HAUSDORFF FIRST COUNTABLE, COUNTABLY COMPACT SPACE IS ω-BOUNDED

NUNO C. FREIRE AND M. F. VEIGA

Abstract. In this paper we obtain an answer to the question in Problem 288, the book Open Problems in Topology by Jan van Mill and George M Reed, p. 131. Namely, we prove that a Hausdorff first countable, countably compact topological space is ω-bounded. We also point out errors occurring in the literature concerning the Ostaszewski spaces.

1. Introduction

We prove that any Hausdorff $(C1)$ countably compact space is in fact ω-bounded. In the Preliminaries, paragraph 2., we state the definitions concerning Problem 288 in [5] and properties that we use to obtain Corollary 1. in paragraph 3., the Results. This work is on Topology and assumes the Axiomatics of Set Theory as in [1], [2]; it does not concern the different, although related, difficult subject of Logic and the Theory of Sets namely, models in **ZFC**. Still in paragraph 3., we prove that it is wrong to assume that the $\gamma\mathbf{N}$ space in [5], p.133 is both Hausdorff and countably compact and that [4], p. 652 contains a contradiction; hence those matters do not really contradict Corollary 1. Also, we show that separation is essential in Corollary 1.

2. Preliminaries

Recall that a topological space $(X, \mathcal{T})$ is first countable or a $(C1)$ space if each point has a countable base of neighborhoods. $(X, \mathcal{T})$ is said a $(T1)$ space if for every pair of distinct points $x, y \in X$ there exist open sets $W_x, W_y \subset X$ such that $x \in W_x$ $(y \in W_y)$ whereas $y \notin W_x$ $(x \notin W_y)$ and, it is Hausdorff space or a $(T2)$ space if two any different points have disjoint neighborhoods. Clearly any $(T2)$ space is a $(T1)$ space.

Definition 1. *(Following [1]) We say that the Hausdorff space $(X, \mathcal{T})$ is countably compact if any countable open cover of X has a finite subcover.*

Remark 1. *According to [1], Chap. XI, Sec. 3 (pp. 233) if $(X, \mathcal{T})$ is metrizable and countably compact, then $(X, \mathcal{T})$ is compact.*

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Definition 2. *(Following [5], p. 131) A topological space* $(X, \mathcal{T})$ *is said to be* ω*-bounded if each countable subset of* X *has compact closure.*

Following [2] (pp. 331) the base $\mathcal{B}$ for the topological space $(X, \mathcal{T})$ is called regular if for every point $x \in X$ and any neighborhood U of x there is a neighborhood $V \subset U$ of the point x such that the set of all members of $\mathcal{B}$ that meet both V and $X \backslash U$ is finite. We then have the Arkhangel'skii metrization theorem

Theorem 1. *A topological space is metrizable if and only if it is a* $(T1)$ *space and has a regular base.*

Proof. See [2], 5.4.6., pp. 332. ∎

Recall that a partially ordered set $(\mathfrak{M}, \leq)$ is well-ordered if for each nonempty set $\mathfrak{B} \subset \mathfrak{M}$ there is some $b_0 \in \mathfrak{B}$ such that $b_0 \leq b$ for each $b \in \mathfrak{B}$.

Definition 3. *(Following [1]) An ordinal number is a set* α *with the properties that, for each* $x, y \in \alpha$ *such that* $x \neq y$ *it holds that either* $x \in y$ *or* $y \in x$ *and,* $(x \in y) \wedge (y \in \alpha) \Rightarrow x \in \alpha$*.*

