Near-exact approximations for the likelihood ratio test statistic for testing equality of several variance-covariance matrices

Carlos A. Coelho [1] Filipe J. Marques [1]

Mathematics Department, Faculty of Sciences and Technology
The New University of Lisbon, Portugal

Abstract

While on one hand, the exact distribution of the likelihood ratio test statistic for testing the equality of several variance-covariance matrices, as it also happens with several other likelihood ratio test statistics used in Multivariate Statistics, has a non-manageable form, which does not allow for the computation of quantiles, even for a small number of variables, on the other hand the asymptotic approximations available do not have the necessary quality for small sample sizes. This way, the development of near-exact approximations to the distribution of this statistic is a good goal. Starting from a factorization of the exact characteristic function for the statistic under study and by adequately replacing some of the factors, we obtain a near-exact characteristic function which determines the near-exact distribution for the statistic. This near-exact distribution takes the form of either a GNIG (Generalized Near-Integer Gamma) distribution or a mixture of GNIG distributions. The evaluation of the performance of the near-exact and asymptotic distributions developed is done through the use of two measures based on the charactersitic function with which we are able to obtain good upper-bounds on the absolute value of the difference between the exact and approximate probability density or cumulative distribution functions. As a reference we use the asymptotic distribution proposed by Box.

Key words: Mixtures, Gamma distribution, Generalized Near-Integer Gamma distribution, mixtures.

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1 Introduction

The results presented in this paper, together with the ones already published on the Wilks Λ statistic (Coelho, 2003, 2004; Alberto and Coelho, 2007; Grilo and Coelho, 2007) and the ones on the sphericity likelihood ratio test statistic (Marques and Coelho, 2007), are intended to be used as the basis for two future works: one on a common approach for the more common likelihood ratio test statistics used in Multivariate Analysis (the Wilks Λ statistic, the statistic to test the equality of several variance-covariance matrices, the statistic to test the equality of several multivariate Normal distributions and the sphericity test statistic) which will recall the common traits of these statistics both in terms of their exact and near-exact distributions, and the other on a general approach for two families of generalized sphericity tests, which we may call as multi-sample block-scalar and multi-sample block-matrix spericity tests, their common links and particular cases.

In this paper we will be dealing with the likelihood ratio test statistic to test the equality of several variance-covariance matrices, under the assumption of underlying multivariate normal distributions. We will show how we can see the exact distribution of this statistic as the distribution of a product of independent Gamma random variables and then how we can factorize the characteristic function of the logarithm of this statistic into a term that is the characteristic function of a Generalized Integer Gamma distribution (Coelho, 1998) and another term that is the characteristic function of a sum of independent random variables whose exponentials have Beta distributions. From this decomposition we will be able to build near-exact distributions for the logarithm of the likelihood ratio test statistic as well as for the statistic itself. The closeness of these approximate distributions to the exact distribution will be assessed and measured through the use of two measures, presented in Section 5, derived from inversion formulas, one of which has a link with the Berry-Esseen bound.

Let us suppose that we have q independent samples from q multivariate Normal distributions $N_p(\mu_j, \Sigma_j)$ $(j = 1, \ldots, q)$, the j-th sample having size $n + 1$, and that we want to test the null hypothesis

$$H_0 : \Sigma_1 = \Sigma_2 = \ldots = \Sigma_q \ (= \Sigma) \qquad (\text{with } \Sigma \text{ unspecified}) . \tag{1}$$

By using the modified likelihood ratio test statistic we obtain an unbiased test (Bartlet, 1937; Muirhead, 1982, Sec. 8.2). The modified likelihood ratio test statistic (where the sample sizes are replaced by the number of degrees of freedom of the Wishart distributions) may be written as (Bartlett, 1937;

Anderson, 1958; Muirhead, 1982)

$$\lambda^* = \frac{(nq)^{npq/2}}{\prod_{j=1}^{q} n^{pn/2}} \frac{\prod_{j=1}^{q} |A_j|^{n/2}}{|A|^{nq/2}}, \tag{2}$$

where A_j is the matrix of corrected sums of squares and products formed from the j-th sample and $A = A_1 + \ldots + A_q$.

The h-th moment of λ^* in (2) is (Muirhead, 1982)

$$E\left[(\lambda^*)^h\right] = q^{npqh/2} \prod_{j=1}^{p} \frac{\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{j}{2}\right)}{\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{j}{2}+\frac{nq}{2}h\right)} \times \prod_{j=1}^{p}\prod_{k=1}^{q} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{j}{2}+\frac{n}{2}h\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{j}{2}\right)} \tag{3}$$

$$\left(h > \frac{p-n-1}{n}\right).$$

From (3) we may write the c.f. (characteristic function) for the r.v. (random variable) $W = -\log(\lambda^*)$ as

$$\begin{aligned}\Phi_W(t) &= E\left[e^{Wit}\right] = E\left[e^{-\log(\lambda^*)it}\right] = E\left[(\lambda^*)^{-it}\right] \\ &= q^{-npqit/2} \prod_{j=1}^{p} \frac{\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{j}{2}\right)}{\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{j}{2}-\frac{nq}{2}it\right)} \\ &\quad\times \prod_{j=1}^{p}\prod_{k=1}^{q} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{j}{2}-\frac{n}{2}it\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{j}{2}\right)}.\end{aligned} \tag{4}$$

It will be based on this expression that we will obtain, in Section 3, decompositions of the c.f. of W that will be used to build near-exact distributions for W and λ^*.

2 Asymptotic distributions

Box (1949) proposes for the statistic $W = -\log(\lambda^*)$ an asymptotic distribution based on an expansion of the form

$$P(2\rho W \leq z) = (1-\omega)P(\chi^2_g \leq z) + \omega P(\chi^2_{g+4} \leq z) + O\left((nq)^{-3}\right)$$

where

$$g = \frac{1}{2}(q-1)p(p+1)\,, \qquad \rho = 1 - \frac{q+1}{nq}\,\frac{2p^2+3p-1}{6(p+1)}$$

and

$$\omega = \frac{1}{48\rho^2}p(p+1)\left\{(p-1)(P-2)\frac{q^3-1}{n^2q^2} - 6(q-1)(1-\rho)^2\right\}$$

and where $P(\chi^2_g \leq z)$ stands for the value of the c.d.f. (cumulative distribution function) of a chi-square r.v. with g degrees of freedom evaluated at $z(>0)$.

However, taking into accout that

$$\text{if} \quad X \sim \chi^2_g \equiv \Gamma\left(\frac{g}{2},\frac{1}{2}\right) \qquad \text{then} \qquad \frac{X}{2\rho} \sim \Gamma\left(\frac{g}{2},\rho\right)\,,$$

where we use the notation

$$X \sim \Gamma(r,\lambda)$$

to denote the fact that the r.v. X has pdf (probability density function)

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)}\, e^{-\lambda x}\, x^{r-1}\,, \qquad (x>0;\, r,\lambda>0)$$

we may write

$$P(W \leq z) \;\approx\; (1-\omega)P\left(\Gamma\left(\frac{g}{2},\rho\right) \leq z\right) + \omega\, P\left(\Gamma\left(\frac{g}{2}+2,\rho\right) \leq z\right)\,,$$

where $P\left(\Gamma\left(\nu,\rho\right) \leq z\right)$ stands for the value of the c.d.f. of a $\Gamma(\nu,\rho)$ distributed r.v. evaluated at $z(>0)$.

We may thus write, for the c.f. of W,

$$\Phi_W(t) \approx \Phi_{Box}(t) \;=\; (1-\omega)\rho^{g/2}(\rho-it)^{-g/2} + \omega\,\rho^{2+g/2}(\rho-it)^{-2-g/2}\,. \quad (5)$$

Somehow inspired on this asymptotic approximation due to Box, which ultimately approximates the exact c.f. of W by the c.f. of a mixture of Gamma distributions and also on Box's introduction of his 1949 paper (Box, 1949), where he states that "Although in many cases the exact distributions cannot be obtained in a form which is of practical use, it is usually possible to obtain the moments, and these may be used to obtain approximations. In some cases, for instance, a suitable power of the likelihood statistic has been found to be distributed approximately in the type I form, and good approximations have been obtained by equating the moments of the likelihood statistic to this curve.", we propose two other mixtures of Gamma distributions, all with the same rate parameter, which match the first four or six exact moments to approximate asymptotically the c.f. of $W = -\log(\lambda^*)$, for increasing values of nq, the number of degrees of freedom of the Wishart distribution of matrix A.

These distributions are: the mixture of two Gamma distributions, both with the same rate parameter, with characteristic function

$$\Phi_{M2G}(t) = \sum_{j=1}^{2} p_{2,j}\, \lambda_2^{r_{2,j}} (\lambda_2 - \mathrm{i}t)^{-r_{2,j}}\,, \tag{6}$$

where $p_{2,2} = 1 - p_{2,1}$ with $p_{2,j},\, r_{2,j},\, \lambda_2 > 0$, and the mixture of three Gamma distributions, all with the same rate parameter, with characteristic function

$$\Phi_{M3G}(t) = \sum_{j=1}^{3} p_{3,j}\, \lambda_3^{r_{3,j}} (\lambda_3 - \mathrm{i}t)^{-r_{3,j}}\,, \tag{7}$$

where $p_{3,3} = 1 - p_{3,1} - p_{3,2}$, with $p_{3,j},\, r_{3,j},\, \lambda_3 > 0$.

The parameters in (6) and (7) are respectively obtained by solving the systems of equations

$$\mathrm{i}^h \sum_{j=1}^{k} p_{k,j}\, \frac{\Gamma(r_{k,j}+h)}{\Gamma(r_{k,j})}\, \lambda_k^{-h} = \left.\frac{\partial^h \Phi_{MkG}(t)}{\partial t^h}\right|_{t=0} = \left.\frac{\partial^h \Phi_{W}(t)}{\partial t^h}\right|_{t=0}\,, \tag{8}$$

with $h = 1, \ldots, 2k$ and making $k = 2$ for the parameters in (6) and $k = 3$ for the parameters in (7).

Actually, since, as shown in Lemma 1 ahead, $\Phi_W(t)$ is the characteristic function of a sum of independent log Beta random variables, the approximation of the distribution of W by a mixture of Gamma distributions is a well justified procedure, since as Coelho et al. (2006) proved, a log Beta distribution may be represented as an infinite mixture of Exponential distributions, and as such a sum of independent log Beta random variables may be represented as an infinite mixture of sums of independent Exponential distributions, which are particular Generalized Integer Gamma distributions (Coelho, 1998). Thus, the use of a finite mixture of Gamma distributions to replace a log Beta distribution seems to be a much adequate simplification.

3 The characteristic function of $W = -\log(\lambda^*)$

In this section we will present several results that will enable us to obtain a decomposition of the c.f. of W that will be used to build near-exact distributions for W and λ^*.

In a first step we will show how the characteristic function of W may be split in two parts, one of them being the c.f. of the sum of independent Logbeta r.v.'s and the other the c.f. of the sum of independent Exponential r.v.'s. In a second step we will identify the Exponential distributions involved and devise a method to count and obtain the corresponding analytic expressions for the number of different Exponential distributions involved.

