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Normally ordered semigroups

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Abstract. In this paper we introduce the notion of normally ordered block-group as a natural extension of the notion of normally ordered inverse semigroup considered previously by the author. We prove that the class NOS of all normally ordered block-groups forms a pseudovariety of semigroups and, by using the Munn representation of a block-group, we deduce the decompositions in Mal'cev products $\mathsf{NOS} = \mathsf{EI} \circledm \mathsf{POI}$ and $\mathsf{NOS} \cap \mathsf{A} = \mathsf{N} \circledm \mathsf{POI}$, where A, EI and N denote the pseudovarieties of all aperiodic semigroups, all semigroups with just one idempotent and all nilpotent semigroups, respectively, and POI denotes the pseudovariety of semigroups generated all semigroups of injective order-preserving partial transformations on a finite chain. These relations are obtained after showing that $\mathsf{BG} = \mathsf{EI} \circledm \mathsf{Ecom} = \mathsf{N} \circledm \mathsf{Ecom}$, where BG and Ecom denote the pseudovarieties of all block-groups and all semigroups with commuting idempotents, respectively.

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Introduction and preliminaries

Let X be a set. We denote by $\mathcal{PT}(X)$ the monoid (under composition) of all partial transformations on X, by $\mathcal{T}(X)$ the submonoid of $\mathcal{PT}(X)$ of all full transformations on X and by $\mathcal{I}(X)$ the *symmetric inverse semigroup* on X, i.e. the inverse submonoid of $\mathcal{PT}(X)$ of all injective partial transformations on X. If X is a finite set with n elements, we denote $\mathcal{PT}(X)$, $\mathcal{T}(X)$ and $\mathcal{I}(X)$ simply by $\mathcal{PT}_n$, $\mathcal{T}_n$ and $\mathcal{I}_n$, respectively. Now, suppose that X is a finite chain with n element, say $X = \{1 < 2 < \cdots < n\}$. We say that a transformation s in $\mathcal{PT}_n$ is *order-preserving* if $x \le y$ implies $xs \le ys$, for all $x, y \in \mathrm{Dom}(s)$, and denote by $\mathcal{PO}_n$ the submonoid of $\mathcal{PT}_n$ of all partial order-preserving transformations. As usual, $\mathcal{O}_n$ denotes the monoid $\mathcal{PO}_n \cap \mathcal{T}_n$ of all full transformations of X_n that preserve the order and the injective counterpart of $\mathcal{O}_n$, i.e. the inverse monoid $\mathcal{PO}_n \cap \mathcal{I}_n$, is denoted by $\mathcal{POI}_n$.

A pseudovariety of [inverse] semigroups is a class of finite [inverse] semigroups closed under homomorphic images of [inverse] subsemigroups and finitary direct products.

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In the 1987 "Szeged International Semigroup Colloquium" J.-E. Pin asked for an *effective* description of the pseudovariety (i.e. an algorithm to decide whether or not a finite semigroup belongs to the pseudovariety) of semigroups O generated by the semigroups $\mathcal{O}_n$, with $n \in \mathbb{N}$. This problem only had essential progresses after 1995. First, Higgins [10] proved that O is self-dual and does not contain all $\mathcal{R}$-trivial semigroups (and so O is properly contained in A, the pseudovariety of all finite aperiodic semigroups, i.e. $\mathcal{H}$-trivial semigroups), although every finite band belongs to O. Next, Vernitskii and Volkov [17] generalized Higgins's result by showing that every finite semigroup whose idempotents form an ideal is in O and in [5] the author proved that the pseudovariety of semigroups POI generated by the semigroups $\mathcal{POI}_n$, with $n \in \mathbb{N}$, is a (proper) subpseudovariety of O. On the other hand, Almeida and Volkov [2] showed that the interval $[\mathsf{O}, \mathsf{A}]$ of the lattice of all pseudovarieties of semigroups has the cardinality of the continuum and Repnitskiĭ and Volkov [15] proved that O is not finitely based. In fact, moreover, Repnitskiĭ and Volkov proved in [15] that any pseudovariety of semigroups V such that $\mathsf{POI} \subseteq \mathsf{V} \subseteq \mathsf{O} \vee \mathsf{R} \vee \mathsf{L}$, where R and L are the pseudovarieties of semigroups of all $\mathcal{R}$-trivial semigroups and of all $\mathcal{L}$-trivial semigroups, respectively, is not finitely based. Another contribution to the resolution of Pin's problem was given by the author [7] who showed that O contains all semidirect products of a chain (considered as a semilattice) by a semigroup of injective order-preserving partial transformations on a finite chain. Nevertheless, Pin's question is still unanswered.