Following [1] we say that a bijection $f : (\mathfrak{M}, \leq) \to (\mathfrak{N}, \preceq)$ between two well-ordered sets is an isomorphism if it holds that $a \leq b$ implies $f(a) \preceq f(b)$. For each $a \in \mathfrak{M}$, the set $\mathfrak{M}(a) = \{x \in \mathfrak{M} : (x \leq a) \wedge (x \neq a)\}$ is the initial interval determined by a. Also if α is an ordinal number, the initial interval $\mathfrak{C}(\alpha) = \alpha$ where, $\mathfrak{C}$ is the well-ordered class of all ordinal numbers, well-ordered putting $\alpha \leq \beta$ if and only if $\alpha \subset \beta$. Each well-ordered set is isomorphic to a suitable $\mathfrak{C}(\alpha)$. (See [1], pp. 36, 42, 43 and Theorem 6.4, the same page). If we say that two sets A, B are equipotent meaning for there is a bijective map on A onto B, it follows by Zermelo's theorem (Theorem 2. 1 (3) in [1], pp. 32) that in the class of all equipotent sets to A, there exists an ordinal number and, also a smallest such ordinal number, wich is called the cardinal number of A ([1], pp. 46).

The first ordinal number is the empty set ϕ. For each ordinal number α, its sucessor is $\alpha \cup \{\alpha\} = \alpha + 1$. We denote $\phi = 0$, $\{\phi\} = 1$, $\{\phi\} \cup \{\{\phi\}\} = 2, ...$ These are the finite ordinal numbers, which can be viewed as the natural numbers. The first infinite ordinal number is ω, which is the cardinal number of the set of all natural numbers. ω is a limit ordinal number that is, it is not a sucessor of an ordinal number. We say that ω is an infinite countable ordinal number and we denote by ω_1 the first uncountable ordinal number; it holds that, $\omega_1 \leq c$ where, c is the cardinal number of the set of real numbers when viewed as an ordinal number. We shall also consider the ordinal number ω_2 namely, we denote by ω_2 the smallest cardinal number viewed as an ordinal number, that is greater than ω_1.

Remark 2. *We say that a set* A *has cardinality no greater than the continuum if there exists an injective map on* A *to a set which has the cardinality of the continuum* c*. In the following we consider* c *viewed as an ordinal number; the continuum hypothesis, which is the assumption that* $\omega_1 = c$ *is not required in what follows.*

Definition 4. *The ordinal space* $[0, c[$ *is the set of all ordinal numbers* $\alpha \prec c$ *(we denote by* $\alpha \prec \beta$ *meaning that* $\alpha \subset \beta$, $\alpha \neq \beta$, $\alpha, \beta \in \mathfrak{C}$ *as above), equipped with the topology that is generated by the sets of the form* $\{x : x \succ \alpha\}$ *and* $\{x : x \prec \beta\}$.

Remark 3. *The set of all limit ordinal numbers in* $[0, c[$ *has the cardinality of the continuum, in the sense of the preceding remark. Hence also the set of all limit ordinal numbers* α *in* $[0, c[$ *such that* $\omega \prec \alpha$ *has the cardinality of the continuum.*

Remark 4. *It holds that* $\{\alpha + 1\} =]\alpha, (\alpha + 1) + 1[$ *is an open subset of* $[0, c[$ *for each* $\alpha \in [0, c[$.

Remark 5. *In the sense of the above remarks, if the set* A *has cardinality no greater than the continuum, then there exists an injection from* A *to* $[1, \omega[\cup \mathcal{I}$, $\mathcal{I}$ *a subset of* $]\omega, c[$ *constituted by different limit ordinal numbers.*

3. The Results

In the following, we consider a Hausdorff first countable, countably compact topological space $(X, \mathcal{T}_X)$.

Lemma 1. *Let* $C = \{x_n : n = 1, 2, ...\} \subset (X, \mathcal{T}_X)$ *be countably infinite.Then each point* $x \in \overline{C}$ *is the limit* $x = \lim_k x_{n(k)}$ *of a sequence* $(x_{n(k)})$ *in* C.

