Semigroup Forum OF1–OF15

DOI: 10.1007/s00233-006-0605-6

RESEARCH ARTICLE

Derived Rees Matrix Semigroups as Semigroups of Transformations

T. S. Blyth and M. H. Almeida Santos

Communicated by Francis J. Pastijn

Abstract

An ordered pair (e,f) of idempotents of a regular semigroup is called a *skew pair* if ef is not idempotent whereas fe is idempotent. Previously [1] we have established that there are four distinct types of skew pairs of idempotents. We have also described (as quotient semigroups of certain regular Rees matrix semigroups [2]) the structure of the smallest regular semigroups that contain precisely one skew pair of each of the four types, there being to within isomorphism ten such semigroups. These we call the *derived Rees matrix semigroups.* In the particular case of full transformation semigroups we proved in [3] that T_X contains all four skew pairs of idempotents if and only if $|X| \geqslant 6$. Here we prove that T_X contains all ten derived Rees matrix semigroups if and only if $|X| \geqslant 7$.

Keywords: Full transformation semigroups, idempotents, skew pairs.

AMS Classification: 20M20, 20M17.

If S is a regular semigroup and $e,f \in E(S)$ then the ordered pair (e,f) is said to be a *skew pair* if $ef \notin E(S)$ whereas $fe \in E(S)$. In a previous publication [1] we showed that there are four distinct types of skew pairs (e,f) of idempotents, namely those that are

(1) *strong* in the sense that $fe = efe = fef = (ef)^2$;

(2) *left regular* in the sense that $fe = fef \neq efe = (ef)^2$;

(3) *right regular* in the sense that $fe = efe \neq fef = (ef)^2$;

(4) *discrete* in the sense that fe, efe, fef, $(ef)^2$ are distinct.

If we let

$$\widehat{S} = \{a \in S \mid a \neq a^2 = a^3,\ V(a) \cap E(S) \neq \emptyset\}$$

then, by [2, Theorem 5] for every skew pair (e,f) of idempotents we have $ef \in \widehat{S}$; and conversely for every $a \in \widehat{S}$ there exists a skew pair (e,f) of idempotents such that $ef = a$. A skew pair (e,f) of idempotents is said to be *fundamental* [3] if $e\,\mathcal{R}\,ef\,\mathcal{L}\,f$. By [3, Theorem 5] for every skew pair (e,f) of idempotents in a regular semigroup S there is a fundamental skew pair $(e^\star,f^\star)$ of idempotents such that $e^\star f^\star = ef$. Consequently, the fundamental skew pairs of idempotents in S are located in the $\mathcal{D}$-classes of $\widehat{S}$.

In the case of the full transformation semigroup T_X on a set X (and herein *all calculations are based on mappings being written on the left*) the fundamental skew pairs of idempotents are of the form $(\vartheta\varphi,\varphi\vartheta)$ where $\vartheta\in\widehat{T_X}$ and $\varphi\in V(\vartheta)\cap E(T_X)$. The following result (see [3, Theorems 2, 3]) then describes the various types.

Theorem 1. *If* $\vartheta\in\widehat{T_X}$ *then* $\operatorname{Im}\vartheta^2 = \operatorname{Fix}\vartheta^2 = \operatorname{Fix}\vartheta$. *Moreover, if* $\varphi\in V(\vartheta)\cap E(T_X)$ *then the fundamental skew pair* $(\vartheta\varphi,\varphi\vartheta)$ *is*

(1) *strong if and only if* $\operatorname{Ker}\varphi\subseteq\operatorname{Ker}\vartheta^2$ *and* $\operatorname{Im}\vartheta^2\subseteq\operatorname{Im}\varphi$;

(2) *left regular if and only if* $\operatorname{Ker}\varphi\subseteq\operatorname{Ker}\vartheta^2$ *and* $\operatorname{Im}\vartheta^2\nsubseteq\operatorname{Im}\varphi$;

(3) *right regular if and only if* $\operatorname{Ker}\varphi\nsubseteq\operatorname{Ker}\vartheta^2$ *and* $\operatorname{Im}\vartheta^2\subseteq\operatorname{Im}\varphi$;

(4) *discrete if and only if* $\operatorname{Ker}\varphi\nsubseteq\operatorname{Ker}\vartheta^2$ *and* $\operatorname{Im}\vartheta^2\nsubseteq\operatorname{Im}\varphi$.

Skew pairs (e,f) and $(e^\star,f^\star)$ are said to be *associated* if $ef = e^\star f^\star$. The ubiquity of skew pairs of idempotents in T_X is revealed in the following theorem which summarises the main results of [3].

Theorem 2. *To every skew pair* (e,f) *of idempotents in* T_X *there exists in* $\mathcal{D}_{ef}$ *an associated fundamental skew pair* $(e^\star,f^\star)$ *of idempotents of the same type as* (e,f). *Moreover,* T_X *contains skew pairs of idempotents that are*

(1) *strong if and only if* $|X|\geqslant 3$;

(2) *left regular if and only if* $|X|\geqslant 4$;

(3) *right regular if and only if* $|X|\geqslant 5$;

(4) *discrete if and only if* $|X|\geqslant 6$.

It follows from the above that T_X contains all four types of skew pairs of idempotents if and only if $|X|\geqslant 6$. Now we have determined to within isomorphism and dual isomorphism the regular semigroups of smallest cardinality that contain precisely one of each of these four types. These can be described as follows. Let **2** be the semilattice $\{0,1\}$ and consider the regular Rees matrix semigroups of the form $R_{3,P} = M(\mathbf{2};3,3;P)$. These consist of two $\mathcal{D}$-classes, namely D_1 that consists of elements of the form $(i,1,j)$, and D_2 that consists of elements of the form $(i,0,j)$. Let $\Delta_{1,2}$ be the smallest congruence that identifies the first two rows of D_2, let $\Delta^{1,2}$ be the smallest congruence that identifies the first two columns of D_2, and let $\Delta = \Delta_{1,2}\vee\Delta^{1,2}$. Then the quotient semigroup $R_{3,P}/\Delta$ has 13 elements. We denote this by ∂P and call it a *derived Rees matrix semigroup*. The required description of the semigroups in question is now given by the following paraphrase of the main results of [2].

Theorem 3. *To within isomorphism there are ten regular semigroups of minimum cardinality that contain precisely one of each of the four types of skew pairs of idempotents. These are given by the derived Rees matrix semigroups* ∂P *where* P *is one of the following matrices or its transpose:*

$$M_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \quad M_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}; \quad M_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};$$

$$M_4 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \quad M_5 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad M_6 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In what follows our objective is to investigate, in full transformation semigroups, the existence of subsemigroups that are isomorphic to these ten derived Rees matrix semigroups. For this purpose, we note from Theorem 2(4) that we need consider only those T_X for which $|X| \geqslant 6$. Of course, as is well known, by the extended right regular representation every semigroup S can be embedded in the full transformation semigroup T_{S^1}. Consequently all ten derived Rees matrix semigroups can be embedded in T_{14}. Here our main result (see Theorem 8 below) is that T_X contains copies of all ten derived Rees matrix semigroups if and only if $|X| \geqslant 7$.

