Numerical evaluation of continuous time ruin probabilities for a risk process with annually varying premiums

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Abstract

In this paper we present a method for the numerical evaluation of the ruin probability in continuous and finite time for a classical risk process where the premium can change from year to year. A major consideration in the development of this methodology is that it should be easily applicable to large portfolios. Our method is based on the simulation of the annual aggregate claims and then on the calculation of the ruin probability for a given surplus at the start and at the end of each year. We calculate the within-year ruin probability assuming first a Brownian motion approximation and, secondly, a translated gamma distribution approximation for aggregate claim amounts.

We illustrate our method by studying the case where the premium at the start of each year is a function of the surplus level at that time or at an earlier time.

1 Introduction

This paper presents a method for calculating the probability of ruin in continuous and finite time for a compound Poisson risk process where the premium rate is constant throughout each year but can change at the start of each year. We are interested in the case where the premium depends on the past aggregate annual claims experience and, in particular, where it depends on the surplus level at the end of the previous year or at some earlier time.

Our method involves simulating the aggregate claims for each year, calculating the premium to be charged each year given the past aggregate claim amounts, and then calculating the within year probability of ruin assuming either a Brownian motion or translated gamma distribution approximation to the surplus process. The Brownian motion approximation is well established, see for example Sections 8.6 and 8.7 of Klugman et al. (2004), and should work well if the expected number of claims in each year is reasonably large. The translated gamma approximation uses ideas which go back at least to Seal (1978a) and has been used more recently by Dickson and Waters (1993, 2006). It has its roots in Bohman and Esscher (1963).

In Section 2 we set out our model and general procedure for calculating ruin probabilities in continuous and finite time. In Section 3 we give details of the Brownian motion and the translated gamma approximations we use to calculate the probability of ruin within each year, given the surplus at the start and at the end of the year. Details of our procedure to simulate ruin probabilities are given in Section 4.

Wikstad (1971) and Seal (1978b) give values for the probability of ruin in finite and continuous time for some examples of a classical risk process (with constant premiums). In Section 5 we check the accuracy of our methodology by applying it to these examples and comparing our values with theirs.

In Section 7 we apply our method to a risk process where the premium at the start of each year depends on the current or a past level of the surplus. We consider several cases. We start by assuming that the premium for the coming year depends on the surplus at the end of the preceding year, *i.e.* on the current surplus. This is intuitively appealing but may not be practicable, since it requires the insurer to determine and charge the new premium instantaneously. In practice there may be some delay in setting a new premium rate so we also consider the case where the premium in the coming year depends on the surplus one year ago. In both cases, the higher the surplus, the lower will be the premium. The premium rate is set each year so that the probability of ultimate ruin from that time is always (approximately) equal to a pre-determined value. We do this using De Vylder (1978)'s approximation; details are given in Section 6.

The problem of calculating the probability of ruin when the premium is a function of the surplus level at the end of the year has been studied by many authors, in most cases in infinite time. For example, Davidson (1969) let the safety loading decrease with an increasing risk reserve. Taylor (1980) and Jasiulewicz (2001) considered the case where the premium rate varies continuously as a function of the surplus. Petersen (1989) illustrates with a simple numerical method how the probability of ruin can be calculated when the general premium rate depends on the reserve. Dickson (1991) considered the case where the premium rate changes when the surplus crosses an upper barrier. More recently, Cardoso and Waters (2005) presented a numerical method for calculating finite time ruin probabilities for the same problem.

The use of simulation to estimate ruin probabilities is not new. Dufresne and Gerber (1989) observed that the probability of ruin is related to the stationary distribution of a certain associated process and estimated it using simulation. Michaud (1996) simulated the jumps and interjump times for two models in order to approximate the probability of ultimate ruin. In his first model the surplus earns interest; in his second model the premium changes continuously and depends on the current level of the surplus.

2 The model

We consider a risk process over an n-year period. We denote by $S(t)$ the aggregate claims up to time t, so that $S(0) = 0$, and by Y_i the aggregate claims in year i, $i = 1, \ldots, n$, so that $Y_i = S(i) - S(i-1)$. We assume that $\{Y_i\}_{i=1}^{n}$ is a sequence of *i.i.d.* random variables, each with a compound Poisson distribution whose first three moments exist. We denote by λ the Poisson parameter for the expected number of claims each year and by $f(\cdot, s)$ the probability density function (*pdf*) of $S(s)$ for $0 < s \leq 1$.

Let P_i denote the premium charged in year i and let $U(t)$ denote the insurer's surplus at time t, $0 \leq t \leq n$. We assume premiums are received continuously at a constant rate throughout each year. The initial surplus, u $(= U(0))$, and the initial premium, P_1, are known. For $i = 2, \ldots, n$, we assume that P_i is a function of $\{U(j)\}_{j=1}^{i-1}$, the surplus at the end of each of the preceding years. For any time t, $0 \leq t \leq n$, $U(t)$ is calculated as follows:

$$U(t) = u + \sum_{j=1}^{i-1} P_j + (t - i + 1)P_i - S(t) \tag{2.1}$$

where i is such that $t \in [i-1, i)$, and where $\sum_{j=1}^{0} P_j = 0$.

For $i \geq 2$, the premium P_i and surplus level $U(i)$ are random variables since they both depend on the claims experience in previous years. Where we wish to refer to a particular realisation of these variables, we will use the lower case letters p_i and $u(i)$, respectively.

The probability of ruin in continuous time within n years is denoted by $\psi(u, n)$ and defined as follows:

$$\psi(u, n) \stackrel{def}{=} \Pr(U(t) < 0 \text{ for some } t \in (0, n])$$

Let $\psi(u(i-1), 1, u(i))$ be the probability of ruin within the year i, given the surplus $u(i-1)$ at the start of the year and the surplus $u(i)$ at the end. We will develop a formula for $\psi(u(i-1), 1, u(i))$ following methods in Dickson and Waters (2006, Section 3.2).