Proof. In fact, if x is the limit of a net (x_α) in C it follows that, $\{V_k : k = 1, 2, ...\}$ being a countable base of neighborhoods of x such that $V_k \supset V_{k+1}$ for each k, there exists some $\alpha(k)$ such that, $x_\alpha \in V_k$ whenever $\alpha \succ \alpha(k)$. Clearly the sequence $(x_{\alpha(k)})$ converges to x and the lemma follows. ∎

C being as above, if $p \in \overline{C} \backslash C$ then there is an infinite sequence of natural numbers $(n(k))$ such that the sequence $(x_{n(k)})$ in C converges to p. Let $S(p)$ be the set of all sequences $(n(k))$ such that $\lim_k x_{n(k)} = p$. Consider $\varphi(\{S(p) : p \in \overline{C} \backslash C\})$ where, φ is the selector of Zermelo in the axiom of Choice assigning a fixed $[n(k)] = (n(k))$ to $S(p)$ and, let $\mathcal{S} = \{[n(k)] : p \in \overline{C} \backslash C\}$.The cardinality of the set of all sequences of natural numbers being the cardinality of the continuum, it follows from Remark 5. that, there exists an injective map $f : \mathbf{N} \cup \mathcal{S} \rightarrow [1, c[$ such that $f([n(k)]) = \alpha[n(k)]$ is the sucessor of a limit ordinal in $[1, c[$ ($[n(k)] \in \mathcal{S}$) and, $f(n) = n$ ($n = 1, 2, ...$).

Consider the set

$E = \{(n, x_n), (\alpha[n(k)], \lim_k x_{n(k)}) : n = 1, 2, ..., [n(k)] \in \mathcal{S}\}$ equipped with the induced topology $\mathcal{T}$ by the product topology of $[1, c[\times X$, X as above. We have

Theorem 2. *The topological space* $(E, \mathcal{T})$ *is metrizable.*

Proof. $\mathcal{B} = \{E\cap(\{n\}\times V_j(x_n)), E\cap(\{\alpha[n(k)]\}\times V_j(p)) : n, j \in \mathbf{N}, p = \lim x_{n(k)}, [n(k)] \in \mathcal{S}\}$ where, $\{V_j(x_n) : j = 1, 2, ...\}$ (respectively $\{V_j(p) : j = 1, 2, ...\}$) is a countable base of neighborhoods of x_n such that $V_j(x_n) \supset V_{j+1}(x_n)$ (respectively of p, such that $V_j(p) \supset V_{j+1}(p)$) is a base for the topology $\mathcal{T}$. The base $\mathcal{B}$ is regular. In fact, let $(n, x_n) \in E$ (resp. $(\alpha[(n(k))], p) \in E$). If U is a neighborhood of (n, x_n) (resp. of $(\alpha[n(k)], p)$), there is a neighborhood V of the point, $V = E\cap(\{n\}\times V_j(x_n))$ (resp. $V = E \cap (\{\alpha[n(k)]\} \times V_j(p)))$ such that $V \subset U$; if then $B \in \mathcal{B}$ and, both sets $B\cap V$ and $B\cap(E\backslash V)$ are nonempty, it follows that we must have $B = E\cap(\{n\}\times V_m(x_n))$ $(B = E \cap (\{\alpha[n(k)]\} \times V_m(p)))$.

Therefore, also $B\cap((([1, c[\backslash\{n\})\times V_j(x_n))\cup(\{n\}\times(X\backslash V_j(x_n))) \neq \phi$ implies that $V_j(x_n) \subset V_m(x_n)$ that is, $1 \leq m \gneqq j$. Analogously for B intersecting $(([1, c[\backslash\{n\}) \times V_j(p)) \cup (\{\alpha[n(k)]\} \times (X\backslash V_j(p))$, whose nonempty intersections are less than j and, we conclude that the set of the nonempty intersections is always finite. Clearly $(E, \mathcal{T})$ is a $(T1)$ space (it is a Hausdorff space). The theorem follows by Theorem 1. and the proof is complete. ∎

Corollary 1. *If $(X, \mathcal{T}_X)$ is a Hausdorff first countable, countably compact topological space then it is ω-bounded.*

