C^*-algebras of Singular Integral Operators with Shifts Having the Same Nonempty Set of Fixed Points

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To the memory of Professor G.S. Litvinchuk

Abstract. The C^*-subalgebra $\mathfrak{B}$ of $\mathcal{B}(L^2(\mathbb{T}))$ generated by all multiplication operators by slowly oscillating and piecewise continuous functions, by the Cauchy singular integral operator and by the range of a unitary representation of an amenable group of diffeomorphisms $g : \mathbb{T} \to \mathbb{T}$ with any nonempty set of common fixed points is studied. A symbol calculus for the C^*-algebra $\mathfrak{B}$ and a Fredholm criterion for its elements are obtained. For the C^*-algebra $\mathcal{A}$ composed by all functional operators in $\mathfrak{B}$, an invertibility criterion for its elements is also established. Both the C^*-algebras $\mathfrak{B}$ and $\mathcal{A}$ are investigated by using a generalization of the local-trajectory method for C^*-algebras associated with C^*-dynamical systems which is based on the notion of spectral measure.

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1. Introduction

The aim of this paper is to construct a symbol calculus and a Fredholm criterion for the nonlocal C^*-algebra $\mathfrak{B} := \text{alg}\,(\mathfrak{A}, U_G)$ generated by a C^*-algebra $\mathfrak{A}$, for which we know a symbol calculus, and by a group $U_G := \{U_g : g \in G\}$ of unitary operators U_g associated to an amenable (see [14]) discrete group G.

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Let $\mathcal{B}(L^2(\mathbb{T}))$ be the C^*-algebra of all bounded linear operators acting on the Lebesgue space $L^2(\mathbb{T})$ where $\mathbb{T}$ is the unit circle in $\mathbb{C}$ with the length measure and the usual anticlockwise orientation. Consider the C^*-algebra

$$\mathfrak{A} := \mathrm{alg}\,(PSO(\mathbb{T}), S_{\mathbb{T}}) \subset \mathcal{B}(L^2(\mathbb{T})) \tag{1.1}$$

generated by all multiplications operators by piecewise slowly oscillating functions, $PSO(\mathbb{T})$ (see definition in Section 2.1), and by the Cauchy singular integral operator $S_{\mathbb{T}}$ defined on $L^2(\mathbb{T})$ by

$$(S_{\mathbb{T}}\varphi)(t) := \lim_{\varepsilon\to 0} \frac{1}{\pi i} \int_{\mathbb{T}\setminus\mathbb{T}(t,\varepsilon)} \frac{\varphi(\tau)}{\tau - t}\, d\tau, \quad t \in \mathbb{T}, \quad \mathbb{T}(t,\varepsilon) = \{\tau \in \mathbb{T} : |\tau - t| < \varepsilon\}.$$

Let G be an amenable discrete group of orientation-preserving diffeomorphisms of $\mathbb{T}$ onto itself, with the group operation given by $(gh)(t) = h(g(t))$ for $g, h \in G,\ t \in \mathbb{T}$. We will denote by e the identity map on $\mathbb{T}$. To each $g \in G$ we assign the unitary shift operator U_g defined on the space $L^2(\mathbb{T})$ by

$$(U_g\varphi)(t) := |g'(t)|^{1/2}\varphi(g(t)), \quad \text{for} \ \ t \in \mathbb{T}. \tag{1.2}$$

The present paper continues investigations in [5]. In contrast to [5], where the C^*-algebra $\mathfrak{B} = \mathrm{alg}\,(\mathfrak{A}, U_G)$ was investigated under the condition that all shifts $g \in G \setminus \{e\}$ have the same finite set of fixed points, we now suppose only that the shifts $g \in G \setminus \{e\}$ have the same nonempty set Λ of fixed points. In particular, Λ can have limit points, be a Cantor set of measure $\mathrm{mes}\,\Lambda \geq 0$, have a nonempty interior (see [18], [21]). This brings new difficulties in studying functional and singular integral operators with shifts. Obviously, we have the partitions

$$\mathbb{T} = \mathbb{T}_{arc} \cup \Lambda^\circ \cup \partial\Lambda, \quad \partial\Lambda = \mathrm{Is}\,\Lambda \cup (\Lambda' \setminus \Lambda^\circ) \tag{1.3}$$

where $\mathbb{T}_{arc} := \mathbb{T} \setminus \Lambda,\ \Lambda^\circ := \mathrm{Int}\,\Lambda$ is the interior of $\Lambda,\ \partial\Lambda$ is the boundary of Λ, $\mathrm{Is}\,\Lambda$ is the (at most countable) set of all isolated points of Λ and Λ' is the set of all limit points of Λ. The sets $\mathbb{T}_{arc}$ and Λ° are at most countable unions of open arcs. If Λ° is nonempty, then the action of the group of shifts G on $\mathbb{T}$ is not topologically free, in contrast to [5]. According to [1], the group G acts topologically freely on the contour $\mathbb{T}$ if for each finite set $F \subset G$ and each open arc $\gamma \subset \mathbb{T}$ there exists a point $t \in \gamma$ such that the points $g(t)$ for $g \in F$ are pairwise distinct.

To study the C^*-algebra $\mathfrak{B} = \mathrm{alg}\,(\mathfrak{A}, U_G)$, we apply the local-trajectory method and its generalization based on the notion of spectral measure and developed for the case when the action of an amenable discrete group G on the maximal ideal space of a central C^*-algebra $\mathcal{Z} \subset \mathfrak{A}$ is not topologically free (see [15], [17], [5]). This C^*-algebra approach, in contrast to the methods of [1]–[2], is related to the Allan-Douglas local principle (see, e.g., [10]) and is essentially different of those applied for studying singular integral operators with shifts and discontinuous coefficients on Banach spaces (see [19]–[21] and the references therein). C^*-algebras of singular integral operators with discontinuous coefficients and amenable discrete groups of shifts acting freely was studied in [16], [7]. The presence of fixed points of shifts qualitatively changes symbol calculi for such C^*-algebras (see [16], [21], [3, Section 53], [4], [5]).

Studying singular integral operators with shifts was initiated and always supported by G.S. Litvinchuk (see [22], [21]).

To study the C^*-algebra $\mathfrak{B}$, we also need to investigate the invertibility in the C^*-algebra of functional operators

$$\mathcal{A} := \mathrm{alg}\,(PSO(\mathbb{T}), U_G) \subset \mathcal{B}(L^2(\mathbb{T})) \tag{1.4}$$

generated by all multiplication operators aI with $a \in PSO(\mathbb{T})$ and by all shift operators U_g $(g \in G)$ given by (1.2). Since the action of the group G on $\mathbb{T}$ in general is not topologically free, studying the C^*-algebra $\mathcal{A}$ is more difficult than in [5]. To investigate the C^*-algebra $\mathcal{A}$ and the quotient C^*-algebra $\mathfrak{B}^\pi := \mathfrak{B}/\mathcal{K}$, where $\mathcal{K} := \mathcal{K}(L^2(\mathbb{T}))$ is the ideal of all compact operators in $\mathcal{B}(L^2(\mathbb{T}))$, we decompose these C^*-algebras in orthogonal sums of operator C^*-algebras obtained with the help of spectral projections related to G-invariant subsets of the maximal ideal space of appropriate commutative C^*-subalgebras of $\mathcal{A}$ and $\mathfrak{B}^\pi$, respectively. Studying the invertibility in these operator C^*-algebras leads to an invertibility criterion for the functional operators $A \in \mathcal{A}$ in Section 3 and to a Fredholm criterion for the operators $B \in \mathfrak{B}$ in Section 5.

The paper is organized as follows. Section 2 is devoted to important requisites to subsequent sections. In Subsection 2.1 we describe the C^*-algebra $PSO(\mathbb{T})$ of piecewise slowly oscillating function and its maximal ideal space $M(PSO(\mathbb{T}))$. In Subsection 2.2 we present a symbol calculus for the C^*-algebra $\mathfrak{A}$ and define a central C^*-subalgebra $\mathcal{Z}^\pi$ of the C^*-algebra $\mathfrak{A}^\pi := \mathfrak{A}/\mathcal{K}$. In Subsections 2.3–2.4 we recall the local-trajectory method and its generalization based on the notion of spectral measure.

In Section 3 we study the invertibility in the C^*-algebra $\mathcal{A}$ (see (1.4)) of the functional operators with shifts having an arbitrary nonempty set of common fixed points. Making use of the local-trajectory method and its generalization, we establish an invertibility criterion for the operators $A \in \mathcal{A}$. To this end we study the invertibility of the operators $\chi^\circ A$, $\chi_{arc} A$ and $\chi_* A$, respectively, on the spaces $L^2(\Lambda^\circ)$, $L^2(\mathbb{T}^*_{arc})$ and $L^2(\Lambda_*)$ where χ°, χ_{arc} and χ_* are the characteristic functions of the sets Λ°, $\mathbb{T}^*_{arc} := \mathbb{T} \setminus \overline{\Lambda^\circ}$ and $\Lambda_* := \overline{\Lambda^\circ} \setminus \Lambda^\circ$.

Sections 4 and 5 are devoted to studying the Fredholmness in C^*-algebra $\mathfrak{B}$ or, equivalently, the invertibility in the C^*-algebra $\mathfrak{B}^\pi = \mathfrak{B}/\mathcal{K}$ considered as $\mathfrak{B}^\pi = \mathrm{alg}\,(\mathfrak{A}^\pi, U_G^\pi)$, the C^*-algebra generated by all the cosets A^π $(A \in \mathfrak{A})$ and U_g^π $(g \in G)$, where $B^\pi := B + \mathcal{K}$ for every $B \in \mathcal{B}(L^2(\mathbb{T}))$. In Section 4, using the spectral measure associated to the central C^*-subalgebra $\mathcal{Z}^\pi$ of $\mathfrak{A}^\pi$ and to a faithful representation φ of $\mathfrak{B}^\pi$ in a Hilbert space, and considering an appropriate G-invariant decomposition of the maximal ideal space $M(\mathcal{Z}^\pi)$ of $\mathcal{Z}^\pi$, we decompose the C^*-algebra $\varphi(\mathfrak{B}^\pi)$ into the direct sum of some operator C^*-algebras $\mathfrak{B}_{arc}$, $\mathfrak{B}_{\mathrm{Is}}$, $\mathfrak{B}^\circ$ and $\mathfrak{B}^\infty$ such that any operator $B \in \mathfrak{B}$ is Fredholm if and only if all the "projections" of the coset B^π are invertible in these C^*-algebras. As a result, we obtain an abstract Fredholm criterion for the operators $B \in \mathfrak{B}$ in terms of the invertibility of corresponding operators in the C^*-algebras $\mathfrak{B}_{arc}$, $\mathfrak{B}_{\mathrm{Is}}$, $\mathfrak{B}^\circ$ and $\mathfrak{B}^\infty$.

In Section 5 we establish explicit invertibility criteria for the C^*-algebras $\mathfrak{B}_{arc}$, $\mathfrak{B}_{\rm Is}$ and $\mathfrak{B}^\circ$ and show that the invertibility in the C^*-algebras $\mathfrak{B}_{arc}$ and $\mathfrak{B}^\circ$ implies the invertibility in the C^*-algebra $\mathfrak{B}^\infty$, which does not have influence on the Fredholm criterion for the C^*-algebra $\mathfrak{B}$. The invertibility conditions and the methods applied for these C^*-algebras are qualitatively different. Using the symbol calculus for the C^*-algebra $\mathfrak{A}$ and the local-trajectory method, we get in Subsection 5.1 an invertibility criterion for the operators in the C^*-algebra $\mathfrak{B}_{arc}$ associated to the set $\mathbb{T}_{arc} = \mathbb{T} \setminus \Lambda$. In Subsection 5.2, applying [5, Section 9], we obtain an invertibility criterion for the operators in the C^*-algebra $\mathfrak{B}_{\rm Is}$ related to the set ${\rm Is}\,\Lambda$ of all isolated points of Λ. The invertibility in the C^*-algebra $\mathfrak{B}^\circ$ associated to the set Λ' of all limit points of Λ is investigated in Section 5.3 on the basis of local-trajectory method. In Subsection 5.4 devoted to the C^*-algebra $\mathfrak{B}^\infty$ we establish a general form of operators in the C^*-algebra $\mathfrak{B}$ and show that the invertibility in the C^*-algebras $\mathfrak{B}_{arc}$ and $\mathfrak{B}^\circ$ implies the invertibility in the C^*-algebra $\mathfrak{B}^\infty$. Finally, in Subsection 5.5, collecting the results of Subsections 5.1–5.4, we construct a symbol calculus for the C^*-algebra $\mathfrak{B}$ and obtain an explicit Fredholm criterion for the operators $B \in \mathfrak{B}$.

2. Preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{K}(\mathcal{H})$ be the ideal of all compact operators on $\mathcal{H}$. If $S, T \in \mathcal{B}(\mathcal{H})$ and $S - T \in \mathcal{K}(\mathcal{H})$, we will use the notation $S \simeq T$. For an operator $A \in \mathcal{B}(\mathcal{H})$ we denote by $A^\pi := A + \mathcal{K}(\mathcal{H})$ the coset of A in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \cong \mathcal{B}$ if they are isometrically *-isomorphic.

2.1. Function spaces

Let $C(\mathbb{T})$, $PC(\mathbb{T})$ and $SO(\mathbb{T})$ denote the C^*-subalgebras of $L^\infty(\mathbb{T})$ consisting, respectively, of the functions continuous on $\mathbb{T}$, of the functions which have one-sided limits at each point of $\mathbb{T}$, and of the functions slowly oscillating at each point of $\mathbb{T}$. A function $f \in L^\infty(\mathbb{T})$ is called *slowly oscillating at a point* $\lambda \in \mathbb{T}$ (cf. [4], [5]) if

$$\lim_{\varepsilon \to 0} \operatorname{ess\,sup} \Big\{ |f(z_1) - f(z_2)| : \, z_1, z_2 \in \mathbb{T}_\varepsilon(\lambda) \Big\} = 0,$$

where $\mathbb{T}_\varepsilon(\lambda) := \big\{ z \in \mathbb{T} : \, \varepsilon/2 \le |z - \lambda| \le \varepsilon \big\}$. Let $PSO(\mathbb{T}) := {\rm alg}(SO(\mathbb{T}), PC(\mathbb{T}))$ be the C^*-subalgebra of $L^\infty(\mathbb{T})$ generated by the C^*-algebras $SO(\mathbb{T})$ and $PC(\mathbb{T})$.

Given a commutative unital C^*-algebra $\mathcal{A}$, we denote by $M(\mathcal{A})$ the maximal ideal space of $\mathcal{A}$. As is well known, $M(C(\mathbb{T})) = \mathbb{T}$ and $M(PC(\mathbb{T})) = \mathbb{T} \times \{0, 1\}$, respectively, where the points $t \in \mathbb{T}$ are identified with the evaluation functionals δ_t given by $\delta_t(f) = f(t)$ for $f \in C(\mathbb{T})$, and the pairs $(t, 0)$ and $(t, 1)$ are the multiplicative linear functionals defined for $a \in PC(\mathbb{T})$ by $(t, 0)a = a(t - 0)$ and $(t, 1)a = a(t + 0)$, where $a(t - 0)$ and $a(t + 0)$ are the left and right one-sided limits of a at the point $t \in \mathbb{T}$. It is also known (see [4]) that

$$M(SO(\mathbb{T})) = \bigcup_{t\in\mathbb{T}} M_t(SO(\mathbb{T})), \quad M(PSO(\mathbb{T})) = \bigcup_{\xi\in M(SO(\mathbb{T}))} M_\xi(PSO(\mathbb{T})), \tag{2.1}$$

where the corresponding fibers are given for $t \in \mathbb{T}$ and $\xi \in M(SO(\mathbb{T}))$ by

$$M_t(SO(\mathbb{T})) = \big\{\xi \in M(SO(\mathbb{T})) : \xi|_{C(\mathbb{T})} = t\big\},$$
$$M_\xi(PSO(\mathbb{T})) = \big\{y \in M(PSO(\mathbb{T})) : y|_{SO(\mathbb{T})} = \xi\big\}.$$

The fibers $M_\xi(PSO(\mathbb{T}))$ for $\xi \in M(SO(\mathbb{T}))$ can be characterized as follows.

Theorem 2.1. [4, Theorem 4.6] *If $\xi \in M_t(SO(\mathbb{T}))$ with $t \in \mathbb{T}$, then*

$$M_\xi(PSO(\mathbb{T})) = \{(\xi,0), (\xi,1)\}, \tag{2.2}$$

where, for $\mu \in \{0,1\}$, $(\xi,\mu)|_{SO(\mathbb{T})} = \xi$, $(\xi,\mu)|_{C(\mathbb{T})} = t$, $(\xi,\mu)|_{PC(\mathbb{T})} = (t,\mu)$.

By (2.1) and (2.2), $M(PSO(\mathbb{T})) = M(SO(\mathbb{T}))\times\{0,1\}$. The Gelfand topology on $M(PSO(\mathbb{T}))$ can be described as follows. If $\xi \in M_t(SO(\mathbb{T}))$ $(t \in \mathbb{T})$, a base of neighborhoods for $(\xi,\mu) \in M(PSO(\mathbb{T}))$ consists of all open sets of the form

$$U_{(\xi,\mu)} = \begin{cases} \big(U_{\xi,t}\times\{0\}\big)\cup\big(U^-_{\xi,t}\times\{0,1\}\big) & \text{if } \mu=0,\\ \big(U_{\xi,t}\times\{1\}\big)\cup\big(U^+_{\xi,t}\times\{0,1\}\big) & \text{if } \mu=1,\end{cases} \tag{2.3}$$

where $U_{\xi,t} = U_\xi \cap M_t(SO(\mathbb{T}))$, U_ξ is an open neighborhood of ξ in $M(SO(\mathbb{T}))$, and $U^-_{\xi,t}$, $U^+_{\xi,t}$ consist of all $\zeta \in U_\xi$ such that $\tau = \zeta|_{C(\mathbb{T})}$ belong, respectively, to the sets $(-t,t) := \{z \in \mathbb{T} : -\pi < \arg(z/t) < 0\}$ and $(t,-t) := \{z \in \mathbb{T} : 0 < \arg(z/t) < \pi\}$.