The inverse counterpart of Pin's problem can be formulated by asking for an effective description of the pseudovariety of inverse semigroups PCS generated by $\{\mathcal{POI}_n \mid n \in \mathbb{N}\}$. In [3] Cowan and Reilly proved that PCS is properly contained in A and also that the interval $[\mathsf{PCS}, \mathsf{A}]$ of the lattice of all pseudovarieties of inverse semigroups has the cardinality of the continuum. From Cowan and Reilly's results it can be deduced that a finite inverse semigroup with n elements belongs to PCS if and only if it can be embedded into the semigroup $\mathcal{POI}_n$. This is in fact an effective description of PCS. On the other hand, in [6] the author introduced the class NO of all normally ordered inverse semigroups. This notion is deeply related with the Munn representation of an inverse semigroup S, an idempotent-separating homomorphism that may be defined by

$$\begin{array}{rcccl} \phi: & S & \to & \mathcal{I}(E) & \\ & s & \mapsto & \phi_s: & \begin{array}{rcl} Ess^{-1} & \to & Es^{-1}s \\ e & \mapsto & s^{-1}es \end{array} \end{array},$$

with E the semilattice of all idempotents of S. Notice that, the kernel of ϕ is μ, the maximum idempotent-separating congruence on S. Therefore, ϕ is an injective homomorphism if and only if S is a fundamental semigroup, (see [11] or [12], for more details). Observe that by a fundamental semigroup we mean any semigroup without non-trivial idempotent-separating congruences. Now, a finite inverse semigroup S is said to be *normally ordered* if there exists a linear order $\sqsubseteq$ in the semilattice E of the idempotents of S preserved by all partial injective mappings ϕ_s (i.e. for $e, f \in Ess^{-1}$, $e \sqsubseteq f$ implies $e\delta_s \sqsubseteq f\delta_s$),

$s \in S$. It was proved in [6] that NO is a pseudovariety of inverse semigroups and also that the class of all fundamental normally ordered inverse semigroups consists of all aperiodic normally ordered inverse semigroups. Moreover, the author showed that $\mathsf{PCS} = \mathsf{NO} \cap \mathsf{A}$, giving this way a Cowan and Reilly alternative (effective) description of PCS. In fact, this also led the author [6] to the following refinement of Cowan and Reilly's description of PCS: a finite inverse semigroup with n *idempotents* belongs to PCS if and only if it can be embedded into $\mathcal{POI}_n$. Another refinement (in fact, the best possible) will be give in this paper. Notice that, in [6] it was also proved that $\mathsf{NO} = \mathsf{PCS} \vee \mathsf{G}$ (the join of PCS and G, the pseudovariety of all groups).

The work presented in this paper was strongly motivated by the author's attempt to obtain an effective description for the pseudovariety of semigroups POI, generalizing the ideas of [6]. Notice that POI is a subpseudovariety of Ecom, the pseudovariety of all idempotent commuting semigroups, whence in order to accomplish this aim, a Munn type representation for, at least, idempotent commuting semigroups is required. Such representation was constructed by the author [8] for a wider class of semigroups: BG, the class of all block-groups. Recall that a block-group is a finite semigroup whose elements have at most one inverse. Clearly, a finite semigroup is a block-group if and only if each $\mathcal{L}$-class and each $\mathcal{R}$-class contains at most one idempotent. Observe that BG is a pseudovariety of semigroups, which plays a main role in the following celebrated result: $\Diamond\mathsf{G} = \mathsf{PG} = \mathsf{J} * \mathsf{G} = \mathsf{J} ⓜ \mathsf{G} = \mathsf{BG} = \mathsf{EJ}$, where J denotes the pseudovariety of all $\mathcal{J}$-trivial semigroups, PG and $\Diamond\mathsf{G}$ denote the pseudovarieties generated by all power monoids of groups and by all Schützenberger products of groups, respectively, and, finally, EJ denotes the pseudovariety of all semigroups whose idempotents generate a $\mathcal{J}$-trivial semigroup. See [14] for precise definitions and for a complete story of these equalities.