For presentational convenience and subsequent reference we list the ten derived Rees matrix semigroups in a particularly useful way. In these descriptions, (e, f) denotes the strong skew pair, (γ, δ) the discrete skew pair, $\mathrm{RR}(x, y)$ the right regular skew pair, and $\mathrm{LR}(x, y)$ the left regular skew pair. In each case we fix e and f at, respectively, the $(2, 1)$ and $(1, 2)$ positions in the $\mathcal{D}$-class D_1 of ∂P. We also use the label $\bullet$ to indicate an element of $\widehat{\partial P}$. Note that, by the fundamental isomorphism of [2, Theorem 10], each $\bullet$ corresponds to an entry 0 in the transpose of the corresponding Rees matrix.

∂M_1

α	f	$\alpha\gamma$
e	$\bullet ef$	γ
$\delta\alpha$	δ	$\delta\alpha\gamma$

$(\gamma\delta)^2$	$\gamma\delta\gamma$
$\delta\gamma\delta$	$\delta\gamma$

RR (γ, f) LR (e, δ)

∂M_2

α	f	$\alpha\delta$
$e = \gamma$	$\bullet ef$	$\bullet \gamma\delta$
$\delta\alpha$	δf	δ

$\gamma\delta\gamma$	$(\gamma\delta)^2$
$\delta\gamma$	$\delta\gamma\delta$

RR $(\gamma, \alpha\delta)$ LR $(\gamma, \delta f)$

∂M_2^t

α	$f = \delta$	$\alpha\gamma$
e	$\bullet ef$	$e\gamma$
$\gamma\alpha$	$\bullet \gamma\delta$	γ

$\delta\gamma\delta$	$\delta\gamma$
$(\gamma\delta)^2$	$\gamma\delta\gamma$

RR $(e\gamma, f)$ LR $(\gamma\alpha, f)$

∂M_3

$f\gamma$	f	$\bullet f\gamma\delta$
e	$\bullet ef$	δ
γ	γf	$\bullet \gamma\delta$

$\delta\gamma$	$\delta\gamma\delta$
$\gamma\delta\gamma$	$(\gamma\delta)^2$

RR $(f\gamma, \delta)$ LR $(e, \gamma f)$

∂M_3^t

δe	f	δ
e	$\bullet ef$	$e\delta$
$\bullet \gamma\delta e$	γ	$\bullet \gamma\delta$

$\delta\gamma$	$\delta\gamma\delta$
$\gamma\delta\gamma$	$(\gamma\delta)^2$

RR $(e\delta, f)$ LR $(\gamma, \delta e)$

∂M_4

γ	f	$\bullet \gamma\delta$
e	$\bullet ef$	$e\delta$
δe	$\bullet \delta ef$	δ

$\gamma\delta\gamma$	$(\gamma\delta)^2$
$\delta\gamma$	$\delta\gamma\delta$

RR $(\gamma, e\delta)$ LR $(\delta e, f)$

∂M_4^t

δ	f	$f\gamma$
e	$\bullet ef$	$\bullet ef\gamma$
$\bullet \gamma\delta$	γf	γ

$\delta\gamma\delta$	$\delta\gamma$
$(\gamma\delta)^2$	$\gamma\delta\gamma$

RR $(e, f\gamma)$ LR $(\gamma f, \delta)$

∂M_5

α	f	δ
e	$\bullet ef$	$\bullet e\delta$
γ	$\bullet \gamma f$	$\bullet \gamma\delta$

$\delta\gamma$	$\delta\gamma\delta$
$\gamma\delta\gamma$	$(\gamma\delta)^2$

RR (e, δ) LR (γ, f)

∂M_6

α	f	$\bullet\,\alpha\delta$
e	$\bullet\, ef$	δ
γ	$\bullet\,\gamma f$	$\bullet\,\gamma\delta$

$\delta\gamma$	$\delta\gamma\delta$
$\gamma\delta\gamma$	$(\gamma\delta)^2$

RR (α,δ) LR (γ,f)

∂M_6^t

α	f	δ
e	$\bullet\, ef$	$\bullet\, e\delta$
$\bullet\,\gamma\alpha$	γ	$\bullet\gamma\delta$

$\delta\gamma$	$\delta\gamma\delta$
$\gamma\delta\gamma$	$(\gamma\delta)^2$

RR (e,δ) LR (γ,α)

We begin by investigating the existence of each of these semigroups as a subsemigroup of T_X where $|X|=6$. For this purpose we require the following important observations.

Theorem 4. *If $|X|=6$ and if (γ,δ) is a discrete skew pair of idempotents in T_X then $|\operatorname{Im}\gamma\delta|=3$ and $|\operatorname{Im}(\gamma\delta)^2|=2$.*

Proof. Let $X=\{x,y,z,u,v,w\}$ and let $\vartheta=\gamma\delta$. Then $\vartheta\in\widehat{T_X}$ and so $\vartheta\neq\vartheta^2=\vartheta^3$. We may therefore assume that

$$\vartheta=\begin{pmatrix} x\,y\,z\,u\,v\,w \\ y\,z\,z\,*\,*\,* \end{pmatrix}\equiv\big(\,y\,z\,z\,*\,*\,*\,\big),\quad \vartheta^2=\big(\,z\,z\,z\,*\,*\,*\,\big).$$

Now since (γ,δ) is discrete we have, by Theorem 1(4), that $\vartheta^2\varphi\neq\vartheta^2$ and $\varphi\vartheta^2\neq\vartheta^2$ for some $\varphi\in V(\vartheta)\cap E(T_X)$. Then ϑ^2 is not a constant mapping. Furthermore, by Theorem 1, we have $\operatorname{Im}\vartheta^2=\operatorname{Fix}\vartheta^2=\operatorname{Fix}\vartheta$. We may therefore assume that

$$\vartheta=\big(\,y\,z\,z\,u\,*\,*\,\big),\quad \vartheta^2=\big(\,z\,z\,z\,u\,*\,*\,\big).$$

We now observe that

(1) $v\notin\operatorname{Fix}\vartheta$ *and* $w\notin\operatorname{Fix}\vartheta$.

In fact, if we suppose for example that $v\in\operatorname{Fix}\vartheta$ then, with φ as above, we have the following five possibilities for ϑ, and correspondingly for ϑ^2 and φ:

$$\begin{array}{rccccc}
\vartheta\colon & (\,y\,z\,z\,u\,v\,y\,) & (\,y\,z\,z\,u\,v\,z\,) & (\,y\,z\,z\,u\,v\,u\,) & (\,y\,z\,z\,u\,v\,v\,) & (\,y\,z\,z\,u\,v\,w\,)\\
\vartheta^2\colon & (\,z\,z\,z\,u\,v\,z\,) & (\,z\,z\,z\,u\,v\,z\,) & (\,z\,z\,z\,u\,v\,u\,) & (\,z\,z\,z\,u\,v\,v\,) & (\,z\,z\,z\,u\,v\,w\,)\\
\varphi\colon & (\,*\,*\,z\,u\,v\,*\,) & (\,x\,x\,*\,u\,v\,*\,) & (\,x\,x\,z\,*\,v\,*\,) & (\,x\,x\,z\,u\,*\,*\,) & (\,x\,x\,z\,u\,v\,w\,)
\end{array}$$

Now since (γ,δ) is discrete we have $\operatorname{Im}\vartheta^2\nsubseteq\operatorname{Im}\varphi$ by Theorem 1(4). The first and last possibilities above are therefore excluded. In the remaining three cases we must have $y\notin\operatorname{Im}\varphi$ since otherwise, φ being idempotent, we would have $\varphi(y)=y$ whence the contradiction $y=\vartheta(x)=\vartheta\varphi\vartheta(x)=\vartheta\varphi(y)=\vartheta(y)=z$. It follows from these observations that in all three possibilities for φ the entries marked $*$ must all be w. But this gives $\operatorname{Ker}\varphi\subseteq\operatorname{Ker}\vartheta^2$ which contradicts the fact that (γ,δ) is discrete. We conclude that $v\notin\operatorname{Fix}\vartheta$. Similarly it can be shown that $w\notin\operatorname{Fix}\vartheta$.