Let $\Delta(u(i-1), 1, z)$ denote the probability that, starting from a surplus of $u(i-1)$, ruin does not occur in the year and the surplus at the end of the year is greater than z. Then:

$$\Delta(u(i-1), 1, y) = \int_{y}^{\infty} (1 - \psi(u(i-1), 1, z)\, f(u(i-1) + p_i - z, 1)\, dz$$

and so:

$$\psi(u(i-1), 1, y) = 1 + \frac{1}{f(u(i-1) + p_i - y, 1)} \frac{d}{dy} \Delta(u(i-1), 1, y)$$

Using formulae (3.11) and (3.13) from Dickson and Waters (2006) and writing $y = u(i)$, we have:

$$\psi(u(i-1),1,u(i)) = \frac{\int_{s=0}^{1-u(i)/p_i} f(u(i-1)+p_i s, s)\frac{u(i)}{(1-s)} f(p_i(1-s)-u(i),1-s)ds}{f(u(i-1)+p_i-u(i),1)} + \frac{f(u(i-1)+p_i-u(i),1-u(i)/p_i)\exp(-\lambda u(i)/p_i)}{f(u(i-1)+p_i-u(i),1)} \qquad (2.2)$$

In principle, formula (2.2) can be used to calculate values of $\psi(u(i-1),1,u(i))$. We require values of the *pdf* $f(\cdot,s)$ for values of s from 0 to 1. Although these values can be calculated using well known recursive formulae, the number of values required can be prohibitively large, particularly if λ is large, and so some approximate method of calculation is required. In the next section we give details of two approximate methods for calculating this probability.

3 Approximate calculation of the within year probability of ruin

In this section we give details of two methods for approximating the within year probability of ruin, $\psi(u(i-1),1,u(i))$. The two methods, a Brownian motion approximation and a translated gamma approximation, are both moment based; the former matching two moments and the latter matching three.

3.1 The Brownian motion approximation

Let $\{W(s); s \geq 0\}$ be a Brownian motion process with drift and variance per unit time denoted μ and σ^2, respectively. For the given values $u(i-1)$ and p_i, we approximate the surplus process, $\{U(t)\}$ over the time interval $i-1 \leq t \leq i$ by the Brownian motion process $\{u(i-1)+W(s)\}$ over the time interval $0 \leq s \leq 1$, with $s = t-i+1$ and with:

$$\mu = p_i - \mathrm{E}[Y_i] \quad \text{and} \quad \sigma^2 = \mathrm{Var}[Y_i]$$

so that for $0 \leq s \; (= t-i+1) \leq 1$:

$$\mathrm{E}[u(i-1)+W(s)] = u(i-1) + s(p_i - \mathrm{E}[Y_i]) = \mathrm{E}[U(t) \mid U(i-1) = u(i-1)]$$

$$\mathrm{Var}[u(i-1)+W(s)] = s.\sigma^2 = \mathrm{Var}[U(t)] \mid U(i-1) = u(i-1)]$$

Let T denote the time until ruin for this approximating process, so that:

$$T = \inf(\tau > 0 : u(i-1) + W(\tau) < 0)$$

with the convention that $T = \infty$ if ruin never occurs. Klugman et al. (2004, Corollaries 8.25 and 8.27) show that the probability that ruin ever occurs, denoted $\psi_{BM}(u(i-1))$, is given by:

$$\psi_{BM}(u(i-1)) = \exp\left(-\frac{2\mu u(i-1)}{\sigma^2}\right) \tag{3.1}$$

and the probability density of the time to ruin, given that ruin occurs, denoted $f_T(s)$, is:

$$f_T(s) = \frac{u(i-1)}{\sqrt{2\pi\sigma^2}} s^{-3/2} \exp\left(-\frac{(u(i-1)-\mu s)^2}{2\sigma^2 s}\right) \tag{3.2}$$

Hence, the probability density of the time to ruin, for finite s, without conditioning on whether ruin occurs, is the product of (3.1) and (3.2), that is:

$$f_T(s)\psi_{BM}(u(i-1)) = \frac{u(i-1)}{\sqrt{2\pi\sigma^2}} s^{-3/2} \exp\left(-\frac{(u(i-1)-\mu s)^2 + 4\mu s u(i-1)}{2\sigma^2 s}\right) \tag{3.3}$$

Klugman et al. (2004, page 259) also show that, for $0 < s < 1$, the conditional probability density of $u(i-1) + W(1)$ at y, given that $T = s$, denoted $f(y|T = s)$, is:

$$f(y|T = s) = \exp\left(\frac{y\mu}{\sigma^2}\right) \frac{\exp\left(-\frac{y^2+\mu^2(1-s)^2}{2\sigma^2(1-s)}\right)}{\sqrt{2\pi\sigma^2(1-s)}} \tag{3.4}$$

Hence, for $0 < T < 1$, the joint probability density of $u(i-1) + W(1)$ and T is given by the product of (3.3) and (3.4) and the conditional probability density of T, given that $u(i-1) + W(1) = u(i)$, denoted $f_W(s)$, is the product of (3.3) and (3.4) divided by the (marginal) density of $u(i-1)+W(1)$ at the point $u(i)$. Since $W(1)$ has a normal distribution, we have:

$$f_W(s) = \frac{\exp\left(\frac{u(i)\mu}{\sigma^2}\right)\frac{u(i-1)s^{-3/2}}{2\pi\sigma^2}\frac{1}{\sqrt{1-s}}\exp\left(-\frac{u(i)^2+\mu^2(1-s)^2}{2\sigma^2(1-s)} - \frac{(u(i-1)-\mu s)^2}{2\sigma^2 s} - \frac{2\mu u(i-1)}{\sigma^2}\right)}{n(u(i)-u(i-1),\mu,\sigma^2)}$$

where $n(\cdot, \mu, \sigma^2)$ is the density function of the normal distribution.