Next, we recall our extension of the Munn representation for block-groups. Let S be a semigroup. We denote by $E(S)$ the set of all idempotents of S and by $\mathrm{Reg}(S)$ the set of all regular elements of S. Recall the definition of the quasi-orders $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ associated to the Green relations $\mathcal{R}$ and $\mathcal{L}$, respectively: for all $s,t \in S$, $s \leq_{\mathcal{R}} t$ if and only if $sS^1 \subseteq tS^1$ and $s \leq_{\mathcal{L}} t$ if and only if $S^1s \subseteq S^1t$, where S^1 denotes the monoid obtained from S through the adjoining of an identity if S has none and denotes S otherwise. To each element $s \in S$, we associate the following two subsets of $E(S)$: $\mathcal{R}(s) = \{e \in E(S) \mid e \leq_{\mathcal{R}} s\}$ and $\mathcal{L}(s) = \{e \in E(S) \mid e \leq_{\mathcal{L}} s\}$. Clearly, if $e \in \mathcal{R}(s)$ then $es \in \mathrm{Reg}(S)$ and, dually, if $e \in \mathcal{L}(s)$ then $se \in \mathrm{Reg}(S)$. Now, let S be a block-group and let s^{-1} denote the unique inverse of a regular element $s \in S$. Then, given $s \in S$, the maps $\delta_s : \mathcal{R}(s) \to \mathcal{L}(s)$, $e \mapsto (es)^{-1}(es)$, and $\bar{\delta}_s : \mathcal{L}(s) \to \mathcal{R}(s)$, $e \mapsto (se)(se)^{-1}$, are mutually inverse bijections that preserve $\mathcal{D}$-classes. Moreover, being $E = E(S)$, the mapping

$$\begin{array}{rccccc} \delta: & S & \to & \mathcal{I}(E) & & \\ & s & \mapsto & \delta_s: & \mathcal{R}(s) \to & \mathcal{L}(s) \\ & & & & e \mapsto & (es)^{-1}(es) \end{array}$$

is an idempotent-separating homomorphism, which we call the *Munn representation* of S. Notice that δ coincides with the (usual) Munn representation of an inverse semigroup S. Furthermore, as for inverse semigroups, the kernel of the Munn representation of a block-group is the maximum idempotent-separating congruence of S (see [8] for details). Now, we can extend, naturally, the concept of "normally ordered" from inverse semigroups to block-groups. We say that a block-group is *normally ordered* if there exists a *normal order* in S, i.e. a linear order $\sqsubseteq$ in $E(S)$ preserved by all partial injective mappings δ_s, $s \in S$, of the Munn representation of S. We denote by NOS the class of all normally ordered block-groups.

The remaining of this paper is organized as follows. In Section 1 we study the class NOS; in particular, we show that NOS is a (decidable) pseudovariety of semigroups. Also in this section we present a refinement of the descriptions of PCS mentioned above. In the next and last section, by using the Munn representation of a block-group, we show the following decompositions in Mal'cev products of the pseudovariety of block-groups: $\mathsf{BG} = \mathsf{EI} \circledm \mathsf{Ecom} = \mathsf{N} \circledm \mathsf{Ecom}$, where EI and N denote the pseudovarieties of all semigroups with just one idempotent and all nilpotent semigroups, respectively. Furthermore, in Section 2, we deduce also the equalities $\mathsf{NOS} = \mathsf{EI} \circledm \mathsf{POI}$ and $\mathsf{NOS} \cap \mathsf{A} = \mathsf{N} \circledm \mathsf{POI}$.

We assume some knowledge on semigroups, namely on Green's relations, regular elements and inverse semigroups. Possible references are [11, 12]. For general background on pseudovarieties, pseudoidentities and other stuff on finite semigroups, we refer the reader to Almeida's book [1]. All semigroups considered in this paper are finite.

1 Normally ordered block-groups

In this section we study the class NOS of all normally ordered block-groups. In particular, we show that NOS is a pseudovariety of semigroups. Notice that, an inverse semigroup belongs to the class NOS if and only if it belongs to the pseudovariety of inverse semigroups NO.

We begin by recalling the following lemma, which proof can be found in [16].

Lemma 1.1 *Let $\varphi : S \longrightarrow T$ be an onto homomorphism of semigroups and let J' be a $\mathcal{J}$-class of T. Then $J'\varphi^{-1} = J_1 \cup \cdots \cup J_k$, for some $\mathcal{J}$-classes $J_1, \ldots, J_k$ of S, and if J_i ($1 \le i \le k$) is $\le_{\mathcal{J}}$-minimal among $J_1, \ldots, J_k$, then $J_i\varphi = J'$. Furthermore, if J' is regular, then the index i is uniquely determined (i.e. J_i is $\le_{\mathcal{J}}$-minimum among $J_1, \ldots, J_k$), and J_i is itself regular.* ■

Next, recall that, given two elements a and b of an arbitrary semigroup S, it is well known that $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent. Moreover, if S is finite and $a \,\mathcal{J}\, b$, then $ab \in R_a \cap L_b$ if and only if $ab \,\mathcal{J}\, a$ (see [13]).