Lemma 1: The characteristic function of $W = -\log\lambda^*$ may be written as

$$\begin{aligned}\Phi_W(t) = {} & \underbrace{\prod_{j=1}^{\lfloor p/2\rfloor}\prod_{k=1}^{q}\frac{\Gamma(a_j+b^*_{jk})}{\Gamma(a_j)}\frac{\Gamma(a_j-nit)}{\Gamma(a_j+b^*_{jk}-nit)} \times\left(\prod_{k=1}^{q}\frac{\Gamma(a_p+b^*_{pk})}{\Gamma(a_p)}\frac{\Gamma\left(a_p-\frac{n}{2}\text{i}t\right)}{\Gamma(a_p+b^*_{pk}-\frac{n}{2}it)}\right)^{p\perp\!\!\!\perp 2}}_{\Phi_1(t)} \\ & \underbrace{\times\prod_{j=1}^{\lfloor p/2\rfloor}\prod_{k=1}^{q}\frac{\Gamma(a_j+b_{jk})}{\Gamma(a_j+b^*_{jk})}\frac{\Gamma(a_j+b^*_{jk}-nit)}{\Gamma(a_j+b_{jk}-nit)} \times\left(\prod_{k=1}^{q}\frac{\Gamma(a_p+b_{pk})}{\Gamma(a_p+b^*_{pk})}\frac{\Gamma(a_p+b^*_{pk}-\frac{n}{2}it)}{\Gamma\left(a_p+b_{pk}-\frac{n}{2}it\right)}\right)^{p\perp\!\!\!\perp 2}}_{\Phi_2(t)},\end{aligned} \tag{9}$$

where $p \perp\!\!\!\perp 2$ represents the remainder of the integer division of p by 2,

$$a_j = n+1-2j\,, \qquad b_{jk} = 2j-1+\frac{k-2j}{q}\,, \tag{10}$$

$$a_p = \frac{n+1-p}{2}\,, \qquad b_{pk} = \frac{pq-q-p+2k-1}{2q}\,, \tag{11}$$

$$b^*_{jk} = \lfloor b_{jk}\rfloor \quad \text{and} \quad b^*_{pk} = \lfloor b_{pk}\rfloor\,. \tag{12}$$

Proof: Using the fact that

$$\Gamma(2z) = \pi^{-1/2}\,2^{2z-1}\,\Gamma(z)\,\Gamma(z+1/2)$$

we may write the c.f. of $W = -\log \lambda^*$ as

$$\Phi_W(t) = q^{-npqit/2} \prod_{j=1}^{\lfloor p/2 \rfloor} \left\{ \frac{\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{2j-1}{2}\right)\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{2j}{2}\right)}{\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{2j-1}{2}-\frac{nq}{2}it\right)\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{2j}{2}-\frac{nq}{2}it\right)} \right.$$
$$\left. \prod_{k=1}^{q} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{2j-1}{2}-\frac{n}{2}it\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{2j}{2}-\frac{n}{2}it\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{2j-1}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{2j}{2}\right)} \right\}$$
$$\times \left(\frac{\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{p}{2}\right)}{\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{p}{2}-\frac{nq}{2}it\right)} \prod_{k=1}^{q} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{p}{2}-\frac{n}{2}it\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{p}{2}\right)} \right)^{p \perp\!\!\!\perp 2}$$
$$= q^{-npqit/2} \prod_{j=1}^{\lfloor p/2 \rfloor} \frac{\Gamma(nq+1-2j)}{\Gamma(nq+1-2j-nqit)} \prod_{k=1}^{q} \frac{\Gamma(n+1-2j-nit)}{\Gamma(n+1-2j)}$$
$$\times \left(\frac{\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{p}{2}\right)}{\Gamma\left(\frac{nq}{2}+\frac{1}{2}-\frac{p}{2}-\frac{nq}{2}it\right)} \prod_{k=1}^{q} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{p}{2}-\frac{n}{2}it\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{p}{2}\right)} \right)^{p \perp\!\!\!\perp 2} .$$

Then, using

$$\Gamma(mz) = (2\pi)^{-\frac{m-1}{2}} m^{mz-1/2} \prod_{k=1}^{m} \Gamma\left(z+\frac{k-1}{m}\right)$$

we may write

$$\Phi_W(t) = q^{-npqit/2}$$
$$\times \prod_{j=1}^{\lfloor p/2 \rfloor} q^{nqit} \prod_{k=1}^{q} \frac{\Gamma\left(n+\frac{1}{q}-\frac{2j}{q}+\frac{k-1}{q}\right)}{\Gamma\left(n+\frac{1}{q}-\frac{2j}{q}+\frac{k-1}{q}-nit\right)} \frac{\Gamma(n+1-2j-nit)}{\Gamma(n+1-2j)}$$
$$\times \left(q^{\frac{nq}{2}it} \prod_{k=1}^{q} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2q}-\frac{p}{2q}+\frac{k-1}{q}\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2q}-\frac{p}{2q}+\frac{k-1}{q}-\frac{n}{2}it\right)} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{p}{2}-\frac{n}{2}it\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{p}{2}\right)} \right)^{p \perp\!\!\!\perp 2}$$
$$= \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \frac{\Gamma\left(n+\frac{1}{q}-\frac{2j}{q}+\frac{k-1}{q}\right)}{\Gamma\left(n+\frac{1}{q}-\frac{2j}{q}+\frac{k-1}{q}-nit\right)} \frac{\Gamma(n+1-2j-nit)}{\Gamma(n+1-2j)}$$
$$\times \left(\prod_{k=1}^{q} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2q}-\frac{p}{2q}+\frac{k-1}{q}\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2q}-\frac{p}{2q}+\frac{k-1}{q}-\frac{n}{2}it\right)} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{p}{2}-\frac{n}{2}it\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\frac{p}{2}\right)} \right)^{p \perp\!\!\!\perp 2}$$

considering that, since for any $p \in I\!N$, $\lfloor p/2 \rfloor + \lfloor (p+1)/2 \rfloor = p$,

$$q^{-npqit/2} \left(q^{\frac{nq}{2}\mathrm{it}}\right)^{\lfloor \frac{p+1}{2} \rfloor - \lfloor \frac{p}{2} \rfloor} \prod_{j=1}^{\lfloor p/2 \rfloor} q^{nq\mathrm{it}} = q^{-npqit/2+nq\lfloor p/2 \rfloor \mathrm{it}/2+nq\lfloor \frac{p+1}{2} \rfloor \mathrm{it}/2} = 1 .$$

Taking then a_j, b_{jk}, a_p and b_{pk} defined as in (10) and (11), we may write

$$\Phi_W(t) = \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \frac{\Gamma(a_j+b_{kj})}{\Gamma(a_j+b_{kj}-nit)} \frac{\Gamma(a_j-nit)}{\Gamma(a_j)} \\ \times \left(\prod_{k=1}^{q} \frac{\Gamma(a_p+b_{pk})}{\Gamma\left(a_p+b_{pk}-\frac{n}{2}it\right)} \frac{\Gamma\left(a_p-\frac{n}{2}it\right)}{\Gamma(a_p)} \right)^{p \perp\!\!\!\perp 2}$$

that taking b^*_{jk} and b^*_{pk} given by (12) may be written as

$$\Phi_W(t) = \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \left\{ \frac{\Gamma(a_j+b^*_{kj})}{\Gamma(a_j)} \frac{\Gamma(a_j-nit)}{\Gamma(a_j+b^*_{kj}-nit)} \right. \\ \left. \times \frac{\Gamma(a_j+b_{kj})}{\Gamma(a_j+b^*_{kj})} \frac{\Gamma(a_j+b^*_{kj}-nit)}{\Gamma(a_j+b_{kj}-nit)} \right\} \\ \times \left(\prod_{k=1}^{q} \frac{\Gamma(a_p+b_{pk})}{\Gamma(a_p+b^*_{pk})} \frac{\Gamma(a_p+b^*_{pk}-\frac{n}{2}it)}{\Gamma\left(a_p+b_{pk}-\frac{n}{2}it\right)} \right)^{p \perp\!\!\!\perp 2} \\ \times \left(\prod_{k=1}^{q} \frac{\Gamma(a_p+b^*_{pk})}{\Gamma(a_p)} \frac{\Gamma\left(a_p-\frac{n}{2}it\right)}{\Gamma(a_p+b^*_{pk}-\frac{n}{2}it)} \right)^{p \perp\!\!\!\perp 2},$$

what, after some small rearrangements yields (9). ∎

Let us now take

$$\Phi_1(t) = \underbrace{\prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \frac{\Gamma(a_j+b^*_{kj})}{\Gamma(a_j)} \frac{\Gamma(a_j-nit)}{\Gamma(a_j+b^*_{kj}-nit)}}_{\Phi_{1,1}(t)} \\ \underbrace{\left(\prod_{k=1}^{q} \frac{\Gamma(a_p+b^*_{pk})}{\Gamma(a_p)} \frac{\Gamma\left(a_p-\frac{n}{2}it\right)}{\Gamma(a_p+b^*_{pk}-\frac{n}{2}it)} \right)^{p \perp\!\!\!\perp 2}}_{\Phi_{1,2}(t)}. \tag{13}$$

We will now show that $\Phi_1(t)$ is indeed the c.f. of the sum of independent Exponential r.v.'s and we will identify the different Exponential distributions involved, by adequately decomposing first $\Phi_{1,1}(t)$ and then $\Phi_{1,2}(t)$.

Lemma 2: We may write

$$\Phi_{1,1}(t) = \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \prod_{l=1-\lfloor \frac{k-2j}{q} \rfloor}^{2j-1} (n-l)\ (n-l-nit)^{-1} \ .$$

Proof: Applying now

$$\frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{l=0}^{n-1} (a+l) \tag{14}$$

and noticing that

$$b^*_{kj} = \left\lfloor 2j-1+\frac{k-2j}{q} \right\rfloor = 2j-1+\left\lfloor \frac{k-2j}{q} \right\rfloor ,$$

we may write

$$\begin{aligned}
\Phi_{1,1}(t) &= \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \frac{\Gamma(a_j+b^*_{kj})}{\Gamma(a_j)} \ \frac{\Gamma(a_j-nit)}{\Gamma(a_j+b^*_{kj}-nit)} \\
&= \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \prod_{l=0}^{b^*_{kj}-1} (n-2j+1+l)\,(n-2j+1+l-nit)^{-1} \\
&= \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \prod_{l=1}^{b^*_{kj}} (n-2j+l)\,(n-2j+l-nit)^{-1} \qquad (15) \\
&= \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \prod_{l=1}^{2j-1+\lfloor \frac{k-2j}{q} \rfloor} (n-2j+l)\,(n-2j+l-nit)^{-1} \\
&= \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \prod_{l=1}^{2j-1+\left\lfloor \frac{k-2j}{q} \right\rfloor} \left(n-l+\left\lfloor \frac{k-2j}{q} \right\rfloor\right) \\
&\qquad\qquad \times \left(n-l+\left\lfloor \frac{k-2j}{q} \right\rfloor - nit\right)^{-1} ,
\end{aligned}$$

that by a simple change in the limits of the last product yields the desired result. ∎

In the next two Lemmas we identify the different Exponential distributions involved in $\Phi_{1,1}(t)$ and obtain analytic expressions for their counts. The first Lemma refers to even q and the second to odd q.