Finally, the probability of ruin in the year, given that the surplus at the end of the year is $u(i)$, is given by the integral of this last conditional density from $s = 0$ to $s = 1$. We denote this probability $\psi_{BM}(u(i-1), 1, u(i))$, so that:

$$\psi_{BM}(u(i-1), 1, u(i)) = \int_{s=0}^{1} f_W(s)\, ds \tag{3.5}$$

In later sections we will use $\psi_{BM}(u(i-1), 1, u(i))$ as an approximation to $\psi(u(i-1), 1, u(i))$.

3.2 Translated gamma approximation

The translated gamma approximation to the probability $\psi(u(i-1), 1, u(i))$ is based on replacing each *pdf* in formula (2.2) by an approximating gamma density chosen to match the first three moments. This is based on the methodology in Dickson and Waters (2006, formulae (4.7) & (4.9)).

Let α, β and κ be such that:

$$\begin{aligned} \frac{2}{\sqrt{\alpha}} &= \frac{\mathrm{E}[(Y_i - \mathrm{E}[Y_i])^3]}{\mathrm{Var}[Y_i]^{3/2}} \\ \frac{\alpha}{\beta^2} &= \mathrm{Var}[Y_i] \\ \frac{\alpha}{\beta} + \kappa &= \mathrm{E}[Y_i] \end{aligned}$$

and let $H(s)$ be a random variable with a gamma distribution with parameters αs and β. Then for $0 < s \leq 1$ the random variable $H(s) + \kappa s$ has a translated gamma distribution whose first three moments match those of $S(s)$.

We can approximate $\psi(u(i-1), 1, u(i))$ by replacing each compound Poisson *pdf* in formula (2.2) by the *pdf* of $H(s) + \kappa s$, with the appropriate value of s. Let $F_G(\cdot, s)$ and $f_G(\cdot, s)$ denote the cumulative distribution function and *pdf* of $H(s)$, respectively. In particular, we need to replace:

$$\begin{aligned} f(x, s) &\quad \text{by} \quad f_G(x - \kappa s, s) \\ \exp(-\lambda t) &\quad \text{by} \quad F_G(-\kappa t, t) \end{aligned}$$

For this last relationship, note that for the compound Poisson process $\exp(-\lambda t)$ is the probability of no claims in a time interval of length t. We approximate this by the probability that $H(t) + \kappa t$ is negative, which is $F_G(-\kappa t, t)$.

Our translated gamma approximation to $\psi(u(i-1), 1, u(i))$, which we denote by $\psi_{TG}(u(i-1), 1, u(i))$, is given by:

$$\begin{aligned} \psi_{TG}(u(i-1), 1, u(i)) \quad = \quad & \frac{\int_{s=0}^{1-u(i)/p_i} f_G(u(i-1) + (p_i - \kappa)s, s) \frac{u(i)}{(1-s)} f_G((p_i - \kappa)(1-s) - u(i), 1-s) ds}{f_G(u(i-1) + p_i - \kappa - u(i), 1)} \\ & + \frac{f_G(u(i-1) + (p_i - \kappa)(1 - \frac{u(i)}{p_i}), 1 - \frac{u(i)}{p_i}) F_G(-\kappa u(i)/p_i, u(i)/p_i)}{f_G(u(i-1) + p_i - \kappa - u(i), 1)} \end{aligned} \quad (3.6)$$

The advantage of using $\psi_{TG}(u(i-1), 1, u(i))$ as an approximation to $\psi(u(i-1), 1, u(i))$ is that there are well established and fast algorithms for calculating gamma densities so that the former can be calculated far more quickly and easily than the latter.

4 Simulation of ruin probabilities

Our goal is to estimate $\psi(u, n)$. To achieve this we simulate N paths of the surplus process (2.1) at integer intervals, *i.e.* for $t = 1, 2, \ldots, n$. Each path starts at u $(= U(0))$. The premium in the first year, P_1 $(= p_1)$, is given.

Let $\psi_j(u,n)$, $j = 1,2,\ldots,N$, denote the estimate of $\psi(u,n)$ from the $j-th$ simulation. Our procedure for calculating $\psi_j(u,n)$ is as follows:

(i) Simulate the values of $\{Y_i\}_{i=1}^n$ by assuming each Y_i is approximately distributed as $H(1)+\kappa$, where $H(1)$ and κ are as defined in Section 3.2, so that $H(1)$ has a translated gamma distribution with parameters α, β and κ.

(ii) From the simulated values of $\{Y_i\}_{i=1}^n$, say $\{y_i\}_{i=1}^n$, calculate successively $u(1)$ $(= u + p_1 - y_1)$, then p_2 (as a function of u and $u(1)$), $u(2)$ $(= u(1)+p_2-y_2)$ and so on until the surplus at the end of each year, $\{u(i)\}_{i=0}^n$, has been calculated.

(iii) If $u(i) < 0$ for any $i, i = 1,2,\ldots,n$, then we set $\psi_j(u,n) = 1$ and we start simulation $j+1$.

(iv) If $u(i) \geq 0$ for all $i, i = 1,2,\ldots,n$, we calculate $\psi_j(u(i-1),1,u(i))$, using either a Brownian motion or translated gamma approximation, as in Sections 3.1 and 3.2, and then calculate $\psi_j(u,n)$ as follows:

$$\begin{aligned}\psi_j(u,n) &= 1 - \prod_{i=1}^{n}(1-\psi_{BM}(u(i-1),1,u(i))) \\ \psi_j(u,n) &= 1 - \prod_{i=1}^{n}(1-\psi_{TG}(u(i-1),1,u(i)))\end{aligned}$$

The mean of our N estimates, $\{\psi_j(u,n)\}_{j=1}^N$, is then our estimate of $\psi(u,n)$ and we can use the sample standard deviation of the N estimates to calculate approximate confidence intervals for the estimate. We will denote our estimates $\hat{\psi}_{BM}(u,n)$ and $\hat{\psi}_{TG}(u,n)$.

A point to note about this procedure is that the size of the portfolio, as measured by the Poisson parameter, λ, does not affect the scale of the calculations. Hence, this methodology can be applied as easily to large as to small portfolios. In fact, the larger the value of λ, the more accurate the approximations are likely to be.