The next two lemmas help us to show that NOS is closed under homomorphic images.

Lemma 1.2 *Let S and T be two block-groups and let $\varphi : S \longrightarrow T$ be an onto homomorphism. Let J' be a regular $\mathcal{J}$-class of T and J the $\mathcal{J}$-class of S $\leq_{\mathcal{J}}$-minimum among the $\mathcal{J}$-classes Q of S such that $Q\varphi \subseteq J'$. Then φ induces a bijection from $J \cap E(S)$ onto $J' \cap E(T)$.*

Proof. First, notice that J is regular and $J\varphi = J'$. Let $e' \in J' \cap E(T)$ and let $x \in J$ be such that $x\varphi = e'$. Take $e = x^\omega$. Then $e\varphi = e'$ and $J_e\varphi \subseteq J'$. By the minimality of J, we have $J \leq_{\mathcal{J}} J_e$. On the other hand $J_e \leq_{\mathcal{J}} J_x = J$ and so $J_e = J$. Hence $e \in J \cap E(S)$. Thus $J' \cap E(T) \subseteq (J \cap E(S))\varphi$ and, since the other inclusion is clear, it follows that $(J \cap E(S))\varphi = J' \cap E(T)$. In order to prove that φ is injective in $J \cap E(S)$, let $e, f \in J \cap E(S)$ be such that $e\varphi = f\varphi = e'$. Then $(ef)\varphi = e'$, and so, again by the minimality of J, we have $J \leq_{\mathcal{J}} J_{ef} \leq_{\mathcal{J}} J_e = J$. Hence $ef \in J$. As $e, f, ef \in J$, then $ef \in R_e \cap L_f$, whence $L_e \cap R_f$ contains an idempotent g. Now, since each $\mathcal{R}$-class and each $\mathcal{L}$-class of S contains at most one idempotent, we conclude that $e = g = f$, as required. ■

Let S and T be two block-groups and let $\varphi : S \longrightarrow T$ be an onto homomorphism. Denote by $E_\varphi(S)$ the subset of $E(S)$ of all idempotents e such that the $\mathcal{J}$-class J_e is $\leq_{\mathcal{J}}$-minimum among the $\mathcal{J}$-classes Q of S such that $Q\varphi \subseteq J_{e\varphi}$. Therefore, by the previous lemma, the restriction $\varphi|_{E_\varphi(S)} : E_\varphi(S) \longrightarrow E(T)$ is a bijection from $E_\varphi(S)$ onto $E(T)$. Furthermore, given $s \in S$ and $e \in \mathcal{R}(s)$, as $e \,\mathcal{J}\, (es)^{-1}(es)$, we have $e \in E_\varphi(S)$ if and only if $(es)^{-1}(es) \in E_\varphi(S)$.

Next, observe that, since any homomorphism maps an inverse of a regular element into an inverse of its image, in particular given a homomorphism $\varphi : S \longrightarrow T$ between block-groups, we have $(s^{-1})\varphi = (s\varphi)^{-1}$, for any regular element $s \in S$.

Lemma 1.3 *Let S and T be two block-groups and let $\varphi : S \longrightarrow T$ be an onto homomorphism. Let $s \in S$, $t = s\varphi$, $a \in \mathcal{R}(t)$ and $e \in E_\varphi(S) \cap a\varphi^{-1}$. Then $e \in \mathcal{R}(s)$.*

Proof. Since $a \in \mathcal{R}(t)$ then at is regular and $a = t(at)^{-1} = (at)(at)^{-1}$. Moreover, $at \in J_a$ and $(es)\varphi = at$. Then, by the minimality of J_e, we have $J_e \leq_{\mathcal{J}} J_{es}$, whence $J_e = J_{es}$. In particular, es is regular and so $(es)^{-1}\varphi = ((es)\varphi)^{-1} = (at)^{-1}$. Then, we have $e\varphi = a = t(at)^{-1} = s\varphi(es)^{-1}\varphi = (s(es)^{-1})\varphi$ and so $e\varphi = (s(es)^{-1})^\omega\varphi$. Thus, again by the minimality of J_e, it follows that $J_e \leq_{\mathcal{J}} J_{(s(es)^{-1})^\omega}$ and, on the other hand, $J_{(s(es)^{-1})^\omega} \leq_{\mathcal{J}} J_{s(es)^{-1}} = J_{s(es)^{-1}(es)(es)^{-1}} \leq_{\mathcal{J}} J_e$. Then $J_e = J_{(s(es)^{-1})^\omega}$ and thence $e = (s(es)^{-1})^\omega$. Therefore $e \in \mathcal{R}(s)$, as required. ■