Since $\operatorname{Fix}\vartheta=\operatorname{Im}\vartheta^2$ it follows from (1) that $\operatorname{Im}\vartheta^2=\{z,u\}$ and therefore $|\operatorname{Im}(\gamma\delta)^2|=2$. We complete the proof by showing that $\operatorname{Im}\vartheta=\{y,z,u\}$.

For this purpose, we note from the above that

$$\vartheta = \begin{pmatrix} y\,z\,z\,u\,p\,q \end{pmatrix} \quad \text{where} \quad p \neq v,\ q \neq w.$$

Since $x \notin \operatorname{Im} \vartheta$ it suffices therefore to show that $p \neq w$ and $q \neq v$.

Suppose, by way of obtaining a contradiction, that $p = w$. Then we have

$$\vartheta = \begin{pmatrix} y\,z\,z\,u\,w\,q \end{pmatrix} \quad \text{where} \quad q \notin \{x, w\}.$$

Now since for every $a \in \operatorname{Im} \vartheta$ we have $\varphi(a) \in \{b \in X \mid \vartheta(b) = a\}$, we see that

$$\varphi(y) \in \{x, w\}, \quad \varphi(z) \in \{y, z, w\}, \quad \varphi(u) \in \{u, w\}, \quad \varphi(w) \in \{v, w\}.$$

But $w = \vartheta(v)$ gives $w = \vartheta\varphi\vartheta(v) = \vartheta\varphi(w)$; and $\vartheta(w) = q \neq w$. Then $\varphi(w) \neq w$ and so, since φ is idempotent, we have $w \notin \operatorname{Im} \varphi$. Since, as observed above, $y \notin \operatorname{Im} \varphi$ it follows that $\varphi(y) = x$, $\varphi(z) = z$, $\varphi(u) = u$, $\varphi(w) = v$ and therefore, φ being idempotent, we must have $\varphi(x) = x$ and $\varphi(v) = v$. Thus we see that $\varphi = \begin{pmatrix} x\,x\,z\,u\,v\,v \end{pmatrix}$ whence $\operatorname{Im} \vartheta^2 \subseteq \operatorname{Im} \varphi$, which contradicts the fact that (γ, δ) is discrete.

Similarly we can show that $q \neq v$. We conclude that $\operatorname{Im} \vartheta = \{y, z, u\}$ as required. ∎

We have seen in the proof of Theorem 4 that if $X = \{x, y, z, u, v, w\}$ and (γ, δ) is a discrete skew pair of idempotents in T_X then $\vartheta = \gamma\delta \in \widehat{T_X}$ is of the form $\vartheta = \begin{pmatrix} y\,z\,z\,u\,*\,* \end{pmatrix}$ with $\operatorname{Im} \vartheta = \{y, z, u\}$ and $\operatorname{Im} \vartheta^2 = \{z, u\}$. By Theorem 2 there is no loss in generality in supposing that (γ, δ) is fundamental, so that in T_X we have the egg-box situation

φ	δ
γ	$\bullet\vartheta = \gamma\delta$

where $\varphi \in V(\vartheta) \cap E(T_X)$. Consequently,

$$\operatorname{Im} \varphi = \operatorname{Im} \delta, \quad \operatorname{Im} \gamma = \operatorname{Im} \vartheta, \quad \operatorname{Ker} \varphi = \operatorname{Ker} \gamma, \quad \operatorname{Ker} \delta = \operatorname{Ker} \vartheta.$$

More particularly, we now observe that

Theorem 5. *In the above situation we have*

(1) $y \notin \operatorname{Im} \delta$;

(2) *precisely one of* z, u *belongs to* $\operatorname{Im} \delta$.

Proof. (1) Since $y \in \operatorname{Im} \vartheta = \operatorname{Im} \gamma$ and γ is idempotent we have $\gamma(y) = y$. Suppose now that $y \in \operatorname{Im} \delta$. Then likewise $\delta(y) = y$ and there follows the contradiction $z = \vartheta(y) = \gamma\delta(y) = \gamma(y) = y$. Hence we see that $y \notin \operatorname{Im} \delta$.

(2) Suppose, by way of obtaining a contradiction, that neither z nor u belongs to $\mathrm{Im}\ \delta$. Since ϑ, δ are $\mathcal{D}$-related we deduce from Theorem 4 that $|\mathrm{Im}\ \delta| = |\mathrm{Im}\ \vartheta| = 3$. It follows by the hypothesis and (1) that

$$\mathrm{Im}\ \delta = \{x, v, w\}.$$

Now $\mathrm{Ker}\ \delta = \mathrm{Ker}\ \vartheta$ and $\delta^2 = \delta$ give $\delta(x) = x$, $\delta(y) = \delta(z) = \delta[\delta(z)]$, and $\delta(u) = \delta[\delta(u)]$ from which we see that

$$\mathrm{Im}\ \delta = \{x, \delta(z), \delta(u)\}.$$

Clearly, we may choose $\delta(z) = v$ and $\delta(u) = w$. We then have

$$X = \{x, y, z, u, \delta(z), \delta(u)\}, \quad \delta = \big(\ x\ v\ v\ w\ v\ w\ \big)\,.$$

We next observe that $\big(z, \delta(z)\big) \in \mathrm{Ker}\ \delta = \mathrm{Ker}\ \vartheta$ and $\big(u, \delta(u)\big) \in \mathrm{Ker}\ \delta = \mathrm{Ker}\ \vartheta$, whence $\vartheta(x) = y$, $\vartheta(y) = \vartheta[\delta(z)] = \vartheta(v)$ and $\vartheta(u) = \vartheta[\delta(u)] = \vartheta(w)$. It follows that $\vartheta^2(x) = \vartheta^2(y) = \vartheta^2(z) = \vartheta^2(v)$ and $\vartheta^2(u) = \vartheta^2(w)$, from which we deduce that

$$\vartheta^2 = \big(\ z\ z\ z\ u\ z\ u\ \big)\,.$$

Finally, $\mathrm{Im}\ \gamma = \mathrm{Im}\ \vartheta = \{y, z, u\}$ and $\gamma^2 = \gamma$ give

$$\begin{aligned}
\delta\gamma(x) &= \delta\gamma\delta(x) = \delta\vartheta(x) = \delta(y) = \delta(z);\\
\delta\gamma(y) &= \delta(y) = \delta(z);\\
\delta\gamma(z) &= \delta(z);\\
\delta\gamma(u) &= \delta(u);\\
\delta\gamma[\delta(z)] &= \delta\vartheta(z) = \delta(z);\\
\delta\gamma[\delta(u)] &= \delta\vartheta(u) = \delta(u),
\end{aligned}$$

and therefore

$$\delta\gamma = \big(\ \delta(z)\ \delta(z)\ \delta(z)\ \delta(u)\ \delta(z)\ \delta(u)\ \big)\,.$$