5 Accuracy of the procedure

5.1 General comments

Our methodology for estimating the probability of ruin in finite and continuous time is based on two approximations:

(i) We simulate the annual aggregate claims using a translated gamma approximation.

(ii) We estimate the within year probability of ruin, given the starting and final surplus, using a Brownian motion or a translated gamma approximation.

We would expect both to be reasonable approximations if the expected number of claims each year, λ, is large and the individual claim size distribution does not have too fat a tail. Note that if $u = 0$ we cannot use the Brownian motion approximation to the within year probability of ruin since each simulation will give $\psi_j(0, 1, u(1)) = 1$.

Wikstad (1971) and Seal (1978b) provide values of ruin probabilities in finite and continuous time for some compound Poisson risk processes, in all cases with a fixed premium rate. We can test the accuracy of our method by applying it to their examples. The ruin probabilities in their examples range from practically zero to almost 1. We restrict our comparisons to cases where their value for the probability of ruin lies between 0.001 and 0.05, which we consider covers all values of practical interest.

In all the examples in this section we use 50 000 simulations. For a given combination of term (n), claim size distribution and target probability of ruin (see Section 6) we use the same 50 000 sets of simulated aggregate annual claims. This makes it easier to compare results within each example.

5.2 Examples

Example 1:
Wikstad (1971) in his case IA considered exponentially distributed individual claims with mean 1 and with one claim expected each year ($\lambda = 1$). Table 5.1 shows his values for selected cases and our estimates, $\hat{\psi}_{BM}(u, n)$ and $\hat{\psi}_{TG}(u, n)$, of these values together with the standard errors of these estimates.

In Table 5.1, ζ denotes the premium loading factor, so that the premium rate each year is $(1 + \zeta)$, and $\psi(u, n)$ denotes Wikstad's value.

u	n	ζ	$\psi(u,n)$	$\hat{\psi}_{TG}(u,n)$	SD$[\hat{\psi}_{TG}(u,n)]$	$\hat{\psi}_{BM}(u,n)$	SD$[\hat{\psi}_{BM}(u,n)]$
10	10	0.05	0.03670	0.03487	0.0008	0.03621	0.0008
10	10	0.15	0.02770	0.02832	0.0007	0.02808	0.0007
10	10	0.25	0.02090	0.02011	0.0006	0.01897	0.0006

Table 5.1: Values and estimates of $\psi(u, n)$. Wikstad (1971, Case IA).

Example 2:
Wikstad (1971) in his case IIA considered a compound Poisson surplus model where individual claim amounts have the following distribution:

$$P(x) = 1 - 0.0039793 \exp(-0.014631x) - 0.1078392 \exp(-0.19206x) - 0.8881815 \exp(-5.514588x)$$

This is described by Wikstad as an attempt to model Swedish non-industrial fire insurance data from 1948-1951. He also describes the distribution as 'extremely skew'. Table 5.2 shows values of the probability of ruin, with $\lambda = 1$, in the same format as Tables 5.1.

u	n	ζ	$\psi(u,n)$	$\hat{\psi}_{TG}(u,n)$	SD$[\hat{\psi}_{TG}(u,n)]$	$\hat{\psi}_{BM}(u,n)$	SD$[\hat{\psi}_{BM}(u,n)]$
10	1	0.05	0.0190	0.00831	0.0004	0.01706	0.0004
10	1	0.15	0.0188	0.00935	0.0004	0.01787	0.0004
10	1	0.25	0.0187	0.00861	0.0004	0.01672	0.0004
100	10	0.05	0.00940	0.01124	0.0005	0.01217	0.0005
100	10	0.15	0.00930	0.00876	0.0004	0.00950	0.0004
100	10	0.25	0.00920	0.00908	0.0004	0.00983	0.0004

Table 5.2: Values and estimates of $\psi(u,n)$. Wikstad (1971, Case IIA).

Example 3:
Seal (1978b) also considered exponentially distributed individual claims with mean 1 and with one claim expected each year ($\lambda = 1$). Table 5.3 shows his values for selected cases and our estimates using the same format as in Tables 5.1 and 5.2.

u	n	ζ	$\psi(u,n)$	$\hat{\psi}_{TG}(u,n)$	SD$[\hat{\psi}_{TG}(u,n)]$	$\hat{\psi}_{BM}(u,n)$	SD$[\hat{\psi}_{BM}(u,n)]$
10	10	0.10	0.03190	0.03105	0.0008	0.03491	0.0008
22	50	0.10	0.01562	0.01448	0.0005	0.01577	0.0005
44	600	0.10	0.01348	0.01328	0.0005	0.01379	0.0005
66	600	0.10	0.00135	0.00162	0.0002	0.00172	0.0002

Table 5.3: Values and estimates of $\psi(u,n)$. Seal (1978b).

Example 4:
Closed formulas for the probability of ultimate ruin, denoted $\psi(u)$, exist when individual claims have either an exponential or a mixed exponential distribution. See, for example, Gerber (1979), pages 115-117. We can test our methodology by calculating $\psi_{TG}(u,n)$ and $\psi_{BM}(u,n)$ for very large values of n and λ for both claim size distributions and comparing the results with the exact values of $\psi(u)$. Results are shown in Tables 5.4 and 5.5 for $n = \lambda = 1000$ for the following two claim size density functions:

$$\begin{aligned} f(x) &= \exp(-x) \quad x > 0 \qquad \text{(exponential, mean = 1)} \\ f(x) &= \frac{3}{2}\exp(-3x) + \frac{7}{2}\exp(-7x) \quad x > 0 \\ &\qquad\qquad \text{(mixed exponential, mean = 238.095, variance = 131.519)} \end{aligned}$$