Now, we can prove:

Proposition 1.4 *Any homomorphic image of a normally ordered block-group is a normally ordered block-group.*

Proof. Let T be a semigroup, let S be a normally ordered block-group and let $\varphi : S \longrightarrow T$ be an onto homomorphism. Denote by $\sqsubseteq$ the normal order of S. As φ is a bijection from $E_\varphi(S)$ onto $E(T)$, we may define a linear order $\sqsubseteq$ in $E(T)$ by $e\varphi \sqsubseteq f\varphi$ if and only if $e \sqsubseteq f$, for all $e, f \in E_\varphi(S)$.

Now, let $t \in T$ and consider $a, b \in \mathcal{R}(t)$ such that $a \sqsubseteq b$. We aim to show that $(at)^{-1}(at) \sqsubseteq (bt)^{-1}(bt)$. Take $e, f \in E_\varphi(S)$ such that $a = e\varphi$ and $b = f\varphi$. Then $e \sqsubseteq f$, by definition. Let $s \in t\varphi^{-1}$. By Lemma 1.3, it follows that $e, f \in \mathcal{R}(s)$ and, as $\sqsubseteq$ is a normal order of S, we have $(es)^{-1}(es) \sqsubseteq (fs)^{-1}(fs)$. Since also $(es)^{-1}(es), (fs)^{-1}(fs) \in E_\varphi(S)$, then $(at)^{-1}(at) = (es)^{-1}\varphi(es)\varphi = ((es)^{-1}(es))\varphi \sqsubseteq ((fs)^{-1}(fs))\varphi = (fs)^{-1}\varphi(fs)\varphi = (bt)^{-1}(bt)$, as required. ■

Let S be a normally ordered block-group and let T be a subsemigroup of S. Then, it is clear that the order induced on $E(T)$ by the normal order of S is a normal order in T. Hence T is also a normally ordered block-group.

On the other hand, consider n normally ordered block-groups $S_1, S_2, \ldots, S_n$. For $i \in \{1, 2, \ldots n\}$, denote by $\sqsubseteq_i$ the normal order of S_i. Take $S = S_1 \times S_2 \times \cdots \times S_n$. Since $E(S) = E(S_1) \times E(S_2) \times \cdots \times E(S_n)$, we may consider in $E(S)$ the *lexicographic order* $\sqsubseteq_{\text{lex}}$ induced by the orders $\sqsubseteq_1, \sqsubseteq_2, \ldots, \sqsubseteq_n$, i.e. given $e = (e_1, e_2, \ldots, e_n), f = (f_1, f_2, \ldots, f_n) \in E(S)$, we have $e \sqsubseteq_{\text{lex}} f$ if and only if $e = f$ or, for some $p \in \{1, 2, \ldots n\}$, $e_i = f_i$, with $1 \leq i \leq p-1$, and $e_p \sqsubset_p f_p$. It is routine to show that $\sqsubseteq_{\text{lex}}$ is a normal order in S, whence the direct product of $S_1, S_2, \ldots, S_n$ is also a normally ordered block-group.

The previous two observations together with Proposition 1.4 allow us to conclude:

Theorem 1.5 *The class* NOS *is a pseudovariety of semigroups.* ■

Observe that, as $\mathcal{POI}_n \in \mathsf{NO}$ [6], for all $n \in \mathbb{N}$, we have:

Corollary 1.6 $\mathsf{POI} \subseteq \mathsf{NOS} \cap \mathsf{Ecom} \cap \mathsf{A}$. ■

As for inverse semigroups [6], we have:

Proposition 1.7 *Let S and T be two block-groups and let $\varphi : S \longrightarrow T$ be an onto idempotent-separating homomorphism. Then, $S \in \mathsf{NOS}$ if and only $T \in \mathsf{NOS}$.*

Proof. By Proposition 1.4, it remains to prove that $T \in \mathsf{NOS}$ implies $S \in \mathsf{NOS}$. Then, suppose that $T \in \mathsf{NOS}$ and let $\sqsubseteq$ be the normal order of T. Define a relation $\sqsubseteq$ in $E(S)$ by $e \sqsubseteq f$ if and only if $e\varphi \sqsubseteq f\varphi$, for all $e, f \in E(S)$. As φ separates idempotents, then φ induces a bijection from $E(S)$ onto $E(T)$ and thence $\sqsubseteq$ is a linear order of $E(S)$. Moreover, $\sqsubseteq$ is a normal order in S. Indeed, take $s \in S$ and $e, f \in \mathcal{R}(s)$ such that $e \sqsubseteq f$. Then $e\varphi, f\varphi \in \mathcal{R}(s\varphi)$ and, by definition, $e\varphi \sqsubseteq f\varphi$. Hence, $(e\varphi s\varphi)^{-1}(e\varphi s\varphi) \sqsubseteq (f\varphi s\varphi)^{-1}(f\varphi s\varphi)$, i.e., $((es)^{-1}(es))\varphi \sqsubseteq ((fs)^{-1}(fs))\varphi$, since es and fs are regular elements of S. Thus, we have $(es)^{-1}(es) \sqsubseteq (fs)^{-1}(fs)$, as required. ■