It follows from the above observations that $\mathrm{Ker}\ \delta\gamma = \mathrm{Ker}\ \vartheta^2$ and so $(\delta\gamma, \vartheta^2) \in \mathcal{L}$. But we always have $(\delta\gamma, \gamma\delta\gamma) \in \mathcal{L}$. Thus $\vartheta^2 = \gamma\delta\gamma$ and this contradicts the fact that (γ, δ) is discrete. Consequently at least one of z, u must belong to $\mathrm{Im}\ \delta$.

However, since (γ, δ) is discrete, we have again by Theorem 1 that $\mathrm{Im}\ \vartheta^2 \nsubseteq \mathrm{Im}\ \varphi$ and therefore either $z \notin \mathrm{Im}\ \varphi$ or $u \notin \mathrm{Im}\ \varphi$. We conclude that precisely one of z, u must be in $\mathrm{Im}\ \varphi = \mathrm{Im}\ \delta$. ■

Theorem 6. *If S is a derived Rees matrix subsemigroup of T_6 then*

$$\Big|\bigcup_{t \in S} \mathrm{Im}\ t\Big| = 5.$$

Proof. Let $|X| = 6$, say $X = \{x, y, z, u, v, w\}$, and suppose that S is a derived Rees matrix subsemigroup of T_X. If (e, f) is the strong skew pair of idempotents in S then we can represent S in the $\mathcal{D}$-class form

α	f	a
e	$\bullet\, ef$	b
c	d	g

fe	

Let (γ, δ) be the discrete skew pair in S. Then, as in the proof of Theorem 4, we may assume that

$$\vartheta = \gamma\delta = \big(\, y\, z\, z\, u - - \,\big), \quad \vartheta^2 = \big(\, z\, z\, z\, u - - \,\big)$$

with $\text{Im}\ \vartheta = \{y, z, u\}$ and $\text{Im}\ \vartheta^2 = \{z, u\}$.

Since $\delta\gamma$ is $\mathcal{D}$-related to ϑ^2 we have $|\,\text{Im}\ \delta\gamma| = 2$. Now $z \in \text{Im}\ \vartheta \subseteq \text{Im}\ \gamma$ gives $z = \gamma(z)$ and so $\delta(z) \in \text{Im}\ \delta\gamma$; and likewise $\delta(u) \in \text{Im}\ \delta\gamma$. Moreover, $\delta(z) \neq \delta(u)$, for otherwise $(z, u) \in \text{Ker}\ \delta \subseteq \text{Ker}\ \vartheta$ which gives the contradiction $z = \vartheta(z) = \vartheta(u) = u$. It follows from this and the fact that $(\delta\gamma\delta, \delta\gamma) \in \mathcal{R}$ that

$$\text{Im}\ \delta\gamma = \{\delta(z), \delta(u)\} = \text{Im}\ \delta\gamma\delta.$$

There are two cases to consider.

(1) $(\vartheta^2, fe) \notin \mathcal{R}$.

[Note from the above descriptions that this case covers all semigroups except ∂M_1, ∂M_2, ∂M_4.]
In this case we have $(fe, \delta\gamma\delta) \in \mathcal{R}$ and so $\text{Im}\ fe = \text{Im}\ \delta\gamma\delta = \{\delta(z), \delta(u)\}$. Then $\text{Im}\ fe \subseteq \text{Im}\ f$ and $\text{Im}\ fe = \text{Im}\,(ef)^2 \subseteq \text{Im}\ e$ give $\{\delta(z), \delta(u)\} \subseteq \text{Im}\ e \cap \text{Im}\ f$.

Since, by Theorem 5(2), either $z = \delta(z)$ or $u = \delta(u)$ but not both, it follows that $\text{Im}\ e \neq \text{Im}\ \vartheta$ and $\text{Im}\ f \neq \text{Im}\ \vartheta$. Consequently $(e, \vartheta) \notin \mathcal{R}$ and $(f, \vartheta) \notin \mathcal{R}$. Thus we see that $\vartheta \in \{c, d, g\}$; and since $(\gamma, \vartheta) \in \mathcal{R}$ we also have $\gamma \in \{c, d, g\}$.

In summary, therefore, we have

$$\text{Im}\ \alpha = \text{Im}\ f = \text{Im}\ a = \{\delta(z), \delta(u), p\};$$

$$\text{Im}\ e = \text{Im}\ ef = \text{Im}\ b = \{\delta(z), \delta(u), q\};$$

$$\text{Im}\ c = \text{Im}\ d = \text{Im}\ g = \text{Im}\ \gamma = \text{Im}\ \vartheta = \{y, z, u\},$$

where $p \neq q$ since $\text{Im}\ \alpha \neq \text{Im}\ e$. Moreover, since α is idempotent, $(\delta(z), p) \notin \text{Ker}\ \alpha = \text{Ker}\ e$ and so $e(p) \neq e\delta(z) = \delta(z)$. Likewise, $e(p) \neq \delta(u)$ and therefore we must have $e(p) = q$. A similar argument shows that $\alpha(q) = p$.

Now since $(\vartheta, \delta) \in \mathcal{L}$ we have $\text{Ker}\ \vartheta = \text{Ker}\ \delta$. Consequently, $\delta(x) \neq \delta(z)$ and $\delta(x) \neq \delta(u)$. Since δ is idempotent it follows that $x \neq \delta(z)$ and $x \neq \delta(u)$. By Theorem 5(2) we thus see that either $\delta(z) = z$ and $\delta(u) \notin \{x, y, z, u\}$, or

$\delta(u) = u$ and $\delta(z) \notin \{x, y, z, u\}$. We may therefore assume, without loss of generality, that

$$v = \begin{cases} \delta(u) & \text{if } z \in \operatorname{Im} \delta; \\ \delta(z) & \text{if } u \in \operatorname{Im} \delta. \end{cases}$$

If we define $T = \operatorname{Im} f \cup \operatorname{Im} e \cup \operatorname{Im} \vartheta$, we thus have

$$T = \{y, z, u, v, p, q\}.$$

Since $\vartheta, \gamma \in \{c, d, g\}$ we must have $\delta \in R_e \cup R_f$. Consequently, $T = \bigcup_{t \in S} \operatorname{Im} t$.

Our objective is therefore to show that T has precisely 5 elements. For this purpose, we require the following observations.

(1.1) $\{\delta(z), \delta(u)\} \subseteq \operatorname{Fix} a \cap \operatorname{Fix} b$.

