known or available.

5 Numerical studies

At this stage we intend to evaluate the quality of the near-exact approximations developed in Theorem 2 in section 3 (which will be here denoted by GNIG, M2GNIG and M3GNIG) and of the asymptotic distributions with characteristic functions $\phi_{M2G}(t)$ and $\phi_{M3G}(t)$, by using the proximity measures Δ_1, Δ_2 and Δ_3 and the asymptotic approximation with characteristic function $\phi_{Box}(t)$ as a reference.

In Tables 1 through 4 we try to illustrate what happens when the values of n and p are close. In these tables we have the values of the measures Δ_1, Δ_2 and Δ_3 computed between the exact distribution of $W = -\log\Lambda$, by using the characteristic function in (4) or (5), and the six approximations under study.

Table 1 – Values of Δ_1, Δ_2 and Δ_3 for the asymptotic and near-exact distributions for $W = -\log\Lambda$ in the paper, for $p = 4$ and $n = 6$

	Δ_1	Δ_2	Δ_3
Box	2.400347×10^{-2}	5.296631×10^{-3}	5.293539×10^{-1}
M2G	5.629880×10^{-3}	3.073746×10^{-4}	1.267215×10^{-2}
M3G	6.755321×10^{-4}	3.042502×10^{-5}	4.806794×10^{-4}
GNIG	8.611649×10^{-4}	5.629021×10^{-5}	1.282105×10^{-3}
M2GNIG	1.856344×10^{-6}	8.455091×10^{-8}	8.635209×10^{-7}
M3GNIG	1.511358×10^{-8}	5.406071×10^{-10}	1.117393×10^{-9}

Table 2 – Values of Δ_1, Δ_2 and Δ_3 for the asymptotic and near-exact distributions for $W = -\log\Lambda$ in the paper, for $p = 5$ and $n = 7$

	Δ_1	Δ_2	Δ_3
Box	5.292725×10^{-2}	1.247970×10^{-2}	22.94659×10^{-1}
M2G	5.863905×10^{-3}	4.144532×10^{-4}	4.734755×10^{-2}
M3G	0.859891×10^{-3}	5.093622×10^{-5}	2.109841×10^{-3}
GNIG	2.229895×10^{-4}	1.867244×10^{-5}	1.153609×10^{-3}
M2GNIG	1.119780×10^{-6}	6.443755×10^{-8}	1.401748×10^{-6}
M3GNIG	3.018537×10^{-9}	1.248176×10^{-10}	1.619620×10^{-10}

Table 3 – Values of Δ_1, Δ_2 and Δ_3 for the asymptotic and near-exact distributions for $W = -\log \Lambda$ in the paper, for $p = 7$ and $n = 9$

	Δ_1	Δ_2	Δ_3
Box	1.435125×10^{-1}	3.513675×10^{-2}	2.334934×10^{1}
M2G	0.613287×10^{-2}	5.540876×10^{-4}	3.472426×10^{-1}
M3G	0.108668×10^{-2}	8.283219×10^{-5}	1.583907×10^{-2}
GNIG	5.452705×10^{-5}	5.875449×10^{-6}	2.225609×10^{-3}
M2GNIG	6.964433×10^{-8}	5.299830×10^{-9}	7.481820×10^{-7}
M3GNIG	4.888577×10^{-11}	3.026994×10^{-12}	7.925231×10^{-11}

Table 4 – Values of Δ_1, Δ_2 and Δ_3 for the asymptotic and near-exact distributions for $W = -\log \Lambda$ in the paper, for $p=10$ and $n=12$

	Δ_1	Δ_2	Δ_3
Box	3.537171×10^{-1}	8.762356×10^{-2}	4.109209×10^{2}
M2G	0.062413×10^{-1}	6.707073×10^{-4}	3.392193×10^{0}
M3G	0.125224×10^{-2}	1.134158×10^{-4}	1.283925×10^{-1}
GNIG	8.939694×10^{-6}	1.170968×10^{-6}	5.029142×10^{-3}
M2GNIG	3.393735×10^{-9}	3.189230×10^{-10}	4.703995×10^{-7}
M3GNIG	3.601155×10^{-12}	2.706019×10^{-13}	5.008025×10^{-11}

The first conclusion we can extract from the observation of Tables 1, 2, 3, and 4 is that the near-exact distributions are more precise (closer to the exact distribution) than the asymptotic distributions, with only the M3G distribution beating the GNIG distribution for $p = 4$ and $n = 6$. Also, as p increases, while the near-exact distributions become even better approximations to the exact distribution, the asymptotic distributions go the other way around.