2.2. The C^*-algebra $\mathfrak{A}$

Consider the C^*-algebra $\mathfrak{A} = \mathrm{alg}\,(PSO(\mathbb{T}), S_\mathbb{T})$ of singular integral operators on $L^2(\mathbb{T})$ with $PSO(\mathbb{T})$ coefficients. Let $\dot{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ and $\overline{\mathbb{R}} = [-\infty,+\infty]$ be the one and two-point compactifications of the real line $\mathbb{R} = (-\infty,+\infty)$. Define the set

$$\mathfrak{M} := M(SO(\mathbb{T}))\times\overline{\mathbb{R}} \tag{2.4}$$

and equip it with the discrete topology. Let $BC(\mathfrak{M},\mathbb{C}^{2\times2})$ be the C^*-algebra of all bounded continuous matrix functions $f : \mathfrak{M} \to \mathbb{C}^{2\times2}$. According to [9, Section 7] and [5, Theorem 5.1] we have the following symbol calculus for the C^*-algebra $\mathfrak{A}$.

Theorem 2.2. *The map* $\mathrm{Sym} : \{aI : a \in PSO(\mathbb{T})\}\cup\{S_\mathbb{T}\} \to BC(\mathfrak{M},\mathbb{C}^{2\times2})$ *given by the matrix functions*

$$(\mathrm{Sym}\,aI)(\xi,x) := \begin{pmatrix} a(\xi,1) & 0\\ 0 & a(\xi,0)\end{pmatrix}, \quad (\mathrm{Sym}\,S_\mathbb{T})(\xi,x) := \begin{pmatrix} u(x) & -v(x)\\ v(x) & -u(x)\end{pmatrix}, \tag{2.5}$$

where $a(\xi,\mu)$ is the Gelfand transform of a at the point $(\xi,\mu) \in M(PSO(\mathbb{T}))$ and $u(x) := \tanh(\pi x)$, $v(x) := -i/\cosh(\pi x)$ for $x \in \overline{\mathbb{R}}$, extends to a C^*-algebra homomorphism $\mathrm{Sym} : \mathfrak{A} \to BC(\mathfrak{M},\mathbb{C}^{2\times2})$ *whose kernel consists of all compact operators on $L^2(\mathbb{T})$. An operator $A \in \mathfrak{A}$ is Fredholm on the space $L^2(\mathbb{T})$ if and only if*

$$\det\big((\mathrm{Sym}\,A)(\xi,x)\big) \neq 0 \quad \textit{for all} \quad (\xi,x) \in \mathfrak{M}.$$

To each point $t \in \mathbb{T}$ we assign the operator $V_t \in \mathcal{B}(L^2(\mathbb{T}))$ given by

$$(V_t\varphi)(z) := \frac{\chi_t^+(z)}{\pi i}\int_{\mathbb{T}}\frac{\varphi(y)\chi_t^+(y)}{y+z-2t}\,dy - \frac{\chi_t^-(z)}{\pi i}\int_{\mathbb{T}}\frac{\varphi(y)\chi_t^-(y)}{y+z-2t}\,dy, \quad \text{for } z \in \mathbb{T}, \tag{2.6}$$

where $\chi_t^\pm$ are the characteristic functions of the arcs $\gamma_t^\pm$ such that $\gamma_t := \gamma_t^+ \cup \gamma_t^-$ is a neighborhood of t, $\gamma_t^+ \cap \gamma_t^- = \{t\}$, γ_t is separated from $-t$, and $\gamma_t^+ \cap (-t, t) = \varnothing$, $\gamma_t^- \cap (t, -t) = \varnothing$. By [5, Lemma 5.3], every operator V_t ($t \in \mathbb{T}$) with a fixed singularity at t belongs to the C^*-algebra $\mathfrak{A}$. Let $\mathcal{P}$ denote the set of all polynomials $\sum_{k=0}^n a_k u^k$ with $a_k \in \mathbb{C}$ and $n = 0, 1, \dots$. Consider the C^*-algebra

$$\mathcal{Z} := \operatorname{alg}\big\{aI,\ H_{P,t}:\ a \in SO(\mathbb{T}),\ P \in \mathcal{P},\ t \in \mathbb{T}\big\} \subset \mathcal{B}(L^2(\mathbb{T})) \tag{2.7}$$

generated by all multiplication operators aI with $a \in SO(\mathbb{T})$ and by all operators

$$H_{P,t} := P(\chi_t^+ S_{\mathbb{T}} \chi_t^+ I - \chi_t^- S_{\mathbb{T}} \chi_t^- I)V_t \in \mathfrak{A} \quad (P \in \mathcal{P},\ t \in \mathbb{T}). \tag{2.8}$$

By [5, (4.11) and (6.3)], for all $a \in PSO(\mathbb{T})$, $b \in SO(\mathbb{T})$, $P \in \mathcal{P}$ and $t \in \mathbb{T}$, we get

$$aH_{P,t} \simeq H_{P,t}aI, \quad S_{\mathbb{T}}H_{P,t} \simeq H_{P,t}S_{\mathbb{T}}, \quad bS_{\mathbb{T}} \simeq S_{\mathbb{T}}bI. \tag{2.9}$$

Thus, $\mathcal{Z}^\pi := (\mathcal{Z} + \mathcal{K})/\mathcal{K}$ is a central C^*-subalgebra of the C^*-algebra $\mathfrak{A}^\pi := \mathfrak{A}/\mathcal{K}$, where $\mathcal{K} = \mathcal{K}(L^2(\mathbb{T}))$ is the ideal of all compact operators on $L^2(\mathbb{T})$.

Consider now the compact Hausdorff space

$$\dot{\mathfrak{M}} := M(SO(\mathbb{T})) \times \dot{\mathbb{R}}, \tag{2.10}$$

where $M(SO(\mathbb{T}))$ is equipped with the Gelfand topology, and the neighborhood base of the topology on $\dot{\mathfrak{M}}$ consists of the open sets of the form

$$W_{(\xi,x)} = \begin{cases} U_{\xi,t} \times (x-\varepsilon, x+\varepsilon) & \text{if } (\xi,x) \in M(SO(\mathbb{T})) \times \mathbb{R}, \\ \big((U_\xi \setminus U_{\xi,t}) \times \dot{\mathbb{R}}\big) \cup \big(U_{\xi,t} \times (\dot{\mathbb{R}} \setminus [-\varepsilon,\varepsilon])\big) & \text{if } (\xi,x) \in M(SO(\mathbb{T})) \times \{\infty\}, \end{cases} \tag{2.11}$$

where $(\xi, x) \in \dot{\mathfrak{M}}$, $\varepsilon > 0$, U_ξ is an open neighborhood of a point $\xi \in M(SO(\mathbb{T}))$, and $U_{\xi,t} = U_\xi \cap M_t(SO(\mathbb{T}))$ with $t = \xi|_{C(\mathbb{T})} \in \mathbb{T}$.

Theorem 2.3. [5, Theorem 6.3] *The maximal ideal space $M(\mathcal{Z}^\pi)$ of the C^*-algebra $\mathcal{Z}^\pi$ coincides with the compact $\dot{\mathfrak{M}}$ given by* (2.10), *and the Gelfand transform of $\mathcal{Z}^\pi$ is defined by $\Gamma : \mathcal{Z}^\pi \to C(\dot{\mathfrak{M}})$, $Z^\pi \mapsto z(\cdot,\cdot)$, where $z(\xi,x) = (\operatorname{Sym} Z)_{11}(\xi,x)$ for $(\xi,x) \in M(SO(\mathbb{T})) \times \mathbb{R}$ and $z(\xi,\infty) = (\operatorname{Sym} Z)_{11}(\xi,\pm\infty)$ for $\xi \in M(SO(\mathbb{T}))$.*

2.3. The local-trajectory method

Let us recall the statements of the local-trajectory method (see [15], [17]).

Let $\mathfrak{A}$ be a unital C^*-algebra and let $\mathcal{Z}$ be a central C^*-subalgebra of $\mathfrak{A}$ with the same identity I. For a discrete group G with unit e, let $U : g \mapsto U_g$ be a unitary representation of G, that is, a homomorphism of the group G onto a group $U_G = \{U_g : g \in G\}$ of unitary elements, where $U_{g_1g_2} = U_{g_1}U_{g_2}$ and $U_e = I$. We denote by $\mathfrak{B} := \operatorname{alg}(\mathfrak{A}, U_G)$ the minimal C^*-algebra containing the C^*-algebra $\mathfrak{A}$ and the group U_G. Assume that

(A1) *for every* $g \in G$ *the mappings* $\alpha_g : a \mapsto U_g\, a\, U_g^*$ *are* **-automorphisms of the* C^**-algebras* $\mathfrak{A}$ *and* $\mathcal{Z}$*.*

According to (A1), $\mathfrak{B}$ is the closure of the set $\mathfrak{B}^0$ consisting of all elements of the form $b = \sum a_g U_g$ where $a_g \in \mathfrak{A}$ and g runs through finite subsets of G.

Since the C^*-algebra $\mathcal{Z}$ is commutative, it follows that $\mathcal{Z} \cong C(M(\mathcal{Z}))$ where $C(M(\mathcal{Z}))$ is the C^*-algebra of all continuous complex-valued functions on the maximal ideal space $M(\mathcal{Z})$ of $\mathcal{Z}$. Furthermore, in view of (A1), each *-automorphism $\alpha_g : \mathcal{Z} \to \mathcal{Z}$ induces a homeomorphism $\beta_g : M(\mathcal{Z}) \to M(\mathcal{Z})$ given by the rule

$$z[\beta_g(m)] = [\alpha_g(z)](m), \quad z \in \mathcal{Z},\ m \in M(\mathcal{Z}),\ g \in G, \tag{2.12}$$

where $z(\cdot) \in C(M(\mathcal{Z}))$ is the Gelfand transform of the element $z \in \mathcal{Z}$. The set $G(m) := \{\beta_g(m) : g \in G\}$ is called the G-*orbit* of a point $m \in M(\mathcal{Z})$.

In what follows we also assume that

(A2) G *is an amenable discrete group.*

Let us equip the set $\mathcal{P}_{\mathfrak{A}}$ of all pure states (see, e.g., [11]) of the C^*-algebra $\mathfrak{A}$ with the induced weak* topology. For each maximal ideal $m \in M(\mathcal{Z})$ of the central C^*-algebra $\mathcal{Z} \subset \mathfrak{A}$, let J_m be the closed two-sided ideal of $\mathfrak{A}$ generated by m. By [8, Lemma 4.1], if $\mu \in \mathcal{P}_{\mathfrak{A}}$, then $\operatorname{Ker}\mu \supset J_m$ where $m := \mathcal{Z} \cap \operatorname{Ker}\mu \in M(\mathcal{Z})$. Furthermore, assume that

(A3) *there is a set* $M_0 \subset M(\mathcal{Z})$ *such that for every finite set* $G_0 \subset G$ *and for every nonempty open set* $W \subset \mathcal{P}_{\mathfrak{A}}$ *there exists a state* $\nu \in W$ *such that* $\beta_g(m_\nu) \neq m_\nu$ *for all* $g \in G_0 \setminus \{e\}$*, where the point* $m_\nu = \mathcal{Z} \cap \operatorname{Ker}\nu$ *belongs to the* G*-orbit* $G(M_0) := \{\beta_g(m) : g \in G,\ m \in M_0\}$ *of the set* M_0*.*

For every $m \in M(\mathcal{Z})$, let $\widetilde{\pi}_m$ be an isometric representation

$$\widetilde{\pi}_m : \mathfrak{A}/J_m \to \mathcal{B}(\mathcal{H}_m) \tag{2.13}$$

of the quotient C^*-algebra $\mathfrak{A}/J_m$ in a Hilbert space $\mathcal{H}_m$ (see [12, Theorem 2.6.1]). Let Ω be the set of all G-orbits of the points $m \in M_0$ with $M_0 \subset M(\mathcal{Z})$ taken from (A3), let $\mathcal{H}_\omega = \mathcal{H}_m$ where $m = m_\omega$ is an arbitrary fixed point of an orbit $\omega \in \Omega$, and let $l^2(G, \mathcal{H}_\omega)$ be the Hilbert space of all functions $f : G \mapsto \mathcal{H}_\omega$ such that $f(g) \neq 0$ for at most countable set of points $g \in G$ and $\sum \|f(g)\|^2_{\mathcal{H}_\omega} < \infty$. For every $\omega \in \Omega$, we consider the representation $\pi_\omega : \mathfrak{B} \to \mathcal{B}(l^2(G, \mathcal{H}_\omega))$ defined for all $a \in \mathfrak{A}$, all $g, h \in G$ and all $f \in l^2(G, \mathcal{H}_\omega)$ by

$$[\pi_\omega(a)f](g) = \widetilde{\pi}_{m_\omega}(\alpha_g(a) + J_{m_\omega})f(g), \quad [\pi_\omega(U_h)f](g) = f(gh). \tag{2.14}$$

Consider the representation $\pi = \bigoplus_{\omega \in \Omega} \pi_\omega$ of the C^*-algebra $\mathfrak{B}$ in the Hilbert space $\bigoplus_{\omega \in \Omega} l^2(G, H_\omega)$. If (A1)–(A3) hold, then π is a *-isomorphism of the C^*-algebra $\mathfrak{A}$ onto the C^*-algebra $\pi(\mathfrak{A})$ (see [17, Theorem 4.1] and [5, Theorem 3.1]), which implies the following due to the inverse closedness of C^*-algebras.

Theorem 2.4. *If assumptions* (A1)–(A3) *are satisfied, then an element* $b \in \mathfrak{B}$ *is invertible in the* C^**-algebra* $\mathfrak{B}$ *if and only if for every orbit* $\omega \in \Omega$ *the operator* $\pi_\omega(b)$ *is invertible on the space* $l^2(G, \mathcal{H}_\omega)$ *and, in the case of infinite* Ω,

$$\sup\left\{\|(\pi_\omega(b))^{-1}\| : \omega \in \Omega\right\} < \infty.$$

2.4. A generalization of the local-trajectory method based on spectral measures

Now we consider a generalization of the local-trajectory method for the case when condition (A3) is not fulfilled. Such generalization, based on the notion of spectral measures, was developed in [17] and [5].

Consider the C^*-algebra $\mathfrak{B} = \operatorname{alg}(\mathfrak{A}, U_G)$ under the only condition (A1) of the local-trajectory method for the C^*-algebras $\mathfrak{A}$ and $\mathcal{Z} \subset \mathfrak{A}$. Let $\mathfrak{R}(M(\mathcal{Z}))$ denote the σ-algebra of all Borel subsets of $M(\mathcal{Z})$, and let

$$\mathfrak{R}_G(M(\mathcal{Z})) = \big\{\Delta \in \mathfrak{R}(M(\mathcal{Z})) : \beta_g(\Delta) = \Delta \ \text{ for all } g \in G\big\},$$

where the homeomorphisms β_g are given by (2.12).

As is known (see, e.g., [12, Theorem 2.6.1]), there exists an isometric representation $\pi : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ of the C^*-algebra $\mathfrak{B}$ in a Hilbert space $\mathcal{H}$. According to [24, § 17], for the representation $\pi|_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{B}(\mathcal{H})$ of a unital commutative C^*-algebra $\mathcal{Z}$, there is a unique spectral measure $P_\pi(\cdot)$ which commutes with all operators in the C^*-algebra $\pi(\mathcal{Z})$ and in its commutant $\pi(\mathcal{Z})'$, and such that

$$\pi(z) = \int_{M(\mathcal{Z})} z(m)\, dP_\pi(m) \quad \text{for all } \ z \in \mathcal{Z},$$

where $z(\cdot) \in C(M(\mathcal{Z}))$ is the Gelfand transform of an element $z \in \mathcal{Z}$.

Since (A1) holds, it follows from [17, Lemma 4.6]) that

$$\pi(b)P_\pi(\Delta) = P_\pi(\Delta)\pi(b) \quad \text{for all } \ b \in \mathfrak{B} \ \text{ and all } \ \Delta \in \mathfrak{R}_G(M(\mathcal{Z})). \tag{2.15}$$

Given $\Delta \in \mathfrak{R}_G(M(\mathcal{Z}))$ such that $P_\pi(\Delta) \neq 0$, we define the Hilbert space $\mathcal{H}_\Delta := P_\pi(\Delta)\mathcal{H}$ and introduce the following three C^*-subalgebras of $\mathcal{B}(\mathcal{H}_\Delta)$:

$$\mathfrak{B}_\Delta := \{P_\pi(\Delta)\pi(b) : \, b \in \mathfrak{B}\},$$

$$\mathfrak{A}_\Delta := \{P_\pi(\Delta)\pi(a) : \, a \in \mathfrak{A}\} \quad \text{and} \quad \mathcal{Z}_\Delta := \{P_\pi(\Delta)\pi(z) : \, z \in \mathcal{Z}\}.$$

Since $\mathcal{Z}$ is a central C^*-subalgebra of $\mathfrak{A}$, from (2.15) it follows that $\mathcal{Z}_\Delta$ is a central C^*-subalgebra of $\mathfrak{A}_\Delta$, where $\mathfrak{A}_\Delta \subset \mathfrak{B}_\Delta$.

For each Borel set $\Delta \in \mathfrak{R}(M(\mathcal{Z}))$, let $\operatorname{Int}\Delta$ and $\overline{\Delta}$ denote the interior and the closure of Δ, respectively, and let $\widetilde{\Delta}$ be the closed subset of $\overline{\Delta}$ given by

$$\widetilde{\Delta} = \big\{m \in M(\mathcal{Z}) : \, P_\pi(W_m \cap \Delta) \neq 0 \text{ for every open neighborhood } W_m \text{ of m}\big\}.$$

Lemma 2.5. [17, Lemmas 5.1–5.2] *If* $\Delta \in \mathfrak{R}(M(\mathcal{Z}))$ *and* $\operatorname{Int}\Delta \neq \varnothing$*, then:* (i) $P_\pi(\Delta) \neq 0$*;* (ii) $\mathcal{Z}_\Delta \cong C(\widetilde{\Delta})$*;* (iii) $\overline{\operatorname{Int}(\Delta)} \subset \widetilde{\Delta} \subset \overline{\Delta}$.

Fix $\Delta \in \mathfrak{R}_G(M(\mathcal{Z}))$. For every $g \in G$, we consider the unitary operator $U_{g,\Delta} := P_\pi(\Delta)\pi(U_g)$ on $\mathcal{H}_\Delta$. As condition (A1) holds, the mappings

$$\alpha_{g,\Delta} : P_\pi(\Delta)\pi(a) \mapsto U_{g,\Delta}P_\pi(\Delta)\pi(a)U_{g,\Delta}^* = P_\pi(\Delta)\pi(U_g a U_g^*)P_\pi(\Delta) \quad (g \in G)$$

are *-automorphisms of the C^*-algebras $\mathcal{Z}_\Delta$ and $\mathfrak{A}_\Delta$. Since $\mathcal{Z}_\Delta \cong C(\widetilde{\Delta})$ where $\widetilde{\Delta} \in \mathfrak{R}_G(M(\mathcal{Z}))$ and the isomorphism is given by $P_\pi(\Delta)\pi(z) \mapsto z(\cdot)|_{\widetilde{\Delta}}$, it follows that each *-automorphism $\alpha_{g,\Delta}$ induces on $\widetilde{\Delta}$ the homeomorphism $\beta_{g,\Delta} := \beta_g|_{\widetilde{\Delta}}$, where β_g is defined by (2.12).