In fact, by [2, Theorem 8], one of a, b must be idempotent. Suppose, for example, that a is idempotent. Then since $b = ea$ and $\delta(z) \in \operatorname{Im} a \cap \operatorname{Im} e = \operatorname{Fix} a \cap \operatorname{Fix} e$ we have $b\delta(z) = ea\delta(z) = e\delta(z) = \delta(z)$ so $\delta(z) \in \operatorname{Fix} b$; and likewise for $\delta(u)$. A similar argument holds if b is idempotent.

(1.2) $c(p) = c(q) = d(p) = y$.

We have $g = ca$, $d = cf$ and, by (1.1), $\{\delta(z), \delta(u)\} \subseteq \operatorname{Fix} a$. Using the fact that $\vartheta \in \{c, d, g\}$ and $\{\delta(z), \delta(u)\} \subseteq \operatorname{Im} f$ we then have $g\delta(z) = c\delta(z) = d\delta(z) = \vartheta\delta(z) = \vartheta(z) = z$, and similarly for u. Since $\big(p, \delta(z)\big) \notin \operatorname{Ker} \alpha = \operatorname{Ker} c$ we have $c(p) \neq c\delta(z) = z$; and similarly $c(p) \neq u$. It follows that $c(p) = y$. Moreover, $e(p) = q$ gives $e(p) = e(q)$ whence $(p, q) \in \operatorname{Ker} e = \operatorname{Ker} c$ and so $c(p) = c(q)$. Finally, $d(p) = cf(p) = c(p)$.

(1.3) $(\forall t \in D_2) \quad t(y) \in \{\delta(z), z\}$.

The $\mathcal{D}$ class D_2 is

$$D_2 = \{\vartheta^2 = (\gamma\delta)^2, \delta\gamma, \delta\gamma\delta, \gamma\delta\gamma\}$$

and $\vartheta^2(y) = \vartheta(z) = z$; $\delta\gamma(y) = \delta(y) = \delta(z)$; $\delta\gamma\delta(y) = \delta\vartheta(y) = \delta(z)$; $\gamma\delta\gamma(y) = \gamma\delta(y) = \vartheta(y) = z$.

(1.4) $(\forall t \in D_2) \quad t\delta(x) \in \{\delta(z), z\}$.

In fact the proof is similar to that of (1.3).

Consider now the set X. Since $\delta(x) \neq \delta(z)$ and $\delta(x) \neq \delta(u)$, the fact that δ is idempotent gives $\delta(x) \neq z$ and $\delta(x) \neq u$. Moreover, by Theorem 5(1), $\delta(x) \neq y$. We can therefore also represent X in the form

$$X = \{\delta(x), y, z, u, v, \star\}$$

for some undetermined element $\star$. Since $\operatorname{Im} \gamma = \operatorname{Im} \vartheta$ and γ is idempotent, we have $\operatorname{Fix} \gamma = \{y, z, u\}$ and simple calculations show that

$$\big(\forall k \in X \setminus \{\star\}\big) \quad \delta\gamma(k) = \delta\gamma\delta(k).$$

Since $\delta\gamma \neq \delta\gamma\delta$ we deduce that $\star$ is such that $\delta\gamma(\star) \neq \delta\gamma\delta(\star)$.

There are two situations to consider.

(A) $\delta \in R_e$. [This occurs in $\partial M_3, \partial M_6$.]

Here we have $\delta = b$, $\gamma = c$, and a is not idempotent. Then

$$\text{Im } e = \text{Im } b = \text{Im } \delta = \{\delta(x), \delta(z), \delta(u)\}$$

whence we see that $q = \delta(x)$. Now since $(y,z) \in \text{Ker } \vartheta = \text{Ker } \delta$ we have, by (1.2), $\delta\gamma(p) = \delta c(p) = \delta(y) = \delta(z)$. Moreover, $c(y) = \gamma(y) = y = c(p)$ gives $(p,y) \in \text{Ker } c = \text{Ker } \alpha$ whence $\alpha(y) = \alpha(p) = p$ and consequently $\delta\alpha(y) = \delta(p)$. Since a is not idempotent, (α, b) is a skew pair and so $b\alpha \in D_2$. Consequently, we have $\delta\gamma\delta(p) = \delta\gamma\delta\alpha(y) = \delta\gamma b\alpha(y)$. But, by (1.3), $b\alpha(y) \in \{\delta(z), z\}$ and so

$$\delta\gamma\delta(p) \in \{\delta\gamma\delta(z), \delta\gamma(z)\} = \{\delta\vartheta(z), \delta(z)\} = \{\delta(z)\}.$$

Hence $\delta\gamma\delta(p) = \delta(z) = \delta\gamma(p)$ and therefore $p \neq \star$. It follows that $p \in \{y,z,u,v\}$ and consequently $|T| = 5$.

(B) $\delta \in R_f$. [This occurs in $\partial M_2^t, \partial M_3^t, \partial M_4^t, \partial M_5, \partial M_6^t$.]

In this case we have $p = \delta(x)$. There are three situations to consider.

(B_1) $\delta = \alpha$. [∂M_4^t]

Here $\gamma = g$, $\vartheta = c$, a is idempotent and b is not idempotent. Since $\gamma\delta(z) = z$, $\gamma\delta(u) = u$, and $\gamma(y) = y$ we have $\big(y, \delta(z)\big), \big(y, \delta(u)\big) \notin \text{Ker } \gamma = \text{Ker } b$. Now, by (1.1), $\delta(z), \delta(u) \in \text{Fix } b$ whence it follows that $b(y) = q$. But since b is not idempotent we have $b^2 \in D_2$. By (1.3) and the fact that $z \notin \text{Im } b$ we therefore have $b^2(y) = \delta(z)$. Consequently, $b(q) = \delta(z)$ and $\big(q, \delta(z)\big) \in \text{Ker } b = \text{Ker } g = \text{Ker } \gamma$ whence $\gamma(q) = \gamma\delta(z) = z$. Thus we have, using (1.2),

$$\delta\gamma\delta(q) = \delta\vartheta(q) = \delta c(q) = \delta(y) = \delta(z) = \delta\gamma(q)$$

whence $q \neq \star$. It follows that $q \in \{y,z,u,v\}$ and consequently $|T| = 5$.

(B_2) $\delta = f$. [∂M_2^t]

Here a, b are idempotents and $a = \alpha\gamma$. Since a is idempotent we have $\text{Fix } a = \text{Fix } \alpha = \{p, \delta(z), \delta(u)\}$; and since $a = \alpha\gamma$ we have $a(y) = \alpha(y)$, $a(z) = \alpha(z)$, $a(u) = \alpha(u)$. Then for all $k \in X \setminus \{\star\}$ we have $a(k) = \alpha(k)$. Since $a \neq \alpha$ it follows that $a(\star) \neq \alpha(\star)$. Now since $\big(p, \delta(z)\big), \big(p, \delta(u)\big) \notin \text{Ker } a = \text{Ker } b$ we see that $b(p) = q$; and $b(q) = q$ since b is idempotent. Thus $(p,q) \in \text{Ker } b = \text{Ker } a$ whence we obtain $a(q) = a(p) = p = \alpha(q)$. This shows that $q \neq \star$ whence $q \in \{y,z,u,v\}$ and consequently $|T| = 5$.