Lemma 3: For even q we may write

$$\Phi_{1,1}(t) = \prod_{j=\alpha+2}^{2\lfloor p/2\rfloor-1} (n-j)^{q(\lfloor p/2\rfloor-\lfloor j/2\rfloor)} \, (n-j-nit)^{-q(\lfloor p/2\rfloor-\lfloor j/2\rfloor)} \times \prod_{k=1}^{\alpha+1} (n-k)^{a_k+\gamma_k} \, (n-k-nit)^{-(a_k+\gamma_k)} \tag{16}$$

where $\alpha = \left\lfloor \frac{p-1}{q} \right\rfloor$,

$$\gamma_k = \lfloor q/2\rfloor((k-1)q - 2\lfloor k/2\rfloor) \qquad (k=1,\dots,\alpha+1) \tag{17}$$

and

$$a_k = \begin{cases} q^2/4 & k=1,\dots,\alpha \\ (q-(\lfloor p/2\rfloor-\alpha\lfloor q/2\rfloor))\,(\lfloor p/2\rfloor-\alpha\lfloor q/2\rfloor) & k=\alpha+1\,. \end{cases} \tag{18}$$

Proof: Since for $k=1,\dots,q$ and $j=1,\dots,\lfloor p/2\rfloor$, $\nu = -\left\lfloor \frac{k-2j}{q} \right\rfloor$ takes the values $0,1,2,\dots,\alpha$, with $\alpha = \left\lfloor \frac{p-1}{q} \right\rfloor$, we may write, from the result in Lemma 2,

$$\begin{aligned} \Phi_{1,1}(t) &= \prod_{j=1}^{\lfloor p/2\rfloor} \prod_{k=1}^{q} \prod_{l=1-\left\lfloor \frac{k-2j}{q} \right\rfloor}^{2j-1} (n-l)\,(n-l-n\mathrm{i}t)^{-1} \\ &= \prod_{j=1}^{\lfloor p/2\rfloor} \prod_{k=1}^{q} \left(\prod_{l=1-\left\lfloor \frac{k-2j}{q} \right\rfloor}^{\min(\alpha+1,2j-1)} (n-l)\,(n-l-n\mathrm{i}t)^{-1} \times \prod_{l=\alpha+2}^{2j-1} (n-l)\,(n-l-n\mathrm{i}t)^{-1} \right) \\ &= \prod_{j=1}^{\lfloor p/2\rfloor} \prod_{l=\alpha+2}^{2j-1} (n-l)^{q}\,(n-l-n\mathrm{i}t)^{-q} \times \prod_{j=1}^{\lfloor p/2\rfloor} \prod_{k=1}^{q} \prod_{l=1-\left\lfloor \frac{k-2j}{q} \right\rfloor}^{\min(\alpha+1,2j-1)} (n-l)\,(n-l-n\mathrm{i}t)^{-1} \end{aligned} \tag{19}$$

where

$$
\prod_{j=1}^{\lfloor p/2\rfloor}\prod_{l=\alpha+2}^{2j-1}(n-l)^q\,(n-l-nit)^{-q}
$$

$$
=\prod_{j=\alpha+2}^{2\lfloor p/2\rfloor-1}(n-j)^{q(\lfloor p/2\rfloor-\lfloor j/2\rfloor)}\,(n-j-nit)^{-q(\lfloor p/2\rfloor-\lfloor j/2\rfloor)} \tag{20}
$$

and, for even q (for which although $\lfloor q/2\rfloor = q/2$, we will use the notation $\lfloor q/2\rfloor$ to make it more uniform with the notation for odd q)

$$
\begin{aligned}
&\prod_{j=1}^{\lfloor p/2\rfloor}\prod_{k=1}^{q}\prod_{l=1-\left\lfloor\frac{k-2j}{q}\right\rfloor}^{\min(\alpha+1,2j-1)}(n-l)\,(n-l-nit)^{-1}\\
&=\prod_{\nu=0}^{\alpha-1}\prod_{j=1+\nu\lfloor q/2\rfloor}^{(\nu+1)\lfloor q/2\rfloor}\left(\prod_{l=1+\nu}^{\min(\alpha+1,2j-1)}(n-l)^{q-(2(j-\nu\lfloor q/2\rfloor)-1)}\right.\\
&\qquad\times(n-l-nit)^{-(q-(2(j-\nu\lfloor q/2\rfloor)-1))}\\
&\qquad\times\prod_{l=2+\nu}^{\min(\alpha+1,2j-1)}(n-l)^{2(j-\nu\lfloor q/2\rfloor)-1}\\
&\qquad\left.\times(n-l-nit)^{-(2(j-\nu\lfloor q/2\rfloor)-1)}\right)\\
&\quad\times\prod_{j=1+\alpha\lfloor q/2\rfloor}^{\lfloor p/2\rfloor}\left(\prod_{l=1+\alpha}^{\min(\alpha+1,2j-1)}(n-l)^{q-(2(j-\alpha\lfloor q/2\rfloor)-1)}\right.\\
&\qquad\times(n-l-nit)^{-(q-(2(j-\alpha\lfloor q/2\rfloor)-1))}\\
&\qquad\times\prod_{l=2+\alpha}^{\min(\alpha+1,2j-1)}(n-l)^{2(j-\alpha\lfloor q/2\rfloor)-1}\\
&\qquad\left.\times(n-l-nit)^{-(2(j-\alpha\lfloor q/2\rfloor)-1)}\right)\\
&=\left\{\prod_{\nu=0}^{\alpha-1}\prod_{j=1}^{\lfloor q/2\rfloor}\left(\prod_{l=1+\nu}^{\min(\alpha+1,2(j+\nu\lfloor q/2\rfloor)-1)}(n-l)^{q-(2j-1)}\right.\right.\\
&\qquad\left.\times(n-l-nit)^{-(q-(2j-1))}\right)\\
&\quad\times\left(\prod_{l=2+\nu}^{\min(\alpha+1,2(j+\nu\lfloor q/2\rfloor)-1)}(n-l)^{2j-1}\right.\\
&\qquad\left.\left.\times(n-l-nit)^{-(2j-1)}\right)\right\}\\
&\quad\times\prod_{j=1}^{\lfloor p/2\rfloor-\alpha\lfloor q/2\rfloor}(n-(\alpha+1))^{q-(2j-1)}\,(n-(\alpha+1)-nit)^{-(q-(2j-1))}
\end{aligned}
$$

$$= \left\{ \prod_{j=1}^{\lfloor q/2 \rfloor} \prod_{\nu=0}^{\alpha-1} \prod_{l=2+\nu}^{\min(\alpha+1, 2j+\nu q-1)} (n-l)^q \, (n-l-n\mathrm{i}t)^{-q} \right\}$$

$$\times \prod_{j=1}^{\lfloor q/2 \rfloor} \prod_{\nu=0}^{\alpha-1} (n-(\nu+1))^{q-(2j-1)} \; (n-(\nu+1)-n\mathrm{i}t)^{-(q-(2j-1))}$$

$$\times \prod_{j=1}^{\lfloor p/2 \rfloor - \alpha \lfloor q/2 \rfloor} (n-(\alpha+1))^{q-(2j-1)} \; (n-(\alpha+1)-n\mathrm{i}t)^{-(q-(2j-1))}$$

where, for even q,

i) $\displaystyle \prod_{j=1}^{\lfloor q/2 \rfloor} \prod_{\nu=0}^{\alpha-1} \prod_{l=2+\nu}^{\min(\alpha+1, 2j+\nu q-1)} (n-l)^q \, (n-l-n\mathrm{i}t)^{-q}$

$$= \prod_{k=2}^{\alpha+1} (n-k)^{\gamma_k} \, (n-k-n\mathrm{i}t)^{-\gamma_k} \, ,$$

for $\gamma_k = \lfloor q/2 \rfloor \left((k-1)q - 2\lfloor k/2 \rfloor\right)$ $(k = 2, \ldots, \alpha+1)$, as defined in (17),

ii) $\displaystyle \prod_{j=1}^{\lfloor q/2 \rfloor} \prod_{\nu=0}^{\alpha-1} (n-(\nu+1))^{q-(2j-1)} \; (n-(\nu+1)-n\mathrm{i}t)^{-(q-(2j-1))}$

$$= \prod_{k=1}^{\alpha} (n-k)^{a_k} \, (n-k-n\mathrm{i}t)^{-a_k} \, ,$$

for a_k given in (18), that is, for $a_k = q^2/4$ $(k = 1, \ldots, \alpha)$,

iii) $\displaystyle \prod_{j=1}^{\lfloor p/2 \rfloor - \alpha \lfloor q/2 \rfloor} (n-(\alpha+1))^{q-(2j-1)} \; (n-(\alpha+1)-n\mathrm{i}t)^{-(q-(2j-1))}$

$$= (n-(\alpha+1))^{a_{\alpha+1}} \; (n-(\alpha+1)-n\mathrm{i}t)^{-a_{\alpha+1}} \, ,$$

for $a_{\alpha+1} = \left(q - (\lfloor p/2 \rfloor - \alpha \lfloor q/2 \rfloor)\right) \left(\lfloor p/2 \rfloor - \alpha \lfloor q/2 \rfloor\right)$, as given in (18),

so that we may write $\Phi_{1,1}(t)$ as in (16). ∎

Lemma 4: For odd q, and once again for $\alpha = \left\lfloor \frac{p-1}{q} \right\rfloor$, we may write

$$\Phi_{1,1}(t) = \prod_{j=\alpha+2}^{2\lfloor p/2 \rfloor - 1} (n-j)^{q(\lfloor p/2 \rfloor - \lfloor j/2 \rfloor)} \, (n-j-nit)^{-q(\lfloor p/2 \rfloor - \lfloor j/2 \rfloor)} \prod_{k=1}^{\alpha+1} (n-k)^{a_k+\gamma_k} \, (n-k-nit)^{-(a_k+\gamma_k)} \qquad (21)$$

where

$$\gamma_k = \lfloor q/2 \rfloor (k-1) q \qquad (k = 1, \ldots, \alpha + 1) \tag{22}$$

and

$$a_k = \begin{cases} \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q+k \perp\!\!\!\perp 2}{2} \right\rfloor & k = 1, \ldots, \alpha \\ \left(q - \left(\left\lfloor \frac{p}{2} \right\rfloor - \alpha \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{\alpha}{2} \right\rfloor\right)\right) \left(\left\lfloor \frac{p}{2} \right\rfloor - \alpha \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{\alpha+1}{2} \right\rfloor\right) & k = \alpha + 1\,. \end{cases} \tag{23}$$

Proof: Following the same lines used in the proof of Lemma 3, we obtain once again expressions (19) and (20). Then, taking into account that $\left\lfloor 1 + \frac{\nu}{2} \right\rfloor - \left\lfloor \frac{\nu+1}{2} \right\rfloor = (\nu+1) \perp\!\!\!\perp 2$ and that for odd q, $2 \left\lfloor \frac{q}{2} \right\rfloor = q - 1$, we have,

$$\begin{aligned} &\prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \prod_{l=1-\left\lfloor \frac{k-2j}{q} \right\rfloor}^{\min(\alpha+1, 2j-1)} (n-l)\,(n-l-n\mathrm{i}t)^{-1} \\ &\quad = \prod_{\nu=0}^{\alpha-1} \prod_{j=1+\nu\lfloor q/2 \rfloor + \lfloor (1+\nu)/2 \rfloor}^{(\nu+1)\lfloor q/2 \rfloor + \lfloor 1+\nu/2 \rfloor} \left(\prod_{l=1+\nu}^{\min(\alpha+1,2j-1)} (n-l)^{q-(2(j-\nu\lfloor q/2 \rfloor)-(\nu+1))} \right. \\ &\qquad\qquad \times (n-l-n\mathrm{i}t)^{-(q-(2(j-\nu\lfloor q/2 \rfloor)-(\nu+1)))} \\ &\qquad\qquad \times \prod_{l=2+\nu}^{\min(\alpha+1,2j-1)} (n-l)^{2(j-\nu\lfloor q/2 \rfloor)-(\nu+1)} \\ &\qquad\qquad \left. \times (n-l-n\mathrm{i}t)^{-(2(j-\nu\lfloor q/2 \rfloor)-(\nu+1))} \right) \\ &\qquad \times \prod_{j=1+\alpha\lfloor q/2 \rfloor + \lfloor (1+\alpha)/2 \rfloor}^{\lfloor p/2 \rfloor} \left(\prod_{l=1+\alpha}^{\min(\alpha+1,2j-1)} (n-l)^{q-(2(j-\alpha\lfloor q/2 \rfloor)-(\alpha+1))} \right. \\ &\qquad\qquad \times (n-l-n\mathrm{i}t)^{-(q-(2(j-\alpha\lfloor q/2 \rfloor)-(\alpha+1)))} \\ &\qquad\qquad \times \prod_{l=2+\alpha}^{\min(\alpha+1,2j-1)} (n-l)^{2(j-\alpha\lfloor q/2 \rfloor)-(\alpha+1)} \\ &\qquad\qquad \left. \times (n-l-n\mathrm{i}t)^{-(2(j-\alpha\lfloor q/2 \rfloor)-(\alpha+1))} \right) \\ &\quad = \left\{ \prod_{\nu=0}^{\alpha-1} \prod_{j=1+\lfloor (1+\nu)/2 \rfloor}^{\lfloor q/2 \rfloor + \lfloor 1+\nu/2 \rfloor} \left(\prod_{l=1+\nu}^{\min(\alpha+1,2(j+\nu\lfloor q/2 \rfloor)-1)} (n-l)^{q-(2j-(\nu+1))} \right. \right. \\ &\qquad\qquad \left. \times (n-l-n\mathrm{i}t)^{-(q-(2j-(\nu+1)))} \right) \end{aligned}$$