5.3 Conclusions

It can be seen that in our examples in this section $\psi_{TG}(u,n)$ is generally closer to $\psi(u,n)$ than is $\psi_{BM}(u,n)$. This is as expected since the former approximation is based

	u	$\psi(u)$	$\psi_{TG}(u,1000)$	$\text{SD}[\hat{\psi}_{TG}(u,1000)]$	$\psi_{BM}(u,1000)$	$\text{SD}[\hat{\psi}_{BM}(u,1000)]$
$\zeta = 0.05$	1300	0.043109	0.044273	0.0007	0.042056	0.0007
$\zeta = 0.15$	500	0.033352	0.033448	0.0002	0.023904	0.0002
	700	0.009050	0.009124	0.0001	0.005506	0.0001
	900	0.002456	0.002500	0.0001	0.001304	0.0000
$\zeta = 0.25$	300	0.039830	0.039976	0.0001	0.023567	0.0001
	500	0.005390	0.005463	0.000033	0.001953	0.000015

Table 5.4: Values and estimates of $\psi(u)$: exponentially distributed claim amounts.

u	$\psi(u)$	$\psi_{TG}(u,1000)$	$\text{SD}[\hat{\psi}_{TG}(u,1000)]$	$\psi_{BM}(u,1000)$	$\text{SD}[\hat{\psi}_{BM}(u,1000)]$
3	0.03414	0.03428	0.0001	0.01298	0.0000
4	0.01256	0.01272	0.0000	0.00305	0.0000
5	0.00462	0.00472	0.0000	0.00072	0.0000

Table 5.5: Values and estimates of $\psi(u)$: mixed exponential claims.

on matching three moments and the latter is based on matching only two. The notable exceptions are values for $u = 10,\ n = 1$ in Table 5.2. These are extreme cases since one claim is expected each year and the probability that this claim on its own exceeds the initial surplus is 0.0192, which is almost the same as the probability of ruin over 10 years in each case.

The standard errors of our estimates are almost identical for the two approximations to the within year probability of ruin. This is not surprising since the major source of randomness comes from the simulation of the aggregate annual claims and the same simulations are used for the two approximations.

Since $\psi_{TG}(u,n)$ generally produces more accurate values than $\psi_{BM}(u,n)$, with no significant difference in the standard errors, we will use the former approximation throughout the rest of this paper.

6 The premium as a function of the surplus

In some of the applications of our methodology in Section 7 we will determine the premium rate at the start of each year as a function of the surplus level. The higher the surplus, the lower will be the premium. More precisely, for $i \geq 1$ we write P_i, the premium rate to be charged in the $i-th$ year, as $h(u_{\tau_i})$, where h is some function which we will specify below and τ_i takes one of two values:

$$\tau_i = i - 1; \quad \text{or } \tau_i = \max(i-2, 0)$$

In the first case, P_i depends on the surplus at the end of the preceding year, *i.e.* at the current time. This is the most intuitively appealing case. However, it may not be reasonable to expect the insurer to adjust the premium rate instantaneously, as this case requires. The other case allows for this by determining the premium as a function of the level of surplus one year earlier. We will also show results for $\tau_i = 0$. In this case the premium is fixed throughout the term.

Given the surplus u_{τ_i}, we will determine P_i so that the probability of ultimate ruin, assuming the premium rate does not change, is approximately some pre-determined level, for example 0.005. We will use De Vylder's (1978) approximation to achieve this. For $k = 1, 2, 3$, let m_k denote $\mathrm{E}[Y_i^k]$ and let:

$$\tilde{a} = \frac{3m_2}{m_3}, \quad \tilde{\lambda} = \frac{9\lambda m_2^3}{2m_3^2}, \quad \text{and} \quad \tilde{P} = P_i - \lambda m_1 + \frac{\tilde{\lambda}}{\tilde{a}}$$

Then De Vylder's approximation to the probability of ultimate ruin given initial surplus u_{τ_i}, denoted $\psi_{DV}(u_{\tau_i})$, is given by:

$$\psi_{DV}(u_{\tau_i}) = \frac{\tilde{\lambda}}{\tilde{a}\tilde{P}} \exp\left\{ - \left(\tilde{a} - \frac{\tilde{\lambda}}{\tilde{P}} \right) u_{\tau_i} \right\} \tag{6.1}$$

Given a pre-determined value for $\psi_{DV}(u_{\tau_i})$, we can calculate numerically for any initial surplus the corresponding value of $\tilde{P}$ and hence P_i and hence the premium loading factor, ζ $(= P_i/(\lambda m_1) - 1)$.

Formula (6.1) does not give a closed form solution for P_i. Since we are going to have to calculate the premium for each year of each (of many) simulations, it is convenient to have a simple formula for ζ in terms of u_{τ_i}. We achieve this using Excel by calculating the values of u_{τ_i} for a range of values of the safety loading using formula (6.1) and then fitting a power function (using *Add trendline*), so that;

$$\zeta \approx Au^B$$

for some parameters A and B.

For small values of u_{τ_i} De Vylder's approximation can give uncomfortably large values for the premium loading factor. De Vylder (1978, page 118) says that for very small values of u 'the accuracy *(of his approximation)* is not so good'. For this reason we put an upper bound of 100% on the premium loading factor, so that:

$$h(u_{\tau_i}) = (1 + \min(Au_{\tau_i}^B, 1))\lambda m_1 \tag{6.2}$$

In our applications in Section 7 we will use two 'target' probabilities of ultimate ruin, 0.01 and 0.005, which we will denote T1 and T2, respectively, and three different individual claim size distributions: exponential, gamma and lognormal with moments as shown in Table 6.1.

Table 6.2 shows for each of the three distributions and for the two target probabilities of ultimate ruin the values of the parameters A and B to be used in formula (6.2).

	Exponential	Gamma	Lognormal
Mean	1	1	1
Variance	1	3	3
Skewness	2	3.46	10.39

Table 6.1: Mean, variance and skewness: Exponential, gamma and lognormal.

	$\psi(u) = 0.01$			$\psi(u) = 0.005$	
Distribution	A	B	Distribution	A	B
Exponential	12.26914	−1.22917	Exponential	15.38387	−1.24137
Gamma	33.33404	−1.25689	Gamma	42.79712	−1.27121
Lognormal	95.87145	−1.44538	Lognormal	141.02398	−1.47958

Table 6.2: Parameters for the power function for formula (6.2).