Below we need the following decomposition result.

Proposition 2.6. [5, Proposition 3.3] *Let $\pi : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ be an isometric representation of the C^*-algebra $\mathfrak{B} = \operatorname{alg}(\mathfrak{A}, U_G)$ in a Hilbert space $\mathcal{H}$ and let $\{\Delta_i\}$ be an at most countable family of disjoint Borel sets in $\mathfrak{R}_G(M(\mathcal{Z}))$ such that $P_\pi(\Delta_i) \neq 0$ for all i and $P_\pi(M(\mathcal{Z}) \setminus \bigcup_i \Delta_i) = 0$. If condition* (A1) *is fulfilled, then the mapping*

$$\Theta : \mathfrak{B} \to \bigoplus\nolimits_i \mathfrak{B}_{\Delta_i}, \quad b \mapsto \bigoplus\nolimits_i P_\pi(\Delta_i)\pi(b)$$

is an isometric C^*-algebra homomorphism from the C^*-algebra $\mathfrak{B}$ into the C^*-algebra $\widetilde{\mathfrak{B}} := \bigoplus_i \mathfrak{B}_{\Delta_i}$. Then an element $b \in \mathfrak{B}$ is invertible if and only if for each i the operator $P_\pi(\Delta_i)\pi(b)$ is invertible on the Hilbert space $\mathcal{H}_{\Delta_i}$ and

$$\sup_i \|(P_\pi(\Delta_i)\pi(b))^{-1}\| < \infty \quad \textit{in case } \{\Delta_i\} \textit{ is countable.}$$

Proposition 2.6 allows us to study the C^*-algebras $\mathfrak{B}_{\Delta_i}$ separately. If some of these algebras satisfy conditions (A1)–(A3), we can apply Theorem 2.4 (for more general situations see [17, Section 5]).

Finally we enunciate a crucial result for studying the C^*-algebras $\mathfrak{B}_{\Delta_i}$ when Δ_i is an open subset of $M(\mathcal{Z})$.

Lemma 2.7. [5, Lemma 3.5] *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{Z}$ a central C^*-subalgebra of $\mathcal{A}$ with the same unit. Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$. Given an open set $\Delta \subset M(\mathcal{Z})$, let $\mathcal{Z}(\Delta)$ denote the subset of $\mathcal{Z}$ composed by the elements $z \in \mathcal{Z}$ whose Gelfand transforms $z(\cdot)$ are real functions in $C(M(\mathcal{Z}))$ with support in $\overline{\Delta}$ and values in the segment* $[0, 1]$. *Then*

$$\|P_\pi(\Delta)\pi(a)\|_{\mathcal{B}(\mathcal{H})} = \sup_{z \in \mathcal{Z}(\Delta)} \|\pi(az)\|_{\mathcal{B}(\mathcal{H})} \quad \textit{for all } \ a \in \mathcal{A}.$$

3. Invertibility in the C^*-algebra $\mathcal{A}$

Using the generalization of the local-trajectory method related to Proposition 2.6, we devote this section to studying the invertibility of functional operators in the C^*-algebra $\mathcal{A} = \operatorname{alg}(PSO(\mathbb{T}), U_G) \subset \mathcal{B}(L^2(\mathbb{T}))$.

Let $\widetilde{\mathcal{Z}} := \{aI : a \in PSO(\mathbb{T})\}$. As $\widetilde{\mathcal{Z}} \cong PSO(\mathbb{T})$, we get $M(\widetilde{\mathcal{Z}}) = M(PSO(\mathbb{T}))$, where $M(PSO(\mathbb{T})) = M(SO(\mathbb{T})) \times \{0, 1\}$ is the Hausdorff compact space with the topology (2.3). For each $a \in PSO(\mathbb{T})$ and each $g \in G$, from [4, Lemma 4.2] it follows that $a \circ g \in PSO(\mathbb{T})$. Consequently, the mapping $\widetilde{\alpha}_g : aI \mapsto U_g a U_g^{-1} = (a \circ g)I$ is a *-automorphism of the commutative C^*-algebra $\widetilde{\mathcal{Z}}$. Since G is an amenable group, conditions (A1)–(A2) of Subsection 2.3 are satisfied for the C^*-algebra $\mathcal{A}$.

For every $g \in G$, the *-automorphism $\widetilde{\alpha}_g$ induces the homeomorphism

$$\widetilde{\beta}_g : \ M(PSO(\mathbb{T})) \to M(PSO(\mathbb{T})), \quad (\xi, \mu) \mapsto (g(\xi), \mu), \tag{3.1}$$

where we save notation g for the homeomorphism $\xi \mapsto g(\xi)$ on $M(SO(\mathbb{T}))$ given by

$$a(g(\xi)) = (a \circ g)(\xi) \quad \text{for all } \ a \in SO(\mathbb{T}) \ \text{ and } \ \xi \in M(SO(\mathbb{T})) \tag{3.2}$$

(as usual, $a(\xi) := \xi(a)$). If the ideal $\xi \in M(SO(\mathbb{T}))$ belongs to the fiber $M_t(SO(\mathbb{T}))$, then $g(\xi) \in M_{g(t)}(SO(\mathbb{T}))$. Moreover, [5, Theorem 6.4] implies the following.

Lemma 3.1. *If Λ is the set of common fixed points of all shifts $g \in G \setminus \{e\}$, then*

$$\mathcal{M}_\Lambda := \bigcup_{t\in\Lambda} M_t(SO(\mathbb{T})) \times \{0,1\} \subset M(PSO(\mathbb{T})) \tag{3.3}$$

is the set of common fixed points of all homeomorphisms $\widetilde{\beta}_g$ ($g \in G \setminus \{e\}$).

Since the C^*-algebra $\widetilde{\mathcal{Z}}$ is commutative and hence its maximal ideal space $M(PSO(\mathbb{T}))$ coincides with the set $\mathcal{P}_{\widetilde{\mathcal{Z}}}$ of its pure states, choosing $\widetilde{\mathcal{Z}}$ as the central C^*-subalgebra of itself, we can rewrite condition (A3) of Subsection 2.3 in the form:

(A3') *there is a set $M_0 \subset M(PSO(\mathbb{T}))$ such that for every finite set $G_0 \subset G$ and for every nonempty open set $W \subset M(PSO(\mathbb{T}))$ there exists an ideal $m_0 \in W \cap G(M_0)$ such that $\widetilde{\beta}_g(m_0) \neq m_0$ for all $g \in G_0 \setminus \{e\}$.*

Obviously, if $\Lambda^\circ = \operatorname{Int}\Lambda \neq \varnothing$, then condition (A3') is not fulfilled.

Let $\Lambda_\pm$ be the set of all $t \in \partial\Lambda$ that are limit points of the sets $\gamma_t^\pm \cap \Lambda^\circ$, respectively, where γ_t^+ (γ_t^-) is a right (left) semi-neighborhood of t on $\mathbb{T}$. Clearly, $\overline{\Lambda^\circ} = \Lambda^\circ \cup \Lambda_+ \cup \Lambda_-$ and $\Lambda_r^\circ \subset \Lambda_-$, $\Lambda_l^\circ \subset \Lambda_+$, where Λ_l° and Λ_r° denotes, respectively, the at most countable set of the initial and final points of all open arcs which compose the set Λ°.

Let χ°, χ_{arc} and χ_* be the characteristic functions of the sets Λ°, $\mathbb{T}^*_{arc} := \mathbb{T} \setminus \overline{\Lambda^\circ} \supset \mathbb{T}_{arc}$ and $\Lambda_* := \Lambda_+ \cup \Lambda_-$, respectively. Since $\Lambda^\circ \cup \mathbb{T}^*_{arc} \cup \Lambda_*$ is a G-invariant partition of $\mathbb{T}$, we immediately obtain the following decomposition result.

Lemma 3.2. *An operator $A \in \mathcal{A}$ is invertible on $L^2(\mathbb{T})$ if and only if:*

(i) *the operator $\chi^\circ A$ is invertible on the Hilbert space $L^2(\Lambda^\circ)$,*

(ii) *the operator $\chi_{arc} A$ is invertible on the Hilbert space $L^2(\mathbb{T}^*_{arc})$,*

(iii) *in case* $\operatorname{mes}\Lambda_* > 0$, *the operator $\chi_* A$ is invertible on the Hilbert space $L^2(\Lambda_*)$.*

Consider now the following subsets of $M(\widetilde{\mathcal{Z}})$:

$$\mathcal{M}^\circ := \bigcup_{t\in\Lambda^\circ} M_t(SO(\mathbb{T})) \times \{0,1\}, \quad \mathcal{M}^*_{arc} := \bigcup_{t\in\mathbb{T}^*_{arc}} M_t(SO(\mathbb{T})) \times \{0,1\}. \tag{3.4}$$

The sets $\mathcal{M}^\circ$ and $\mathcal{M}^*_{arc}$ are invariant under the action of all homeomorphisms $\widetilde{\beta}_g$ ($g \in G$). Since these sets are open, from Lemma 2.5(iii) it follows that

$$\begin{aligned}\widetilde{\mathcal{M}}^\circ := \widetilde{\mathcal{M}^\circ} = \overline{\mathcal{M}^\circ} &= \Big(\bigcup_{t\in\Lambda^\circ} M_t(SO(\mathbb{T})) \times \{0,1\}\Big)\\ &\cup \Big(\bigcup_{t\in\Lambda_l^\circ} M_t(SO(\mathbb{T})) \times \{1\}\Big) \cup \Big(\bigcup_{t\in\Lambda_r^\circ} M_t(SO(\mathbb{T})) \times \{0\}\Big)\\ &\cup \Big(\bigcup_{t\in\Lambda_+\setminus\Lambda_l^\circ} M^\circ_{t,+}) \times \{1\}\Big) \cup \Big(\bigcup_{t\in\Lambda_-\setminus\Lambda_r^\circ} M^\circ_{t,-}(SO(\mathbb{T})) \times \{0\}\Big),\end{aligned} \tag{3.5}$$

$$\begin{aligned}\widetilde{\mathcal{M}}^*_{arc} := \widetilde{\mathcal{M}^*_{arc}} = \overline{\mathcal{M}^*_{arc}} &= \Big(\bigcup_{t\in\mathbb{T}^*_{arc}} M_t(SO(\mathbb{T})) \times \{0,1\}\Big)\\ &\cup \Big(\bigcup_{t\in\Lambda_l^\circ\setminus\Lambda_-} M_t(SO(\mathbb{T})) \times \{0\}\Big) \cup \Big(\bigcup_{t\in\Lambda_r^\circ\setminus\Lambda_+} M_t(SO(\mathbb{T})) \times \{1\}\Big)\\ &\cup \Big(\bigcup_{t\in\Lambda_+\setminus\Lambda_l^\circ} M^{arc}_{t,+}) \times \{1\}\Big) \cup \Big(\bigcup_{t\in\Lambda_-\setminus\Lambda_r^\circ} M^{arc}_{t,-}(SO(\mathbb{T})) \times \{0\}\Big).\end{aligned} \tag{3.6}$$

where $M^\circ_{t,+}$ (resp. $M^{arc}_{t,+}$) for $t \in \Lambda_+ \setminus \Lambda^\circ_l$ denotes the closed set of all $\xi \in M(SO(\mathbb{T}))$ which are limits of nets δ_{t_α} where $t_\alpha \to t$ and $t_\alpha \in \gamma_t^+ \cap \Lambda^\circ$ (resp. $t_\alpha \in \gamma_t^+ \cap \mathbb{T}_{arc}$); $M^\circ_{t,-}$ (resp. $M^{arc}_{t,-}$) for $t \in \Lambda_- \setminus \Lambda^\circ_r$ is the closed set of all $\xi \in M(SO(\mathbb{T}))$ which are limits of nets δ_{t_α} where $t_\alpha \to t$ and $t_\alpha \in \gamma_t^- \cap \Lambda^\circ$ (resp. $t_\alpha \in \gamma_t^- \cap \mathbb{T}_{arc}$). Note that the limits of nets δ_{t_α}, with $t_\alpha \to t$ and $t_\alpha \in \gamma_t^\pm \cap (\partial\Lambda \setminus \{t\})$, belong to $M^\circ_{t,\pm} \cap M^{arc}_{t,\pm}$.

Let $\mathfrak{R}(M(\widetilde{\mathcal{Z}}))$ be the σ-algebra of all Borel subsets of $M(PSO(\mathbb{T}))$ and let $P_I : \mathfrak{R}(M(\widetilde{\mathcal{Z}})) \to \mathcal{B}(L^2(\mathbb{T}))$ be the spectral measure associated to the identity representation I of the C^*-algebra $\widetilde{\mathcal{Z}}$ in the Hilbert space $L^2(\mathbb{T})$. Then we get

$$P_I(\mathcal{M}^\circ) = \widetilde{\chi}^\circ I, \quad P_I(\mathcal{M}^*_{arc}) := \widetilde{\chi}_{arc} I,$$

where $\widetilde{\chi}^\circ$, $\widetilde{\chi}_{arc}$ are the characteristic functions of the sets $\mathcal{M}^\circ, \mathcal{M}^*_{arc} \in \mathfrak{R}_G(M(\widetilde{\mathcal{Z}}))$. With the sets $\mathcal{M}^\circ$ and $\mathcal{M}^*_{arc}$ given by (3.4) we associate the C^*-algebras

$$\mathcal{A}^\circ := \operatorname{alg}\big\{\chi^\circ aI, \chi^\circ U_g : a \in PSO(\mathbb{T}),\ g \in G\big\} \subset \mathcal{B}(L^2(\Lambda^\circ)), \tag{3.7}$$

$$\mathcal{A}_{arc} := \operatorname{alg}\big\{\chi_{arc}\, aI, \chi_{arc} U_g : a \in PSO(\mathbb{T}),\ g \in G\big\} \subset \mathcal{B}(L^2(\mathbb{T}^*_{arc})). \tag{3.8}$$

Let us study the invertibility in these C^*-algebras.

Since Λ° consists of fixed points of all shifts $g \in G$, then $\chi^\circ U_g$ can be identified with the identity operator on $L^2(\Lambda^\circ)$. Thus, if $A = \sum_{g\in F} a_g U_g \in \mathcal{A}$ where F is a finite set of G, then $\chi^\circ A = \chi^\circ \sum a_g I$, whence $\big\|\chi^\circ \sum a_g\big\| \le \|A\|$. Hence the map

$$\sum\nolimits_{g\in F} a_g U_g \mapsto \chi^\circ \sum\nolimits_{g\in F} a_g \tag{3.9}$$

extends by continuity to a C^*-algebra homomorphism ν° of $\mathcal{A}$ onto the C^*-algebra $\chi^\circ PSO(\mathbb{T})$ (as $\widetilde{\mathcal{M}}^\circ$ is a closed subset of $M(\widetilde{\mathcal{Z}})$ by (3.5), every function $\nu^\circ(A)$ extends to a function $a \in PSO(\mathbb{T})$). Consequently, we deduce from (3.7) that

$$\mathcal{A}^\circ = \{\chi^\circ aI : a \in PSO(\mathbb{T})\} \cong \widetilde{\chi}^\circ C(M(\widetilde{\mathcal{Z}})). \tag{3.10}$$

By (3.10) and Lemma 2.5(ii), we obtain $\mathcal{A}^\circ \cong C(\widetilde{\mathcal{M}}^\circ)$. Thus, due to (3.9), for each functional operator $A \in \mathcal{A}$ there exists a function $a \in PSO(\mathbb{T})$ such that $\chi^\circ A = \nu^\circ(A) I = \chi^\circ aI$. If we denote by $\widehat{A}$ the restriction of the Gelfand transform of the function a to the set $\widetilde{\mathcal{M}}^\circ$, then $\widehat{A}(\xi,\mu) = a(\xi,\mu)$ for all $(\xi,\mu) \in \widetilde{\mathcal{M}}^\circ$. In particular, for $A = \sum_{g\in F} a_g U_g$, we get $\widehat{A}(\xi,\mu) = \sum_{g\in F} a_g(\xi,\mu)$ for $(\xi,\mu) \in \widetilde{\mathcal{M}}^\circ$. With the previous notation, the mapping

$$\Gamma^\circ : \mathcal{A}^\circ \to C(\widetilde{\mathcal{M}}^\circ),\ \chi^\circ A \mapsto \widehat{A} \quad (A \in \mathcal{A}) \tag{3.11}$$

is the C^*-algebra isomorphism of the C^*-algebras $\mathcal{A}^\circ$ and $C(\widetilde{\mathcal{M}}^\circ)$, which implies the following invertibility criterion for $\mathcal{A}^\circ$.

Theorem 3.3. *For each functional operator $A \in \mathcal{A}$, the operator $A^\circ := \chi^\circ A \in \mathcal{A}^\circ$ is invertible on the space $L^2(\Lambda^\circ)$ if and only if $\widehat{A}(\xi,\mu) \ne 0$ for all $(\xi,\mu) \in \widetilde{\mathcal{M}}^\circ$.*

Using the local-trajectory method, we establish now an invertibility criterion for the operators in the C^*-algebra $\mathcal{A}_{arc}$. Consider the commutative C^*-subalgebra $\mathcal{Z}_{arc} := \{\chi_{arc}\, aI : a \in PSO(\mathbb{T})\}$ of $\mathcal{A}_{arc}$. From Lemma 2.5(ii) and (3.6) it follows that $\mathcal{Z}_{arc} \cong C(\widetilde{\mathcal{M}}_{arc})$. It is clear that $\mathcal{A}_{arc} = \mathrm{alg}(\mathcal{Z}_{arc}, \widetilde{U}_{arc}(G))$, the C^*-algebra generated by $\mathcal{Z}_{arc}$ and $\widetilde{U}_{arc}(G) := \{\widetilde{U}_{g,arc} := \chi_{arc} U_g : \ g \in G\}$. All the mappings

$$\widetilde{\alpha}_{g,arc} : \chi_{arc}\, aI \mapsto \widetilde{U}_{g,arc}(\chi_{arc}\, a)\widetilde{U}^*_{g,arc} = \chi_{arc}(a \circ g)I \quad (g \in G)$$

are *-automorphisms of the C^*-algebra $\mathcal{Z}_{arc}$ that induce on $\widetilde{\mathcal{M}}^*_{arc}$ the homeomorphisms $\widetilde{\beta}_{g,arc}$ being restrictions on $\widetilde{\mathcal{M}}^*_{arc}$ of the homeomorphisms $\widetilde{\beta}_g$ given by (3.1).