Proof. In fact, we know by Theorem 2. that E is metrizable for a metric d. It holds that $\overline{C} = pr_2(E)$ by Lemma 1. and the definition of the set $\mathcal{S}$ where, the map $pr_2 : E \to X$, $pr_2(u, v) = v$ is a homeomorphism. In fact injectivity holds, due of we assume that the x_n are all different in Lemma 1; also the assignment $[n(k)] \mapsto p = \lim_k x_{n(k)}$ is a bijection on the set $\mathcal{S}$ of the $[n(k)]$ to the set of limits $\lim_k x_{n(k)}$, for $v = \lim_k x_{j(k)} \neq \lim_k x_{n(k)} = p$ implies that $S(v) \cap S(p) = \phi$ in the above notation, hence $[j(k)] \neq [n(k)]$, $\alpha[n(k)] \neq \alpha[j(k)]$. Hence the induced topology on $\overline{C}$ by $\mathcal{T}_X$ is the topology for the metric d_2 defined through $d_2(v, v') = d((u, v), (u', v'))$ iff $pr_2(u, v) = v, pr_2(u', v') = v'$ $(v, v' \in \overline{C})$. The fact that the closure $\overline{C}$ is compact follows from being countably compact ([1], Theorem 3.6 (2), pp. 230) and by Remark 1. and the corollary is proved. ∎

Remark 6. *Any space which topology is strictly finer than a Hausdorff first countable topology fails to be countably compact.*

Proof. Let $(X, \mathcal{T})$ be first countable Hausdorff and let σ be a topology on X strictly finer than $\mathcal{T}$. This implies that there is a set A whose σ-closure A^σ is a proper subset of its $\mathcal{T}$-closure $A^\mathcal{T}$. Let $p \in A^\mathcal{T}\backslash A^\sigma$. By first countability, there is a $\mathcal{T}$-sequence in A converging to p. The range of the sequence has p as its only $\mathcal{T}$-acumulation point. Since p is not in the σ-closure of the range, the range is an infinite closed discrete subspace of (X, σ) and the remark follows. ∎

Remark 7. *The space $\gamma\mathbf{N}$ as defined at p. 133 and characterized as in Example 2.2. in [5] is not first countable Hausdorff.*

Proof. Just before Example 2.2. the author states that he considers a definition of $\mathbf{N}$ that makes it disjoint from ω_1, so that he considers the Franklin-Rojagopalan space $\gamma\mathbf{N}$ in such a way that, he identifies $\gamma\mathbf{N}\backslash\mathbf{N}$ with ω_1. In Example 2.2. the author states namely "Let $\{A_\alpha : \alpha \in \omega_1\}$ be a $\subset^*$-ascending sequence of infinite subsets of $\mathbf{N}$. (An easy "diagonal" argument allows one to construct such a sequence in **ZFC**). Set $A_{-1} = \phi$. On the set $\mathbf{N} \cup \omega_1$ we impose the topology $\mathcal{T}$ which has the sets of the form $\{n\}$ ($n \in \mathbf{N}$) and $U_n(\beta, \alpha]$ ($n \in \mathbf{N}, \beta \in \omega_1 \cup \{-1\}, \alpha \in \omega_1$) as a base, where $(\beta, \alpha]$ means $\{\gamma \in \omega_1 : \beta \prec \gamma \leq \alpha\}$ and $U_n(\beta, \alpha] = (\beta, \alpha] \cup (A_\alpha \backslash A_\beta) \backslash \{1, ..., n\}$ where $\{1, ..., n\} \subset \mathbf{N}$ ... Thus this gives a $\gamma\mathbf{N}$ ". Here, the symbol $\subset^*$stands for $B \subset^* A$ meaning that, $B \backslash A$ is finite and $A \backslash B$ is infinite where, A, B are subsets of $\mathbf{N}$ (line 7). Now we have the following: clearly that in the notation as above, the sets $W_n(\beta, \alpha] = (\beta, \alpha] \backslash \{1, ..., n\}$ together with the sets $\{n\}$ constitute a base for a topology σ that is strictly finer than the topology $\mathcal{T}$ as defined in Example 2.2. as above. It holds that, σ is a countably compact topology ([1], Ex. 1, Chap. XI, Sec. 3, pp. 228/9) hence, by the preceding Remark, $\mathcal{T}$ cannot be first countable Hausdorff and the remark follows. ∎