Formula (6.2) is an attempt to control the surplus process by adjusting the premium each year so that the probability of ultimate ruin has a given target value. This is necessarily a somewhat crude attempt since a two parameter function does not fully reflect the behavior of the premium loading, ζ, as the surplus varies. We can check the accuracy of this formula by calculating the (approximate) probability of ultimate ruin using formula (6.1) for selected values of the initial surplus, u, and the fitted premium loading, $\zeta(u)$ $(= Au^B)$. The results should be close to the target value for $\psi(u)$, either 0.01 or 0.005. Some results are shown in Tables 6.3 and 6.4 for some values of u of interest in our examples. The differences between the target and calculated probabilities of ultimate ruin arise from the inaccuracy of the fit of the two parameter power function (over a large range of values of u) and the sensitivity of $\psi(u)$ to the premium loading, ζ.

u	T1	T2
40	0.0037	0.0084
50	0.0043	0.0096
60	0.0049	0.0109
70	0.0056	0.0121
80	0.0063	0.0134
90	0.0070	0.0147

Table 6.3: Exponential: Values for $\psi(u)$ calculated using formula (6.1) and the safety loadings of Table 6.2.

u	T1 Gamma	T1 Lognormal	u	T2 Gamma	T2 Lognormal
120	0.0047	0.0034	80	0.0081	0,0064
130	0.0051	0.0038	90	0.0087	0,0068
140	0.0054	0.0041	100	0.0093	0,0073
150	0.0058	0.0045	110	0.0100	0,0079
160	0.0062	0.0049	120	0.0107	0,0085
170	0.0066	0.0054	130	0.0113	0,0093

Table 6.4: Gamma & Lognormal: Values for $\psi(u)$ calculated using formula (6.1) and the safety loadings of Table 6.2.

7 Applications

7.1 Scenarios

In our numerical examples, we will consider:

(i) a finite term of 10 years, which we consider to be a reasonable planning horizon,

(ii) two target probabilities of ultimate ruin: 0.005 (T1) and 0.01 (T2),

(iii) three methods for calculating the premium, all of them with the same target for the probability of ultimate ruin:

- P1: $\tau_i = 0$, so that the premium is fixed throughout the 10 years at the level which would achieve (approximately) the target probability of ultimate ruin,
- P2: $\tau_i = i-1$, so that the premium is adjusted at the start of each year according to the current level of the surplus, and,
- P3: $\tau_i = \max(i-2, 0)$, so that the premium is adjusted at the start of each year according to the level of the surplus one year ago.

(iv) two algorithms for calculating the Poisson parameter for the expected number of claims. These are:

- N1: $\lambda = 1000$ so that the Poisson parameter is constant from year to year.
- N2: λ varies from year to year between 800 and 1200. More precisely, let λ_{ij} denote the Poisson parameter in the $i-th$ year for the $j-th$ simulation. Then $\{\lambda_{ij}\}$ is a set of independent and identically distributed random variables, each with a uniform distribution on the interval [800, 1200]. In this case, the premium is calculated using the mean value of λ, so that:

$$P_i = h(u_{\tau_i}) = (1 + \min(Au_{\tau_i}^B, 1))\mathrm{E}[\lambda]m_1$$

In our examples the Poisson parameter is simulated for each of the 10 years for each of the 50 000 simulations and then these simulations of the aggregate annual claims are used for a given combination of T and P.

Scenario N1 is the classical model for claim numbers. However, Daykin et al. (1996) suggest that this scenario may not capture the full variability of the claim number process in practice. They say on page 329, 'each *(claim number)* process is a superimposition of trends, cycles and short–term fluctuations', and also that, 'It seems clear that business cycles are so common in general insurance, and their impact so profound, that any risk theory model which claims to describe real–life situations must permit the user to evaluate the impact of any cycles which may be present.' Figure 12.2.1 of Daykin et al. (1996) shows some examples of the variability of the claim ratio (claims/premiums) for general insurance. Our scenario N2 is an attempt to produce this variability through a variable Poisson parameter.

7.2 Exponential claim amounts

Tables 7.1 and 7.2 show results for exponentially distributed claim amounts separately for the two target probabilities of ruin, T1 and T2, and within each table for each combination of scenarios N and P.

	N1			N2		
	P1	P2	P3	P1	P2	P3
u	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$
40	0.00370	0.00418	0.00388	0.11270	0.27753	0.23432
50	0.00422	0.00496	0.00467	0.18125	0.31909	0.30875
60	0.00497	0.00543	0.00584	0.23619	0.34073	0.35653
70	0.00569	0.00532	0.00693	0.27984	0.34818	0.38332
80	0.00630	0.00473	0.00769	0.31357	0.34834	0.39867
90	0.00686	0.00389	0.00804	0.33766	0.34342	0.40581

Table 7.1: Exponential: Values of $\psi(u, 10)$, scenario T1.

We make the following comments about Tables 7.1 and 7.2:

(i) Standard deviations of our estimates of $\psi_{TG}(u, 10)$ (not shown) are all very small. For a given combination of scenarios T and N, the standard deviations increase as the initial surplus u increases and also increase in the order P1 → P2 → P3. For the combination T1/N1, the range of values is 3.42E-09 to 9.33E-08. The ranges for the other combinations are: 1.48E-06 to 4.43E-06 (T1/N2), 1.79E-08 to 2.19E-07 (T2/N1), 2.37E-06 to 4.56E-06 (T2/N2).

(ii) For the combination N1/P1, the values of $\psi_{TG}(u, 10)$ are very close to the values in Table 6.3, as we would expect them to be.

u	N1 P1 $\psi_{TG}(u,10)$	N1 P2 $\psi_{TG}(u,10)$	N1 P3 $\psi_{TG}(u,10)$	N2 P1 $\psi_{TG}(u,10)$	N2 P2 $\psi_{TG}(u,10)$	N2 P3 $\psi_{TG}(u,10)$
40	0.00848	0.01038	0.00942	0.17737	0.35190	0.31620
50	0.00976	0.01202	0.01177	0.24598	0.38497	0.38072
60	0.01116	0.01236	0.01419	0.29748	0.39903	0.41523
70	0.01247	0.01144	0.01606	0.33850	0.40168	0.43396
80	0.01394	0.00985	0.01715	0.36714	0.39799	0.44354
90	0.01532	0.00808	0.01736	0.38733	0.39043	0.44648

Table 7.2: Exponential: Values of $\psi(u,10)$, scenario T2.