As the kernel of the Munn representation of a block-group S is the (maximum) idempotent-separating congruence μ of S, we have, by Proposition 1.7, $S \in \mathsf{NOS}$ if and only if $S/\mu \in \mathsf{NOS}$. On the other hand, if $S \in \mathsf{NOS}$, then S/μ is, up to an isomorphism, a subsemigroup of $\mathcal{I}(E(S))$ whose elements preserve the normal order of S (a linear order in $E(S)$). Therefore, we have:

Corollary 1.8 *Let S be a block-group and let μ be the maximum idempotent-separating congruence of S. Then, $S \in \mathsf{NOS}$ if and only if $S/\mu \in \mathsf{POI}$.* ■

And so, we have:

Corollary 1.9 *Every fundamental normally ordered block-group belongs to POI.* ■

Notice that any aperiodic inverse semigroup is fundamental. Moreover, a normally ordered inverse semigroup is aperiodic if and only if it is fundamental [6]. Unfortunately, in general, an aperiodic normally ordered block-group must not be fundamental; for instance, this is the case of a non-trivial zero semigroup. Nevertheless, it seems reasonable to make the following guess:

Conjecture 1.10 $\mathsf{POI} = \mathsf{NOS} \cap \mathsf{Ecom} \cap \mathsf{A}$.

Observe that, if $S \in \mathsf{NOS} \cap \mathsf{Ecom} \cap \mathsf{A}$, then clearly $\mathrm{Reg}(S) \in \mathsf{POI}$.

We finish this section by presenting a refinement of the author's description [6] (and of Cowan and Reilly's description [3]) of the pseudovariety of inverse semigroups PCS.

First, recall the following refinement of the Munn representation of a block-group S presented by the author in [8]: the mapping

$$\begin{array}{rccl} \vartheta: & S & \to & \mathcal{I}(\mathfrak{Irr}(E(S))) \\ & s & \mapsto & \vartheta_s: \quad \begin{array}{ccc} \mathfrak{Irr}(\mathcal{R}(s)) & \to & \mathfrak{Irr}(\mathcal{L}(s)) \\ e & \mapsto & (es)^{-1}(es) \end{array}, \end{array}$$

is an idempotent-separating homomorphism, where $\mathfrak{Irr}(X)$ denotes the set of all join irreducible idempotents belonging to X, for any subset X of $E(S)$.

Theorem 1.11 *A finite inverse semigroup S with n join irreducible idempotents belongs to PCS if and only if S is isomorphic to a subsemigroup of $\mathcal{POI}_n$.*

Proof. If S is isomorphic to a subsemigroup of $\mathcal{POI}_n$, then it is clear that $S \in \mathsf{PCS}$. Conversely, if $S \in \mathsf{PCS}$, then the author showed in [6] that there exists a linear order $\sqsubseteq$ in $E(S)$ preserved by the mappings $\phi_s\,(=\delta_s)$, $s \in S$, of the Munn representation of S. Thus, for all $s \in S$, the mapping ϑ_s is an injective order-preserving partial transformation on the subchain $\mathfrak{Irr}(E(S))$ of $(E(S), \sqsubseteq)$. Since $\mathfrak{Irr}(E(S))$ has n elements, we may consider $\mathcal{POI}_n$ built over this chain and look at ϑ_s as an element of $\mathcal{POI}_n$, for all $s \in S$. On the other hand, as S is aperiodic, then S is fundamental, whence the homomorphism $\vartheta: S \to \mathcal{POI}_n$, $s \mapsto \vartheta_s$, is injective, and the result follows. ■

Observe that Easdown showed in [4] that the least non-negative integer n such that a fundamental inverse semigroup S embeds in $\mathcal{PT}_n$ is the number of join irreducible idempotents of S, whence Theorem 1.11 gives us the best possible refinement of the prior descriptions of PCS.