(B_3) $\delta = a$. [$\partial M_3^t, \partial M_5, \partial M_6^t$]

Here $\vartheta = g$ and $(ef)^2 = \delta\gamma$. Since $f(p) = p$ we have $ef(p) = e(p) = q$ and so $(ef)^2(p) = ef(q)$. Hence

$$\delta\gamma(q) = (ef)^2(q) = (ef)^3(p) = (ef)^2(p) = \delta\gamma(p) = \delta\gamma\delta(x) = \delta(y) = \delta(z).$$

Now in ∂M_3^t we have $\delta\gamma\delta(q) = \delta\gamma\delta e(p) = \delta\gamma\alpha(p) = \delta\gamma(q)$. As for ∂M_5 and ∂M_6^t, using (1.4) we have

$$\delta\gamma\delta(q) = \delta\gamma\delta e\delta(x) = \delta\gamma\delta(z) = \delta\vartheta(z) = \delta(z) = \delta\gamma(q).$$

We deduce from this that $q \neq \star$. It follows that $q \in \{y, z, u, v\}$ and consequently $|T| = 5$.

(2) $(\vartheta^2, fe) \in \mathcal{R}$.

[Note from the above descriptions that this case covers the remaining semigroups ∂M_1, ∂M_2, ∂M_4.]

In this case we have $(\vartheta^2, \gamma\delta\gamma) \in \mathcal{R}$ with $\vartheta^2 \neq \gamma\delta\gamma$. Moreover, $\delta \in \{d, g\}$ and $\vartheta \in R_e \cup R_f$. Clearly, we may take

$$\text{Im } c = \text{Im } d = \text{Im } g = \text{Im } \delta = \{\delta(x), \delta(z), \delta(u)\};$$

$$\text{Im } \alpha = \text{Im } f = \text{Im } a = \{z, u, p\};$$

$$\text{Im } e = \text{Im } ef = \text{Im } b = \{z, u, q\},$$

where $p \neq q$. Since $\vartheta \in R_e \cup R_f$ with $\text{Im } \vartheta = \{y, z, u\}$ it follows that $y \in \{p, q\}$.

Corresponding to the set T in (1) above, consider the set

$$W = \text{Im } e \cup \text{Im } f \cup \text{Im } \delta.$$

With v as before, we then have

$$W = \{\delta(x), z, u, v, p, q\} = \bigcup_{t \in S} \text{Im } t,$$

and our objective is to show that $|W| = 5$.

For this purpose, we may as before represent the set X in the form

$$X = \{\delta(x), y, z, u, v, \star\}.$$

Simple calculations show that

$$\big(\forall k \in X \setminus \{\star\}\big) \qquad \vartheta^2(k) = \gamma\delta\gamma(k).$$

Since $\vartheta^2 \neq \gamma\delta\gamma$ it follows that $\vartheta^2(\star) \neq \gamma\delta\gamma(\star)$.

There are two cases to consider.

(C) $\vartheta \in R_\alpha$. [This occurs in ∂M_4.]

Here $\mathrm{Im}\,\vartheta = \mathrm{Im}\,\alpha$ whence $p = y$. Also, $\vartheta = a$, $\gamma = \alpha$, $\delta = g$, and b is idempotent. Then $\vartheta = a = \alpha b = \gamma b$, and $b(q) = q$. Thus

$$\vartheta^2(q) = \vartheta\gamma b(q) = \vartheta\gamma(q) = \gamma\delta\gamma(q)$$

whence we see that $q \neq \star$, and consequently $|W| = 5$.

(D) $\vartheta \in R_e$ [This occurs in $\partial M_1, \partial M_2$.]

Here $\mathrm{Im}\,\vartheta = \mathrm{Im}\,e$ and so $q = y$. Also, a is idempotent and $\gamma\delta \in \{\gamma f, \gamma a\}$. Consequently,

$$\vartheta(p) = \gamma\delta(p) \in \{\gamma f(p), \gamma a(p)\} = \{\gamma(p)\}.$$

It follows that $\vartheta^2(p) = \vartheta\gamma(p) = \gamma\delta\gamma(p)$ whence $p \neq \star$ and again $|W| = 5$. ■

Corollary. *or a subsemigroup S of T_X with $|X| = 6$ to be a derived Rees matrix semigroup it is necessary that there exist some element of X that is not in the image of any element of S.*

With the above results to hand, we can now determine which of the ten derived Rees matrix semigroups can be represented as semigroups of transformations. For this purpose we take $X = \{1, 2, 3, 4, 5, 6\}$.

Consider $\vartheta \in T_X$ given by $\vartheta = (\,1\,1\,6\,5\,6\,6\,)$. We have $\vartheta \neq \vartheta^2 = \vartheta^3$, and simple calculations show that the mappings

$$\alpha = (\,1\,1\,6\,4\,4\,6\,), \qquad \beta = (\,1\,1\,1\,4\,4\,6\,);$$
$$\gamma = (\,2\,2\,6\,4\,4\,6\,), \qquad \delta = (\,2\,2\,2\,4\,4\,6\,),$$

are idempotent inverses of ϑ in T_X, so that $\vartheta \in \widehat{T_X}$. Moreover, $\mathrm{Im}\,\alpha = \mathrm{Im}\,\beta$ and so $\alpha\,\mathcal{R}\,\beta$ [mappings on the left!], and similarly $\gamma\,\mathcal{R}\,\delta$. Likewise, $\mathrm{Ker}\,\alpha = \mathrm{Ker}\,\gamma$ and so $\alpha\,\mathcal{L}\,\gamma$, and similarly $\beta\,\mathcal{L}\,\delta$. Since we have $\alpha \in V(\vartheta) \cap E(T_X)$, the idempotents $\vartheta\alpha = (\,1\,1\,6\,5\,5\,6\,)$ and $\alpha\vartheta = (\,1\,1\,6\,4\,6\,6\,)$ are such that $(\vartheta\alpha, \alpha\vartheta)$ is a fundamental skew pair. Since $\alpha\vartheta^2 = \vartheta^2 = \vartheta^2\alpha$, we see by Theorem 1(1) that $(\vartheta\alpha, \alpha\vartheta)$ is a strong skew pair in T_X.

Repeating this process with β, γ, δ we see similarly that $(\vartheta\beta, \beta\vartheta)$ is right regular, that $(\vartheta\gamma, \gamma\vartheta)$ is left regular, and that $(\vartheta\delta, \delta\vartheta)$ is discrete. Consequently in T_6 we have (displayed as previously) the following representation of

∂M_1

(1 1 6 4 4 6)	(1 1 6 4 6 6)	(1 1 1 4 4 6)
(1 1 6 5 5 6)	• (1 1 6 5 6 6)	(1 1 1 5 5 6)
(2 2 6 4 4 6)	(2 2 6 4 6 6)	(2 2 2 4 4 6)

(1 1 6 6 6 6)	(1 1 1 6 6 6)
(2 2 6 6 6 6)	(2 2 2 6 6 6)

Likewise we have the following representations:

∂M_2

(1 1 6 4 4 6)	(1 1 6 4 6 6)	(1 1 1 4 6 6)
(1 1 6 5 5 6)	• (1 1 6 5 6 6)	• (1 1 1 5 6 6)
(2 2 6 4 4 6)	(2 2 6 4 6 6)	(2 2 2 4 6 6)