(iii) For lower values of u we have the following ordering for ruin probabilities as functions of the premium scenario: P2 > P3 > P1. As u increases, the ordering switches to: P3 > P2 > P1, then to: P3 > P1 > P2, and eventually to: P1 > P3 > P2. We can see this in detail in Afonso (2007).

(iv) Varying the Poisson parameter, scenario N2, increases the probability of ruin considerably, as we would expect.

We can gain some insight into the causes of ruin, or at least end of year, as opposed to within year, ruin, by recording some information from those simulations which result in end of year ruin, i.e. for which $u(i) < 0$ for some i. Table 7.3 shows the following information for scenarios T1/N1 and T2/N2 and for a low and a high value of the initial surplus:

P The premium scenario.

Prop The proportion of the probability of ruin in Table 7.1 or 7.2 attributable to end of year ruin.

Avg i The mean of the year in which end of year ruin occurs.

Avg $u(i-1)$ The mean of the surplus at the start of the year of ruin.

Avg λ The mean Poisson parameter in the year of ruin. This is always 1 000 for N1.

Avg p_i The mean premium in the year of ruin.

Avg y_i The mean aggregate claims in the year of ruin.

r_{y_{i-1},y_i} The correlation between the aggregate claims in the year before ruin and the year of ruin.

All these items, except r_{y_{i-1},y_i}, are recorded from the 50 000 simulations used as the basis for Tables 7.1 and 7.2. The correlation coefficient, r_{y_{i-1},y_i}, has been calculated from a separate set of 200 000 simulations using data from simulations where end of year ruin occurs after the first year. Also, we used 20 000 simulations from the referred 50 000 only for the N2 case because of the size of the file.

	u	P	Prop	Avg i	Avg $u(i-1)$	Avg λ	Avg p_i	Avg y_i	r_{y_{i-1},y_i}
		1	0.024	1.0	50.0	1000.0	1119.7	1184.5	-
N1	50	2	0.081	3.0	98.0	1000.0	1065.6	1173.3	-0.47
		3	0.043	4.4	61.2	1000.0	1074.5	1166.5	-0.57
		1	0.254	2.0	55.6	1000.0	1057.7	1128.4	-0.58
N1	90	2	0.242	2.0	103.1	1000.0	1052.0	1166.5	-0.40
		3	0.266	2.9	58.0	1000.0	1051.3	1125.7	-0.78
		1	0.721	1.5	50.0	1162.9	1119.7	1207.6	-
N2	50	2	0.770	3.3	92.9	1163.0	1078.1	1209.0	-0.37
		3	0.798	3.7	72.1	1153.9	1072.5	1192.0	-0.21
		1	0.902	2.5	77.0	1151.7	1057.7	1184.3	-0.58
N2	90	2	0.836	2.9	104.1	1159.5	1055.6	1200.7	-0.28
		3	0.870	3.1	83.5	1149.6	1046.2	1181.9	-0.19

Table 7.3: Exponential: Statistical information for ruin cases, scenario T1.

These statistics show the following for scenarios N1/T1 and N2/T1 (results for T2 are similar):

(a) The proportion of the probability of ruin due to end of year ruin does not depend very much on the premium calculation scenario, but does depend on the initial surplus. This proportion will depend on the Poisson parameter: a smaller value for lambda, and hence fewer expected claims each year, will increase the proportion of the probability of ruin due to end of year ruin.

(b) The average aggregate claims in the year of ruin is significantly higher than the expected aggregate claims.

(c) The correlation between aggregate claims in the year before ruin, y_{i-1}, and y_i is negative for all premium scenarios.

(d) The average surplus at the start of the year of ruin does not depend to any great extent on the initial surplus. It is comparable for P1 and P3, but noticeably higher for P2.

(e) The average premium in the year of ruin is comparable for all three premium scenarios.

(f) The average number of years until ruin is higher for P3 than for P2 and almost always higher for P2 than for P1.

An additional point for scenario N2 is that:

(g) The average value of the Poisson parameter in the year of ruin, around 1 160, is near the upper end of its range.

It is not surprising that, however the premium is calculated, heavier than expected claims are associated with ruin (point (b)), or that, for N2, a large value for the Poisson parameter can lead to a large value for the aggregate claims (point (g)) and then to ruin.

Where the premium depends on the surplus at the start of the current year, scenario P2, the implication of points (b), (c) and (d) is that a major factor causing (end of year) ruin is a year of relatively light claims, and hence a lower premium in the following year, followed by a year of heavier than expected claims.

The 'ruin profile' statistics for P1 and P3 are very similar. This is not surprising since, typically, the average number of years until end of year ruin for P3 is four or less and for the first two years the premiums will be the same for these two scenarios.

More details can be found in Afonso (2007).

7.3 Gamma and lognormal claim amounts

Tables 7.4, 7.5, 7.6 and 7.7 show results for gamma and lognormally distributed claim amounts in the same format as Tables 7.1 and 7.2.

	N1			N2		
	P1	P2	P3	P1	P2	P3
u	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$
120	0.00493	0.00370	0.00595	0.14893	0.14487	0.20305
130	0.00527	0.00325	0.00617	0.16322	0.14204	0.20865
140	0.00558	0.00278	0.00626	0.17487	0.13771	0.21148
150	0.00591	0.00232	0.00621	0.18443	0.13242	0.21222
160	0.00624	0.00189	0.00605	0.19247	0.12664	0.21132
170	0.00660	0.00150	0.00578	0.19877	0.12064	0.20894

Table 7.4: Gamma: Values of $\psi(u,10)$, scenario T1.