Both the asymptotic and near-exact distributions based on mixtures and that also equate a few of the first exact moments (M2G, M3G, M2GNIG, M3GNIG) show a much better performance than their counterparts which are only mixtures (Box) or only equate moments (GNIG).

The cases in Tables 5 through 8 show us what happens when n is large and fixed and p varies.

Table 5 – Values of Δ_1, Δ_2 and Δ_3 for the asymptotic and near-exact distributions for $W = -\log \Lambda$ in the paper, for $p = 4$ and $n = 50$

	Δ_1	Δ_2	Δ_3
Box	1.316307×10^{-4}	2.308134×10^{-6}	6.916870×10^{-7}
M2G	8.810475×10^{-6}	3.902971×10^{-8}	8.435886×10^{-14}
M3G	7.465405×10^{-8}	2.565611×10^{-10}	9.519035×10^{-19}
GNIG	1.581990×10^{-4}	9.826998×10^{-7}	9.018162×10^{-10}
M2GNIG	3.671823×10^{-9}	1.627832×10^{-11}	3.490914×10^{-17}
M3GNIG	8.927223×10^{-13}	3.215025×10^{-15}	1.445208×10^{-23}

Table 6 – Values of Δ_1, Δ_2 and Δ_3 for the asymptotic and near-exact distributions for $W = -\log \Lambda$ in the paper, for $p = 5$ and $n = 50$

	Δ_1	Δ_2	Δ_3
Box	3.307800×10^{-4}	7.431952×10^{-6}	2.936049×10^{-6}
M2G	1.040064×10^{-5}	6.831249×10^{-8}	5.005316×10^{-13}
M3G	6.778877×10^{-8}	3.566580×10^{-10}	6.579225×10^{-18}
GNIG	5.326060×10^{-5}	4.776306×10^{-7}	9.920014×10^{-10}
M2GNIG	4.101531×10^{-8}	2.637224×10^{-10}	1.743970×10^{-15}
M3GNIG	8.748147×10^{-11}	4.432099×10^{-13}	6.449350×10^{-21}

Table 7 – Values of Δ_1, Δ_2 and Δ_3 for the asymptotic and near-exact distributions for $W = -\log \Lambda$ in the paper, for $p = 7$ and $n = 50$

	Δ_1	Δ_2	Δ_3
Box	1.153656×10^{-3}	3.678555×10^{-5}	2.442663×10^{-5}
M2G	1.479446×10^{-5}	1.544824×10^{-7}	7.885513×10^{-12}
M3G	7.818462×10^{-8}	6.711671×10^{-10}	1.618277×10^{-16}
GNIG	2.711777×10^{-5}	3.795430×10^{-7}	2.889724×10^{-9}
M2GNIG	4.099824×10^{-9}	4.211031×10^{-11}	1.969911×10^{-15}
M3GNIG	7.829089×10^{-12}	6.509655×10^{-14}	1.254690×10^{-20}

Table 8 – Values of Δ_1, Δ_2 and Δ_3 for the asymptotic and near-exact distributions for $W = -\log \Lambda$ in the paper, for $p = 10$ and $n = 50$

	Δ_1	Δ_2	Δ_3
Box	4.106258×10^{-3}	1.875948×10^{-4}	2.774119×10^{-4}
M2G	2.378513×10^{-5}	3.804352×10^{-7}	2.066435×10^{-10}
M3G	1.646943×10^{-7}	2.186226×10^{-9}	1.119725×10^{-14}
GNIG	8.777086×10^{-6}	1.868771×10^{-7}	7.281156×10^{-9}
M2GNIG	7.456426×10^{-10}	1.176134×10^{-11}	5.873697×10^{-15}
M3GNIG	7.626329×10^{-13}	9.891727×10^{-15}	4.184783×10^{-20}

Observing tables 5 through 8 and comparing with tables 1 through 4 we can verify that:

- the near-exact approximations are still, in general, more precise than the asymptotic approximations, with only the asymptotic mixture of three Gamma distributions (M3G) beating the near-exact GNIG distribution in every case, and the asymptotic mixture of two Gamma distributions also beating the GNIG distribution for $p = 4$ and $p = 5$;
- large values of n improve the quality of both the asymptotic and near-exact approximations, what shows the asymptotic character of these latter distributions;
- when referring to the measures Δ_1 and Δ_2, in general, when p increases the near-exact approximations seem to become even more precise (only between $p = 4$ and $p = 5$ this is not true).