2 Mal'cev decompositions

Given a pseudovariety of semigroups V, a semigroup S is called a V-*extension* of a semigroup T if there exists an onto homomorphism $\varphi : S \longrightarrow T$ such that, for every idempotent e of T, the subsemigroup $e\varphi^{-1}$ of S belongs to V. Let W be another pseudovariety of semigroups. The *Mal'cev product* $\mathsf{V} \circledm \mathsf{W}$ is the pseudovariety of semigroups generated by all V-extensions of elements of W. One can define alternatively the Mal'cev product by using "relational morphisms". Recall that a *relational morphism* $\tau : S \longrightarrow\!\!\!\!\!\!\circ\;\; T$ from a semigroup S into a semigroup T is a function τ from S into the power set $\mathcal{P}(T)$ of T such that: (1) $a\tau \neq \emptyset$, for $a \in S$; and $a\tau b\tau \subseteq (ab)\tau$, for $a, b \in S$. Observe that, for each idempotent e of T, the set $e\tau^{-1}$ is either empty or a subsemigroup of S. Then, a semigroup S belongs to $\mathsf{V} \circledm \mathsf{W}$ if and only if there exists a relational morphism τ from S into a member T of W such that, for each idempotent e of T, if $e\tau^{-1}$ is nonempty then $e\tau^{-1} \in \mathsf{V}$ (see [13, 9]).

Now, recall that the pseudovarieties BG, Ecom, EI and N can be defined by just one pseudoidentity: $\mathsf{Ecom} = [\![x^\omega y^\omega = y^\omega x^\omega]\!]$, $\mathsf{BG} = [\![(x^\omega y^\omega)^\omega = (y^\omega x^\omega)^\omega]\!]$, $\mathsf{EI} = [\![x^\omega = y^\omega]\!]$ and $\mathsf{N} = [\![x^\omega = 0]\!]$. Notice also that EI is equal to the join $\mathsf{G} \vee \mathsf{N}$. See [1].

Let $S \in \mathsf{BG}$ and $E = E(S)$. Since the Munn representation $\delta : S \to \mathcal{I}(E)$ of S is an idempotent-separating homomorphism and $\mathcal{I}(E) \in \mathsf{Ecom}$, we immediately have $S \in \mathsf{EI} \circledm \mathsf{Ecom}$. Hence $\mathsf{BG} \subseteq \mathsf{EI} \circledm \mathsf{Ecom}$. Next, by recalling that $\mathsf{BG} = \mathsf{J} \circledm \mathsf{G}$, we can consider a relational morphism ξ from S into some group G such that $1\xi^{-1} \in \mathsf{J}$. Define a function τ from S into $\mathcal{P}(\mathcal{I}(E) \times G)$ by $s\tau = \{(s\delta, g) \in \mathcal{I}(E) \times G \mid g \in s\xi\}$, for all $s \in S$. It is easy to show that τ is a relational morphism and, given an idempotent e of $\operatorname{Im}\delta$, $(e,1)\tau^{-1} = e\delta^{-1} \cap 1\xi^{-1} \in \mathsf{EI} \cap \mathsf{J}$. Since $\mathcal{I}(E) \times G$ is an idempotent commuting semigroup and $\mathsf{EI} \cap \mathsf{J} = \mathsf{N}$ (in fact, we also have $\mathsf{EI} \cap \mathsf{A} = \mathsf{N}$: recall that $\mathsf{J} = [\![(xy)^\omega = (yx)^\omega, x^{\omega+1} = x^\omega]\!]$ and $\mathsf{A} = [\![x^{\omega+1} = x^\omega]\!]$ [1]), we deduce that $S \in \mathsf{N} \circledm \mathsf{Ecom}$ and so we also have $\mathsf{BG} \subseteq \mathsf{N} \circledm \mathsf{Ecom}$.

On the other hand, let S be an EI-extension of an idempotent commuting semigroup T and let $\varphi : S \longrightarrow T$ be an onto homomorphism such that, for every idempotent e of T, $e\varphi^{-1} \in \mathsf{EI}$ (i.e. S is an arbitrary generator of $\mathsf{EI} \circledm \mathsf{Ecom}$). Take $x, y \in S$. Then $x^\omega\varphi, y^\omega\varphi \in E(T)$, whence $e = (x^\omega y^\omega)\varphi = x^\omega\varphi y^\omega\varphi = y^\omega\varphi x^\omega\varphi = (y^\omega x^\omega)\varphi$ is an idempotent of T. Therefore $(x^\omega y^\omega)^\omega, (y^\omega x^\omega)^\omega \in e\varphi^{-1}$ and, since $e\varphi^{-1} \in \mathsf{EI}$, we have $(x^\omega y^\omega)^\omega = (y^\omega x^\omega)^\omega$. Thus $S \in \mathsf{BG}$ and so $\mathsf{EI} \circledm \mathsf{Ecom} \subseteq \mathsf{BG}$.