$$\times \left(\prod_{l=2+\nu}^{\min(\alpha+1,2(j+\nu\lfloor q/2\rfloor)-1)} (n-l)^{2j-(\nu+1)} \times (n-l-n\mathrm{i}t)^{-(2j-(\nu+1))} \right) \Bigg\}$$

$$\times \prod_{j=1+\lfloor(1+\alpha)/2\rfloor}^{\lfloor p/2\rfloor-\alpha\lfloor q/2\rfloor} (n-(\alpha+1))^{q-(2j-(\alpha+1))} \, (n-(\alpha+1)-n\mathrm{i}t)^{-(q-(2j-(\alpha+1)))}$$

$$= \left\{ \prod_{\nu=0}^{\alpha-1} \prod_{j=1+\lfloor(1+\nu)/2\rfloor}^{\lfloor q/2\rfloor+\lfloor 1+\nu/2\rfloor} \prod_{l=2+\nu}^{\min(\alpha+1,2j+\nu q-\nu-1)} (n-l)^q \, (n-l-n\mathrm{i}t)^{-q} \right\}$$

$$\times \prod_{\nu=0}^{\alpha-1} \prod_{j=1+\lfloor(1+\nu)/2\rfloor}^{\lfloor q/2\rfloor+\lfloor 1+\nu/2\rfloor} (n-(\nu+1))^{q-2j+\nu+1} \, (n-(\nu+1)-n\mathrm{i}t)^{-(q-2j+\nu+1)}$$

$$\times \prod_{j=1+\lfloor(1+\alpha)/2\rfloor}^{\lfloor p/2\rfloor-\alpha\lfloor q/2\rfloor} (n-(\alpha+1))^{q-2j+\alpha+1} \, (n-(\alpha+1)-n\mathrm{i}t)^{-(q-2j+\alpha+1)}$$

where, for odd q,

i) $$\prod_{\nu=0}^{\alpha-1} \prod_{j=1+\lfloor(1+\nu)/2\rfloor}^{\lfloor q/2\rfloor+\lfloor 1+\nu/2\rfloor} \prod_{l=2+\nu}^{\min(\alpha+1,2j+\nu q-\nu-1)} (n-l)^q \, (n-l-n\mathrm{i}t)^{-q}$$

$$= \prod_{\nu=0}^{\alpha-1} \prod_{j=1}^{\lfloor q/2\rfloor+((\nu+1)\perp\!\!\!\perp 2)} \prod_{l=2+\nu}^{\min(\alpha+1,2j+\nu q-(\nu+1)\perp\!\!\!\perp 2)} (n-l)^q \, (n-l-n\mathrm{i}t)^{-q}$$

$$= \prod_{k=2}^{\alpha+1} (n-k)^{\gamma_k} \, (n-k-n\mathrm{i}t)^{-\gamma_k} ,$$

for $\gamma_k = \lfloor q/2\rfloor(k-1)q$ $(k=2,\ldots,\alpha+1)$, as defined in (22),

ii) $$\prod_{\nu=0}^{\alpha-1} \prod_{j=1+\lfloor(1+\nu)/2\rfloor}^{\lfloor q/2\rfloor+\lfloor 1+\nu/2\rfloor} (n-(\nu+1))^{q-2j+\nu+1} \, (n-(\nu+1)-n\mathrm{i}t)^{-(q-2j+\nu+1)}$$

$$= \prod_{\nu=0}^{\alpha-1} \prod_{j=1}^{\lfloor q/2\rfloor+(\nu+1)\perp\!\!\!\perp 2} (n-(\nu+1))^{q-2j-(\nu\perp\!\!\!\perp 2)+1} \, (n-(\nu+1)-n\mathrm{i}t)^{-(q-2j-(\nu\perp\!\!\!\perp 2)+1)}$$

$$= \prod_{k=1}^{\alpha} (n-k)^{a_k} \, (n-k-n\mathrm{i}t)^{-a_k} ,$$

for a_k given by (23), that is, for $a_k = \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q+k\perp\!\!\!\perp 2}{2} \right\rfloor$ $(k=1,\ldots,\alpha)$,

iii) $$\prod_{j=1+\left\lfloor \frac{\alpha+1}{2} \right\rfloor}^{\lfloor p/2\rfloor-\alpha\lfloor q/2\rfloor} (n-(\alpha+1))^{q-2j+\alpha+1} \, (n-(\alpha+1)-n\mathrm{i}t)^{-(q-2j+\alpha+1)}$$

$$= \prod_{j=1}^{\lfloor p/2\rfloor-\alpha\lfloor q/2\rfloor-\left\lfloor\frac{\alpha+1}{2}\right\rfloor} (n-(\alpha+1))^{q-2j-\alpha\perp\!\!\!\perp 2+1} \qquad (n-(\alpha+1)-nit)^{-(q-2j-\alpha\perp\!\!\!\perp 2+1)}$$

$$= (n-(\alpha+1))^{a_{\alpha+1}}\,(n-(\alpha+1)-nit)^{-a_{\alpha+1}}\,,$$

for $a_{\alpha+1} = \left(q-\left(\left\lfloor\frac{p}{2}\right\rfloor-\alpha\left\lfloor\frac{q}{2}\right\rfloor-\left\lfloor\frac{\alpha}{2}\right\rfloor\right)\right)\left(\left\lfloor\frac{p}{2}\right\rfloor-\alpha\left\lfloor\frac{q}{2}\right\rfloor-\left\lfloor\frac{\alpha+1}{2}\right\rfloor\right)$, as given by (23),

so that we may write $\Phi_{1,1}(t)$ as in (21). ∎

Equalities i), ii) and iii) in the proofs of Lemmas 3 and 4 are quite straightforward to verify and may be proven by induction but the proofs are not shown because they are a bit long and tedious.

In the next Corollary we show how we can put together in one single expression the results from Lemmas 3 and 4, for both even and odd q.

Corollary 1: For both even and odd q we may write

$$\Phi_{1,1}(t) = \prod_{j=\alpha+2}^{2\lfloor p/2\rfloor-1}(n-j)^{q(\lfloor p/2\rfloor-\lfloor j/2\rfloor)}\,(n-j-nit)^{-q(\lfloor p/2\rfloor-\lfloor j/2\rfloor)} \prod_{k=1}^{\alpha+1}(n-k)^{a_k+\gamma_k}\,(n-k-nit)^{-(a_k+\gamma_k)} \tag{24}$$

where

$$\gamma_k = \left\lfloor\frac{q}{2}\right\rfloor\left((k-1)q-2\left((q+1)\perp\!\!\!\perp 2\right)\left\lfloor\frac{k}{2}\right\rfloor\right) \qquad (k=1,\dots,\alpha+1) \tag{25}$$

and

$$a_k = \begin{cases} \left\lfloor\frac{q}{2}\right\rfloor\left\lfloor\frac{q+k\perp\!\!\!\perp 2}{2}\right\rfloor & k=1,\dots,\alpha \\ \left(q-\left(\left\lfloor\frac{p}{2}\right\rfloor-\alpha\left\lfloor\frac{q}{2}\right\rfloor\right)\right)\left(\left\lfloor\frac{p}{2}\right\rfloor-\alpha\left\lfloor\frac{q}{2}\right\rfloor\right) & \\ \quad +(q\perp\!\!\!\perp 2)\left(\alpha\left\lfloor\frac{p}{2}\right\rfloor-\alpha^2\left\lfloor\frac{q}{2}\right\rfloor-\frac{\alpha^2}{4}+\frac{\alpha\perp\!\!\!\perp 2}{4}-q\left\lfloor\frac{\alpha+1}{2}\right\rfloor\right) & k=\alpha+1\,. \end{cases} \tag{26}$$

Proof: Since for $k\in I\!N_0$ and even q we have $\left\lfloor\frac{q+k\perp\!\!\!\perp 2}{2}\right\rfloor=\left\lfloor\frac{q}{2}\right\rfloor$, and since for even q we also have

$$\frac{q^2}{4}=\left\lfloor\frac{q}{2}\right\rfloor\left\lfloor\frac{q}{2}\right\rfloor\,,$$

from (18) and (23), for any q, even or odd, we may write

$$a_k = \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q + k \perp\!\!\!\perp 2}{2} \right\rfloor , \qquad k = 1, \ldots, \alpha \, ,$$

while from (17) and (22) we may write (25). Finally, since

$$\left\lfloor \frac{\alpha+1}{2} \right\rfloor \left\lfloor \frac{\alpha}{2} \right\rfloor = \alpha$$

and

$$\left\lfloor \frac{\alpha}{2} \right\rfloor \left\lfloor \frac{\alpha+1}{2} \right\rfloor = \begin{cases} \frac{\alpha^2}{4} & \alpha \text{ even} \\ \frac{\alpha^2-1}{4} & \alpha \text{ odd} \end{cases}$$

we have

$$\begin{aligned} &\left(q - \left(\left\lfloor \frac{p}{2} \right\rfloor - \alpha \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{\alpha}{2} \right\rfloor\right)\right) \left(\left\lfloor \frac{p}{2} \right\rfloor - \alpha \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{\alpha+1}{2} \right\rfloor\right) \\ &\quad = \left(q - \left(\left\lfloor \frac{p}{2} \right\rfloor - \alpha \left\lfloor \frac{q}{2} \right\rfloor\right)\right) \left(\left\lfloor \frac{p}{2} \right\rfloor - \alpha \left\lfloor \frac{q}{2} \right\rfloor\right) - q \left\lfloor \frac{\alpha+1}{2} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{\alpha+1}{2} \right\rfloor \\ &\qquad\qquad -\alpha \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{\alpha+1}{2} \right\rfloor - \left\lfloor \frac{\alpha}{2} \right\rfloor \left\lfloor \frac{\alpha+1}{2} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{\alpha}{2} \right\rfloor - \alpha \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{\alpha}{2} \right\rfloor \\ &\quad = \left(q - \left(\left\lfloor \frac{p}{2} \right\rfloor - \alpha \left\lfloor \frac{q}{2} \right\rfloor\right)\right) \left(\left\lfloor \frac{p}{2} \right\rfloor - \alpha \left\lfloor \frac{q}{2} \right\rfloor\right) - q \left\lfloor \frac{\alpha+1}{2} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor \alpha \\ &\qquad\qquad - \left\lfloor \frac{q}{2} \right\rfloor \alpha^2 - \frac{\alpha^2}{4} + \frac{\alpha \perp\!\!\!\perp 2}{4} \, . \blacksquare \end{aligned}$$

In the next Lemma we show how $\Phi_{1,2}(t)$ may also be seen as the c.f. of the sum of independent Exponential r.v.'s. We identify the different Exponential distributions involved and obtain analytic expressions for their counts.