As an overall remark we may say that the measure Δ_3 seems to penalize a bit the distributions that do not equate any moments and overbenefit the distributions that equate more moments.

Finally, we intend to analyze how precise the near-exact approximations are for the computation of quantiles. To evaluate the precision of the approximate quantiles we will use the measure

$$\Delta_4 = -\log_{10} |\text{exact} - \text{approx.}|$$

used by Alberto & Coelho (2005).

In tables 9 through 12 we have the exact and approximate (asymptotic and near-exact) 0.05 and 0.01 quantiles of $\Lambda = \lambda^{2/N}$, for λ given by (2).

Table 9 – Comparison between the exact and approximate 0.05 and 0.01 quantiles for $\Lambda = \lambda^{2/N}$, for p=4 and $n = 6$

Distribuição	quant. 0.05	Δ_4	quant. 0.01	Δ_4
exact	0.0168675905941638	—	0.0050311233877392	—
Box	0.0178416505363502	3.0	0.0056387672070156	3.2
M2G	0.0168753311183583	5.1	0.0050176930650620	4.9
M3G	0.0168656225900511	5.7	0.0050319376157784	6.1
GNIG	0.0168693683122978	5.8	0.0050332151150509	5.7
M2GNIG	0.0168675885068604	8.7	0.0050311217115163	8.8
M3GNIG	0.0168675905988773	11.3	0.0050311233945564	11.2

Table 10 – Comparison between the exact and approximate 0.05 and 0.01 quantiles for $\Lambda = \lambda^{2/N}$, for p=4 and $n = 50$

Distribuição	quant. 0.05	Δ_4	quant. 0.01	Δ_4
exact	0.7049677092298717	—	0.6390788053204583	—
Box	0.7049688987100046	5.9	0.6390807866253055	5.7
M2G	0.7049677098372609	9.2	0.6390787892143052	7.8
M3G	0.7049677092046101	10.6	0.6390788053921197	10.1
GNIG	0.7049677606261154	7.3	0.6390793225105899	6.3
M2GNIG	0.7049677092301036	12.6	0.6390788053137241	11.2
M3GNIG	0.7049677092298712	15.3	0.6390788053204593	15.0

Table 11 – Comparison between the exact and approximate 0.05 and 0.01 quantiles for $\Lambda = \lambda^{2/N}$, for $p = 5$ and $n = 7$

Distribuição	quant. 0.05	Δ_4	quant. 0.01	Δ_4
exact	0.0064001285238792	—	0.0018281122216092	—
Box	0.0072695203302933	3.1	0.0023087004587565	3.3
M2G	0.0064016650127828	5.8	0.0018190894487171	5.0
M3G	0.0063987734959247	5.9	0.0018290332400471	6.0
GNIG	0.0064004575589233	6.5	0.0018284034242836	6.5
M2GNIG	0.0064001274775452	9.0	0.0018281119552464	9.6
M3GNIG	0.0064001285232339	11.1	0.0018281122221749	12.0

Table 12 – Comparison between the exact and approximate 0.05 and 0.01 quantiles for $\Lambda = \lambda^{2/N}$, for $p = 5$ and $n = 50$

Distribuição	quant. 0.05	Δ_4	quant. 0.01	Δ_4
exact	0.6109257783234166	—	0.5453271467029756	—
Box	0.6109294330577949	5.4	0.5453326755425309	5.3
M2G	0.6109257736916504	8.3	0.5453271117271034	7.5
M3G	0.6109257783121262	10.9	0.5453271468448589	9.8
GNIG	0.6109258303258338	7.3	0.5453274453932077	6.5
M2GNIG	0.6109257782978391	10.6	0.5453271465834711	9.9
M3GNIG	0.6109257783234295	13.9	0.5453271467025087	13.3

The results show that in most cases the near-exact distributions provide quantile values with greater precision than the asymptotic distributions. However, for $n = 50$, the asymptotic distributuions developed in this paper give very precise quantiles, outperforming the GNIG distribution for both $p = 4$ and $p = 5$ and with the asymptotic M3G distribution outperforming the M2GNIG for $p = 5$ for the 0.05 quantile.