The author goes on in Example 2.2., showing that each $\gamma\mathbf{N}$ space can be obtained through a $\subset^*$-ascending sequence of infinite sets as he starts explaining in Example 2.2. Now at p. 134, Observation 2.3., the author asserts that namely, "$\gamma\mathbf{N}$ is countably compact if and only if, no infinite subset of $\mathbf{N}$ is almost disjoint from all the A_α. (Two sets are said to be almost disjoint if their intersection is finite). The existence of a $\subset^*$-ascending sequence $\{A_\alpha : \alpha \prec \omega_1\}$ of subsets of $\mathbf{N}$, such that no infinite subset of $\mathbf{N}$ is almost disjoint from all A_α, is equivalent to $t = \omega_1$". The cardinal number t above is defined at lines 8/9 p. 133 ([5]). Line 21 p. 135 in [5] asserts that it is not known if a Ostaszewski-van Dowen space exists in **ZFC**. It follows from above that it leads to a contradiction the method for the proof of 3.3. Construction, pp. 135/6 using the extra hypothesis to **ZFC.**

Concerning p. 652 in [4], we now prove that there is a flaw in the proof. The author considers spaces $X_\alpha = \omega \cup \{p_\beta : \beta \prec \alpha\}$ where, $\alpha \in \omega_1$, $X_{\omega_1} = \cup\{X_\alpha : \alpha \prec \omega_1\}$. He then defines a topology $\mathcal{T}$ on X_{ω_1} such that X_{ω_1} has ω as a dense subspace and, according to the proof, ω is dense in each space X_α.

At line 21, we can read that the X_α are open in X_{ω_1}; further, each X_α is metrizable (line 9) hence it is a Hausdorff space. If $\beta \prec \alpha$ then X_β is a subspace of X_α (line 7) hence the topology of X_β is the induced topology on X_β by X_α. X_α being locally compact (line 6) we may take the point p_α as ∞ concerning the Alexandroff one point compactification $\widehat{X_\alpha}$ so that $\widehat{X_\alpha} = X_\alpha \cup \{p_\alpha\} = X_{\alpha+1}$ as sets. Clearly the identity injection $X_{\alpha+1} \to \widehat{X_\alpha}$ is continuous (since that $X_{\alpha+1}$ is separated Hausdorff). We prove that the point $p_{\alpha+2}$ is not the limit, in X_{ω_1}, of a sequence in ω that is, ω is not dense in X_{ω_1}, contradicting the conclusion at lines 20, 21. Let the $n(k)$ be in ω, $n(k) \to p$. We have that $n(k) \to p$ in some space X_γ, $\alpha + 2 \prec \gamma \prec \omega_1$. Also, a subsequence $n(k(j)) \to q$ in $\widehat{X_\alpha}$ hence $q \in X_{\alpha+1}$. It follows $n(k(j)) \to p$ in the topology of $\widehat{X_\gamma}$ and, $n(k(j)) \to q$ in the topology of $\widehat{X_\alpha}$, hence $n(k(j)) \to q$ also in the topology of $\widehat{X_\gamma}$; therefore $p = q \in X_{\alpha+1}$ which is a contradiction as we wished to point out. We also obtain the contradiction that, ω is not dense in X_γ.

We now obtain an example of a (non separated) first countable, countably compact space that is not ω-bounded.