Lemma 5: We may write, for a_p and b^*_{pk} defined in (11) and (12),

$$\begin{aligned} \Phi^*_{1,2}(t) &= \prod_{k=1}^{q} \frac{\Gamma(a_p + b^*_{pk})}{\Gamma(a_p)} \frac{\Gamma\left(a_p - \frac{n}{2} it\right)}{\Gamma(a_p + b^*_{pk} - \frac{n}{2} it)} \\ &= \left(\frac{n-k^*}{n}\right)^{\gamma(\alpha_2-\alpha_1)} \left(\frac{n-k^*}{n} - it\right)^{-\gamma(\alpha_2-\alpha_1)} \qquad (27) \\ &\qquad\qquad \times \prod_{\substack{l=1+p-2\alpha_1 \\ \text{step } 2}}^{p-1} \left(\frac{n-l}{n}\right)^{q} \left(\frac{n-l}{n} - it\right)^{-q} , \end{aligned}$$

where

$$\gamma = q - \left(\frac{p-1}{2} - \beta\right), \qquad \text{with} \qquad \beta = \left\lfloor \frac{p}{2q} \right\rfloor q\,, \tag{28}$$

and

$$k^* = 1 + p - 2\alpha_2\,, \quad \alpha_1 = \left\lfloor \frac{q-1}{q}\,\frac{p-1}{2} \right\rfloor, \quad \alpha_2 = \left\lfloor \frac{q-1}{q}\,\frac{p+1}{2} \right\rfloor. \tag{29}$$

Proof: Using (14) we may always write,

$$\begin{aligned}\Phi^*_{1,2}(t) &= \prod_{k=1}^{q} \frac{\Gamma(a_p + b^*_{pk})\,\Gamma\left(a_p - \frac{n}{2}it\right)}{\Gamma(a_p)\,\Gamma\left(a_p + b^*_{pk} - \frac{n}{2}it\right)} \\ &= \prod_{k=1}^{q}\prod_{l=1}^{b^*_{pk}} (a_p + l - 1)\,\left(a_p + l - 1 - \frac{n}{2}it\right)^{-1},\end{aligned}$$

where $b^*_{pk} = \lfloor b_{pk} \rfloor$, with

$$b_{pk} = \frac{pq - q - p + 2k - 1}{2q} = \frac{q-1}{q}\,\frac{p-1}{2} + \frac{k-1}{q}$$

so that, given that here p is odd and thus $\frac{p-1}{2}$ is an integer,

$$b^*_{pk} = \left\lfloor \frac{q-1}{q}\,\frac{p-1}{2} + \frac{k-1}{q} \right\rfloor = \begin{cases} \alpha_1 & \text{for} \quad k = 1, \dots, \frac{p-1}{2} - \beta \\ \alpha_2 & \text{for} \quad k = \frac{p+1}{2} - \beta, \dots, q \end{cases}$$

with β given by (28) and α_1 and α_2 given by (29).
We may thus write, taking into account the definitions of a_p in (11), γ in (28) and α_1 and α_2 in (29),

$$\begin{aligned}\Phi^*_{12}(t) &= \prod_{k=1}^{\frac{p-1}{2}-\beta}\prod_{l=1}^{\alpha_1} (a_p + l - 1)\,\left(a_p + l - 1 - \frac{n}{2}it\right)^{-1} \\ &\quad \times \prod_{k=\frac{p+1}{2}-\beta}^{q}\prod_{l=1}^{\alpha_2} (a_p + l - 1)\,\left(a_p + l - 1 - \frac{n}{2}it\right)^{-1} \\ &= \prod_{l=1}^{\alpha_1} (a_p + l - 1)^{\frac{p-1}{2}-\beta}\,\left(a_p + l - 1 - \frac{n}{2}it\right)^{-\left(\frac{p-1}{2}-\beta\right)}\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{l=1}^{\alpha_2}(a_p+l-1)^{q-\left(\frac{p+1}{2}-\beta\right)} \left(a_p+l-1-\frac{n}{2}it\right)^{-q+\left(\frac{p+1}{2}-\beta\right)} \\
=&\prod_{l=1}^{\alpha_1}\left(\frac{n+1-p}{2}+l-1\right)^{q}\left(\frac{n+1-p}{2}+l-1-\frac{n}{2}it\right)^{-q} \\
&\times\left(\frac{n+1-p}{2}+\alpha_2-1\right)^{\gamma(\alpha_2-\alpha_1)} \\
&\qquad\times\left(\frac{n+1-p}{2}+\alpha_2-1-\frac{n}{2}it\right)^{-\gamma(\alpha_2-\alpha_1)} \\
=&\prod_{l=1}^{\alpha_1}\left(\frac{n-p-1+2l}{n}\right)^{q}\left(\frac{n-p-1+2l}{n}-it\right)^{-q} \\
&\times\left(\frac{n-p-1+2\alpha_2}{n}\right)^{\gamma(\alpha_2-\alpha_1)}\left(\frac{n-p-1+2\alpha_2}{n}-it\right)^{-\gamma(\alpha_2-\alpha_1)} \\
=&\prod_{l=1}^{\alpha_1}\left(\frac{n-(p+1-2l)}{n}\right)^{q}\left(\frac{n-(p+1-2l)}{n}-it\right)^{-q} \\
&\times\left(\frac{n-(p+1-2\alpha_2)}{n}\right)^{\gamma(\alpha_2-\alpha_1)} \\
&\quad\times\left(\frac{n-(p+1-2\alpha_2)}{n}-it\right)^{-\gamma(\alpha_2-\alpha_1)} \\
=&\prod_{\substack{l^*=p+1-2\alpha_1\\ \text{step } 2}}^{p-1}\left(\frac{n-l^*}{n}\right)^{q}\left(\frac{n-l^*}{n}-it\right)^{-q} \\
&\times\left(\frac{n-k^*}{n}\right)^{\gamma(\alpha_2-\alpha_1)}\left(\frac{n-k^*}{n}-it\right)^{-\gamma(\alpha_2-\alpha_1)}
\end{aligned}
$$

where the last equality is obtained by taking $l^*=p+1-2l$ and taking into account the definition of k^* in (29). ∎

Corollary 2: Using the results in Lemmas 1 through 5, we may finally write

$$
\begin{aligned}
\Phi_W(t) = &\underbrace{\prod_{k=1}^{p-1}\left(\frac{n-k}{n}\right)^{r_k}\left(\frac{n-k}{n}-it\right)^{-r_k}}_{\Phi_1(t)} \\
&\underbrace{\times\prod_{j=1}^{\lfloor p/2\rfloor}\prod_{k=1}^{q}\frac{\Gamma(a_j+b_{jk})}{\Gamma(a_j+b^*_{jk})}\frac{\Gamma(a_j+b^*_{jk}-nit)}{\Gamma(a_j+b_{jk}-nit)} \\
\times\left(\prod_{k=1}^{q}\frac{\Gamma(a_p+b_{pk})}{\Gamma(a_p+b^*_{pk})}\frac{\Gamma(a_p+b^*_{pk}-\frac{n}{2}it)}{\Gamma\left(a_p+b_{pk}-\frac{n}{2}it\right)}\right)^{p\perp\!\!\!\perp 2}}_{\Phi_2(t)},
\end{aligned}
\tag{30}
$$

where

$$r_k = \begin{cases} r_k^* & \text{for } k = 1, \dots, p-1, \\ & \text{except for } k = p-1-2\alpha_1 \\ r_k^* + (p \perp\!\!\!\perp 2)(\alpha_2 - \alpha_1) & \\ \qquad \left(q - \frac{p-1}{2} + q \left\lfloor \frac{p}{2q} \right\rfloor\right) & \text{for } k = p-1-2\alpha_1 \end{cases} \tag{31}$$

with

$$r_k^* = \begin{cases} a_k + \gamma_k & \text{for} \quad k = 1, \dots, \alpha+1 \\ q\left(\left\lfloor \frac{p}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor\right) & \text{for} \quad k = \alpha+2, \dots, \min(p-2\alpha_1, p-1) \\ & \text{and } k = 2+p-2\alpha_1, \dots, 2\left\lfloor \frac{p}{2} \right\rfloor - 1, \text{ step } 2 \\ q\left(\left\lfloor \frac{p+1}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor\right) & \text{for} \quad k = 1+p-2\alpha_1, \dots, p-1, \text{ step } 2\,, \end{cases} \tag{32}$$

and

$$\alpha = \left\lfloor \frac{p-1}{q} \right\rfloor, \qquad \alpha_1 = \left\lfloor \frac{q-1}{q} \frac{p-1}{2} \right\rfloor, \qquad \alpha_2 = \left\lfloor \frac{q-1}{q} \frac{p+1}{2} \right\rfloor,$$

where, from (25) and (26),

$$a_k + \gamma_k = \left\lfloor \frac{q}{2} \right\rfloor \left((k-1)q - 2\left((q+1) \perp\!\!\!\perp 2\right)\left\lfloor \frac{k}{2} \right\rfloor\right) + \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q + k \perp\!\!\!\perp 2}{2} \right\rfloor \qquad \text{for } k = 1, \dots, \alpha$$

and

$$\begin{aligned} a_{\alpha+1} + \gamma_{\alpha+1} = {} & \left(\left\lfloor \frac{p}{2} \right\rfloor - \alpha \left\lfloor \frac{q}{2} \right\rfloor\right)^2 + q\left(\left\lfloor \frac{p}{2} \right\rfloor - \left\lfloor \frac{\alpha+1}{2} \right\rfloor\right) \\ & + (q \perp\!\!\!\perp 2)\left(\alpha \left\lfloor \frac{p}{2} \right\rfloor + \frac{\alpha \perp\!\!\!\perp 2}{4} - \frac{\alpha^2}{4} - \alpha^2 \left\lfloor \frac{q}{2} \right\rfloor\right). \end{aligned} \tag{33}$$

Proof: First of all we have to notice that

$$(n-k)^{r_k}(n-k-nit)^{-r_k} = \left(\frac{n-k}{n}\right)^{r_k} \left(\frac{n-k}{n} - it\right)^{-r_k}.$$

Then, while for $k = 1, \ldots, \alpha$, r_k is obtained just by adding γ_k in (25) and a_k in the first row of (26) in Corollary 1, for $k = \alpha + 1$ we have, from (26),

$$
\begin{aligned}
a_{\alpha+1} = & \left(q - \left(\left\lfloor\frac{p}{2}\right\rfloor - \alpha\left\lfloor\frac{q}{2}\right\rfloor\right)\right)\left(\left\lfloor\frac{p}{2}\right\rfloor - \alpha\left\lfloor\frac{q}{2}\right\rfloor\right) \\
& + (q \perp\!\!\!\perp 2)\left(\alpha\left\lfloor\frac{p}{2}\right\rfloor - \alpha^2\left\lfloor\frac{q}{2}\right\rfloor - \frac{\alpha^2}{4} + \frac{\alpha \perp\!\!\!\perp 2}{4} - q\left\lfloor\frac{\alpha+1}{2}\right\rfloor\right) \\
= & \; q\left\lfloor\frac{p}{2}\right\rfloor - q\alpha\left\lfloor\frac{q}{2}\right\rfloor - \left(\left\lfloor\frac{p}{2}\right\rfloor - \alpha\left\lfloor\frac{q}{2}\right\rfloor\right)^2 - (q \perp\!\!\!\perp 2)q\left\lfloor\frac{\alpha+1}{2}\right\rfloor \\
& + (q \perp\!\!\!\perp 2)\left(\alpha\left\lfloor\frac{p}{2}\right\rfloor - \alpha^2\left\lfloor\frac{q}{2}\right\rfloor - \frac{\alpha^2}{4} + \frac{\alpha \perp\!\!\!\perp 2}{4}\right)
\end{aligned}
$$

while, from (25),

$$
\gamma_{\alpha+1} = q\alpha\left\lfloor\frac{q}{2}\right\rfloor - 2\left\lfloor\frac{q}{2}\right\rfloor\left\lfloor\frac{\alpha+1}{2}\right\rfloor((q+1) \perp\!\!\!\perp 2) \; ,
$$

where

$$
\begin{aligned}
2\left\lfloor\frac{q}{2}\right\rfloor\left\lfloor\frac{\alpha+1}{2}\right\rfloor((q+1) \perp\!\!\!\perp 2) &= \begin{cases} 0, & \text{odd } q \\ q\left\lfloor\frac{\alpha+1}{2}\right\rfloor, & \text{even } q \end{cases} \\
&= ((q+1) \perp\!\!\!\perp 2)\, q\left\lfloor\frac{\alpha+1}{2}\right\rfloor
\end{aligned}
$$

so that $a_{\alpha+1} + \gamma_{\alpha+1}$ comes given by (33), since

$$
(q \perp\!\!\!\perp 2)q\left\lfloor\frac{\alpha+1}{2}\right\rfloor + ((q+1) \perp\!\!\!\perp 2)\, q\left\lfloor\frac{\alpha+1}{2}\right\rfloor = q\left\lfloor\frac{\alpha+1}{2}\right\rfloor \; .
$$