Yet in order to better evaluate the quality of the asymptotic and near-exact distributions presented in this paper, we compare their performance with the performance of several saddlepoint approximations to the distribution of $\Lambda = \lambda^{2/N}$ in Butler *et all.* (1993).

In Tables 13 and 14 we show the approximate exceedance probabilities for the exact 0.05 and 0.01 quantiles of $\Lambda = \lambda^{2/N}$ rounded to four decimal places, obtained with the asymptotic and near-exact distributions in this paper, using the same exact quantiles used in Butler *et all.* (1993), that is, the ones in Nagarsenker & Pillai (1973). Computations are done exactly for the same combinations of values of n and p used by Butler et all. (1993). Only the case $p = 2$ was left out since for this case the exact distribution of $W = -\log \Lambda$ is easily obtained as a simple Exponential distribution.

Table 13 – Exceedance probabilities for the exact quantile 0.05 of Λ for the asymptotic and near-exact distributions in the paper

n	p	Box	M2G	M3G	GNIG	M2GNIG	M3GNIG
7	3	0.0498	0.0500	0.0500	0.0500	0.0500	0.0500
6	4	0.0464	0.0500	0.0500	0.0500	0.0500	0.0500
6	5	0.0332	0.0499	0.0501	0.0500	0.0500	0.0500
12	6	0.0481	0.0500	0.0500	0.0500	0.0500	0.0500
8	7	0.0192	0.0500	0.0501	0.0500	0.0500	0.0500
13	8	0.0432	0.0500	0.0500	0.0500	0.0500	0.0500
11	9	0.0220	0.0501	0.0500	0.0500	0.0500	0.0500
15	10	0.0377	0.0500	0.0500	0.0500	0.0500	0.0500

Table 14 – Exceedance probabilities for the exact quantile 0.01 of Λ for the asymptotic and near-exact distributions in the paper

n	p	Box	M2G	M3G	GNIG	M2GNIG	M3GNIG
7	3	0.0099	0.0100	0.0100	0.0100	0.0100	0.0100
6	4	0.0085	0.0100	0.0100	0.0100	0.0100	0.0100
6	5	0.0043	0.0102	0.0100	0.0100	0.0100	0.0100
12	6	0.0092	0.0100	0.0100	0.0100	0.0100	0.0100
8	7	0.0017	0.0103	0.0099	0.0100	0.0100	0.0100
13	8	0.0076	0.0100	0.0100	0.0100	0.0100	0.0100
11	9	0.0024	0.0101	0.0100	0.0100	0.0100	0.0100
15	10	0.0061	0.0100	0.0100	0.0100	0.0100	0.0100

We may observe that when rounding to four decimal places, all of the near-exact distributions always give the exact probability, while the saddlepoint approximations, which have a more complicated formulation than the asymptotic distributions proposed in this paper, are, in their best performance cases, only able to match the asymptotic distributions, never outperforming these ones.

In order to better assess the quality of the approximations given by the two asymptotic and the three near-exact distributions developed in this paper, we present in Appendix B two tables, similar to the last ones, where we give the tail probabilities for those distributions, rounded to twelve decimal places, computed using the exact quantiles for Λ rounded to sixteeen decimal places for the cases where such computation was possible to be carried out, that is, for $p \leq 8$.

6 Conclusions and final remarks

The near-exact distributions generally have a much better performance than the asymptotic distributions considered in this paper, with the almost impressive performance of the distribution M3GNIG. The values obtained with the four measures considered (Δ_1, Δ_2 , Δ_3 , Δ_4) indicate that in general the near-exact distributions become even better approximations when p increases, mainly in situations where the sample size is small.

It seems like the three measures Δ_1, Δ_2 and Δ_3, give us a very strong vision of the quality of this approximations, garanting the precision of the computed quantiles.

For larger values of p the computation of exact quantiles becomes so hard and time consuming that the asymptotic and near-exact distributions developed in this paper, namely the distributions M3G, M2GNIG and M3GNIG, may well be used instead of the exact distribution to compute quantiles. Moreover, the parameters in all the proposed asymptotic and near-exact distributions are easy to determine, being that computation even easier in the case of the near-exact distributions. These facts emphasize the need and usefulness of such distributions.