In view of (3.3) and (3.6), $\mathcal{M}_\Lambda \cap \widetilde{\mathcal{M}}^*_{arc}$ is the set of fixed points for all homeomorphisms $\widetilde{\beta}_{g,arc}$ $(g \neq e)$. Setting

$$\mathcal{M}_{arc} := \bigcup_{t \in \mathbb{T}_{arc}} M_t(SO(\mathbb{T})) \times \{0, 1\}, \tag{3.12}$$

we infer from (2.3), (3.6) and (3.12) that $\widetilde{\mathcal{M}_{arc}} = \widetilde{\mathcal{M}}^*_{arc}$. Hence, in view of the topologically free action of the group G on $\mathbb{T}\backslash\Lambda^\circ$ and since $g(\xi) \in M_{g(t)}(SO(\mathbb{T}))$ for $\xi \in M_t(SO(\mathbb{T}))$, we conclude from (3.6) that condition (A3') for $\widetilde{\mathcal{M}}^*_{arc}$ holds with $M_0 := \mathcal{M}_{arc}$. Thus, conditions (A1)–(A3) for the C^*-algebra $\mathcal{A}_{arc}$ are satisfied.

For each operator $A \in \mathcal{A}$, let $A_{arc} := \chi_{arc} A \in \mathcal{A}_{arc}$. With every maximal ideal $(\xi, \mu) \in \mathcal{M}_{arc}$ we associate the representation

$$\Pi_{(\xi,\mu)} : \mathcal{A}_{arc} \to \mathcal{B}(l^2(G)), \quad A_{arc} \mapsto A_{(\xi,\mu)}, \tag{3.13}$$

given for the operators $A_{arc} = \sum_{g \in F} \chi_{arc}\, a_g U_g$ with finite sets $F \subset G$ by

$$(A_{(\xi,\mu)} f)(h) = \sum_{g \in F} [(a_g \circ h)(\xi, \mu)] f(hg) \quad (h \in G, \ f \in l^2(G)). \tag{3.14}$$

Let $\mathcal{O}_{arc}$ be a subset of $\mathbb{T}_{arc}$ containing exactly one point in each G-orbit defined by the action of the group G on $\mathbb{T}_{arc}$, and consider the set

$$\mathfrak{R}_{arc} := \big\{(\xi, \mu) \in M(PSO(\mathbb{T})) : \ \xi \in M_\tau(SO(\mathbb{T})), \ \tau \in \mathcal{O}_{arc}, \ \mu = 0, 1\big\}. \tag{3.15}$$

The set $\mathfrak{R}_{arc}$ contains exactly one point in each G-orbit defined by the action of the group G on $\mathcal{M}_{arc}$, by means of the homeomorphisms $\widetilde{\beta}_{g,arc}$ $(g \in G)$.

Theorem 3.4. *For each functional operator $A \in \mathcal{A}$, the operator $A_{arc} := \chi_{arc} A$ is invertible on the space $L^2(\mathbb{T}^*_{arc})$ if and only if for all $(\xi, \mu) \in \mathfrak{R}_{arc}$ the operators $A_{(\xi,\mu)}$ are invertible on the space $l^2(G)$ and*

$$\sup_{(\xi,\mu) \in \mathfrak{R}_{arc}} \big\|(A_{(\xi,\mu)})^{-1}\big\| < \infty. \tag{3.16}$$

Proof. With each functional $(\xi, \mu) \in \mathcal{M}_{arc}$ we associate the maximal ideal $J_{(\xi,\mu)} := \{\chi_{arc}\, aI : a \in PSO(\mathbb{T}), a(\xi, \mu) = 0\}$ of $\mathcal{Z}_{arc}$. Since $\mathcal{Z}_{arc}$ is a commutative C^*-algebra, the mapping

$$\widetilde{\Pi}_{(\xi,\mu)} : \mathcal{Z}_{arc}/J_{(\xi,\mu)} \to \mathbb{C}, \quad \chi_{arc}\, aI + J_{(\xi,\mu)} \mapsto a(\xi, \mu),$$

is an isometric representation of the C^*-algebra $\mathcal{Z}_{arc}/J_{(\xi,\mu)}$ in $\mathbb{C}$. Following (2.13)–(2.14), for all $(\xi, \mu) \in \mathcal{M}_{arc}$, we construct the representations $\Pi_{(\xi,\mu)}$ of the C^*-algebra $\mathcal{A}_{arc}$ in the Hilbert space $l^2(G)$ by formulas (3.13) and (3.14). Since the C^*-algebra $\mathcal{A}_{arc}$ given by (3.8) satisfies conditions (A1)–(A3) of the local-trajectory method, Theorem 2.4 immediately implies the statement of the theorem. □

Let us study the invertibility of the operator $A_* := \chi_* A$ on the space $L^2(\Lambda_*)$. To this end we need first to construct limit operators (see, e.g., [6] and [25]) associated with operators $A \in \mathcal{A}$ and points $t \in \Lambda_*$. Assume, for example, that $1 \in \Lambda_*$. Representing each diffeomorphism $g \in G$ in the form $g(e^{ix}) = e^{i\widetilde{g}(x)}$ for $x \in [0, 2\pi]$ where $\widetilde{g}$ is an orientation-preserving diffeomorphism of $[0, 2\pi]$ onto itself, we conclude that $\widetilde{g}(0) = 0$, and $\widetilde{g}'(0) = g'(1) = 1$. Hence $\widetilde{g}(x) = x + \varepsilon(x)x$ where $\varepsilon(x) \to 0$ as $x \to 0$. For all $k > 0$, we define the unitary shift operators $V_k \in \mathcal{B}(L^2(\mathbb{T}))$ by $(V_k f)(e^{ix}) = [\eta_k'(x)]^{1/2} f(e^{i\eta_k(x)})$ for $x \in [0, 2\pi]$ where $\eta_k(x) = 2\pi kx/[2\pi + (k-1)x]$. By direct computation, we obtain

$$\big(\eta_k^{-1} \circ \widetilde{g} \circ \eta_k\big)(x) = \frac{2\pi x(1 + \varepsilon[\eta_k(x)])}{2\pi + (1-k)x\varepsilon[\eta_k(x)]}, \tag{3.17}$$

$$\big(\eta_k^{-1} \circ \widetilde{g} \circ \eta_k\big)'(x) = \frac{(2\pi)^2 \widetilde{g}'[\eta_k(x)]}{(2\pi + (1-k)x\varepsilon[\eta_k(x)])^2}. \tag{3.18}$$

Fix $x_0 \in (0, 2\pi)$. Since $\varepsilon[\eta_k(x)] \to 0$ and $\widetilde{g}'[\eta_k(x)] \to 1$ as $k \to 0$ uniformly with respect to $x \in [0, x_0]$, we infer from (3.17) and (3.18) that for any $x_0 \in (0, 2\pi)$,

$$\lim_{k\to 0} \big(\eta_k^{-1} \circ \widetilde{g} \circ \eta_k\big)(x) = x, \quad \lim_{k\to 0} \big(\eta_k^{-1} \circ \widetilde{g} \circ \eta_k\big)'(x) = 1$$

uniformly for $x \in [0, x_0]$. Hence, we infer from the relation

$$\big(V_k U_g V_k^{-1} f\big)(e^{ix}) = \big[\big(\eta_k^{-1} \circ \widetilde{g} \circ \eta_k\big)'(x)\big]^{1/2} f\big(e^{i(\eta_k^{-1} \circ \widetilde{g} \circ \eta_k)(x)}\big) \quad (x \in [0, 2\pi]),$$

that $\text{s-lim}_{k\to 0} V_k U_g V_k^{-1} = I$ for all $g \in G$, which in its turn implies for finite sets $F \subset G$ and $\widehat{\eta}_k(t) = \exp[i\eta_k(-i \ln t)]$ that

$$\operatorname*{s-lim}_{k\to 0} V_k \Big(\sum\nolimits_{g\in F} a_g U_g\Big) V_k^{-1} = \operatorname*{s-lim}_{k\to 0} \Big(\sum\nolimits_{g\in F} (a_g \circ \widehat{\eta}_k) I\Big), \tag{3.19}$$

if the limit on the right exists. Taking now any $\xi \in M_1(SO(\mathbb{T}))$, by analogy with [6, Proposition 4.2], we can choose a positive sequence $\{k_n\}$ such that $k_n \to 0$ as $n \to \infty$ and

$$\operatorname*{s-lim}_{n\to\infty} \Big(\sum\nolimits_{g\in F} (a_g \circ \widehat{\eta}_{k_n}) I\Big) = \sum\nolimits_{g\in F} a_g(\xi, 1) I,$$

which implies due to (3.19) that

$$\operatorname*{s-lim}_{n\to\infty} V_{k_n} \Big(\sum\nolimits_{g\in F} a_g U_g\Big) V_{k_n}^{-1} = \sum\nolimits_{g\in F} a_g(\xi, 1) I. \tag{3.20}$$

By (3.20), for every $\xi \in M_1(SO(\mathbb{T}))$, the mapping $\big(\sum_{g\in F} a_g U_g\big) \mapsto \sum_{g\in F} a_g(\xi, 1)$ extends to a C^*-algebra homomorphism $\nu_{\xi,1} : \mathcal{A} \to \mathbb{C},\ A \mapsto \widehat{A}(\xi, 1) := \nu_{\xi,1}(A)$ where the notation $\widehat{A}$ is consistent with that for A°. Analogously, there exists a sequence $\{k_n\}$ such that $k_n \to \infty$ as $n \to \infty$ and

$$\operatorname*{s-lim}_{n\to\infty} V_{k_n} \Big(\sum\nolimits_{g\in F} a_g U_g\Big) V_{k_n}^{-1} = \sum\nolimits_{g\in F} a_g(\xi, 0) I,$$

which leads to the C^*-algebra homomorphism $\nu_{\xi,0} : \mathcal{A} \to \mathbb{C}$, $A \mapsto \widehat{A}(\xi, 0)$. Thus, we have the C^*-algebra homomorphisms $\nu_{\xi,\mu} : A \mapsto \widehat{A}(\xi, \mu)$ for all (ξ, μ) in the set

$$\mathcal{M}_* := \bigcup_{t\in\Lambda_*} M_t(SO(\mathbb{T})) \times \{0, 1\} \subset M(PSO(\mathbb{T})). \tag{3.21}$$

Since C^*-algebra homomorphisms $h : \mathcal{A} \to \mathcal{B}$ send invertible elements of a C^*-algebra $\mathcal{A}$ to invertible elements of a C^*-algebra $\mathcal{B}$, we at once obtain the following.

Lemma 3.5. *If a functional operator $A \in \mathcal{A}$ is invertible on the space $L^2(\mathbb{T})$, then $\widehat{A}(\xi, \mu) \neq 0$ for all $(\xi, \mu) \in \mathcal{M}_*$, where $\mathcal{M}_*$ is given by* (3.21).

Lemma 3.6. *For any functional operator $A \in \mathcal{A}$, if $\widehat{A}(\xi, \mu) \neq 0$ for all $(\xi, \mu) \in \mathcal{M}_*$, then the operator $A_* := \chi_* A$ is invertible on the space $L^2(\Lambda_*)$.*

Proof. Since $\mathcal{M}_*$ is a closed subset of $M(PSO(T))$, it follows from the lemma condition that the function $\widehat{A} : \mathcal{M}_* \to \mathbb{C}$ is continuous and invertible. Further, for each polynomial functional operator $A = \sum_{g\in F} a_g U_g \in \mathcal{A}$ with a finite set $F \subset G$, we infer that $\chi_* A = \chi_* \sum a_g I$. The latter equality extends by continuity to a C^*-algebra isomorphism $\nu_* : \chi_* \mathcal{A} \to \chi_* PSO(\mathbb{T})$, $\chi_* A = \chi_* aI$ where the Gelfand transform of $a \in PSO(\mathbb{T})$ is obtained by an extension of $\widehat{A} \in C(\mathcal{M}_*)$ to a function continuous on $M(PSO(\mathbb{T}))$. Since $(\chi_* a)(\xi, \mu) = \widehat{A}(\xi, \mu)$ for all $(\xi, \mu) \in \mathcal{M}_*$ and the function $\widehat{A} \in C(\mathcal{M}_*)$ is invertible, we conclude that the operator $A_* := \chi_* A$ is invertible on the space $L^2(\Lambda_*)$. □

Note that the conditions of Lemma 3.6 in general are not necessary for the invertibility of the operator A_*.

Combining Theorems 3.3, 3.4 and Lemmas 3.2, 3.5, 3.6, we get the following invertibility criterion for the functional operators in the C^*-algebra $\mathcal{A}$.

Theorem 3.7. *An operator $A \in \mathcal{A}$ is invertible on the space $L^2(\mathbb{T})$ if and only if*

(i) *$\widehat{A}(\xi, \mu) \neq 0$ for all $(\xi, \mu) \in \mathcal{M}_{\overline{\Lambda^\circ}} := \bigcup_{t\in\overline{\Lambda^\circ}} M_t(SO(\mathbb{T})) \times \{0, 1\}$;*

(ii) *for all $(\xi, \mu) \in \mathfrak{R}_{arc}$, the operators $A_{(\xi,\mu)}$ are invertible on the space $l^2(G)$ and* (3.16) *holds.*

Proof. By Lemma 3.6 and Theorem 3.3, condition (i) implies the invertibility of the operators $\chi_* A$ and $\chi^\circ A$, respectively. Condition (ii) and Theorem 3.4 guarantees the invertibility of the operator $\chi_{arc} A$. Thus, by Lemma 3.2, the operator A is invertible on the space $L^2(\mathbb{T})$. Conversely, if A is invertible, conditions (i) and (ii) follow from Lemmas 3.2, 3.5 and Theorems 3.3–3.4. □

4. Abstract Fredholm criterion for the C^*-algebra $\mathfrak{B}$

Consider the C^*-algebra $\mathfrak{A}$ given by (1.1) and fix an isometric representation

$$\varphi : \mathfrak{B}^\pi \to \mathcal{B}(\mathcal{H}_\varphi), \quad B^\pi \mapsto \varphi(B^\pi) \tag{4.1}$$

of the quotient C^*-algebra $\mathfrak{B}^\pi = \mathfrak{B}/\mathcal{K}$ in an abstract Hilbert space $\mathcal{H}_\varphi$, where

$$\mathfrak{B} := \operatorname{alg}(PSO(\mathbb{T}), S_\mathbb{T}, U_G) = \operatorname{alg}(\mathfrak{A}, U_G) \subset \mathcal{B}(L^2(\mathbb{T}))$$

and $\mathcal{K} = \mathcal{K}(L^2(\mathbb{T}))$. Using the generalization of the local-trajectory method by means of a spectral measure $P_\varphi(\cdot)$, we decompose here the C^*-algebra $\varphi(\mathfrak{B}^\pi)$ in an orthogonal sum of operator C^*-algebras satisfying conditions of Proposition 2.6. As a result, an abstract Fredholm criterion for the operators $B \in \mathfrak{B}$ will be obtained.

Put $\mathfrak{A}^\pi = \mathfrak{A}/\mathcal{K}$. For every orientation-preserving diffeomorphism $g : \mathbb{T} \to \mathbb{T}$, every function $a \in L^\infty(\mathbb{T})$ and every operator $H_{P,t}$ defined in (2.8), it follows that

$$U_g a U_g^{-1} = (a \circ g)I, \quad U_g S_\mathbb{T} U_g^{-1} \simeq S_\mathbb{T}, \quad U_g H_{P,t} U_g^{-1} \simeq H_{P,g^{-1}(t)} \tag{4.2}$$

(see [23, Theorem 4.1] and [5, (6.9)]). Since $a \circ g \in PSO(\mathbb{T})$ for all $a \in PSO(\mathbb{T})$ and all $g \in G$ in view of [4, Lemma 4.2], we infer from (4.2) that the mapping

$$\alpha_g : A^\pi \mapsto U_g^\pi A^\pi (U_g^\pi)^{-1} \tag{4.3}$$

is a *-automorphism of the C^*-algebra $\mathfrak{A}^\pi$ and its central C^*-subalgebra $\mathcal{Z}^\pi := (\mathcal{Z} + \mathcal{K})/\mathcal{K}$, with $\mathcal{Z}$ defined by (2.7). Thus, condition (A1) of the local-trajectory method is satisfied. By (4.3), each diffeomorphism $g : \mathbb{T} \to \mathbb{T}$ ($g \in G$) induces on the compact $M(\mathcal{Z}^\pi) = \dot{\mathfrak{M}}$ (see (2.10)) a homeomorphism β_g acting by the rule

$$\beta_g : \dot{\mathfrak{M}} \to \dot{\mathfrak{M}}, \quad (\xi, x) \mapsto (g(\xi), x), \tag{4.4}$$

where $g(\xi)$ is given by (3.2). By analogy with [5, Theorem 4.2], we get the following.

Lemma 4.1. *All the homeomorphisms β_g ($g \in G \setminus \{e\}$) have the same set $\widehat{\Lambda} = \bigcup_{t\in\Lambda} M_t(SO(\mathbb{T})) \times \dot{\mathbb{R}}$ of fixed points, where Λ is the set of all (common) fixed points of the shifts $g \in G \setminus \{e\}$ on $\mathbb{T}$.*

By Lemma 4.1, for each fixed point $t \in \Lambda$ of all $g \in G$, there is an open subset $M_t(SO(\mathbb{T})) \times \mathbb{R}$ of $\dot{\mathfrak{M}}$ composed by fixed points of all β_g. Thus, the action of the group G on $\dot{\mathfrak{M}}$ is not topologically free.