	N1			N2		
	P1	P2	P3	P1	P2	P3
u	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$
80	0.00840	0.01015	0.01004	0.12681	0.19298	0.21022
90	0.00904	0.01013	0.01121	0.15168	0.20046	0.23212
100	0.00966	0.00965	0.01219	0.17381	0.20308	0.24694
110	0.01029	0.00885	0.01289	0.19262	0.20210	0.25681
120	0.01087	0.00790	0.01327	0.19262	0.20210	0.25681
130	0.01149	0.00691	0.01331	0.22132	0.19304	0.26531

Table 7.5: Gamma: Values of $\psi(u,10)$, scenario T2.

	N1			N2		
	P1	P2	P3	P1	P2	P3
u	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$
120	0.00342	0.00344	0.00451	0.09461	0.10225	0.14942
130	0.00370	0.00325	0.00488	0.11109	0.10321	0.16070
140	0.00401	0.00298	0.00518	0.12593	0.10245	0.16866
150	0.00441	0.00266	0.00540	0.13953	0.10026	0.17373
160	0.00489	0.00233	0.00551	0.15173	0.09717	0.17654
170	0.00542	0.00201	0.00553	0.16210	0.09349	0.17764

Table 7.6: Lognormal: Values of $\psi(u,10)$, scenario T1.

	N1			N2		
	P1	P2	P3	P1	P2	P3
u	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$	$\psi_{TG}(u,10)$
80	0.00678	0.00833	0.00776	0.06606	0.13088	0.13885
90	0.00720	0.00876	0.00878	0.08909	0.14188	0.16751
100	0.00773	0.00892	0.00993	0.11166	0.14912	0.18880
110	0.00834	0.00874	0.01095	0.13368	0.15283	0.20444
120	0.00906	0.00825	0.01177	0.15357	0.15344	0.21603
130	0.00985	0.00758	0.01238	0.17115	0.15155	0.22389

Table 7.7: Lognormal: Values of $\psi(u,10)$, scenario T2.

We make the following comments about Tables 7.4 to 7.7:

(i) Standard deviations of our estimates of $\psi_{TG}(u,10)$ (not shown) are all comparable to those for exponential claims.

(ii) For the combination N1/P1, the values of $\psi_{TG}(u,10)$ are close to the values in Table 6.3, though not as close as in the case of exponential claim amounts. This is to be expected since the values in Table 6.3 are based on De Vylder's approximation, which is exact for exponential claim amounts.

(iii) As for exponential claims, for lower values of u we have the following ordering for ruin probabilities as functions of the premium scenario: P2 > P3 > P1. As u increases, the ordering switches to: P3 > P2 > P1, then to P3 > P1 > P2, and eventually to: P1 > P3 > P2. We can see this in detail in Afonso (2007).

(iv) Varying the Poisson parameter, scenario N2, increases the probability of ruin considerably, as it did for exponential claims.

Tables 7.8 and 7.9 show statistics for end of year ruin for gamma and lognormal claims, respectively, in the same format as Table 7.3. These statistics lead us to the same conclusions about the major causes of ruin as those stated at the end of Section 7.2. Compared to the case with exponential claims, the increase in the average surplus at the start of the year of ruin is linked directly to the increase in the average aggregate claims, which, in turn, is linked to the greater variability of aggregate claims when individual claims have a gamma or lognormal distribution. See Table 6.1.

More details can be found in Afonso (2007).

	u	P	Prop	Avg i	Avg $u(i-1)$	Avg λ	Avg p_i	Avg y_i	r_{y_{i-1},y_i}
N1	120	1	0.211	1.4	99.8	1000.0	1097.4	1217.6	-0.65
		2	0.232	1.5	125.7	1000.0	1094.4	1239.4	-0.37
		3	0.229	2.4	102.6	1000.0	1084.2	1209.9	-0.72
N1	170	1	0.331	3.3	81.2	1000.0	1062.5	1169.7	-0.32
		2	0.253	2.1	164.8	1000.0	1067.4	1256.8	-0.44
		3	0.253	2.2	96.7	1000.0	1059.7	1184.5	-0.73
N2	120	1	0.728	2.2	92.8	1155.9	1097.4	1237.8	-0.39
		2	0.690	2.9	142.5	1163.9	1087.9	1269.9	-0.23
		3	0.722	3.5	104.7	1152.8	1075.2	1232.4	-0.47
N2	170	1	0.835	3.3	101.7	1147.0	1062.5	1218.1	-0.32
		2	0.717	3.1	165.7	1165.3	1072.2	1276.1	-0.24
		3	0.765	3.6	114.5	1148.9	1056.2	1224.6	-0.48

Table 7.8: Gamma: Statistical information for ruin cases, scenario T1.

	u	P	Prop	Avg i	Avg $u(i-1)$	Avg λ	Avg p_i	Avg y_i	r_{y_{i-1},y_i}
N1	120	1	0.134	1.3	106.3	1000.0	1118.3	1243.6	-0.73
		2	0.169	2.2	146.3	1000.0	1100.4	1268.6	-0.45
		3	0.182	3.1	109.0	1000.0	1089.4	1225.4	-0.75
N1	170	1	0.292	2.8	90.9	1000.0	1070.7	1186.1	-0.55
		2	0.259	2.3	178.7	1000.0	1070.5	1272.0	-0.44
		3	0.271	3.0	101.5	1000.0	1068.0	1197.1	-0.80
N2	120	1	0.691	1.9	96.1	1159.0	1118.3	1259.9	-0.47
		2	0.664	2.8	152.0	1163.2	1097.1	1287.4	-0.25
		3	0.695	3.7	107.5	1152.9	1083.8	1242.6	-0.51
N2	170	1	0.809	3.1	103.8	1148.5	1070.7	1228.9	-0.34
		2	0.708	2.9	175.3	1163.3	1075.0	1288.5	-0.25
		3	0.748	3.5	116.3	1149.6	1061.5	1231.2	-0.51

Table 7.9: Lognormal: Statistical information for ruin cases, scenario T1.

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