

# QUOTIENTS ON THE SATO GRASSMANNIAN AND THE MODULI OF VECTOR BUNDLES

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ABSTRACT. It is shown that there exists a geometric quotient of the subscheme of stable points of  $\mathrm{Gr}(\mathbb{C}((z))^{\oplus r})$  under the action of  $\mathrm{Sl}(r, \mathbb{C})$ . The consequences in terms of vector bundles on an algebraic curve are studied.

## 1. INTRODUCTION

In this paper it is shown that there exists a geometric quotient of the subscheme of stable points of  $\mathrm{Gr}(\mathbb{C}((z))^{\oplus r})$  under the action of  $\mathrm{Sl}(r, \mathbb{C})$  following GIT techniques.

It is worth recalling that Sato Grassmannians have shown up as a fruitful tool for many problems from integrable systems to moduli spaces (see [SS, SW, M, AMP] and references therein). Because of the existence of *symmetries* one is led to wonder about the existence of quotients. An standard (and powerful) procedure to carry out such a study is the *geometric invariant theory* ([MF]).

However, the main obstacle when applying GIT to our situation comes from the fact that Sato Grassmannians are not schemes of finite type. The second section of this paper is devoted to provide a way to overcome this problem and shows how Sato Grassmannians can be constructed from schemes of finite type (Theorem 2.1). Section 3, which recalls the notion of stability from [CMP], shows a similar result for the subscheme of the Grassmannian consisting of stable points and finishes with the existence of the geometric quotient (Theorem 3.4). Here, for the restrictions on length and the sake of clarity, we have only dealt with stability but a similar study can be carried out for semistability. Finally, as an application of our results we use the Krichever map to study the relation between our results and the well-known results for the case of vector bundles on algebraic curves.

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Let us finish this introduction by pointing out a future line of research. Once the quotient by  $\mathrm{Sl}(r, \mathbb{C})$  has been constructed, one should develop a theory of stability under  $\mathrm{Sl}(r, \mathbb{C}[[z]])$  and discuss the possible quotients. As an application, one should study the space of invariants of  $H^0(\mathrm{Gr}(\mathbb{C}((z))), \mathrm{Det}^*)$  ( $\mathrm{Det}$  being the determinant line bundle) since these spaces are closely related to the spaces of conformal blocks in Conformal Field Theory.

## 2. PRELIMINARIES ON INFINITE GRASSMANNIANS

In this section we remind some definitions and results about infinite Grassmannians. For more details on this subject we address the reader to [AMP] and [CMP].

Let us begin with the definition of infinite Grassmannians. Let  $V$  a  $\mathbb{C}$ -vector space and  $V_+$  a subspace of  $V$ . We say that a subspace  $A \subset V$  is commensurable with  $V_+$  when  $\dim_k(A + V_+)/ (A \cap V_+) < \infty$  and we denote this by  $A \sim V_+$ . The pair  $(V, V_+)$  is assumed to satisfy

- $\bigcap_{A \sim V_+} A = (0)$
- $V = \varprojlim_{A \sim V_+} V/A$

The infinite Grassmannian  $\mathrm{Gr}(V, V_+)$  (briefly  $\mathrm{Gr}(V)$  if we fix  $V_+$ ) is the  $\mathbb{C}$ -scheme whose rational points are

$$\mathrm{Gr}(V) = \left\{ \begin{array}{l} \mathbb{C}\text{-subspaces } F \subset V \text{ such that} \\ \dim_k V/(V_+ + F) < \infty, \quad \dim_k F \cap V_+ < \infty \end{array} \right\}$$

The index or characteristic of  $F \in \mathrm{Gr}(V)$

$$\chi(F) = \dim_k(F \cap V_+) - \dim_k \left( \frac{V}{F + V_+} \right)$$

is locally constant as function of  $F$ . If  $\mathrm{Gr}^\chi(V)$  denotes the set where the index takes the value  $\chi \in \mathbb{Z}$ , then

$$\mathrm{Gr}(V) = \coprod_{\chi \in \mathbb{Z}} \mathrm{Gr}^\chi(V)$$

is the decomposition in connected components.