For every $t \in \Lambda$, we consider the next subsets of $\dot{\mathfrak{M}}$ and $\mathfrak{M}$ (see (2.10), (2.4)):

$$\begin{aligned} \mathfrak{M}_t^\circ &:= M_t(SO(\mathbb{T})) \times \mathbb{R}, \quad \mathfrak{M}_t^\infty := M_t(SO(\mathbb{T})) \times \{\infty\}, \\ \dot{\mathfrak{M}}_t &:= M_t(SO(\mathbb{T})) \times \dot{\mathbb{R}}, \quad \mathfrak{M}_t \ := M_t(SO(\mathbb{T})) \times \overline{\mathbb{R}}. \end{aligned} \tag{4.5}$$

Further, using (4.5) and the notation of (1.3), we also introduce the sets

$$\dot{\mathfrak{M}}_{arc} := \bigcup_{t\in\mathbb{T}_{arc}} \dot{\mathfrak{M}}_t, \quad \dot{\mathfrak{M}}_{\rm Is} := \bigcup_{{\rm Is}\Lambda} \mathfrak{M}_t^\circ, \tag{4.6}$$

$$\dot{\mathfrak{M}}^\circ := \Big(\bigcup_{t\in\Lambda^\circ} \dot{\mathfrak{M}}_t\Big) \cup \Big(\bigcup_{t\in\Lambda'\setminus\Lambda^\circ} \mathfrak{M}_t^\circ\Big), \quad \dot{\mathfrak{M}}^\infty := \bigcup_{t\in\partial\Lambda} \mathfrak{M}_t^\infty, \tag{4.7}$$

$$\mathfrak{M}_{arc} := \bigcup_{t\in\mathbb{T}_{arc}} \mathfrak{M}_t, \quad \mathfrak{M}^\circ := \bigcup_{t\in\Lambda'} \mathfrak{M}_t. \tag{4.8}$$

Observe that all the sets in the partition

$$\dot{\mathfrak{M}} = \dot{\mathfrak{M}}_{arc} \cup \dot{\mathfrak{M}}_{\rm Is} \cup \dot{\mathfrak{M}}^\circ \cup \dot{\mathfrak{M}}^\infty, \tag{4.9}$$

belong in view of Lemma 4.1 to the set

$$\mathfrak{R}_G(\dot{\mathfrak{M}}) := \Big\{\Delta \in \mathfrak{R}(\dot{\mathfrak{M}}) : \ \beta_g(\Delta) = \Delta \ \text{ for all } g \in G\Big\}, \tag{4.10}$$

where $\mathfrak{R}(\dot{\mathfrak{M}})$ is the σ-algebra of all Borel subsets of $\dot{\mathfrak{M}}$. We also define the sets

$$\mathfrak{T}_{\rm Is} := \bigcup\nolimits_{t\in {\rm Is}\Lambda} M_t(SO(\mathbb{T})), \quad \mathfrak{T}^\circ := \bigcup\nolimits_{t\in\Lambda'} M_t(SO(\mathbb{T})), \tag{4.11}$$

and the Hilbert space

$$\mathcal{H}_\phi := \quad l^2(\mathfrak{M}_{arc}, \mathbb{C}^2) \oplus l^2(\mathfrak{T}_{\rm Is}, L_2^2(\mathbb{R})) \oplus l^2(\mathfrak{T}^\circ, L_2^2(\mathbb{R})). \tag{4.12}$$

Consider the C^*-subalgebra $\phi(\mathfrak{A}^\pi)$ of $\mathcal{B}(\mathcal{H}_\phi)$ consisting of the operators

$$\phi(A^\pi) = \Big(\bigoplus_{(\xi,x)\in\mathfrak{M}_{arc}} ({\rm Sym}\, A)(\xi,x)I\Big) \oplus \Big(\bigoplus_{\xi\in\mathfrak{T}_{\rm Is}} ({\rm Sym}\, A)(\xi,\cdot)I\Big) \oplus \Big(\bigoplus_{\xi\in\mathfrak{T}^\circ} ({\rm Sym}\, A)(\xi,\cdot)I\Big) \tag{4.13}$$

where $A \in \mathfrak{A}$ and $({\rm Sym}\, A)(\xi,\cdot)$ is the matrix function $x \mapsto ({\rm Sym}\, A)(\xi,x)$, with $x \in \overline{\mathbb{R}}$. By Theorem 2.2, the homomorphism

$$\phi : \mathfrak{A}^\pi \to \mathfrak{B}(\mathcal{H}_\phi), \quad A^\pi \mapsto \phi(A^\pi), \tag{4.14}$$

is an isometric representation of $\mathfrak{A}^\pi$ in the Hilbert space $\mathcal{H}_\phi$. Let

$$P_\varphi : \mathfrak{R}(\dot{\mathfrak{M}}) \to \mathcal{B}(\mathcal{H}_\varphi), \quad P_\phi : \mathfrak{R}(\dot{\mathfrak{M}}) \to \mathcal{B}(\mathcal{H}_\phi) \tag{4.15}$$

be the unique spectral measures associated to the representations (4.1) and (4.14) of the commutative unital C^*-algebra $\mathcal{Z}^\pi$ in the Hilbert spaces $\mathcal{H}_\varphi$ and $\mathcal{H}_\phi$, respectively. According to (4.9), we introduce the following C^*-subalgebras of $\varphi(\mathfrak{B}^\pi)$:

$$\mathfrak{B}_{arc} := {\rm alg}\big\{P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(A^\pi),\ P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(U_g^\pi) : \ A\in\mathfrak{A},\ g\in G\big\}, \tag{4.16}$$

$$\mathfrak{B}_{\rm Is} := {\rm alg}\big\{P_\varphi(\dot{\mathfrak{M}}_{\rm Is})\varphi(A^\pi),\ P_\varphi(\dot{\mathfrak{M}}_{\rm Is})\varphi(U_g^\pi) : \ A\in\mathfrak{A},\ g\in G\big\}, \tag{4.17}$$

$$\mathfrak{B}^\circ := {\rm alg}\big\{P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(A^\pi),\ P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(U_g^\pi) : \ A\in\mathfrak{A},\ g\in G\big\}, \tag{4.18}$$

$$\mathfrak{B}^\infty := {\rm alg}\big\{P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi(A^\pi),\ P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi(U_g^\pi) : \ A\in\mathfrak{A},\ g\in G\big\} \tag{4.19}$$

of $\mathcal{B}(P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi)$, $\mathcal{B}(P_\varphi(\dot{\mathfrak{M}}_{\rm Is})\mathcal{H}_\varphi)$, $\mathcal{B}(P_\varphi(\dot{\mathfrak{M}}^\circ)\mathcal{H}_\varphi)$, $\mathcal{B}(P_\varphi(\dot{\mathfrak{M}}^\infty)\mathcal{H}_\varphi)$, respectively.

Since the spectral projections of the open sets $\dot{\mathfrak{M}}_{arc}$, $\dot{\mathfrak{M}}_{\rm Is}$, $\dot{\mathfrak{M}}^\circ$ in $\mathfrak{R}_G(\dot{\mathfrak{M}})$ given by (4.10) are not zero due to Lemma 2.5, we infer from the partition (4.9) and Proposition 2.6 the following result.

Theorem 4.2 (Abstract Fredholm criterion for $\mathfrak{B}$). *An operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$ if and only if the following conditions hold:*

(i) *the operator $P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$,*

(ii) *the operator $P_\varphi(\dot{\mathfrak{M}}_{\rm Is})\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}_{\rm Is})\mathcal{H}_\varphi$,*

(iii) *the operator $P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}^\circ)\mathcal{H}_\varphi$,*

(iv) *if $P_\varphi(\dot{\mathfrak{M}}^\infty) \neq 0$, the operator $P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}^\infty)\mathcal{H}_\varphi$.*

For the spectral measure P_ϕ associated by (4.15) to the concrete representation (4.13)–(4.14), we easily see that

$$P_\phi(\dot{\mathfrak{M}}_{arc}) = I_{arc} \oplus O_{\rm Is} \oplus O^\circ, \quad P_\phi(\dot{\mathfrak{M}}_{\rm Is}) = O_{arc} \oplus I_{\rm Is} \oplus O^\circ, \tag{4.20}$$

$$P_\phi(\dot{\mathfrak{M}}^\circ) = O_{arc} \oplus O_{\rm Is} \oplus I^\circ, \quad P_\phi(\dot{\mathfrak{M}}^\infty) = O_{arc} \oplus O_{\rm Is} \oplus O^\circ, \tag{4.21}$$

where O_{arc} and I_{arc} are, respectively, the zero and identity operators on the Hilbert space $l^2(\mathfrak{M}_{arc}, \mathbb{C}^2)$, $O_{\rm Is}$ and $I_{\rm Is}$ are the zero and identity operators on the Hilbert space $l^2(\mathfrak{T}_{\rm Is}, L_2^2(\mathbb{R}))$, O° and I° denote the zero and identity operator on the Hilbert space $l^2(\mathfrak{T}^\circ, L_2^2(\mathbb{R}))$.

5. Symbol calculus and Fredholmness for the C^*-algebra $\mathfrak{B}$

5.1. The C^*-algebra $\mathfrak{B}_{arc}$

Using the local-trajectory method we establish in this section an invertibility criterion for the C^*-algebra $\mathfrak{B}_{arc}$ defined by (4.16).

Fix a connected component γ of the open set $\mathbb{T} \setminus \Lambda$ and define the sets

$$\dot{\mathfrak{M}}_\gamma := \bigcup_{t\in\gamma} M_t(SO(\mathbb{T})) \times \dot{\mathbb{R}} \subset \dot{\mathfrak{M}}, \quad \mathfrak{M}_\gamma := \bigcup_{t\in\gamma} M_t(SO(\mathbb{T})) \times \overline{\mathbb{R}} \subset \mathfrak{M}. \tag{5.1}$$

The C^*-algebra $\mathfrak{B}_\gamma := P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(\mathfrak{B}^\pi)$ can be viewed as the C^*-subalgebra ${\rm alg}\,(\mathfrak{A}_\gamma, U_\gamma(G))$ of $\mathcal{B}(P_\varphi(\dot{\mathfrak{M}}_\gamma)\mathcal{H}_\varphi)$ generated by the C^*-algebra $\mathfrak{A}_\gamma := P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(\mathfrak{A}^\pi)$ and the group $U_\gamma(G)$ of the unitary operators $U_{g,\gamma} := P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(U_g^\pi)$ $(g \in G)$. The C^*-algebra $\mathcal{Z}_\gamma := P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(\mathcal{Z}^\pi)$ is a central subalgebra of $\mathfrak{A}_\gamma$. By Lemma 2.5(ii), $\mathcal{Z}_\gamma \cong C(\widetilde{\dot{\mathfrak{M}}_\gamma})$ where $\widetilde{\dot{\mathfrak{M}}_\gamma} := \widetilde{\dot{\mathfrak{M}}_\gamma}$. As the set $\dot{\mathfrak{M}}_\gamma$ is open due to (5.1) , Lemma 2.5(iii) implies in view of the topology (2.11) that

$$\widetilde{\dot{\mathfrak{M}}_\gamma} = \overline{\dot{\mathfrak{M}}_\gamma} = \dot{\mathfrak{M}}_\gamma \cup \Big(\bigcup_{t\in\partial\gamma} M_t(SO(\mathbb{T})) \times \{\infty\}\Big). \tag{5.2}$$

For the open arc $\gamma := (t_l, t_r) \subset \mathbb{T} \setminus \Lambda$, along with $\widetilde{\dot{\mathfrak{M}}_\gamma}$, we consider the set

$$\overline{\mathfrak{M}}_\gamma := \mathfrak{M}_\gamma \cup \Big(\bigcup_{t\in\partial\gamma} M_t(SO(\mathbb{T})) \times \{\pm\infty\}\Big) \subset \mathfrak{M}$$

equipped with the discrete topology. Given an operator $A \in \mathfrak{A}$, let us define, with the help of Theorem 2.2, the matrix function ${\rm Sym}_\gamma A : \overline{\mathfrak{M}}_\gamma \to \mathbb{C}^{2\times 2}$ by

$$({\rm Sym}_\gamma A)(\xi, x) :=$$
$$\begin{cases} ({\rm Sym}\, A)(\xi, x) & \text{if } (\xi, x) \in \mathfrak{M}_\gamma, \\ {\rm diag}\,\big\{({\rm Sym}\, A)_{11}(\xi, x), ({\rm Sym}\, A)_{11}(\xi, -x)\big\} & \text{if } (\xi, x) \in M_{t_l}(SO(\mathbb{T})) \times \{\pm\infty\}, \\ {\rm diag}\,\big\{({\rm Sym}\, A)_{22}(\xi, -x), ({\rm Sym}\, A)_{22}(\xi, x)\big\} & \text{if } (\xi, x) \in M_{t_r}(SO(\mathbb{T})) \times \{\pm\infty\} \end{cases} \tag{5.3}$$

where $({\rm Sym}\, A)_{jj}(\xi, x)$ is the (j, j)-entry of the matrix $({\rm Sym}\, A)(\xi, x)$.

Since $\varphi(\mathfrak{A}^\pi) \cong \phi(\mathfrak{A}^\pi)$ and the set $\dot{\mathfrak{M}}_\gamma$ is open in $\dot{\mathfrak{M}}$, we infer similarly to [5, Theorem 8.1] that $P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(\mathfrak{A}^\pi) \cong P_\phi(\dot{\mathfrak{M}}_\gamma)\phi(\mathfrak{A}^\pi)$. Hence, taking into account

(4.13) and (4.20), we conclude that the C^*-algebra $\mathfrak{A}_\gamma \cong P_\phi(\dot{\mathfrak{M}}_\gamma)\phi(\mathfrak{A}^\pi)$ is isometrically *-isomorphic to the C^*-algebra of all matrix functions $\operatorname{Sym} A : \mathfrak{M}_\gamma \to \mathbb{C}^{2\times 2}$ for $A \in \mathfrak{A}$, which, in its turn, is isometrically *-isomorphic to the C^*-algebra of the matrix functions $\operatorname{Sym}_\gamma A : \overline{\mathfrak{M}}_\gamma \to \mathbb{C}^{2\times 2}$ $(A \in \mathfrak{A})$ defined by (5.3), because

$$\sup_{(\xi,x)\in\mathfrak{M}_\gamma} \|(\operatorname{Sym} A)(\xi,x)\|_{sp} = \sup_{(\xi,x)\in\overline{\mathfrak{M}}_\gamma} \|(\operatorname{Sym}_\gamma A)(\xi,x)\|_{sp} \quad (A \in \mathfrak{A})$$

where $\|\cdot\|_{sp}$ is the spectral norm. Thus, by analogy with [5, Theorem 8.3], we obtain the following result.

Theorem 5.1. *The mapping*

$$\operatorname{Sym}_\gamma : \mathfrak{A}_\gamma \to BC(\overline{\mathfrak{M}}_\gamma, \mathbb{C}^{2\times 2}), \quad P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(A^\pi) \mapsto \operatorname{Sym}_\gamma A, \tag{5.4}$$

where $\operatorname{Sym}_\gamma A$ *is given by* (5.3)*, is an isometric C^*-algebra homomorphism.*

Consider now the C^*-algebra $\mathfrak{B}_\gamma$. For every $g \in G$, the mapping

$$\alpha_{g,\gamma} : P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(A^\pi) \mapsto U_{g,\gamma}(P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(A^\pi))U_{g,\gamma}^*$$

is a *-automorphism of the C^*-algebras $\mathcal{Z}_\gamma$ and $\mathfrak{A}_\gamma$. Thus, condition (A1)–(A2) of the local-trajectory method for the C^*-algebra $\mathfrak{B}_\gamma$ are satisfied. Each *-automorphism $\alpha_{g,\gamma}$ $(g \in G)$ induces the homeomorphism

$$\beta_{g,\gamma} : \widetilde{\dot{\mathfrak{M}}}_\gamma \to \widetilde{\dot{\mathfrak{M}}}_\gamma, \quad (\xi,x) \mapsto \beta_g(\xi,x), \tag{5.5}$$

where β_g and $\widetilde{\dot{\mathfrak{M}}}_\gamma = M(\mathcal{Z}_\gamma)$ are given by (4.4) and (5.2), respectively. The set $\Delta_\gamma := \bigcup_{t\in\partial\gamma} M_t(SO(\mathbb{T})) \times \{\infty\}$ is the set of fixed points of all $\beta_{g,\gamma}$ $(g \in G \setminus \{e\})$.

Let us check condition (A3) of the local-trajectory method for the C^*-algebra $\mathfrak{B}_\gamma$. For each $(\xi,x) \in \widetilde{\dot{\mathfrak{M}}}_\gamma$, let $J_{(\xi,x)}$ be the smallest closed two-sided ideal of $\mathfrak{A}_\gamma$ that contains the set $\{P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(Z^\pi) : Z \in \mathcal{Z}, (\operatorname{Sym}_\gamma Z)(\xi,x) = 0_{2\times 2}\}$. The set of all pure states of the C^*-algebra $\mathfrak{A}_\gamma$ has the form $\mathcal{P}_{\mathfrak{A}_\gamma} = \bigcup_{(\xi,x)\in\widetilde{\dot{\mathfrak{M}}}_\gamma} \mathcal{P}_{(\xi,x)}$ where $\mathcal{P}_{(\xi,x)}$ can be identified with the set of all pure states of the quotient C^*-algebra $\mathfrak{A}_\gamma/J_{(\xi,x)}$ (see, e.g., [17]). In its turn, the C^*-algebra $\mathfrak{A}_\gamma/J_{(\xi,x)}$ is isometrically *-isomorphic to $(\operatorname{Sym}\mathfrak{A})(\xi,x)$ if $(\xi,x) \in \bigcup_{t\in\gamma} M_t(SO(\mathbb{T})) \times \mathbb{R}$, and to $(\operatorname{Sym}_\gamma \mathfrak{A})(\xi,+\infty) \oplus (\operatorname{Sym}_\gamma \mathfrak{A})(\xi,-\infty)$ if $(\xi,x) \in \bigcup_{t\in\overline{\gamma}} M_t(SO(\mathbb{T})) \times \{\infty\}$, where the mapping $\operatorname{Sym}_\gamma$ is defined by (5.5). Since $\beta_g(\xi,x) \neq (\xi,x)$ for all $(\xi,x) \in \bigcup_{t\in\gamma} M_t(SO(\mathbb{T}))\times\mathbb{R}$ and all $g \in G\setminus\{e\}$, it remains to prove the approximability (in the weak* topology) of all states in $\mathcal{P}_{(\xi,\infty)}$ with $\xi \in M_{t_l}(SO(\mathbb{T})) \cup M_{t_r}(SO(\mathbb{T}))$ by states in $\mathcal{P}_{(\zeta,\infty)}$ where $\zeta \in \bigcup_{t\in\gamma} M_t(SO(\mathbb{T}))$, which will give (A3) with $M_0 = \dot{\mathfrak{M}}_\gamma$.