Appendix A

Details of the proof of expression (16) in Theorem 1

In Theorem 1 we observed that, for $j = 2, \ldots, p$, with $p \geq 2$,

$$\prod_{h=0}^{b_j^*-1} (a_j + h)(a_j + h - \mathrm{i}t)^{-1} \;=\; a_j^{s_{j,p}}\,(a_j - \mathrm{i}t)^{-s_{j,p}} \tag{26}$$

where the $s_{j,p}$ are given by (13).

This fact may be proven if we notice that

$$a_j = a_{j+2k} + k$$

or, in a more general way, that,

$$a_j + n_0 = a_{j+2k} + k + n_0$$

for

$$n_0 \leq b_j^* - 1 \ , \quad n_0 + k \leq b_{j+2k}^* - 1 \quad \text{and} \quad k \in I\!N \ , \ n_0 \in I\!N_0 \, ,$$

and that, given the definition of b_j ($j = 1, \ldots, p$) in (14) then $b_j^* = \lfloor b_j \rfloor$ will be given by (11) (with $b_1^* = 0$ for any p, $b_2^* = 1$ for $p = 2$ and $b_2^* = 0$ for any other value of $p > 2$) what may lead us to collect the rate parameters $a_j + h$ ($j = 2, \ldots, p$; $h = 0, \ldots, b_j^* - 1$) in the following array

$$\begin{pmatrix}
(a_2)^* & & & & & \\
a_3 & & & & & \\
a_4 & (a_4{+}1)^* & & & & \\
a_5 & a_5{+}1 & & & & \\
a_6 & a_6{+}1 & (a_6{+}2)^* & & & \\
a_7 & a_7{+}1 & a_7{+}2 & & & \\
a_8 & a_8{+}1 & a_8{+}2 & (a_8{+}3)^* & & \\
a_9 & a_9{+}1 & a_9{+}2 & a_9{+}3 & & \\
a_{10} & a_{10}{+}1 & a_{10}{+}2 & a_{10}{+}3 & (a_{10}{+}4)^* & \\
a_{11} & a_{11}{+}1 & a_{11}{+}2 & a_{11}{+}3 & a_{11}{+}4 & \\
a_{12} & a_{12}{+}1 & a_{12}{+}2 & a_{12}{+}3 & a_{12}{+}4 & (a_{12}{+}5)^* \\
 & & & \vdots & & \\
 & & & \text{"last row"} & &
\end{pmatrix}$$

where

$$(a_j + h)^* = \begin{cases} a_j + h & , \quad p \leq j + 2h \\ \text{does not exist} \ , & \quad p > j + 2h \end{cases}$$

and

$$\text{"last row"} = \begin{cases} a_p \quad a_p{+}1 \quad \cdots \quad a_p{+}\frac{p-3}{2} & , \quad p \ \text{ odd} \\ a_p \quad a_p{+}1 \quad \cdots \quad a_p{+}\frac{p-4}{2} \quad a_p{+}\frac{p-2}{2} & , \quad p \ \text{ even} \, . \end{cases}$$

Let then $s_{j,p}$ be the number of times that, for a given value of p, a rate parameter with the value a_j appears in the left hand side of (26). Then $s_{j,p}$ may be defined as in (13), since by analysing well the definition of b_j^* in (11) and the aspect it conffers to the above matrix of rate parameters, we may easily derive the following relations:

i) for $p \geq 2$ (with $s_{2,p-2} = 0$ if $p-2 < 2$),

$$s_{2,p} = \begin{cases} s_{2,p-2}+1 & , \quad p-2 = 4k+0\ ,\ k \in I\!N_0 \\ s_{2,p-2} & , \quad p-2 = 4k+2\ ,\ k \in I\!N_0 \\ s_{2,p-2} & , \quad p-2 = 4k+1\ ,\ k \in I\!N_0 \\ s_{2,p-2}+1 & , \quad p-2 = 4k+3\ ,\ k \in I\!N_0 \end{cases} , \tag{27}$$

ii) for $p \geq 3$ (with $s_{3,p-1} = 0$ if $p-1 < 3$),

$$s_{3,p} = \begin{cases} s_{3,p-1}+1 & \text{for} \quad p \quad \text{odd} \\ s_{3,p-1} & \text{for} \quad p \quad \text{even} \end{cases} \tag{28}$$

iii) for $j > 3$ (and $p \geq j$),

$$s_{j,p} = \begin{cases} s_{j-1,p}-1 & , \quad p \quad \text{even and } j-1 \text{ even} \\ s_{j-1,p} & , \quad p \quad \text{even and } j-1 \text{ odd} \\ s_{j-1,p} & , \quad p \quad \text{odd and } j-1 \text{ even} \\ s_{j-1,p}-1 & , \quad p \quad \text{odd and } j-1 \text{ odd} \end{cases} . \tag{29}$$

These relations imply the definition of $s_{j,p}$ the way it is done in (13). We will summarize this result in Proposition 1 and prove it by induction.