Remark 8. *if* $\alpha \in [0, \omega_1[$ *and,* α *has an immediate predecessor* α^- *(that is,* $\alpha = \alpha^- + 1$*,* α^- *an ordinal number), we have that* $]\alpha^-, \alpha+1[= \{\alpha\}$ *is open in* $[0, \omega_1[$*. If* α *has no immediate predecessor, then the set* $\{]\lambda, \alpha+1[: \lambda \leq \alpha\}$ *is a countable base of neighborhoods of* α*. Hence* $[0, \omega_1[$ *is a first countable space.*

Remark 9. *(Following [1]) The ordinal space* $[0, \omega_1[$ *is countably compact and, it is not compact.*

Proof. See [1], Ex. 1, Chap. XI, Sec. 3, pp. 228/9. ∎

Keeping the notations as in the Preliminairies, we consider the set of ordinal numbers $E = [0, \omega_2[\backslash \{\omega_1\}$.

Example 1. $(E, \mathcal{T}^*)$ *will stand for the set* E *equipped with the topology* $\mathcal{T}^*$ *for which a set* $A \subset E$ *is open if and only if both conditions hold: (1) if* $\alpha \in A \cap [0, \omega_1[$ *then* $A \cap [0, \omega_1[$ *contains an open subset of the ordinal space* $[0, \omega_1[$ *containing* α*; (2) if* $\gamma \in A \cap]\omega_1, \omega_2[$ *then* $A \supset [0, \omega[\cup]\omega_1, \gamma]$*.*

Claim 1. *The class* $\mathcal{T}^*$ *is a non separated topology on* E *such that, the induced topology on* $[0, \omega_1[$ *coincides with the topology of the ordinal space* $[0, \omega_1[$*.*

Proof. This follows immediately. ∎

Claim 2. *The topological space* $(E, \mathcal{T}^*)$ *is first countable, countably compact and it is not* ω*-bounded.*

Proof. If $\gamma \in]\omega_1, \omega_2[$ then the class $\{[0, \omega[\cup]\omega_1, \gamma]\}$ is a base of neighborhoods of γ; hence the space is $(C1)$ follows from Remark 8. and Claim 1. Using Remark 9. and E (d) in [3] (p. 162) we conclude that $(E, \mathcal{T}^*)$ is countably compact if we prove that, each sequence (γ_n) in $]\omega_1, \omega_2[$ has a convergent subsequence. Since that each infinite increasing sequence in $]\omega_1, \omega_2[$ has an upper bound in $]\omega_1, \omega_2[$, we may obtain an increasing subsequence $(\gamma_{n(k)})$ of (γ_n) (both these facts follow easily from $]\omega_1, \omega_2[$ is well ordered). Now $\gamma = \sup\{\gamma_{n(k)} : k = 1, 2, ...\}$ is the limit of $(\gamma_{n(k)})$ in $(E, \mathcal{T}^*)$ according to the definition of the topology (an easy cardinality argument shows that $\gamma \prec \omega_2$) hence $(E, \mathcal{T}^*)$ is countably compact. It remains to prove that the space is not ω-bounded. In fact, each point $\gamma \in]\omega_1, \omega_2[$ is in the closure of the countable set $[0, \omega[$; hence $\overline{[0, \omega[} = E$. We have that E is not compact (clearly the net (β_α) where, $\beta_\alpha = \alpha \in [0, \omega_2[\backslash \{\omega_1\}$ has no convergent subnet) and the claim follows. ∎

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DMAT Universidade de Évora Col. Luís Verney, R. Romão Ramalho 59, 7000 Évora PORTUGAL and Mailing Address Av. República No 975 Lote 2, 2^{o}Dto 2775 - 274 Parede PORTUGAL

E-mail address: freirenuno@2003iol.pt

Current address: Faculdade de Ciências e Tecnologia Universidade Nova de Lisboa, Quinta da Torre 2829 - 516 Monte de Caparica PORTUGAL

E-mail address: mfvo@fct.unl.pt