By (2.5), $(\operatorname{Sym} A)(\xi,\pm\infty)$ are diagonal matrices for all $\xi \in M(SO(\mathbb{T}))$. Hence, we infer from (5.3) that the set $\mathcal{P}_{(\xi,\infty)}$ with $\xi \in M_t(SO(\mathbb{T}))$ consists, respectively, of the pure states $\rho^{(1)}_{\xi,\pm\infty}$, $\rho^{(2)}_{\xi,\pm\infty}$ if $t \in \gamma$, $\rho^{(1)}_{\xi,\pm\infty}$ if $t = t_l$, and $\rho^{(2)}_{\xi,\pm\infty}$ if $t = t_r$, where

$$\rho^{(j)}_{\xi,\pm\infty} : \mathfrak{A}_\gamma \to \mathbb{C}, \quad P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(A^\pi) \mapsto (\operatorname{Sym} A)_{jj}(\xi,\pm\infty) \quad (j = 1,2). \tag{5.6}$$

According to [5, Theorem 5.2], every operator $A \in \mathfrak{A}$ is uniquely represented in the form $A = a_+ P_{\mathbb{T}}^+ + a_- P_{\mathbb{T}}^- + H_A$, where $a_\pm \in PSO(\mathbb{T})$, $P_{\mathbb{T}}^\pm = (I \pm S_{\mathbb{T}})/2$ and $(\operatorname{Sym} H_A)(\xi, \pm\infty) = 0_{2\times 2}$ for all $\xi \in M(SO(\mathbb{T}))$. Thus, for such ξ, by (2.5), we get

$$(\operatorname{Sym} A)_{11}(\xi, \pm\infty) = a_\pm(\xi, 1), \quad (\operatorname{Sym} A)_{22}(\xi, \pm\infty) = a_\mp(\xi, 0). \tag{5.7}$$

Hence, taking into account (2.3), we infer from (5.7) and (5.6) that the pure states $\rho^{(1)}_{\xi,\pm\infty}$ with $\xi \in M_{t_l}(SO(\mathbb{T}))$ and $\rho^{(2)}_{\xi,\pm\infty}$ with $\xi \in M_{t_r}(SO(\mathbb{T}))$ are approximated, respectively, by $\rho^{(1)}_{\zeta,\pm\infty}$ and $\rho^{(2)}_{\zeta,\pm\infty}$ with $\zeta \in \bigcup_{t\in\gamma} M_t(SO(\mathbb{T}))$, which gives (A3).

For each $(\xi, x) \in \mathfrak{M}_{arc}$, we consider the representation

$$\pi_{(\xi,x)} : \mathfrak{B}_{arc} \to \mathcal{B}(l^2(G, \mathbb{C}^2)) \tag{5.8}$$

given on the generators of the C^*-algebra $\mathfrak{B}_{arc}$ by

$$\begin{aligned}
\big[\pi_{(\xi,x)}\big(P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi((aI)^\pi)\big)f\big](g) &= (\operatorname{Sym}((a\circ g)I))(\xi, x)f(g),\\
\big[\pi_{(\xi,x)}\big(P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(S_{\mathbb{T}}^\pi)\big)f\big](g) &= (\operatorname{Sym} S_{\mathbb{T}})(\xi, x)f(g),\\
\big[\pi_{(\xi,x)}\big(P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(U_h^\pi)\big)f\big](g) &= f(gh),
\end{aligned} \tag{5.9}$$

where $a \in PSO(\mathbb{T})$, $g, h \in G$, $f \in l^2(G, \mathbb{C}^2)$ and $\dot{\mathfrak{M}}_{arc}$ is given by (4.6).

Fix now a set $\mathcal{O}_{arc} \subset \mathbb{T}_{arc}$ which contains exactly one point in each orbit defined by the group of shifts G on $\mathbb{T}_{arc} = \mathbb{T} \setminus \Lambda$, and consider the set

$$\mathfrak{N}_{arc} := \bigcup_{\tau\in\mathcal{O}_{arc}} M_\tau(SO(\mathbb{T})) \times \overline{\mathbb{R}}. \tag{5.10}$$

Theorem 5.2. *For each $B \in \mathfrak{B}$, the operator $B_{arc} := P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(B^\pi) \in \mathfrak{B}_{arc}$ is invertible on the space $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$ if and only if for all $(\xi, x) \in \mathfrak{N}_{arc}$ the operators $\pi_{(\xi,x)}(B_{arc})$ are invertible on the space $l^2(G, \mathbb{C}^2)$ and*

$$\sup_{(\xi,x)\in\mathfrak{N}_{arc}} \big\|(\pi_{(\xi,x)}(B_{arc}))^{-1}\big\| < \infty. \tag{5.11}$$

Proof. Since assumptions (A1)–(A3) are fulfilled for the C^*-algebra $\mathfrak{B}_\gamma$, we infer from Theorem 2.4 by analogy with [5, Theorem 8.4] that, for every $B \in \mathfrak{B}$, the operator $B_\gamma := P_\varphi(\dot{\mathfrak{M}}_\gamma)\varphi(B^\pi) \in \mathfrak{B}_\gamma$ is invertible on the space $P_\varphi(\dot{\mathfrak{M}}_\gamma)\mathcal{H}_\varphi$ if and only if for all $(\xi, x) \in \mathfrak{N}_{arc} \cap \mathfrak{M}_\gamma$ (see (5.10) and (5.1)) the operators $\pi_{(\xi,x)}(B_\gamma)$ are invertible on the space $l^2(G, \mathbb{C}^2)$ and the norms of their inverses are uniformly bounded. Hence, taking into account the equality $\mathfrak{B}_{arc} = \bigoplus_\gamma \mathfrak{B}_\gamma$ where γ runs through all the connected components of the open set $\mathbb{T}\setminus\Lambda$, we immediately obtain the desired assertion for the C^*-algebra $\mathfrak{B}_{arc}$. □

5.2. The C^*-algebra $\mathfrak{B}_{\mathrm{Is}}$

By [5], with any isolated point $t \in \Lambda$ and the C^*-algebra $\mathfrak{B}_t^\circ := P_\varphi(\mathfrak{M}_t^\circ)\varphi(\mathfrak{B}^\pi)$ we associate the Hilbert space $\mathcal{H}_t = l^2(M_t(SO(\mathbb{T})), L_2^2(\mathbb{R}))$ and the C^*-algebra

$$\widetilde{\Psi}_t(\mathfrak{B}_t^\circ) := \operatorname{alg}\big\{\Psi_t(A^\pi),\ \Psi_t(U_g^\pi) :\ A \in \mathfrak{A},\ g \in G\big\} \subset \mathcal{B}(\mathcal{H}_t)$$

generated by the operators $\Psi_t(A^\pi)$ $(A\in\mathfrak{A})$ and $\Psi_t(U_g^\pi)$ $(g\in G)$ where

$$\Psi_t(A^\pi):=\bigoplus_{\xi\in M_t(SO(\mathbb{T}))}(\mathrm{Sym}\,A)(\xi,\cdot)I,\quad \Psi_t(U_g^\pi):=\bigoplus_{\xi\in M_t(SO(\mathbb{T}))}e_{\ln g'(t)}(\cdot)I,\quad (5.12)$$

and $e_{\ln g'(t)}(x):=e^{ix\ln g'(t)}$ $(x\in\mathbb{R})$.

Theorem 5.3. [5, Theorem 9.5] *For every $t\in\mathrm{Is}\,\Lambda$, the mapping*

$$P_\varphi(\mathfrak{M}_t^\circ)\,\varphi\Big(\sum\nolimits_{g\in F}A_g^\pi U_g^\pi\Big)\mapsto\Psi_t\Big(\sum\nolimits_{g\in F}A_g^\pi U_g^\pi\Big):=\sum\nolimits_{g\in F}\Psi_t(A_g^\pi)\Psi_t(U_g^\pi),$$

where F is a finite subset of G and $A_g\in\mathfrak{A}$ for $g\in F$, extends to an isometric C^*-algebra homomorphism

$$\widetilde{\Psi}_t:\ \mathfrak{B}_t^\circ\to\mathcal{B}(\mathcal{H}_t),\quad B_t^\circ:=P_\varphi(\mathfrak{M}_t^\circ)\varphi(B^\pi)\mapsto\Psi_t(B^\pi)\quad(B\in\mathfrak{B}),\qquad(5.13)$$

where Ψ_t is a C^*-algebra homomorphism of the quotient C^*-algebra $\mathfrak{B}^\pi$ into $\mathcal{B}(\mathcal{H}_t)$, and $\Psi_t(B^\pi)=\bigoplus_{\xi\in M_t(SO(\mathbb{T}))}B_t^\circ(\xi,\cdot)I$ with $B_t^\circ(\xi,\cdot):\ \mathbb{R}\to\mathbb{C}^{2\times2}$ for any $B\in\mathfrak{B}$.

As $\mathfrak{B}_{\mathrm{Is}}=P_\varphi(\dot{\mathfrak{M}}_{\mathrm{Is}})\varphi(\mathfrak{B}^\pi)=\bigoplus_{t\in\mathrm{Is}\Lambda}\mathfrak{B}_t^\circ$ and $l^2(\mathfrak{T}_{\mathrm{Is}},L_2^2(\mathbb{R}))=\bigoplus_{t\in\mathrm{Is}\Lambda}\mathcal{H}_t$ (see (4.17) and (4.11)), Theorem 5.3 and Proposition 2.6 imply the following corollary.

Theorem 5.4. *The map $\Psi_{\mathrm{Is}}=\bigoplus\limits_{t\in\mathrm{Is}\Lambda}\widetilde{\Psi}_t$ is the isometric C^*-algebra homomorphism*

$$\Psi_{\mathrm{Is}}:\mathfrak{B}_{\mathrm{Is}}\to\mathcal{B}(l^2(\mathfrak{T}_{\mathrm{Is}},L_2^2(\mathbb{R}))),\ \ P_\varphi(\dot{\mathfrak{M}}_{\mathrm{Is}})\varphi(B^\pi)\mapsto\bigoplus_{t\in\mathrm{Is}\Lambda}\ \bigoplus_{\xi\in M_t(SO(\mathbb{T}))}B_t^\circ(\xi,\cdot)I.$$

For each $B\in\mathfrak{B}$, the operator $B_{\mathrm{Is}}:=P_\varphi(\dot{\mathfrak{M}}_{\mathrm{Is}})\varphi(B^\pi)\in\mathfrak{B}_{\mathrm{Is}}$ is invertible on the space $P_\varphi(\dot{\mathfrak{M}}_{\mathrm{Is}})\mathcal{H}_\varphi$ if and only if the operator $\Psi_{\mathrm{Is}}(B_{\mathrm{Is}})$ is invertible on the space $l^2(\mathfrak{T}_{\mathrm{Is}},L_2^2(\mathbb{R}))$, that is, if

$$\inf_{t\in\mathrm{Is}\Lambda}\ \min_{\xi\in M_t(SO(\mathbb{T}))}\ \inf_{x\in\mathbb{R}}\big|\det(B_t^\circ(\xi,x))\big|>0.$$

5.3. The C^*-algebra $\mathfrak{B}^\circ$

Consider now the C^*-algebra $\mathfrak{B}^\circ=\mathrm{alg}\,(\mathfrak{A}^\circ,U^\circ(G))\subset\mathcal{B}(P_\varphi(\dot{\mathfrak{M}}^\circ)\mathcal{H}_\varphi)$ given by (4.18). It is generated by the C^*-algebra $\mathfrak{A}^\circ:=P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(\mathfrak{A}^\pi)$ and by the group of unitary operators $U^\circ(G):=\big\{U_g^\circ:=P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(U_g^\pi):\ g\in G\big\}$.

Consider the Hilbert space $\mathcal{H}^\circ:=l^2(\mathfrak{T}^\circ,L_2^2(\mathbb{R}))$, with $\mathfrak{T}^\circ$ defined by (4.11). Since $\dot{\mathfrak{M}}^\circ$ is an open subset of $\dot{\mathfrak{M}}$, applying Lemma 2.7, (4.12) and (4.21), we infer analogously to [5, Theorem 8.1] that $P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(\mathfrak{A}^\pi)\cong P_\phi(\dot{\mathfrak{M}}^\circ)\phi(\mathfrak{A}^\pi)$. Hence, taking into account (4.8), (4.13) and Theorem 2.2, we get the following result.

Theorem 5.5. *The map $\mathrm{Sym}^\circ:\mathfrak{A}^\circ\to\mathcal{B}\,(\mathcal{H}^\circ)$, defined by*

$$P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(A^\pi)\mapsto\bigoplus_{\xi\in\mathfrak{T}^\circ}(\mathrm{Sym}\,A)(\xi,\cdot)I\quad for\quad A\in\mathfrak{A},\qquad(5.14)$$

is an isometric C^*-algebra homomorphism. An operator $P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(A^\pi)$ for $A\in\mathfrak{A}$ is invertible on the space $P_\varphi(\dot{\mathfrak{M}}^\circ)\mathcal{H}_\varphi$ if and only if

$$\det\big((\mathrm{Sym}\,A)(\xi,x)\big)\neq0\quad for\ all\ \ (\xi,x)\in\mathfrak{M}^\circ.$$

Now we are going to extend the isometric C^*-algebra homomorphism (5.14) to all the C^*-algebra $\mathfrak{B}^\circ$. To this end we define the closed two-sided ideal $\widetilde{\mathfrak{H}}^\pi$ of the quotient C^*-algebra $\mathcal{Z}^\pi$ that is generated by the cosets $H^\pi_{P,t} = H_{P,t} + \mathcal{K}$ with $t \in \Lambda' \setminus \Lambda^\circ$ and $P \in \mathcal{P}$ and by the cosets $(cI)^\pi$ where $c \in C(\mathbb{T})$ and $\operatorname{supp} c \subset \Lambda^\circ$ (see (2.6)–(2.8)). Since $\mathcal{Z}^\pi$ is commutative, we deduce that $\widetilde{\mathfrak{H}}^\pi$ is the closure in $\mathcal{Z}^\pi$ of the set $\big\{ \sum_i Z_i^\pi (c_i I)^\pi + \sum_k \sum_j Z^\pi_{j,k} H^\pi_{P_{j,k},t_k} \big\}$ where $Z_i, Z_{j,k} \in \mathcal{Z}$, $t_k \in \Lambda' \setminus \Lambda^\circ$, $P_{j,k} \in \mathcal{P}$, $c_i \in C(\mathbb{T})$, $c_i = 0$ on $\mathbb{T} \setminus \Lambda^\circ$ and i, j, k run finite subsets of $\mathbb{N}$. Let

$$\mathcal{Z}(\dot{\mathfrak{M}}^\circ) := \Big\{ Z^\pi \in \mathcal{Z}^\pi : \operatorname{supp} z(\cdot,\cdot) \subset \overline{\dot{\mathfrak{M}}^\circ},\ z(\xi,x) \in [0,1] \text{ for all } (\xi,x) \in \dot{\mathfrak{M}} \Big\}, \tag{5.15}$$

where $z(\cdot,\cdot) \in C(\dot{\mathfrak{M}})$ is the Gelfand transform of the coset Z^π and $\overline{\dot{\mathfrak{M}}^\circ} = \mathfrak{T}^\circ \times \dot{\mathbb{R}}$ is the closure in $\dot{\mathfrak{M}}$ of the set $\dot{\mathfrak{M}}^\circ$ defined in (4.7).

Lemma 5.6. *The ideal $\widetilde{\mathfrak{H}}^\pi$ possesses the properties:*

(i) $B^\pi H^\pi = H^\pi B^\pi \in \mathfrak{A}^\pi$ *for each coset* $B^\pi \in \mathfrak{B}^\pi$ *and each coset* $H^\pi \in \widetilde{\mathfrak{H}}^\pi$*;*

(ii) $\mathcal{Z}(\dot{\mathfrak{M}}^\circ) \subset \widetilde{\mathfrak{H}}^\pi$.

Proof. (i) By [5, Lemma 5.4], $U^\pi_g H^\pi_{P,t} = H^\pi_{P,t} = H^\pi_{P,t} U^\pi_g$ for all $g \in G$, all $P \in \mathcal{P}$ and all $t \in \Lambda' \setminus \Lambda^\circ$ because $g'(t) = 1$ for such t. Further, $U^\pi_g (cI)^\pi = (cI)^\pi = (cI)^\pi U^\pi_g$ for all $g \in G$ and all $c \in C(\mathbb{T})$ such that $\operatorname{supp} c \subset \Lambda^\circ$. Since $\mathcal{Z}^\pi$ is a central subalgebra of $\mathfrak{A}^\pi$, from these relations it follows by definition of the ideal $\widetilde{\mathfrak{H}}^\pi$ that

$$U^\pi_g H^\pi = H^\pi = H^\pi U^\pi_g \quad \text{for all } g \in G \text{ and all } H^\pi \in \widetilde{\mathfrak{H}}^\pi, \tag{5.16}$$

which immediately imply (i).

(ii) If $H \in \mathcal{Z}$ and $H^\pi \in \widetilde{\mathfrak{H}}^\pi$, then, by definition of $\widetilde{\mathfrak{H}}^\pi$ and Theorem 2.3,

$$(\operatorname{Sym} H)(\xi,x) := \begin{cases} \operatorname{diag}\{h(\xi,x), h(\xi,x)\} & \text{if } (\xi,x) \in \mathfrak{M}^\circ, \\ 0_{2\times 2} & \text{if } (\xi,x) \in \mathfrak{M} \setminus \mathfrak{M}^\circ, \end{cases} \tag{5.17}$$

where $h(\cdot,\cdot) \in C(\dot{\mathfrak{M}})$ is the Gelfand transform of the coset H^π. Further, as in [5, Lemma 9.4(ii)], we deduce from (5.17), [5, Lemma 6.2] and the relations

$$(cI)^\pi \widetilde{\mathfrak{H}}^\pi = (cI)^\pi \mathcal{Z}^\pi \quad (c \in C(\mathbb{T}),\ \operatorname{supp} c \subset \Lambda^\circ)$$

that the ideal $\widetilde{\mathfrak{H}}^\pi$ is isometrically *-isomorphic to the ideal of all continuous functions on the compact $\dot{\mathfrak{M}}$ which vanish on $\dot{\mathfrak{M}} \setminus \dot{\mathfrak{M}}^\circ$. Hence, making use of (5.15), we obtain (ii). ☐

By analogy with Subsection 5.2, we now consider the C^*-algebra

$$\widetilde{\Psi}^\circ(\mathfrak{B}^\circ) := \operatorname{alg}\big\{ \Psi^\circ(A^\pi),\ \Psi^\circ(U^\pi_g) : A \in \mathfrak{A},\ g \in G \big\} \subset \mathcal{B}(\mathcal{H}^\circ) \tag{5.18}$$

generated by the operators

$$\Psi^\circ(A^\pi) := \bigoplus_{\xi \in \mathfrak{T}^\circ} (\operatorname{Sym} A)(\xi,\cdot) I \ \ (A \in \mathfrak{A}), \quad \Psi^\circ(U^\pi_g) := I^\circ \ \ (g \in G), \tag{5.19}$$

where I° is the identity operator on the space $\mathcal{H}^\circ = l^2(\mathfrak{T}^\circ, L^2_2(\mathbb{R}))$.