As $\mathsf{N} \subseteq \mathsf{EI}$, then $\mathsf{N} \circledm \mathsf{Ecom} \subseteq \mathsf{EI} \circledm \mathsf{Ecom}$ and so we have proved:

Theorem 2.1 $\mathsf{BG} = \mathsf{EI} \circledm \mathsf{Ecom} = \mathsf{N} \circledm \mathsf{Ecom}$. ∎

This result allows us to conclude that block-groups is the largest class of finite semigroups for which one can consider a Munn type representation, i.e. an idempotent-separating representation by partial injective transformations.

Now, let S be a normally ordered block-group and let $\delta : S \longrightarrow \mathcal{I}(E(S))$ be the Munn representation of S. As already observed, the semigroup $S\delta$ is a subsemigroup of $\mathcal{I}(E(S))$ whose elements preserve the normal order of S, which is a linear order in $E(S)$, and so $S\delta \in \mathsf{POI}$. Since δ separates idempotents, it follows that $S \in \mathsf{EI} \textcircled{m} \mathsf{POI}$. Hence, $\mathsf{NOS} \subseteq \mathsf{EI} \textcircled{m} \mathsf{POI}$. On the other hand, let S be an EI-extension of a semigroup $T \in \mathsf{POI}$ and let $\varphi : S \longrightarrow T$ be an onto homomorphism such that, for every idempotent e of T, $e\varphi^{-1} \in \mathsf{EI}$ (i.e. S is an arbitrary generator of $\mathsf{EI} \textcircled{m} \mathsf{POI}$). Then, φ separates idempotents, $T \in \mathsf{POI} \subseteq \mathsf{NOS}$ and $S \in \mathsf{EI} \textcircled{m} \mathsf{POI} \subseteq \mathsf{EI} \textcircled{m} \mathsf{Ecom} = \mathsf{BG}$, whence $S \in \mathsf{NOS}$, by Proposition 1.7. Therefore, $\mathsf{EI} \textcircled{m} \mathsf{POI} \subseteq \mathsf{NOS}$ and so we have proved:

Theorem 2.2 $\mathsf{NOS} = \mathsf{EI} \textcircled{m} \mathsf{POI}$. ■

Next, observe that any aperiodic extension of an aperiodic semigroup is an aperiodic semigroup. In fact, let T be an aperiodic semigroup and let $\varphi : S \longrightarrow T$ be an onto homomorphism such that, for every idempotent e of T, $e\varphi^{-1} \in \mathsf{A}$. Take $x \in S$ and let $e = (x^\omega)\varphi$. Then, as $T \in \mathsf{A}$, we have $e = (x^\omega)\varphi = (x\varphi)^\omega = (x\varphi)^{\omega+1} = (x^{\omega+1})\varphi$, whence $x^{\omega+1} \in e\varphi^{-1}$. Then $(x^{\omega+1})^{\omega+1} = (x^{\omega+1})^\omega$, since $e\varphi^{-1} \in \mathsf{A}$, and so $x^\omega = (x^{\omega+1})^\omega = (x^{\omega+1})^{\omega+1} = x^{\omega+1}$, by definition. Thus $S \in \mathsf{A}$, as required.

Now, as $\mathsf{N} = \mathsf{EI} \cap \mathsf{A}$, we have $\mathsf{N} \textcircled{m} \mathsf{POI} \subseteq \mathsf{A} \cap (\mathsf{EI} \textcircled{m} \mathsf{POI}) = \mathsf{A} \cap \mathsf{NOS}$, by the above observation and Theorem 2.2. On the other hand, let $S \in \mathsf{NOS} \cap \mathsf{A}$. Considering again the Munn representation $\delta : S \longrightarrow \mathcal{I}(E(S))$ of S, we have, as above, $S\delta \in \mathsf{POI}$ and $e\varphi^{-1} \in \mathsf{EI}$, for all $e \in E(T)$. Since S is aperiodic, we have also $e\varphi^{-1} \in \mathsf{A}$, for all $e \in E(T)$, and so $S \in (\mathsf{EI} \cap \mathsf{A}) \textcircled{m} \mathsf{POI} = \mathsf{N} \textcircled{m} \mathsf{POI}$. Thus, we have proved:

Theorem 2.3 $\mathsf{NOS} \cap \mathsf{A} = \mathsf{N} \textcircled{m} \mathsf{POI}$. ■

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