For $k = \alpha + 2, \ldots, \min(p - 2\alpha_1, p - 1)$ and $k = 2 + p - 2\alpha_1, \ldots, 2\left\lfloor\frac{p}{2}\right\rfloor - 1$, with step 2, we have to consider the result in the first row in (24) in Corollary 1, while for $k = 1 + p - 2\alpha_1, \ldots, p - 1$, with step 2, we have to consider this same result together with the result in Lemma 5 and notice that

$$
\Phi_{1,2}(t) = \left(\Phi^*_{1,2}(t)\right)^{p \perp\!\!\!\perp 2}
$$

and that, as such, the exponent q in (27) in Lemma 5 only appears for odd p

and that

$$q\left(\left\lfloor\frac{p}{2}\right\rfloor-\left\lfloor\frac{k}{2}\right\rfloor\right)+q(p \perp\!\!\!\perp 2)=\begin{cases} q\frac{p}{2}-q\left\lfloor\frac{k}{2}\right\rfloor & \text{for even } q\\ q\left(\frac{p-1}{2}+1\right)-q\left\lfloor\frac{k}{2}\right\rfloor & \text{for odd } q\end{cases}$$
$$=q\left(\left\lfloor\frac{p+1}{2}\right\rfloor-\left\lfloor\frac{k}{2}\right\rfloor\right).$$

Finally, from (27) and (29) in Lemma 5, we see that for $k=p+1-2\alpha_2$ we have to add, for odd p, $\gamma(\alpha_2-\alpha_1)$, where $\gamma=q-\left(\frac{p-1}{2}-q\left\lfloor\frac{p}{2q}\right\rfloor\right)$, to the value of r_k to obtain r_k^*. It happens that the only possible values for $\alpha_2-\alpha_1$ are either zero or 1, so that it will be only for $\alpha_2-\alpha_1=1$ or $1+p-2\alpha_2=p-1-2\alpha_1$ that we will have to add $\gamma(\alpha_2-\alpha_1)$ to r_k. ∎

4 Near-exact distributions for W and λ^*

In all cases where we are able to factorize a c.f. into two factors, one of which corresponds to a manageable well known distribution and the other to a distribution that although giving us some problems in terms of being convoluted with the first factor, may however be adequately asymptoticaly replaced by another c.f. in such a way that the overall c.f. obtained by leaving the first factor unchanged and adequately replacing the second factor may then correspond to a known manageable distribution. This way we will be able to obtain what we call a near-exact c.f. for the random variable under study.

This is exactly what happens with the c.f. of W. In this Section we will show how by keeping $\Phi_1(t)$ in the characteristic function of W in (30) unchanged and replacing $\Phi_2(t)$ by the c.f. of a Gamma distribution or the mixture of two or three Gamma distributions, matching the first two, four or six derivatives of $\Phi_2(t)$ in order to t at $t=0$, we will be able to obtain high quality near-exact distributions for W under the form of a GNIG (Generalized Near-Integer Gamma) distribution or mixtures of GNIG distributions. From these distributions we may then easily obtain near-exact distributions for $\lambda^*=e^{-W}$, as it is shown in the next section.

The GNIG distribution of depth $g+1$ (Coelho, 2004) is the distribution of the r.v.

$$Z=Y+\sum_{i=1}^{g}X_i$$

where the $g+1$ random variables Y and X_i $(i=1,\ldots,g)$ are all independent with Gamma distributions, Y with shape parameter r, a positive non-integer, and rate parameter λ and each X_i ($i=1,\dots,g$) with an integer shape parameter r_i and rate parameter λ_i, being all the $g+1$ rate parameters different. The p.d.f. (probability density function) of Z is given by

$$f_Z(z|r_1,\dots,r_g,r;\,\lambda_1,\dots,\lambda_g,\lambda)= K\lambda^r\sum_{j=1}^{g}\mathrm{e}^{-\lambda_j z}\sum_{k=1}^{r_j}\left\{c_{j,k}\frac{\Gamma(k)}{\Gamma(k+r)}z^{k+r-1}\,{}_1F_1(r,k+r,-(\lambda-\lambda_j)z)\right\},\quad (z>0) \tag{34}$$

and the c.d.f. (cumulative distribution function) given by

$$\begin{aligned}F_Z(z|r_1,\dots,r_g,r;\,\lambda_1,\dots,\lambda_g,\lambda)&=\lambda^r\frac{z^r}{\Gamma(r+1)}\,{}_1F_1(r,r+1,-\lambda z)\\&-K\lambda^r\sum_{j=1}^{g}\mathrm{e}^{-\lambda_j z}\sum_{k=1}^{r_j}c^*_{j,k}\sum_{i=0}^{k-1}\frac{z^{r+i}\lambda_j^i}{\Gamma(r+1+i)}\,{}_1F_1(r,r+1+i,-(\lambda-\lambda_j)z)\\&\qquad (z>0)\end{aligned} \tag{35}$$

where

$$K=\prod_{j=1}^{g}\lambda_j^{r_j}\qquad\text{and}\qquad c^*_{j,k}=\frac{c_{j,k}}{\lambda_j^k}\Gamma(k)$$

with $c_{j,k}$ given by (11) through (13) in Coelho (1998). In the above expressions ${}_1F_1(a,b;z)$ is the Kummer confluent hypergeometric function. This function has usually very good convergence properties and is nowadays easily handled by a number of software packages.

In the next Theorem we develop near-exact distributions for W.

Theorem 1: Using for $\Phi_2(t)$ in the characteristic function of $W=-\log\lambda$ the approximations:

i) $\lambda^s(\lambda-\mathrm{i}t)^{-s}$ with $s,\lambda>0$, such that

$$\frac{\partial^h}{\partial t^h}\,\lambda^s(\lambda-it)^{-s}\bigg|_{t=0}=\frac{\partial^h}{\partial t^h}\,\Phi_2(t)\bigg|_{t=0}\qquad\text{for}\quad h=1,2\,; \tag{36}$$

ii) $\sum_{k=1}^{2} \theta_k\, \mu^{s_k} (\mu - \mathrm{i}t)^{-s_k}$, where $\theta_2 = 1 - \theta_1$ with $\theta_k,\ s_k,\ \mu > 0$, such that

$$\frac{\partial^h}{\partial t^h} \sum_{k=1}^{2} \theta_k\, \mu^{s_k} (\mu - \mathrm{i}t)^{-s_k} \bigg|_{t=0} = \frac{\partial^h}{\partial t^h}\, \Phi_2(t) \bigg|_{t=0} \qquad \text{for} \quad h = 1, \ldots, 4\,; \tag{37}$$

iii) $\sum_{k=1}^{3} \theta_k^*\, \nu^{s_k^*} (\nu - \mathrm{i}t)^{-s_k^*}$, where $\theta_3^* = 1 - \theta_1^* - \theta_2^*$ with $\theta_k^*,\ s_k^*,\ \nu > 0$, such that

$$\frac{\partial^h}{\partial t^h} \sum_{k=1}^{3} \theta_k^*\, \nu^{s_k^*} (\nu - \mathrm{i}t)^{-s_k^*} \bigg|_{t=0} = \frac{\partial^h}{\partial t^h}\, \Phi_2(t) \bigg|_{t=0} \qquad \text{for} \quad h = 1, \ldots, 6\,; \tag{38}$$

we obtain as near-exact distributions for W, respectively,

i) a GNIG distribution of depth p with cdf

$$F(w | r_1, \ldots, r_{p-1}, s; \lambda_1, \ldots, \lambda_{p-1}, \lambda)\,, \tag{39}$$

where the r_j $(j = 1, \ldots, p-1)$ are given in (31) and

$$\lambda_j = \frac{n-j}{n}\,, \quad (j = 1, \ldots, p-1), \tag{40}$$

and

$$\lambda = \frac{m_1}{m_2 - m_1^2} \qquad \text{and} \qquad s = \frac{m_1^2}{m_2 - m_1^2} \tag{41}$$

with

$$m_h = i^{-h}\, \frac{\partial^h}{\partial t^h}\, \Phi_2(t) \bigg|_{t=0}\,, \qquad h = 1, 2\,;$$

ii) a mixture of two GNIG distributions of depth p, with cdf

$$\sum_{k=1}^{2} \theta_k\, F(w | r_1, \ldots, r_{p-1}, s_k; \lambda_1, \ldots, \lambda_{p-1}, \mu)\,, \tag{42}$$

where r_j and λ_j $(j = 1, \ldots, p-1)$ are given in (31) and (40) and θ_1, μ, r_1 and r_2 are

obtained from the numerical solution of the system of four equations

$$\sum_{k=1}^{2} \theta_k\, \frac{\Gamma(r_k + h)}{\Gamma(r_k)}\, \mu^{-h} = i^{-h}\, \frac{\partial^h}{\partial t^h}\, \Phi_2(t) \bigg|_{t=0} \qquad (h = 1, \ldots, 4) \tag{43}$$

for these parameters, with $\theta_2 = 1 - \theta_1$;

iii) or a mixture of three GNIG distributions of depth $p-1$, with cdf

$$\sum_{k=1}^{3} \theta_k^* \, F(w|r_1,\dots,r_{p-2},s_k^*;\lambda_1,\dots,\lambda_{p-2},\nu)\,, \tag{44}$$

with r_j and λ_j ($j=1,\dots,p-2$) given by (31) and (40) and θ_1^*, θ_2^*, ν, s_1^*, s_2^* and s_3^* obtained from the numerical solution of the system of six equations

$$\sum_{k=1}^{3} \theta_j^* \frac{\Gamma(r_k^*+h)}{\Gamma(r_k^*)}\,\nu^{-h} \;=\; i^{-h}\,\frac{\partial^h}{\partial t^h}\,\Phi_2(t)\bigg|_{t=0} \qquad (h=1,\dots,6) \tag{45}$$

for these parameters, with $\theta_3^* = 1-\theta_1^*-\theta_2^*$.

Proof: If in the c.f. of W we replace $\Phi_2(t)$ by $\lambda^s(\lambda-\mathrm{i}t)^{-s}$ we obtain

$$\Phi_W(t) \;\approx\; \lambda^s(\lambda-\mathrm{i}t)^{-s}\;\underbrace{\prod_{k=1}^{p-1}\left(\frac{n-k}{n}\right)^{r_k}\left(\frac{n-k}{n}-it\right)^{-r_k}}_{\Phi_1(t)}\,,$$

that is the c.f. of the sum of $p-1$ independent Gamma random variables, $p-2$ of which with integer shape parameters r_j and rate parameters λ_j given by (31) and (40), and a further Gamma random variable with rate parameter $s>0$ and shape parameter λ. This c.f. is thus the c.f. of the GNIG distribution of depth p with distribution function given in (39). The parameters s and λ are determined in such a way that (36) holds. This compels s and λ to be given by (41) and makes the two first moments of this near-exact distribution for W to be the same as the two first exact moments of W.

If in the c.f. of W we replace $\Phi_2(t)$ by $\sum_{k=1}^{2}\theta_k\,\mu^{r_k}(\mu-\mathrm{i}t)^{-r_k}$ we obtain

$$\Phi_W(t) \approx \sum_{k=1}^{2}\theta_k\,\mu^{r_k}(\mu-\mathrm{i}t)^{-r_k}\;\underbrace{\prod_{k=1}^{p-1}\left(\frac{n-k}{n}\right)^{r_k}\left(\frac{n-k}{n}-it\right)^{-r_k}}_{\Phi_1(t)}\,,$$

that is the c.f. of the mixture of two GNIG distributions of depth p with distribution function given in (42). The parameters θ_1, μ, r_1 and r_2 are defined in such a way that (37) holds, giving rise to the evaluation of these parameters as the numerical solution of the system of equations in (37) and to a near-exact distribution that matches the first four exact moments of W.