Theorem 5.7. *The mapping*

$$P_\varphi(\dot{\mathfrak{M}}^\circ)\,\varphi\Big(\sum\nolimits_{g\in F} A_g^\pi U_g^\pi\Big) \mapsto \Psi^\circ\Big(\sum\nolimits_{g\in F} A_g^\pi U_g^\pi\Big) := \sum\nolimits_{g\in F} \Psi^\circ(A_g^\pi)\Psi^\circ(U_g^\pi), \tag{5.20}$$

where F is a finite subset of G and $A_g \in \mathfrak{A}$ for $g \in F$, extends to an isometric C^*-algebra homomorphism

$$\widetilde{\Psi}^\circ : \mathfrak{B}^\circ \to \mathcal{B}(\mathcal{H}^\circ), \quad B^\circ := P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi) \mapsto \Psi^\circ(B^\pi) \quad (B \in \mathfrak{B}), \tag{5.21}$$

where Ψ° is a C^*-algebra homomorphism of the C^*-algebra $\mathfrak{B}^\pi$ into $\mathcal{B}(\mathcal{H}^\circ)$, and $\Psi^\circ(B^\pi) = \bigoplus_{\xi\in\mathfrak{T}^\circ} B^\circ(\xi,\cdot)I$ with $B^\circ(\xi,\cdot) : \overline{\mathbb{R}} \to \mathbb{C}^{2\times 2}$ for any $B \in \mathfrak{B}$. For each $B \in \mathfrak{B}$, the operator $B^\circ \in \mathfrak{B}^\circ$ is invertible on the space $P_\varphi(\dot{\mathfrak{M}}^\circ)\mathcal{H}_\varphi$ if and only if the operator $\Psi^\circ(B^\pi) \in \widetilde{\Psi}^\circ(\mathfrak{B}^\circ)$ is invertible on the space $l^2(\mathfrak{T}^\circ, L_2^2(\mathbb{R}))$, that is, if

$$\det\big(B^\circ(\xi,x)\big) \neq 0 \quad \textit{for all} \ \ (\xi,x) \in \mathfrak{M}^\circ = \mathfrak{T}^\circ \times \overline{\mathbb{R}}. \tag{5.22}$$

Proof. Fix an operator $B = \sum_{g\in F} A_g U_g \in \mathfrak{B}$, where F is a finite subset of G and $A_g \in \mathfrak{A}$ for $g \in F$. Then we deduce from (5.20) and (5.19) that

$$\Psi^\circ(B^\pi) = \sum\nolimits_{g\in F} \Psi^\circ(A_g^\pi)\Psi^\circ(U_g^\pi) = \sum\nolimits_{g\in F} \Psi^\circ(A_g^\pi).$$

Let $\phi^\circ : \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}^\circ)$ be the restriction of the representation (4.14) to the invariant subspace $\mathcal{H}^\circ$ of $\mathcal{H}_\phi$. According to Lemma 5.6 (i), for each coset $H^\pi \in \widetilde{\mathfrak{H}}^\pi$ we get $B^\pi H^\pi \in \mathfrak{A}^\pi$. Hence, from (4.13), (5.16) and (5.19) it follows that

$$\phi^\circ(B^\pi H^\pi) = \Psi^\circ(B^\pi)\phi^\circ(H^\pi) \quad \text{for all} \ \ H^\pi \in \widetilde{\mathfrak{H}}^\pi. \tag{5.23}$$

Since $\dot{\mathfrak{M}}^\circ$ is an open subset of $\dot{\mathfrak{M}}$ and the C^*-algebras $\varphi(\mathfrak{A}^\pi)$ and $\phi(\mathfrak{A}^\pi)$ are isometrically *-isomorphic, using Lemma 2.7 we easily conclude that

$$\|P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(A^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|P_\phi(\dot{\mathfrak{M}}^\circ)\phi(A^\pi)\|_{\mathcal{B}(\mathcal{H}^\circ)} \quad \text{for all } A \in \mathfrak{A}. \tag{5.24}$$

Consequently, from (5.24) and (5.23) it follows that

$$\|P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi H^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|\Psi^\circ(B^\pi)\phi^\circ(H^\pi)\|_{\mathcal{B}(\mathcal{H}^\circ)} \ \text{ for all } H^\pi \in \widetilde{\mathfrak{H}}^\pi. \tag{5.25}$$

Since the set $\dot{\mathfrak{M}}^\circ$ is open and since $P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi) = \varphi(B^\pi)P_\varphi(\dot{\mathfrak{M}}^\circ)$, we infer similarly to the proof of [5, Lemma 3.5] (cf. Lemma 2.7) that

$$\|P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi)\|_{\mathcal{B}(H_\varphi)} = \sup_{Z^\pi\in\mathcal{Z}(\dot{\mathfrak{M}}^\circ)} \|\varphi(B^\pi Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}, \tag{5.26}$$

where $\mathcal{Z}(\dot{\mathfrak{M}}^\circ)$ is the set (5.15). Because $\mathcal{Z}(\dot{\mathfrak{M}}^\circ) \subset \widetilde{\mathfrak{H}}^\pi$ (see Lemma 5.6) and because

$$P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi Z^\pi) = \varphi(B^\pi)P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(Z^\pi) = \varphi(B^\pi Z^\pi) \quad \text{for all} \ \ Z^\pi \in \mathcal{Z}(\dot{\mathfrak{M}}^\circ),$$

we infer from (5.26) and (5.25) that

$$\begin{aligned} \|P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} &= \sup_{Z^\pi\in\mathcal{Z}(\dot{\mathfrak{M}}^\circ)} \|P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi Z^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} \\ &= \sup_{Z^\pi\in\mathcal{Z}(\dot{\mathfrak{M}}^\circ)} \|\Psi^\circ(B^\pi)\phi^\circ(Z^\pi)\|_{\mathcal{B}(\mathcal{H}^\circ)}. \end{aligned} \tag{5.27}$$

On the other hand, if τ is the identical representation of the unital C^*-algebra $\widetilde{\Psi}^\circ(\mathfrak{B}^\circ)$ given by (5.18) in the Hilbert space $\mathcal{H}^\circ$, then, by (5.19), $\phi^\circ(\mathcal{Z}^\pi)$ is a central C^*-subalgebra of $\Psi^\circ(\mathfrak{B}^\circ)$ with the same unit, whose maximal ideal space is $\overline{\dot{\mathfrak{M}}^\circ}$. Since the spectral projection $P_\tau(\dot{\mathfrak{M}}^\circ)$ is the identity operator on Hilbert space $\mathcal{H}^\circ$ and since $\dot{\mathfrak{M}}^\circ$ is an open subset of $\dot{\mathfrak{M}}$, we conclude from Lemma 2.7 that

$$\|\Psi^\circ(B^\pi)\|_{\mathcal{B}(\mathcal{H}^\circ)} = \|P_\tau(\dot{\mathfrak{M}}^\circ)\Psi^\circ(B^\pi)\|_{\mathcal{B}(\mathcal{H}^\circ)} = \sup_{Z^\pi\in\mathcal{Z}(\dot{\mathfrak{M}}^\circ)} \|\Psi^\circ(B^\pi)\phi^\circ(Z^\pi)\|_{\mathcal{B}(\mathcal{H}^\circ)},$$

which together with (5.27) implies that

$$\|P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|\Psi^\circ(B^\pi)\|_{\mathcal{B}(\mathcal{H}^\circ)} \tag{5.28}$$

for all finite sums $B^\pi = \sum_{g\in F} A_g^\pi U_g^\pi \in \mathfrak{B}^\pi$ with $A_g^\pi \in \mathfrak{A}^\pi$. Because the set of such finite sums is dense in $\mathfrak{B}^\pi$, we infer from (5.28) that the mapping Ψ° given by (5.19) uniquely extends to a C^*-algebra homomorphism of $\mathfrak{B}^\pi$ into $\mathcal{B}(\mathcal{H}^\circ)$ and the mapping (5.20) uniquely extends to a C^*-algebra isomorphism $\widetilde{\Psi}^\circ$ of $\mathfrak{B}^\circ$ onto $\widetilde{\Psi}^\circ(\mathfrak{B}^\circ) = \Psi^\circ(\mathfrak{B}^\pi)$ by the rule (5.21).

Thus, for every $B \in \mathfrak{B}$, the operator $B^\circ := P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi) \in \mathfrak{B}^\circ$ is invertible on the space $P_\varphi(\dot{\mathfrak{M}}^\circ)\mathcal{H}_\varphi$ if and only if the operator $\Psi^\circ(B^\pi) \in \Psi^\circ(\mathfrak{B}^\pi)$ is invertible on the space $\mathcal{H}^\circ = l^2(\mathfrak{T}^\circ, L_2^2(\mathbb{R}))$. Finally, using the Allan-Douglas local principle (see, e.g., [10, Theorem 1.35]) for the C^*-algebra $\widetilde{\Psi}^\circ(\mathfrak{B}^\circ) = \Psi^\circ(\mathfrak{B}^\pi)$ with the central subalgebra $\Psi^\circ(\mathcal{Z}^\pi) \cong C(\overline{\dot{\mathfrak{M}}^\circ})$, we easily infer that the operator $\Psi^\circ(B^\pi)$ is invertible on the space $\mathcal{H}^\circ$ if and only if (5.22) holds. □

5.4. The C^*-algebra $\mathfrak{B}^\infty$

Finally we arrive to studying the C^*-algebra $\mathfrak{B}^\infty \subset \mathcal{B}(P_\varphi(\dot{\mathfrak{M}}^\infty)\mathcal{H}_\varphi)$ given by (4.19). In contrast to the algebras $\mathfrak{B}_{arc}$, $\mathfrak{B}_{\rm Is}$ and $\mathfrak{B}^\circ$ associated to the open subsets in decomposition (4.9) of $\dot{\mathfrak{M}}$, the set $\dot{\mathfrak{M}}^\infty$ (see (4.7)) associated to the C^*-algebra $\mathfrak{B}^\infty$ is closed. Therefore, the study of the algebra $\mathfrak{B}^\infty$ requires a methodology different of those used for the previous algebras where Lemma 2.7 was crucial.

In this section we will show that for each operator $B \in \mathfrak{B}$ the invertibility of the operators $B_{arc} = P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(B^\pi) \in \mathfrak{B}_{arc}$ and $B^\circ = P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi) \in \mathfrak{B}^\circ$ on the spaces $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$ and $P_\varphi(\dot{\mathfrak{M}}^\circ)\mathcal{H}_\varphi$, respectively, implies the invertibility of the operator $B^\infty := P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi(B^\pi)$ on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}^\infty)\mathcal{H}_\varphi$. To prove this fact we consider the C^*-algebra $\mathfrak{B} = \operatorname{alg}(PSO(\mathbb{T}), S_\mathbb{T}, U_G)$ as the C^*-algebra $\mathfrak{B} = \operatorname{alg}(\mathcal{A}, S_\mathbb{T})$ generated by the C^*-subalgebra $\mathcal{A} = \operatorname{alg}(PSO(\mathbb{T}), U_G)$ of functional operators and by the Cauchy singular integral operator $S_\mathbb{T}$. Writing the C^*-algebra $\mathfrak{B}$ in the latter form, we start with establishing a general form of the operators in $\mathfrak{B}$ (cf. [5, Theorem 10.3]).

Let $\mathfrak{H}$ denote the closed two-sided ideal of $\mathfrak{B}$ generated by all the commutators $aS_\mathbb{T} - S_\mathbb{T}aI$ with $a \in PC(\mathbb{T})$, that is, the closure of the set

$$\mathfrak{H}^0 := \Big\{\sum\nolimits_{i=1}^n B_iH_iC_i : B_i, C_i \in \mathfrak{B},\ H_i = a_iS_\mathbb{T} - S_\mathbb{T}a_iI,\ a_i \in PC(\mathbb{T}),\ n \in \mathbb{N}\Big\}.$$

The ideal $\mathfrak{H}$ contains the ideal $\mathcal{K}$ of all compact operators on $L^2(\mathbb{T})$ (see, e.g., [13]). Consequently, the commutators $aS_\mathbb{T} - S_\mathbb{T} aI$ ($a \in SO(\mathbb{T})$) and $U_g S_\mathbb{T} - S_\mathbb{T} U_g$ ($g \in G$) belong to the ideal $\mathfrak{H}$ (see (2.9) and (4.2)). Thus, for all $A \in \mathcal{A}$ the commutators $AS_\mathbb{T} - S_\mathbb{T} A$ are in $\mathfrak{H}$.

Let $\widetilde{\mathcal{A}}$ be the C^*-algebra of the 2×2 diagonal matrices with $\mathcal{A}$-valued entries. Similarly to [5], for the C^*-algebra $\mathfrak{B}$ we have the following result.

Theorem 5.8. *Every operator $B \in \mathfrak{B}$ is uniquely represented in the form*

$$B = A^+ P_\mathbb{T}^+ + A^- P_\mathbb{T}^- + H_B, \tag{5.29}$$

where $A^\pm$ are functional operators in the C^*-algebra $\mathcal{A}$, $P_\mathbb{T}^\pm = (I \pm S_\mathbb{T})/2$ are the orthogonal projections associated with the Cauchy singular integral operator $S_\mathbb{T}$, $H_B \in \mathfrak{H}$, the mapping $B \mapsto \mathrm{diag}\{A^+, A^-\}$ is a C^*-algebra homomorphism of the C^*-algebra $\mathfrak{B}$ onto the C^*-algebra $\widetilde{\mathcal{A}}$ with kernel $\mathfrak{H}$, and

$$\|A^\pm\| \le \inf_{H \in \mathfrak{H}} \|B + H\| \le |B| := \inf_{K \in \mathcal{K}} \|B + K\|. \tag{5.30}$$

Proof. Every operator $\widetilde{B} \in \mathfrak{B}$ of the form $\widetilde{B} = \sum_{i=1}^n T_{i1} T_{i2} \dots T_{ij_i}$, where $n, j_i \in \mathbb{N}$ and $T_{i,k}$ are generators of $\mathfrak{B}$, is represented in the form (5.29). Thus, the mapping $B \mapsto \mathrm{diag}\{A^+, A^-\}$, defined on the generators of the algebra $\mathfrak{B}$ by

$$aI \mapsto \mathrm{diag}\{aI, aI\}, \quad U_g \mapsto \mathrm{diag}\{U_g, U_g\}, \quad S_\mathbb{T} \mapsto \mathrm{diag}\{I, -I\},$$

is an algebraic homomorphism of the non-closed algebra $\mathfrak{B}^0$, composed by all operators $\widetilde{B}$, into $\widetilde{\mathcal{A}}$, and the kernel of this map is contained in $\mathfrak{H}$. To complete the proof, it only remains to show (5.30) for all operators $\widetilde{B} \in \mathfrak{B}^0$.

Since the ideal $\mathfrak{H} \subset \mathfrak{B}$ is generated by the commutators $aS_\mathbb{T} - S_\mathbb{T} aI$ with $a \in PC(\mathbb{T})$ and, according to (2.5),

$$(\mathrm{Sym}(aS_\mathbb{T} - S_\mathbb{T} aI))(\xi, \pm\infty) = 0_{2\times 2} \quad \text{for all } a \in PC(\mathbb{T}) \text{ and all } \xi \in M(SO(\mathbb{T})),$$

we infer from (5.9) and (5.19)–(5.20) that for any operator $H \in \mathfrak{H}$,

$$\pi_{(\xi,\pm\infty)}(H_{arc}) = 0 \quad \text{for all } \ (\xi, \pm\infty) \in \mathfrak{M}_{arc}, \tag{5.31}$$

$$H^\circ(\xi, \pm\infty) = 0_{2\times 2} \quad \text{for all } \ (\xi, \pm\infty) \in \mathfrak{M}^\circ, \tag{5.32}$$

where $H_{arc} := P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(H^\pi)$ and $H^\circ := P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(H^\pi)$. From (2.5) we also deduce that, for all $\xi \in M(SO(\mathbb{T}))$,

$$\begin{aligned} &(\mathrm{Sym}\, P_\mathbb{T}^+)(\xi, +\infty) = \mathrm{diag}\{1, 0\}, \quad (\mathrm{Sym}\, P_\mathbb{T}^+)(\xi, -\infty) = \mathrm{diag}\{0, 1\}, \\ &(\mathrm{Sym}\, P_\mathbb{T}^-)(\xi, -\infty) = \mathrm{diag}\{1, 0\}, \quad (\mathrm{Sym}\, P_\mathbb{T}^-)(\xi, +\infty) = \mathrm{diag}\{0, 1\}. \end{aligned} \tag{5.33}$$

Further, from (5.8)–(5.9), (3.13)–(3.14) and (5.19)–(5.20), (3.11) it follows that

$$\pi_{(\xi,x)}(A_{arc}^\pm) = \mathrm{diag}\big\{A_{(\xi,1)}^\pm, A_{(\xi,0)}^\pm\big\} \quad \text{for all } \ (\xi, x) \in \mathfrak{M}_{arc}, \tag{5.34}$$

$$(A^\pm)^\circ(\xi, x) = \mathrm{diag}\big\{\widehat{A^\pm}(\xi, 1), \widehat{A^\pm}(\xi, 0)\big\} \quad \text{for all } \ (\xi, x) \in \mathfrak{M}^\circ. \tag{5.35}$$

Hence, for every operator $\widetilde{B} = A^+P^+_{\mathbb{T}} + A^-P^-_{\mathbb{T}} + H_{\widetilde{B}} \in \mathfrak{B}^0$ we deduce from (5.31)–(5.35) that

$$\pi_{(\xi,\pm\infty)}(B_{arc}) = \operatorname{diag}\big\{A^{\pm}_{(\xi,1)}, A^{\mp}_{(\xi,0)}\big\} \quad \text{for all} \quad \xi \in \bigcup\nolimits_{\tau\in\mathbb{T}\setminus\Lambda} M_\tau(SO(\mathbb{T})), \tag{5.36}$$

$$B^\circ(\xi,\pm\infty) = \operatorname{diag}\big\{\widehat{A^{\pm}}(\xi,1), \widehat{A^{\mp}}(\xi,0)\big\} \quad \text{for all} \quad \xi \in \bigcup\nolimits_{\tau\in\Lambda'} M_\tau(SO(\mathbb{T})). \tag{5.37}$$

Therefore, for every $H \in \mathfrak{H}$ and every $\xi \in \bigcup_{\tau\in\mathbb{T}\setminus\Lambda} M_\tau(SO(\mathbb{T}))$, we obtain

$$\max\big\{\|A^{\pm}_{(\xi,1)}\|, \|A^{\pm}_{(\xi,0)}\|\big\} \le \big\|P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(B^\pi + H^\pi)\big\| \le \|B+H\|, \tag{5.38}$$

and, for every $\xi \in \bigcup_{\tau\in\Lambda'} M_\tau(SO(\mathbb{T}))$,

$$\max\big\{|\widehat{A^{\pm}}(\xi,1)|, |\widehat{A^{\pm}}(\xi,0)|\big\} \le \big\|P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi + H^\pi)\big\| \le \|B+H\|. \tag{5.39}$$

By Lemma 3.2, we obtain

$$\|A\| = \max\big\{\|\chi^\circ A\|,\ \|\chi_{arc}A\|,\ \|\chi_*\, A\|\big\} \quad \text{for all } A \in \mathcal{A}, \tag{5.40}$$

where, by Theorems 3.3, 3.4 and the property $\chi_*\, A = \chi_*\, aI$ ($a \in PSO(\mathbb{T})$),

$$\|\chi^\circ A\| = \max_{(\xi,\mu)\in\widetilde{\mathcal{M}}^\circ} |\widehat{A}(\xi,\mu)|,\ \ \|\chi_{arc}A\| = \sup_{(\xi,\mu)\in\mathfrak{R}_{arc}} \|A_{(\xi,\mu)}\|,\ \ \|\chi_*\, A\| \le \max_{(\xi,\mu)\in\mathcal{M}_*} |\widehat{A}(\xi,\mu)|. \tag{5.41}$$

Finally, since $\widetilde{\mathcal{M}}^\circ \cup \mathcal{M}_*$ is contained in $\bigcup_{\tau\in\Lambda'} M_\tau(SO(\mathbb{T})) \times \{0,1\}$, we infer from (5.38)–(5.41) that $\|A^{\pm}\| \le \|B+H\|$ for all $B \in \mathfrak{B}^0$ and all $H \in \mathfrak{H}$, which immediately implies (5.30). ☐

Similarly to [5, Lemma 10.5], one can prove the following result.