Proposition 3 *Let $s_{j,p}$ be the number of times that, for a given value of p (≥ 2), a rate parameter with the value a_j ($j = 2, \ldots, p$) appears on the left hand side of* (26)*. Then $s_{j,p}$ is given by* (13)*, or equivalently,*

$$s_{j,p} = \begin{cases} \left\lfloor \frac{p-j}{2} + 1 \right\rfloor , & j = 3, \ldots p \\ \left\lfloor \frac{p}{4} + \frac{1}{2} \right\rfloor & , \quad j = 2 \text{ and } \operatorname{mod}(p-2,4) = 0 \\ \left\lfloor \frac{p}{4} \right\rfloor & , \quad j = 2 \text{ and } \operatorname{mod}(p-2,4) \neq 0 , \end{cases} \tag{30}$$

where $\operatorname{mod}(a,b)$ *represents the remainder of the integer division of a by b.*

Proof: We will first prove that for $j = 3, \ldots, p$, $s_{j,p} = \left\lfloor \frac{p-j}{2} + 1 \right\rfloor$:

a) we will first prove that for $p \geq 3$, $s_{3,p} = \left\lfloor \frac{p-3}{2} + 1 \right\rfloor$:

  i) in fact, from (30), we have $s_{3,3} = \left\lfloor \frac{3-3}{2} + 1 \right\rfloor = 1$

ii) assuming then that for $p \geq 4$, $s_{3,p-1} = \left\lfloor \frac{p-1-3}{2} + 1 \right\rfloor$

iii) we have, from (29), that for even p,

$$s_{3,p} = s_{3,p-1} = \left\lfloor \frac{p-1-3}{2} + 1 \right\rfloor = \left\lfloor \frac{p-2}{2} \right\rfloor = \left\lfloor \frac{p-1}{2} \right\rfloor = \left\lfloor \frac{p-3}{2} + 1 \right\rfloor ,$$

while for odd p,

$$\begin{aligned} s_{3,p} = s_{3,p-1} + 1 &= \left\lfloor \frac{p-1-3}{2} + 1 \right\rfloor + 1 = \left\lfloor \frac{p-2}{2} \right\rfloor + 1 = \left\lfloor \frac{p}{2} \right\rfloor \\ &= \left\lfloor \frac{p-1}{2} \right\rfloor = \left\lfloor \frac{p-3}{2} + 1 \right\rfloor , \end{aligned}$$

what proves that for $p \geq 3$, $s_{3,p} = \left\lfloor \frac{p-3}{2} + 1 \right\rfloor$;

b) we will now prove that for $3 \leq j \leq p$, $s_{j,p} = \left\lfloor \frac{p-j}{2} + 1 \right\rfloor$:

i) indeed, for $p \geq 3$, $s_{3,p} = \left\lfloor \frac{p-3}{2} + 1 \right\rfloor$, as it was proven above

ii) assuming that $s_{j-1,p} = \left\lfloor \frac{p-j+1}{2} + 1 \right\rfloor$

iii) from (29), for both p and $j-1$ even or both odd, what makes $p-j$ odd, we have

$$\begin{aligned} s_{j,p} = s_{j-1,p} - 1 &= \left\lfloor \frac{p-j+1}{2} + 1 \right\rfloor - 1 = \left\lfloor \frac{p-j+1}{2} \right\rfloor = \left\lfloor \frac{p-j}{2} + \frac{1}{2} \right\rfloor \\ &= \left\lfloor \frac{p-j}{2} + 1 \right\rfloor \end{aligned}$$

while for p and $j-1$, with different parity, we have $p-j$ even and thus

$$s_{j,p} = s_{j-1,p} = \left\lfloor \frac{p-j+1}{2} + 1 \right\rfloor = \left\lfloor \frac{p-j}{2} + \frac{3}{2} \right\rfloor = \left\lfloor \frac{p-j}{2} + 1 \right\rfloor$$

what concludes the proof that for $j = 3, \ldots, p$, $s_{j,p} = \left\lfloor \frac{p-j}{2} + 1 \right\rfloor$.