(1 1 6 6 6 6)	(1 1 1 6 6 6)
(2 2 6 6 6 6)	(2 2 2 6 6 6)

∂M_2^t

$(1\,1\,1\,4\,4\,6)$	$(1\,1\,1\,4\,6\,6)$	$(1\,1\,4\,4\,4\,6)$
$(1\,1\,1\,5\,5\,6)$	$\bullet(1\,1\,1\,5\,6\,6)$	$(1\,1\,5\,5\,5\,6)$
$(2\,2\,2\,5\,5\,6)$	$\bullet(2\,2\,2\,5\,6\,6)$	$(2\,2\,5\,5\,5\,6)$

$(1\,1\,1\,6\,6\,6)$	$(1\,1\,6\,6\,6\,6)$
$(2\,2\,2\,6\,6\,6)$	$(2\,2\,6\,6\,6\,6)$

∂M_3

$(1\,1\,6\,4\,4\,6)$	$(1\,1\,6\,4\,6\,6)$	$\bullet(1\,1\,1\,6\,4\,6)$
$(1\,1\,6\,5\,5\,6)$	$\bullet(1\,1\,6\,5\,6\,6)$	$(1\,1\,1\,6\,5\,6)$
$(2\,2\,6\,4\,4\,6)$	$(2\,2\,6\,4\,6\,6)$	$\bullet(2\,2\,2\,6\,4\,6)$

$(1\,1\,6\,6\,6\,6)$	$(1\,1\,1\,6\,6\,6)$
$(2\,2\,6\,6\,6\,6)$	$(2\,2\,2\,6\,6\,6)$

∂M_4

$(1\,1\,1\,4\,4\,6)$	$(1\,1\,1\,4\,6\,6)$	$\bullet(1\,1\,6\,6\,4\,6)$
$(1\,1\,1\,5\,5\,6)$	$\bullet(1\,1\,1\,5\,6\,6)$	$(1\,1\,6\,6\,5\,6)$
$(2\,2\,2\,5\,5\,6)$	$\bullet(2\,2\,2\,5\,6\,6)$	$(2\,2\,6\,6\,5\,6)$

$(1\,1\,1\,6\,6\,6)$	$(1\,1\,6\,6\,6\,6)$
$(2\,2\,2\,6\,6\,6)$	$(2\,2\,6\,6\,6\,6)$

∂M_5

$(1\,1\,1\,4\,4\,6)$	$(1\,1\,1\,4\,6\,6)$	$(1\,1\,6\,4\,6\,6)$
$(1\,1\,1\,5\,5\,6)$	$\bullet(1\,1\,1\,5\,6\,6)$	$\bullet(1\,1\,6\,5\,6\,6)$
$(2\,2\,2\,5\,5\,6)$	$\bullet(2\,2\,2\,5\,6\,6)$	$\bullet(2\,2\,6\,5\,6\,6)$

$(1\,1\,1\,6\,6\,6)$	$(1\,1\,6\,6\,6\,6)$
$(2\,2\,2\,6\,6\,6)$	$(2\,2\,6\,6\,6\,6)$

The reader will note that missing from the above list are the remaining four semigroups ∂M_3^t, ∂M_4^t, ∂M_6^t, ∂M_6. In fact, we have the following somewhat surprising result.

Theorem 7. *The derived Rees matrix semigroups ∂M_3^t, ∂M_4^t, ∂M_6^t, ∂M_6 are not isomorphic to any subsemigroup of T_6.*

Proof. Suppose, by way of obtaining contradictions, that T_6 contains a copy of one of these four semigroups. Then, continuing with the previous notation, for $X = \{x, y, z, u, v, w\}$ we have $\vartheta = \gamma\delta = (\, y\, z\, z\, u \star\star\,)$ with $\mathrm{Im}\ \vartheta = \{y, z, u\}$ and $\mathrm{Im}\ \vartheta^2 = \{z, u\}$. Since $\delta\gamma$ is $\mathcal{D}$-related to ϑ^2 we have $|\,\mathrm{Im}\ \delta\gamma| = 2$. Now $z \in \mathrm{Im}\ \vartheta = \mathrm{Im}\ \gamma$ gives $\delta(z) = \delta\gamma(z) \in \mathrm{Im}\ \delta\gamma$; and likewise $\delta(u) \in \mathrm{Im}\ \delta\gamma$. And $\delta(z) \neq \delta(u)$; for otherwise in all cases we would have $(z, u) \in \mathrm{Ker}\ \delta = \mathrm{Ker}\ \vartheta$ whence the contradiction $z = \vartheta(z) = \vartheta(u) = u$. It therefore follows that in all four semigroups

$$\mathrm{Fix}\ ef = \mathrm{Im}\,(ef)^2 = \mathrm{Im}\ fe = \mathrm{Im}\ \delta\gamma = \{\delta(z), \delta(u)\}.$$

Again since $\mathrm{Ker}\ \delta = \mathrm{Ker}\ \vartheta$ it now follows that $\mathrm{Im}\ \delta = \{\delta(x), \delta(z), \delta(u)\}$. Since, by Theorem 5(1), we have $y \notin \mathrm{Im}\ \delta$ and, by Theorem 5(2), precisely one of z, u belongs to $\mathrm{Im}\ \delta$, we may again represent X in the form

$$X = \{\delta(x), y, z, u, v, \star\}$$

in which we can choose

$$v = \begin{cases} \delta(u) & \text{if } z \in \mathrm{Im}\ \delta; \\ \delta(z) & \text{if } u \in \mathrm{Im}\ \delta. \end{cases}$$

There are two situations to consider:

(1) ∂M_3^t, ∂M_4^t, ∂M_6^t.

In each of these three semigroups we have $\delta = f\delta = f\vartheta$ and therefore, using the fact that $\delta(z), \delta(u) \in \text{Fix}\ ef$, we have

$$\text{Im}\ e\delta \supseteq \{e\delta(x), e\delta(z), e\delta(u)\} = \{ef\vartheta(x), ef\delta(z), ef\delta(u)\} = \{ef(y), \delta(z), \delta(u)\}.$$

Consider now the element $ef(y)$. First we observe that $ef(y) \neq \delta(z), \delta(u)$. Indeed, suppose for example that $ef(y) = \delta(z)$. Since in all three semigroups $ef(y) = ef\vartheta(x) = e\delta(x)$, we have $e\delta(x) = \delta(z)$ whence $e\delta(x) = e\delta(z)$. It follows that $(x, z) \in \text{Ker}\ e\delta = \text{Ker}\ \vartheta$ whence we have the contradiction $y = \vartheta(x) = \vartheta(z) = z$. Since $|\text{Im}\ e\delta| = 3$ we deduce from the above that

$$\text{Im}\ e\delta = \{ef(y), \delta(z), \delta(u)\}.$$

To show that $ef(y)$ is the missing element $\star$ of X, we next observe that

(1.1) $ef(y) \neq \delta(x)$.

In fact, if $ef(y) = \delta(x)$ then $\text{Im}\ e\delta = \text{Im}\ \delta$ whence $(e\delta, \delta) \in \mathcal{R}$. But in each of the three semigroups we have $(e\delta, \delta) \in \mathcal{L}$.