If in the c.f. of W we replace $\Phi_2(t)$ by $\sum_{k=1}^{3} \theta_k^* \nu^{r_k^*}(\nu - \mathrm{i}t)^{-r_k^*}$ we obtain

$$\Phi_W(t) \approx \sum_{k=1}^{3} \theta_k^* \nu^{r_k^*}(\nu - \mathrm{i}t)^{-r_k^*} \underbrace{\prod_{k=1}^{p-1} \left(\frac{n-k}{n}\right)^{r_k} \left(\frac{n-k}{n} - it\right)^{-r_k}}_{\Phi_1(t)},$$

that is the characteristic function of the mixture of three GNIG distributions of depth p with distribution function given in (44). The parameters θ_1^*, θ_2^*, ν, r_1^*, r_2^* and r_3^* are defined in such a way that (38) holds, what gives rise to the evaluation of these parameters as the numerical solution of the system of equations in (38), giving rise to a near-exact distribution that matches the first six exact moments of W. ■

We should note here that the replacement of the characteristic function of a sum of Logbeta random variables by the characteristic function of a single Gamma random variable or the characteristic function of a mixture of two or three of such random variables has already been well justified at the end of Section 2.

Corollary 3: Distributions with cdf's given by

i) $1 - F(-\log z | r_1, \ldots, r_{p-1}, s; \lambda_1, \ldots, \lambda_{p-1}, \lambda)$,

ii) $1 - \sum_{k=1}^{2} \theta_k F(-\log z | r_1, \ldots, r_{p-1}, s_k; \lambda_1, \ldots, \lambda_{p-1}, \mu)$, or

iii) $1 - \sum_{k=1}^{3} \theta_k^* F(-\log z | r_1, \ldots, r_{p-1}, s_k^*; \lambda_1, \ldots, \lambda_{p-1}, \nu)$,

where the parameters are the same as in Theorem 1, and $0 < z < 1$ represents the running value of the statistic $\lambda^* = e^{-W}$, may be used as near-exact distributions for this statistic.

Proof: Since the near-exact distributions developed in Theorem 1 were for the random variable $W = -\log(\lambda^*)$ we only need to mind the relation

$$F_{\lambda^*}(z) = 1 - F_W(-\log z)$$

where $F_{\lambda^*}(\cdot)$ is the cumulative distribution function of λ^* and $F_W(\cdot)$ is the cumulative distribution function of W, in order to obtain the corresponding near-exact distributions for λ^*. ■

We should also stress that although we advocate the numerical solution of systems of equations (8), (43) and (45), the remarks in Marques and Coelho

(2007), at the end of Section 3, also apply here.

5 Numerical studies

In order to evaluate the quality of the approximations developed we use two measures of proximity between characteristic functions which are also measures of proximity between distribution functions or densities.

Let Y be a continuous random variable defined on S with distribution function $F_Y(y)$, density function $f_Y(y)$ and characteristic function $\phi_Y(t)$, and let $\phi_n(t)$, $F_n(y)$ and $f_n(y)$ be respectively the characteristic, distribution and density function of a random variable X_n. The two measures are

$$\Delta_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\phi_Y(t) - \phi_n(t)|\, \mathrm{d}t \qquad \text{and} \qquad \Delta_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{\phi_Y(t) - \phi_n(t)}{t}\right| \mathrm{d}t\,,$$

with

$$\max_{y\in S} |f_Y(y) - f_n(y)| \le \Delta_1\,; \quad \max_{y\in S} |F_Y(y) - F_n(y)| \le \Delta_2\,. \tag{46}$$

We should note that for continuous random variables,

$$\lim_{n\to\infty} \Delta_1 = 0 \iff \lim_{n\to\infty} \Delta_2 = 0 \tag{47}$$

and either one of the limits above imply that

$$X_n \xrightarrow{d} Y\,. \tag{48}$$

Indeed both measures and both relations in (46) may be derived directly from inversion formulas, and Δ_2 may be seen as based on the Berry-Esseen upper bound on $|F_Y(y) - F_n(y)|$ (Berry, 1941; Esseen, 1945; Loève, 1977, Chap. VI, Sec. 21; Hwang, 1998) which may, for any $b > 1/(2\pi)$ and any $T > 0$, be written as

$$\max_{y\in S} |F_Y(y) - F_n(y)| \le b\int_{-T}^{T} \left|\frac{\phi_Y(t) - \phi_n(t)}{t}\right| dt + C(b)\,\frac{M}{T} \tag{49}$$

where $M = max_{y\in S} f_n(y)$ and $C(b)$ is a positive constant that only depends of b. If in (49) above we take $T \to \infty$ then we will have Δ_2, since then we may take $b = 1/(2\pi)$.

These measures were used by Grilo and Coelho (2007) to study near-exact approximations to the distribution of the product of independent Beta random variables and by Marques and Coelho (2007) to the study of near-exact distributions for the sphericity likelihood ratio test statistic.

In this section we will denote Box's asymptotic distribution by 'Box', by 'M2G' and 'M3G' respectively the asymptotic mixture of two and three Gamma distributions proposed in Section 2 and by 'GNIG', 'M2GNIG' and 'M3GNIG' the near-exact single GNIG distribution and the mixtures of two and three GNIG distributions.

We show in this section the Tables for values of Δ_2 while in Appendix A are the corresponding Tables for the values of Δ_1.

In Table 1, we may see that for increasing p (number of variables), with the sample size remaining close to p ($n-p=2$), the continuous degradation (increase) of the values of the measure Δ_2 for Box's asymptotic distribution, even with a value which does not make much sense for $p=50$ (since Δ_2 is an upper-bound on the absolute value of the difference between the approximate and the exact c.d.f.). Actually, in many cases where n is close to p, Box's asymptotic distribution does not even correspond to a true distribution (see Appendix B). Also the two asymptotic distributions M2G and M3G show slightly increasing values for Δ_2, although remaining within much low values. For the distribution M3G it was not possible to obtain the convergence for the solutions for $p=20$ and $p=50$, but this distribution, for values of $p \leq 7$ even shows lower values of Δ_2 then the GNIG near-exact distribution.

Opposite to this behavior, all three near-exact distributions show a sharp improvement (decrease) in their values for Δ_2 for increasing values of p, only with some minor fluctuations for consecutive even and odd values of p, showing the asymptotic character of these distributions for increasing values of p.

Table 1 – Values of the measure Δ_2 for increasing values of p, with small sample sizes

p	q	n	Box	M2G	M3G	GNIG	M2GNIG	M3GNIG
3	2	5	1.364×10^{-1}	2.601×10^{-4}	2.826×10^{-5}	3.461×10^{-4}	4.503×10^{-6}	4.503×10^{-6}
4	2	6	2.058×10^{-1}	1.417×10^{-4}	9.342×10^{-6}	9.733×10^{-5}	1.623×10^{-7}	9.621×10^{-10}
5	2	7	2.690×10^{-1}	2.092×10^{-4}	1.940×10^{-5}	1.952×10^{-5}	2.261×10^{-7}	9.605×10^{-9}
6	2	8	3.256×10^{-1}	2.601×10^{-4}	2.826×10^{-5}	6.091×10^{-5}	4.167×10^{-7}	3.404×10^{-9}
7	2	9	3.762×10^{-1}	3.000×10^{-4}	3.579×10^{-5}	1.022×10^{-4}	1.037×10^{-6}	1.537×10^{-8}
8	2	10	4.214×10^{-1}	3.319×10^{-4}	4.216×10^{-5}	3.493×10^{-5}	1.906×10^{-7}	1.492×10^{-9}
9	2	11	4.621×10^{-1}	3.579×10^{-4}	4.755×10^{-5}	2.244×10^{-5}	6.825×10^{-8}	1.940×10^{-10}
10	2	12	4.986×10^{-1}	3.794×10^{-4}	5.213×10^{-5}	2.141×10^{-5}	8.573×10^{-8}	5.282×10^{-10}
20	2	22	7.099×10^{-1}	4.749×10^{-4}	——×—	4.070×10^{-6}	4.854×10^{-9}	9.293×10^{-12}
50	2	52	1.286×10^{0}	4.864×10^{-4}	——×—	4.184×10^{-7}	9.503×10^{-11}	3.325×10^{-14}

Table A.1 in Appendix A, has the corresponding values for Δ_1 and would lead us to draw similar conclusions.

In Tables 2 and A.2 we may see the clear asymptotic character of the near-exact distributions for increasing values of q (the number of matrices being tested), with a similar but less marked behavior of the M2G and M3G asymptotic distributions, opposite to what happens with Box's asymptotic distribution.

In Tables 3 and A.3 we may see how, for increasing sample sizes, the asymptotic character is stronger for the near-exact distributions than for the asymptotic distributions. This asymptotic character being more marked for the near-exact distributions based on mixtures. However, the M3G asymptotic distribution for $p = 7$, $q = 2$ and $n = 50$ even beats the M2GNIG near-exact distribution and the M2G asymptotic distribution performs better than the GNIG near-exact distriution for most of the cases, namely those with larger sample sizes.

Table 2 – Values of the measure Δ_2 for increasing values of q, with small sample sizes

p	q	n	Box	M2G	M3G	GNIG	M2GNIG	M3GNIG
3	2	5	1.364×10^{-1}	2.601×10^{-4}	2.826×10^{-5}	3.461×10^{-4}	4.503×10^{-6}	4.503×10^{-6}
3	5	5	1.669×10^{-1}	3.947×10^{-5}	1.697×10^{-6}	4.234×10^{-4}	8.892×10^{-6}	3.004×10^{-7}
3	7	5	1.869×10^{-1}	3.283×10^{-5}	1.423×10^{-6}	2.870×10^{-4}	4.725×10^{-6}	1.320×10^{-7}
3	10	5	2.143×10^{-1}	2.633×10^{-5}	1.111×10^{-6}	1.796×10^{-4}	2.168×10^{-6}	4.610×10^{-8}
10	2	12	4.986×10^{-1}	3.794×10^{-4}	5.213×10^{-5}	2.141×10^{-5}	8.573×10^{-8}	9.293×10^{-10}
10	5	12	5.820×10^{-1}	1.796×10^{-4}	——×—	8.878×10^{-7}	2.092×10^{-10}	3.988×10^{-14}
10	7	12	6.376×10^{-1}	1.389×10^{-4}	——×—	4.090×10^{-7}	6.378×10^{-11}	1.125×10^{-14}
10	10	12	7.066×10^{-1}	1.056×10^{-4}	——×—	1.657×10^{-7}	6.354×10^{-12}	9.907×10^{-16}

Table 3 – Values of the measure Δ_2 for increasing sample sizes

p	q	n	Box	M2G	M3G	GNIG	M2GNIG	M3GNIG
3	2	5	1.364×10^{-1}	2.601×10^{-4}	2.826×10^{-5}	3.461×10^{-4}	4.503×10^{-6}	4.503×10^{-6}
3	2	20	4.629×10^{-3}	2.186×10^{-8}	——×—	2.010×10^{-5}	1.250×10^{-7}	1.040×10^{-9}
3	2	50	6.684×10^{-4}	4.014×10^{-9}	——×—	3.033×10^{-6}	6.261×10^{-9}	5.697×10^{-12}
7	2	9	3.762×10^{-1}	3.000×10^{-4}	3.579×10^{-5}	1.022×10^{-4}	1.037×10^{-6}	1.537×10^{-8}
7	2	20	3.897×10^{-2}	3.912×10^{-6}	5.604×10^{-8}	2.059×10^{-5}	4.180×10^{-8}	7.620×10^{-11}
7	2	50	4.829×10^{-3}	7.172×10^{-8}	8.263×10^{-11}	3.021×10^{-6}	9.071×10^{-10}	4.811×10^{-14}
10	2	12	4.986×10^{-1}	3.794×10^{-4}	5.213×10^{-5}	2.141×10^{-5}	8.573×10^{-8}	5.282×10^{-10}
10	2	50	1.090×10^{-2}	2.093×10^{-7}	——×—	1.844×10^{-6}	5.352×10^{-10}	1.601×10^{-13}
10	2	100	2.433×10^{-3}	1.159×10^{-8}	——×—	4.595×10^{-7}	3.272×10^{-11}	2.338×10^{-15}

In Tables 4 and A.4, if we compare the values of Δ_2 for different values of p and the same sample size, we may see how, opposite to the asymptotic distributions, the near-exact distributions show a clear asymptotic character for increasing values of p.