Finally we will prove (30), or (13), for $j = 2$.

i) in fact, either from (13) or (30), we have

$$s_{2,2} = 1 \quad \text{and} \quad s_{2,3} = 0;$$

ii) assuming that

$$s_{2,p-2} = \begin{cases} \left\lfloor \frac{p-2}{4} \right\rfloor & , \quad \mathrm{mod}(p-4,4) \neq 0 \\ \left\lfloor \frac{p-2}{4} + \frac{1}{2} \right\rfloor & , \quad \mathrm{mod}(p-4,4) = 0 \end{cases}$$

iii)

- for $p-2=4k+0$ ($k\in I\!N_0$),

$$s_{2,p}=s_{2,p-2}+1=\left\lfloor\frac{p-2}{4}\right\rfloor+1=\left\lfloor\frac{p-2}{4}+1\right\rfloor=\left\lfloor\frac{p+2}{4}\right\rfloor=\left\lfloor\frac{p}{4}+\frac{1}{2}\right\rfloor$$

- for $p-2=4k+2$ ($\Longleftrightarrow p=4k+8$) ($k\in I\!N_0$),

$$s_{2,p}=s_{2,p-2}=\left\lfloor\frac{p-2}{4}+\frac{1}{2}\right\rfloor=\left\lfloor\frac{p}{4}\right\rfloor$$
$$\left(=\left\lfloor\frac{4k+4}{4}\right\rfloor=\left\lfloor\frac{4k+6}{4}\right\rfloor=\left\lfloor\frac{p}{4}+\frac{1}{2}\right\rfloor\right)$$

- for $p-2=4k+1$ ($\Longleftrightarrow p=4k+3$) ($k\in I\!N_0$),

$$s_{2,p}=s_{2,p-2}=\left\lfloor\frac{p-2}{4}\right\rfloor=\left\lfloor\frac{4k+1}{4}\right\rfloor=\left\lfloor\frac{4k+3}{4}\right\rfloor=\left\lfloor\frac{p}{4}\right\rfloor$$

- for $p-2=4k+3$ ($\Longleftrightarrow p=4k+5$) ($k\in I\!N_0$),

$$s_{2,p}=s_{2,p-2}+1=\left\lfloor\frac{p-2}{4}\right\rfloor+1=\left\lfloor\frac{4k+3}{4}\right\rfloor+1=\left\lfloor\frac{4k+5}{4}\right\rfloor=\left\lfloor\frac{p}{4}\right\rfloor$$

what proves (13) or (30) for $j=2$. □

Appendix B

Exceedance probabilities for exact quantiles of $\Lambda=\lambda^{2/N}$ for the asymptotic and near-exact distributions presented in the paper

Table B.1 – Exceedance probabilities for the exact quantile 0.05 of $\Lambda=\lambda^{2/N}$ for the asymptotic and near-exact distributions presented in the paper

n	p	M2G	M3G	GNIG	M2GNIG	M3GNIG
7	3	0.049996325841	0.050000138079	0.049999668406	0.050000004937	0.049999996284
6	4	0.049970822608	0.050007434526	0.049993284475	0.050000007884	0.049999999982
6	5	0.049915626363	0.050064299293	0.049996405450	0.050000009506	0.049999999964
12	6	0.050004943806	0.050000168450	0.049998872166	0.050000001179	0.049999999997
8	7	0.050025390406	0.050090741271	0.049998800136	0.050000000764	0.050000000000
13	8	0.050020919307	0.049999452177	0.049999261995	0.050000000269	0.050000000000

Table B.2 – Exceedance probabilities for the exact quantile 0.01 of $\Lambda=\lambda^{2/N}$ for the asymptotic and near-exact distributions presented in the paper

n	p	M2G	M3G	GNIG	M2GNIG	M3GNIG
7	3	0.010001927692	0.009999986323	0.009993656389	0.010000022541	0.010000002483
6	4	0.010036738923	0.009997772599	0.009994280233	0.010000004585	0.009999999981
6	5	0.010177525578	0.009974134917	0.009998133122	0.010000000481	0.010000000002
12	6	0.010008521891	0.009999593686	0.009999043556	0.010000000439	0.009999999999
8	7	0.010295272227	0.009935293267	0.009999448707	0.009999999967	0.010000000000
13	8	0.010019904708	0.009998541296	0.009999480001	0.010000000055	0.010000000000

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