(1.2) $ef(y) \neq y$.

In fact, if $ef(y) = y$ then in all three cases $y \in \text{Fix}\ ef = \text{Im}\ fe \subseteq \text{Im}\ f = \text{Im}\ \delta$ in contradiction to Theorem 5(1).

(1.3) $ef(y) \neq z, u$.

In ∂M_4^t we have $\text{Ker}\ e = \text{Ker}\ \vartheta$. Suppose, for example, that $ef(y) = z$. Then $e\delta(x) = z = e(z)$ whence $(\delta(x), z) \in \text{Ker}\ e = \text{Ker}\ \vartheta$ which gives the contradiction $y = \vartheta(x) = \vartheta\delta(x) = \vartheta(z) = z$; and similarly for u.

As for ∂M_3^t and ∂M_6^t, observe that here we have $\alpha ef = f$ and $\vartheta f = \gamma$. Thus

(a) $\vartheta\alpha[ef(y)] = \vartheta f(y) = \gamma(y) = y$.

Moreover, in each of these semigroups we have

$$\gamma\delta\gamma = (\vartheta\alpha)^2.$$

Thus

(b) $(\vartheta\alpha)^2[ef(y)] = \gamma\delta\gamma[ef(y)] = \gamma fe[ef(y)] = \gamma\delta\gamma(y) = \gamma\delta(y) = \vartheta(y) = z$.

Then $\vartheta\alpha(y) = z$ and consequently

(c) $\vartheta\alpha(z) = (\vartheta\alpha)^2(y) = \gamma\delta\gamma(y) = z$.

We deduce from (a) and (c) that $ef(y) \neq z$.

Since also

(d) $(\vartheta\alpha)^2(u) = \gamma\delta\gamma(u) = \gamma\delta(u) = \vartheta(u) = u$,

we see from (b) and (d) that $ef(y) \neq u$.

It is clear from the above observations that for $\partial M_3^t, \partial M_4^t, \partial M_6^t$ we have

$$X = \{\delta(x), y, z, u, v, ef(y)\}.$$

Since this contradicts Theorem 6, we conclude that T_6 does not contain copies of the three derived Rees matrix semigroups $\partial M_3^t, \partial M_4^t, \partial M_6^t$.

(2) ∂M_6.

As for ∂M_6, here we have $\text{Im}\ \alpha = \text{Im}\ \alpha\delta \supseteq \{\alpha\delta(x), \alpha\delta(z), \alpha\delta(u)\}$. But $\alpha\delta = \alpha\vartheta$, so $\alpha\delta(x) = \alpha(y)$. Also, in ∂M_6 we have

$$\delta\gamma\delta = (\alpha\delta)^2.$$

Hence $\delta(z) = \delta\vartheta(z) = \delta\gamma\delta(z) = (\alpha\delta)^2(z) \in \text{Im}\ \alpha$ and so $\alpha\delta(z) = \delta(z)$. Likewise, we have $\alpha\delta(u) = \delta(u)$. Consequently, $\text{Im}\ \alpha \supseteq \{\alpha(y), \delta(z), \delta(u)\}$.

Consider now the element $\alpha(y)$. First we observe that $\alpha(y) \neq \delta(z), \delta(u)$. For example, $\alpha(y) = \delta(z)$ gives the contradiction $y = \gamma(y) = \gamma\alpha(y) = \gamma\delta(z) = \vartheta(z) = z$. Since $|\,\text{Im}\ \alpha| = |\,\text{Im}\ \vartheta| = 3$ it therefore follows that

$$\text{Im}\ \alpha = \{\alpha(y), \delta(z), \delta(u)\}.$$

To show that in this case $\alpha(y)$ is the missing element $\star$ of X, we observe that

(2.1) $\alpha(y) \neq \delta(x)$.

In fact, if $\alpha(y) = \delta(x)$ then we have $\text{Im}\ \alpha = \text{Im}\ \delta$ which gives the contradiction $(\alpha, \delta) \in \mathcal{R}$.

(2.2) $\alpha(y) \neq y$.

Since $y, z \in \text{Im}\ \vartheta = \text{Im}\ \gamma$ we have $(\gamma f)^2(y) = \gamma\delta\gamma(y) = \gamma\delta(y) = \vartheta(y) = z$. It follows from this that $\gamma f(y) \neq y$ and consequently $f(y) \neq y$. Thus, since f is idempotent, $y \notin \text{Im}\ f = \text{Im}\ \alpha$, so $\alpha(y) \neq y$.

(2.3) $\alpha(y) \neq z, u$.

Clearly, $(y, z) \notin \text{Ker}\ \gamma = \text{Ker}\ \alpha$ and so $\alpha(y) \neq \alpha(z)$. Since α is idempotent this implies that $\alpha(y) \neq z$. Similarly, $\alpha(y) \neq u$.

It is clear from the above observations that for ∂M_6 we have

$$X = \{\delta(x), y, z, u, v, \alpha(y)\}.$$

Since this contradicts Theorem 6, we conclude that T_6 also does not contain a copy of ∂M_6. ■

Combining the above considerations, we arrive at the following result which is our main conclusion.

Theorem 8. *The full transformation semigroup T_X contains copies of all ten derived Rees matrix semigroups if and only if $|X| \geqslant 7$.*

Proof. It suffices to show that T_7 contains copies of each of ∂M_3^t, ∂M_4^t, ∂M_6^t, ∂M_6. The following examples for $X = \{1,2,3,4,5,6,7\}$ serve this purpose:

∂M_3^t

(1156561)	(1165561)	(1156565)
(1136361)	• (1163361)	(1136363)
• (2246462)	(2264462)	• (2246464)

(1166661)	(1166666)
(2266662)	(2266666)

∂M_4^t

(1136363)	(1133663)	(1133661)
(1156565)	• (1155665)	• (1155661)
• (2246464)	(2244664)	(2244662)

(1166666)	(1166661)
(2266666)	(2266662)

∂M_6^t

(1156561)	(1165561)	(1166565)
(1136361)	• (1163361)	• (1166363)
• (2246462)	(2264462)	• (2266464)

(1166661)	(1166666)
(2266662)	(2266666)

∂M_6

(1133366)	(1136666)	• (1166361)
(1155566)	• (1156666)	(1166561)
(2244466)	• (2246666)	• (2266462)

(1166666)	(1166661)
(2266666)	(2266662)

■

References

[1] Blyth, T. S., and M. H. Almeida Santos, *Regular semigroups with skew pairs of idempotents*, Semigroup Forum **65** (2002), 264–274.

[2] Blyth, T. S., and M. H. Almeida Santos, *The smallest regular semigroups with all four skew pairs of idempotents*, Semigroup Forum **69** (2004), 230–242.

[3] Blyth, T. S., and M. H. Almeida Santos, *Skew pairs of idempotents in transformation semigroups*, Acta Mathematica Sinica, English series (to appear).

Mathematical Institute
University of St Andrews
Scotland
tsblyth.prof@btinternet.com

Departamento de Matemática
F.C.T.
Universidade Nova de Lisboa
Portugal
mhas@fct.unl.pt

Received July 15, 2005
and in final form January 30, 2006
Online publication