Lemma 5.9. *For every $\tau \in \mathbb{T}$, $P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi(V^\pi_\tau) = 0$. Consequently, for all $H \in \mathfrak{H}$,*

$$P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi(H^\pi) = 0. \tag{5.42}$$

It follows from Lemma 5.9 that if an operator $B \in \mathfrak{B}$ is written in the form (5.29), then according to (5.42),

$$B^\infty := P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi(B^\pi) = P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi((A^+P^+_{\mathbb{T}} + A^-P^-_{\mathbb{T}})^\pi), \tag{5.43}$$

which implies that the operators $H \in \mathfrak{H}$ do not have influence on the operators in the C^*-algebra $\mathfrak{B}^\infty$. Finally we get the desired result.

Theorem 5.10. *If $B \in \mathfrak{B}$ is written in the form* (5.29) *and the operators*

$$B_{arc} = P_\varphi(\dot{\mathfrak{M}}_{arc})\varphi(B^\pi) \quad \text{and} \quad B^\circ = P_\varphi(\dot{\mathfrak{M}}^\circ)\varphi(B^\pi)$$

are invertible on the Hilbert spaces $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$ and $P_\varphi(\dot{\mathfrak{M}}^\circ)\mathcal{H}_\varphi$, respectively, then the operator $B^\infty = P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi(B^\pi)$ is invertible on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}^\infty)\mathcal{H}_\varphi$.

Proof. Suppose that the operators B_{arc} and B° are invertible on the Hilbert spaces $P_\varphi(\dot{\mathfrak{M}}_{arc})\mathcal{H}_\varphi$ and $P_\varphi(\dot{\mathfrak{M}}^\circ)\mathcal{H}_\varphi$, respectively. Then, by Theorem 5.2, the operators $\pi_{(\xi,x)}(B_{arc})$ are invertible on the Hilbert space $l^2(G,\mathbb{C}^2)$ for all $(\xi,x) \in \mathfrak{N}_{arc}$ and condition (5.11) is fulfilled. Further, by Theorem 5.7, the matrices $B^\circ(\xi,x)$ are invertible for all $(\xi,x) \in \mathfrak{M}^\circ$. In particular, the operators $\pi_{(\xi,\pm\infty)}(B_{arc})$ are invertible on the space $l^2(G, \mathbb{C}^2)$ for all $(\xi, \pm\infty) \in \mathfrak{R}_{arc}$ (see (3.15)), and $B^\circ(\xi, \pm\infty)$ are invertible for all $\xi \in \bigcup_{t\in\overline{\Lambda^\circ}} M_t(SO(\mathbb{T}))$. Hence, from (5.36) it follows that all the operators $A^{\pm}_{(\xi,\mu)}$ are invertible on the space $l^2(G)$ for $(\xi, \mu) \in \mathfrak{R}_{arc}$ and

$$\sup_{(\xi,\mu)\in\mathfrak{R}_{arc}} \big\|\big(A^{\pm}_{(\xi,\mu)}\big)^{-1}\big\| < \infty.$$

On the other hand, we deduce from (5.37) and (5.22) that $\widehat{A^{\pm}}(\xi, \mu) \neq 0$ for all $(\xi, \mu) \in \bigcup_{t\in\overline{\Lambda^\circ}} M_t(SO(\mathbb{T}))\times\{0, 1\}$. Then, by Theorem 3.7, the functional operators $A^{\pm}$ are invertible on the space $L^2(\mathbb{T})$, which implies the invertibility of the operators $A^{\pm}_{\infty} := P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi((A^{\pm})^\pi)$ on the Hilbert space $P_\varphi(\dot{\mathfrak{M}}^\infty)\mathcal{H}_\varphi$. Let $(A^{\pm}_{\infty})^{-1}$ be the inverses of the operators $A^{\pm}_{\infty}$.

Finally, we only need to observe that the operator

$$(B^\infty)^{-1} := (A^{+}_{\infty})^{-1}P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi((P^{+}_{\mathbb{T}})^\pi) + (A^{-}_{\infty})^{-1}P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi((P^{-}_{\mathbb{T}})^\pi)$$

is the inverse to the operator (5.43). This is a consequence of the equalities

$$P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi(A^\pi)P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi((P^{\pm}_{\mathbb{T}})^\pi) = P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi((P^{\pm}_{\mathbb{T}})^\pi)P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi(A^\pi) \quad (A \in \mathcal{A}),$$
$$P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi((P^{+}_{\mathbb{T}})^\pi)P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi((P^{-}_{\mathbb{T}})^\pi) = P_\varphi(\dot{\mathfrak{M}}^\infty)\varphi((P^{+}_{\mathbb{T}}P^{-}_{\mathbb{T}})^\pi) = 0$$

following from Lemma 5.9. □

5.5. Symbol calculus and a Fredholm criterion for the C^*-algebra $\mathfrak{B}$

Consider the C^*-algebra $\mathfrak{B} = \mathrm{alg}\,(PSO(\mathbb{T}), S_{\mathbb{T}}, U_G) \subset \mathcal{B}(L^2(\mathbb{T}))$ and the sets

$$\mathfrak{N}_{arc} = \bigcup\nolimits_{\tau\in\mathcal{O}_{arc}} M_\tau(SO(\mathbb{T})) \times \overline{\mathbb{R}}, \quad \dot{\mathfrak{M}}_{\mathrm{Is}} = \bigcup\nolimits_{\tau\in\mathrm{Is}\,\Lambda} M_\tau(SO(\mathbb{T})) \times \mathbb{R},$$
$$\mathfrak{M}^\circ = \bigcup\nolimits_{\tau\in\Lambda'} M_\tau(SO(\mathbb{T})) \times \overline{\mathbb{R}},$$

related to the partition $\mathbb{T} = \mathbb{T}_{arc} \cup \mathrm{Is}\,\Lambda \cup \Lambda'$, where the set $\mathcal{O}_{arc} \subset \mathbb{T}_{arc}$ contains exactly one point in each orbit defined by the group of shifts G on $\mathbb{T}_{arc} = \mathbb{T} \setminus \Lambda$.

For each $(\xi, x) \in \mathfrak{N}_{arc}$, we introduce the representation

$$\Phi_{\xi,x} : \mathfrak{B} \to \mathcal{B}(l^2(G, \mathbb{C}^2)), \quad B \mapsto \Phi_{\xi,x}(B) := \pi_{(\xi,x)}(B_{arc}) \tag{5.44}$$

given on the generators of the C^*-algebra $\mathfrak{B}$, according to (5.8)–(5.9) and (2.5), by

$$\begin{aligned} &[\Phi_{\xi,x}(aI)f](g) = \mathrm{diag}\big\{(a\circ g)(\xi, 1), (a\circ g)(\xi, 0)\big\}f(g), \\ &[\Phi_{\xi,x}(S_{\mathbb{T}})f](g) = \begin{pmatrix} \tanh(\pi x) & i/\cosh(\pi x) \\ -i/\cosh(\pi x) & -\tanh(\pi x)\end{pmatrix} f(g), \\ &[\Phi_{\xi,x}(U_h)f](g) = f(gh), \end{aligned} \tag{5.45}$$

where $a \in PSO(\mathbb{T})$, $a(\xi, \mu)$ is the value of the Gelfand transform of a at the point $(\xi, \mu) \in M(PSO(\mathbb{T}))$, $g, h \in G$, and $f \in l^2(G, \mathbb{C}^2)$.

For each $(\xi, x) \in \dot{\mathfrak{M}}_{\mathrm{Is}}$, we introduce the representation

$$\Phi_{\xi,x} : \mathfrak{B} \to \mathcal{B}(\mathbb{C}^2), \quad B \mapsto B^\circ_t(\xi, x)I, \tag{5.46}$$

where $t \in \operatorname{Is}\Lambda$ is such that $\xi \in M_t(SO(\mathbb{T}))$ and, for all $(\xi, x) \in M_t(SO(\mathbb{T})) \times \mathbb{R}$, the 2×2 matrices $B_t^\circ(\xi, x)$ are given for the generators of $\mathfrak{B}$, according to (5.12) and Theorem 5.3, by

$$(aI)_t^\circ(\xi, x) = \begin{pmatrix} a(\xi,1) & 0 \\ 0 & a(\xi,0) \end{pmatrix}, \quad (S_\mathbb{T})_t^\circ(\xi, x) = \begin{pmatrix} \tanh(\pi x) & i/\cosh(\pi x) \\ -i/\cosh(\pi x) & -\tanh(\pi x) \end{pmatrix},$$

$$(U_h)_t^\circ(\xi, x) = \operatorname{diag}\big\{e^{ix\ln h'(t)}, e^{ix\ln h'(t)}\big\}, \quad \text{where } a \in PSO(\mathbb{T}),\ h \in G. \tag{5.47}$$

For each $(\xi, x) \in \mathfrak{M}^\circ$, we introduce the representation

$$\Phi_{\xi,x} : \mathfrak{B} \to \mathcal{B}(\mathbb{C}^2), \quad B \mapsto B^\circ(\xi, x)I, \tag{5.48}$$

where the 2×2 matrices $B^\circ(\xi, x)$ are given for the generators of $\mathfrak{B}$, in view of (5.19) and Theorem 5.7, by

$$(aI)^\circ(\xi, x) = \begin{pmatrix} a(\xi,1) & 0 \\ 0 & a(\xi,0) \end{pmatrix}, \quad (S_\mathbb{T})^\circ(\xi, x) = \begin{pmatrix} \tanh(\pi x) & i/\cosh(\pi x) \\ -i/\cosh(\pi x) & -\tanh(\pi x) \end{pmatrix},$$

$$(U_h)^\circ(\xi, x) = \operatorname{diag}\big\{1, 1\big\}, \quad \text{where } a \in PSO(\mathbb{T}),\ h \in G. \tag{5.49}$$

Combining Theorems 4.2, 5.2, 5.4, 5.7 and 5.10, we get the following criterion.

Theorem 5.11. *An operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$ if and only if the following three conditions are satisfied:*

(i) *for all $(\xi, x) \in \mathfrak{N}_{arc}$, the operators $\Phi_{\xi,x}(B)$ are invertible on the space $l^2(G, \mathbb{C}^2)$ and $\sup_{(\xi,x)\in\mathfrak{N}_{arc}} \big\|(\Phi_{\xi,x}(B))^{-1}\big\| < \infty$;*

(ii) $\inf_{t\in\operatorname{Is}\Lambda} \min_{\xi\in M_t(SO(\mathbb{T}))} \inf_{x\in\mathbb{R}} \big|\det(B_t^\circ(\xi, x))\big| > 0$*;*

(iii) *for all $(\xi, x) \in \mathfrak{M}^\circ$,* $\det\left(B^\circ(\xi, x)\right) \neq 0$.

Consider now the Hilbert space

$$\mathcal{H}_\Phi := \bigoplus_{(\xi,x)\in\mathfrak{N}_{arc}} l^2(G, \mathbb{C}^2) \oplus \bigoplus_{(\xi,x)\in\dot{\mathfrak{N}}_{\rm Is}\cup\mathfrak{M}^\circ} \mathbb{C}^2.$$

Then the mapping

$$\Phi : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_\Phi), \quad B \mapsto \Phi(B) := \bigoplus_{(\xi,x)\in\mathfrak{N}_{arc}\cup\dot{\mathfrak{N}}_{\rm Is}\cup\mathfrak{M}^\circ} \Phi_{\xi,x}(B),$$

where the operators $\Phi_{\xi,x}(B)$ are given by (5.44)–(5.49), is a representation of the C^*-algebra $\mathfrak{B}$ in the Hilbert space $\mathcal{H}_\Phi$, with $\operatorname{Ker}\Phi = \mathcal{K}$. Since $\mathfrak{B}^\pi \cong \Phi(\mathfrak{B})$, we may refer the operator function $\Phi(B)$ defined on the set $\mathfrak{N}_{arc} \cup \dot{\mathfrak{N}}_{\rm Is} \cup \mathfrak{M}^\circ$ by $(\xi, x) \mapsto \Phi_{\xi,x}(B)$ to as the *symbol* of an operator $B \in \mathfrak{B}$. Hence, Theorem 5.11 can be rewritten in the following form.

Theorem 5.12. *An operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$ if and only if its symbol $\Phi(B)$ is invertible, that is, the operator $\Phi(B)$ is invertible on the Hilbert space $\mathcal{H}_\Phi$.*

References

[1] A.B. Antonevich, *Linear Functional Equations. Operator Approach.* Operator Theory: Advances and Appl. **83**, Birkhäuser Verlag, Basel, 1996. Russian original: University Press, Minsk, 1988.

[2] A. Antonevich and A. Lebedev, *Functional Differential Equations:* I. C^**-Theory.* Pitman Monographs and Surveys in Pure and Applied Mathematics **70**, Longman Scientific & Technical, Harlow, 1994.

[3] A. Antonevich, M. Belousov, and A. Lebedev, *Functional Differential Equations:* II. C^**-Applications. Part 2 Equations with Discontinuous Coefficients and Boundary Value Problems.* Pitman Monographs and Surveys in Pure and Applied Mathematics **95**, Longman Scientific & Technical, Harlow, 1998.

[4] M.A. Bastos, C.A. Fernandes, and Yu.I. Karlovich, C^**-algebras of integral operators with piecewise slowly oscillating coefficients and shifts acting freely.* Integral Equations and Operator Theory **55** (2006), 19–67.

[5] M.A. Bastos, C.A. Fernandes, and Yu.I. Karlovich, *Spectral measures in* C^**-algebras of singular integral operators with shifts.* J. Funct. Analysis **242** (2007), 86–126.

[6] A. Böttcher, Yu.I. Karlovich, and V.S. Rabinovich, *The method of limit operators for one-dimensional singular integrals with slowly oscillating data.* J. Operator Theory **43** (2000), 171–198.

[7] A. Böttcher, Yu.I. Karlovich, and B. Silbermann, *Singular integral equations with PQC coefficients and freely transformed argument.* Math. Nachr. **166** (1994), 113–133.

[8] A. Böttcher, Yu.I. Karlovich, and I.M. Spitkovsky, *The* C^**-algebra of singular integral operators with semi-almost periodic coefficients.* J. Funct. Analysis **204** (2003), 445–484.

[9] A. Böttcher, S. Roch, B. Silbermann, and I.M. Spitkovsky, *A Gohberg-Krupnik-Sarason symbol calculus for algebras of Toeplitz, Hankel, Cauchy, and Carleman operators.* In: Operator Theory: Advances and Appl. **48**, Birkhäuser, Basel, 1990, pp. 189–234.

[10] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, 2nd edition. Springer, Berlin, 2006.

[11] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics. I:* C^**- and* W^**-algebras, Symmetry Groups, Decomposition of States.* Springer, New York, 1979.

[12] J. Dixmier, C^**-Algebras.* North-Holland, Amsterdam, 1977.

[13] I. Gohberg and N. Krupnik, *On the algebra generated by the one-dimensional singular integral operators with piecewise continuous coefficients.* Funct. Anal. Appl. **4** (1970), 193–201.

[14] F.P. Greenleaf, *Invariant Means on Topological Groups and Their Representations.* Van Nostrand-Reinhold, New York, 1969.

[15] Yu.I. Karlovich, *The local-trajectory method of studying invertibility in* C^**-algebras of operators with discrete groups of shifts.* Soviet Math. Dokl. **37** (1988), 407–411.

[16] Yu.I. Karlovich, C^**-algebras of operators of convolution type with discrete groups of shifts and oscillating coefficients.* Soviet Math. Dokl. **38** (1989), 301–307.

[17] Yu.I. Karlovich, *A local-trajectory method and isomorphism theorems for nonlocal C^*-algebras.* In: Operator Theory: Advances and Appl. **170**, Birkhäuser, Basel, 2006, pp. 137–166.

[18] Yu.I. Karlovich and V.G. Kravchenko. *A Noether theory for a singular integral operator with a shift having periodic points.* Soviet Math. Dokl. **17** (1976), 1547–1551.

[19] Yu.I. Karlovich and V.G. Kravchenko, *An algebra of singular integral operators with piecewise-continuous coefficients and piecewise-smooth shift on a composite contour.* Math. USSR-Izv. **23** (1984), 307–352.

[20] Yu.I. Karlovich and B. Silbermann, *Fredholmness of singular integral operators with discrete subexponential groups of shifts on Lebesgue spaces.* Math. Nachr. **272** (2004), 55–94.

[21] V.G. Kravchenko and G.S. Litvinchuk. *Introduction to the Theory of Singular Integral Operators with Shift.* Mathematics and its Applications **289**, Kluwer Academic Publishers, Dordrecht, 1994.

[22] G.S. Litvinchuk. *Boundary Value Problems and Singular Integral Equations with Shift.* Nauka, Moscow, 1977 [Russian].

[23] P.S. Muhly and J. Xia, *Calderón-Zygmund operators, local mean oscilation and certain automorphisms of the Toeplitz algebra.* Amer. J. Math. **117** (1995), 1157–1201.

[24] M.A. Naimark, *Normed Algebras.* Wolters-Noordhoff, Groningen, The Netherlands, 1972.

[25] V. Rabinovich, S. Roch and B. Silbermann, *Limit Operators and Their Applications in Operator Theory.* Operator Theory: Advances and Appl. **150**, Birkhäuser, Basel, 2004.

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