Table 4 – Values of the measure Δ_2 for large q and large sample sizes

p	q	n	Box	M2G	M3G	GNIG	M2GNIG	M3GNIG
3	10	20	8.659×10^{-3}	7.104×10^{-8}	1.313×10^{-9}	5.882×10^{-6}	9.890×10^{-9}	3.224×10^{-11}
3	10	50	1.283×10^{-3}	1.852×10^{-9}	1.330×10^{-11}	8.294×10^{-7}	4.831×10^{-10}	6.378×10^{-13}
7	10	20	6.879×10^{-2}	1.099×10^{-6}	——×—	1.226×10^{-6}	3.699×10^{-10}	1.942×10^{-13}
7	10	50	8.980×10^{-3}	2.574×10^{-8}	——×—	2.172×10^{-7}	1.753×10^{-11}	5.605×10^{-15}
10	10	50	2.007×10^{-2}	6.342×10^{-8}	——×—	3.343×10^{-8}	1.458×10^{-13}	——×—
10	10	100	4.565×10^{-3}	4.263×10^{-9}	——×—	9.210×10^{-9}	4.031×10^{-14}	——×—

6 Conclusions

All the near-exact distributions show a very good performance, with the ones based on mixtures showing an outstanding behaviour. For the approximate distributions developed in this paper, the near-exact distributions, which also have a more elaborate structure, clearly outperform their asymptotic counterparts, for a given number of exact moments matched.

Moreover, opposite to the usual asymptotic distributions, the near-exact distributions developed show a marked asymptotic behavior not only for increasing sample sizes but also for increasing values of p (the number of variables) and for increasing values of q (the number of matrices being tested). Yet, all the near-exact distributions proposed may be easily used to compute near-exact quantiles.

We should stress here that also the two new asymptotic distributions proposed show an asymptotic behavior for increasing values of q (the number of matrices being tested).

Thus, as a final comment, and given the values of the measures Δ_1 and Δ_2 obtained for the distributions, we would say that we may use the asymptotic distributions proposed in this paper in practical applications that may need a not so high degree of precision, although higher than the one that the usual asymptotic distributions deliver. For applications that may need a high degree of precision in the computation of quantiles we may then use the more elaborate near-exact distributions, mainly those based on mixtures of GNIG distributions, which anyway allow for an easy computation of quantiles. The distribution M3GNIG, given its excellent performance and manageability, may even be used as a replacement of the exact distribution.

Appendix A
Tables with values of Δ_1

Table A.1 – Values of the measure Δ_1 for increasing values of p, with small sample sizes

p	q	n	Box	M2G	M3G	GNIG	M2GNIG	M3GNIG
3	2	5	5.100×10^{-2}	1.057×10^{-4}	1.378×10^{-5}	3.926×10^{-4}	8.756×10^{-6}	8.756×10^{-6}
4	2	6	5.151×10^{-2}	1.134×10^{-4}	9.118×10^{-6}	6.086×10^{-5}	1.413×10^{-7}	1.085×10^{-9}
5	2	7	5.094×10^{-2}	1.128×10^{-4}	1.261×10^{-5}	8.648×10^{-6}	1.461×10^{-7}	8.152×10^{-9}
6	2	8	4.955×10^{-2}	1.057×10^{-4}	1.378×10^{-5}	1.991×10^{-5}	1.904×10^{-7}	1.932×10^{-9}
7	2	9	4.773×10^{-2}	9.760×10^{-5}	1.394×10^{-5}	2.653×10^{-5}	3.726×10^{-7}	6.828×10^{-9}
8	2	10	4.571×10^{-2}	8.984×10^{-5}	1.365×10^{-5}	7.580×10^{-6}	5.747×10^{-8}	5.580×10^{-10}
9	2	11	4.362×10^{-2}	8.276×10^{-5}	1.314×10^{-5}	4.152×10^{-6}	1.751×10^{-8}	6.154×10^{-11}
10	2	12	4.152×10^{-2}	7.642×10^{-5}	1.256×10^{-5}	3.431×10^{-6}	1.898×10^{-8}	1.447×10^{-10}
20	2	22	2.377×10^{-2}	4.022×10^{-5}	——×—	2.666×10^{-7}	4.324×10^{-10}	1.012×10^{-12}
50	2	52	1.523×10^{-2}	1.382×10^{-5}	——×—	8.971×10^{-9}	2.741×10^{-12}	1.161×10^{-15}

Table A.2 – Values of the measure Δ_1 for increasing values of q, with small sample sizes

p	q	n	Box	M2G	M3G	GNIG	M2GNIG	M3GNIG
3	2	5	5.100×10^{-2}	1.057×10^{-4}	1.378×10^{-5}	3.926×10^{-4}	8.756×10^{-6}	8.756×10^{-6}
3	5	5	2.929×10^{-2}	1.938×10^{-5}	1.007×10^{-6}	1.566×10^{-4}	4.447×10^{-6}	1.830×10^{-7}
3	7	5	2.687×10^{-2}	1.280×10^{-5}	6.685×10^{-7}	8.435×10^{-5}	1.872×10^{-6}	6.352×10^{-8}
3	10	5	2.524×10^{-2}	8.247×10^{-6}	4.191×10^{-7}	4.238×10^{-5}	6.879×10^{-7}	1.771×10^{-8}
10	2	12	4.152×10^{-2}	7.642×10^{-5}	1.256×10^{-5}	3.431×10^{-6}	1.898×10^{-8}	1.447×10^{-10}
10	5	12	2.563×10^{-2}	2.008×10^{-5}	——×—	7.559×10^{-8}	2.407×10^{-11}	5.573×10^{-15}
10	7	12	2.257×10^{-2}	1.292×10^{-5}	——×—	2.875×10^{-8}	6.034×10^{-12}	1.288×10^{-15}
10	10	12	1.974×10^{-2}	8.127×10^{-6}	——×—	9.588×10^{-9}	4.935×10^{-13}	9.298×10^{-17}

Table A.3 – Values of the measure Δ_1 for increasing sample sizes

p	q	n	Box	M2G	M3G	GNIG	M2GNIG	M3GNIG
3	2	5	5.100×10^{-2}	1.057×10^{-4}	1.378×10^{-5}	3.926×10^{-4}	8.756×10^{-6}	8.756×10^{-6}
3	2	20	2.355×10^{-3}	5.564×10^{-8}	——×—	3.040×10^{-5}	2.849×10^{-7}	3.113×10^{-9}
3	2	50	3.581×10^{-4}	9.763×10^{-9}	——×—	4.826×10^{-6}	1.498×10^{-8}	1.768×10^{-11}
7	2	9	4.773×10^{-2}	9.760×10^{-5}	1.394×10^{-5}	2.653×10^{-5}	3.726×10^{-7}	6.828×10^{-9}
7	2	20	6.963×10^{-3}	1.980×10^{-6}	3.425×10^{-8}	7.832×10^{-6}	2.157×10^{-8}	4.763×10^{-11}
7	2	50	9.812×10^{-4}	4.175×10^{-8}	5.771×10^{-11}	1.309×10^{-6}	5.310×10^{-10}	3.406×10^{-14}
10	2	12	4.152×10^{-2}	7.642×10^{-5}	1.256×10^{-5}	3.431×10^{-6}	1.898×10^{-8}	1.447×10^{-10}
10	2	50	1.506×10^{-3}	7.965×10^{-8}	——×—	5.258×10^{-7}	2.051×10^{-10}	7.399×10^{-14}
10	2	100	3.559×10^{-4}	4.684×10^{-9}	——×—	1.387×10^{-7}	1.326×10^{-11}	1.141×10^{-15}

Table A.4 – Values of the measure Δ_1 for large q and large sample sizes

p	q	n	Box	M2G	M3G	GNIG	M2GNIG	M3GNIG
3	10	20	1.268×10^{-3}	2.872×10^{-8}	6.017×10^{-10}	1.776×10^{-6}	4.018×10^{-9}	1.588×10^{-11}
3	10	50	1.951×10^{-4}	7.791×10^{-10}	6.342×10^{-12}	2.602×10^{-7}	2.040×10^{-10}	3.269×10^{-13}
7	10	20	4.219×10^{-3}	1.794×10^{-7}	——×—	1.503×10^{-7}	6.064×10^{-11}	3.831×10^{-14}
7	10	50	6.052×10^{-4}	4.654×10^{-9}	——×—	2.946×10^{-8}	3.176×10^{-12}	1.222×10^{-15}
10	10	50	9.398×10^{-4}	7.933×10^{-9}	——×—	3.137×10^{-9}	1.828×10^{-14}	——×—
10	10	100	2.229×10^{-4}	5.569×10^{-10}	——×—	9.023×10^{-10}	5.274×10^{-15}	——×—

Appendix B
Plots of p.d.f.'s and c.d.f.'s of Box's asymptotic distribution for $W = -\log(\lambda^*)$

In this Appendix we show some plots of p.d.f.'s and c.d.f.'s of Box's asymptotic distribution corresponding to the c.f. in (5) for $W = -\log(\lambda^*)$, for a few combinations of p, q and n for which this distribution is not a proper distribution. In all the four cases presented the p.d.f. goes below zero and for $p = 5, 7$ and 10 the c.d.f. goes above 1, while for $p = 50$ it has negative values for smaller values of the argument, as it is shown in Figures B.1 and B.2.

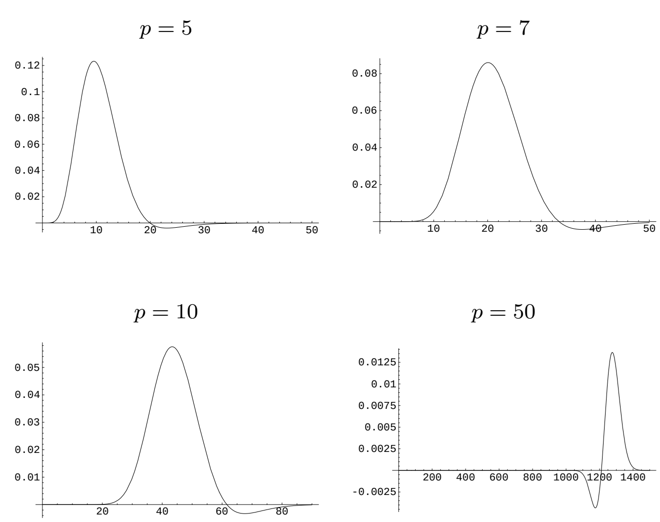

Figure B.1 – Plots of p.d.f.'s of Box's asymptotic distribution for $W = -\log(\lambda^*)$ for $q = 2$ and $n = p + 2$.

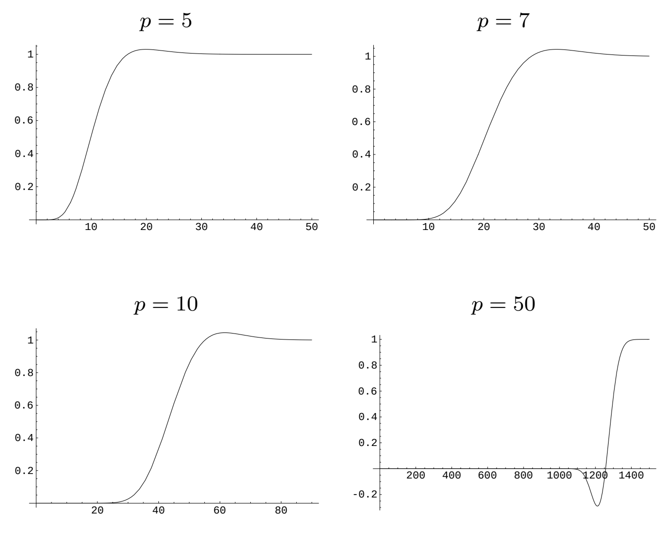

Figure B.2 – Plots of c.d.f.'s of Box's asymptotic distribution for $W = -\log(\lambda^*)$ for $q = 2$ and $n = p + 2$.

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