In particular, if  $V$  is a finite dimensional vector space, the points of  $\mathrm{Gr}^\chi(V)$  are those subspaces  $F$  where  $\dim_k F = \chi(F) + \dim_k(V/V_+)$ .

Henceforth, we will work with the case  $V := \mathbb{C}((z))^{\oplus r}$  and  $V_+ := \mathbb{C}[[z]]^{\oplus r}$  ( $r \geq 0$ ) and fix  $\chi \in \mathbb{Z}$ . Let us recall how the infinite Grassmannian  $\mathrm{Gr}^\chi(V)$  can be expressed in terms of finite Grassmannians  $\mathrm{Gr}^\chi(V_{[-m, m]})$  where  $V_{[-m, m]} := (z^{-m}V_+)/ (z^mV_+)$ ,  $m \in \mathbb{N}$ . More general, we introduce the notation  $V_{[-m, i]} := (z^{-m}V_+)/ (z^iV_+)$ , with  $m, i \in \mathbb{N}$ .

Let us consider

$$\begin{aligned} \tilde{U}_{m,m} &:= \text{Gr}^X(V_{[-m,m]}) \\ \tilde{U}_{m,m+1} &:= \left\{ \begin{array}{l} F_{m+1} \in \text{Gr}^X(V_{[-(m+1),m+1]}) \text{ such that} \\ F_{m+1} + V_{[-m,m+1]} = V_{[-(m+1),m+1]} \text{ and } F_{m+1} \cap V_{[m,m+1]} = (0) \end{array} \right\} \\ \tilde{U}_{m,i} &:= \Phi_{i-1}^{-1}(\tilde{U}_{m,i-1}), \quad i > m+1 \end{aligned}$$

where  $\Phi_m$  is the rational map  $\text{Gr}^X(V_{[-(m+1),m+1]}) \dashrightarrow \text{Gr}^X(V_{[-m,m]})$  defined by

$$\Phi_m(F_{m+1}) := \frac{(F_{m+1} \cap V_{[-m,m+1]}) + V_{[m,m+1]}}{V_{[m,m+1]}}$$

whose domain of definition is the open subscheme  $\tilde{U}_{m,m+1}$ .

The schemes  $\tilde{U}_{m,i}$ 's fit into the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \tilde{U}_{0,3} & \xrightarrow{\Phi_2} & \tilde{U}_{0,2} & \xrightarrow{\Phi_1} & \tilde{U}_{0,1} \xrightarrow{\Phi_0} \text{Gr}^X(V_{[-0,0]}) & (2.1) \\ & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & \longrightarrow & \tilde{U}_{1,3} & \xrightarrow{\Phi_2} & \tilde{U}_{1,2} & \xrightarrow{\Phi_1} & \text{Gr}^X(V_{[-1,1]}) & \\ & & \downarrow & & \downarrow & & & \\ \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & & & \end{array}$$

whose squares are cartesian. Furthermore, note that  $\{(\tilde{U}_{m,i}, \Phi_{i-1})\}_{i \geq m}$  is an inverse system for each  $m$ . From Proposition 1.5.1 of [EGA-II] one obtains that  $\Phi_{i-1} : \tilde{U}_{m,i} \rightarrow \tilde{U}_{m,i-1}$  is an affine morphism for all  $i$  and, hence, the inverse limit  $U^m := \varprojlim_{i \geq m} \tilde{U}_{m,i}$  is an open subscheme of  $\text{Gr}^X(V)$ . Explicitly, one has the following description

$$U^m = \{F \in \text{Gr}^X(V) \text{ s. t. } F + z^{-m}V = V \text{ and } F \cap z^mV_+ = (0)\}$$

Now, §2 of [CMP] yields the following

**Theorem 2.1.** *For every  $m > 0$ ,  $U^m$  is an open subscheme of  $U^{m+1}$ . Moreover, the open sets  $U^m$  are a covering of  $\text{Gr}^X(V)$*

$$\text{Gr}^X(V) = \bigcup_{m>0} U^m = \bigcup_{m>0} \varprojlim_{i \geq m} \tilde{U}_{m,i} \quad (2.2)$$

In particular, a subspace  $F \in \text{Gr}^X(V)$  corresponds to a family of finite dimensional subspaces,  $\{F_{[-i,i]}\}_{i \geq m_0}$ , where  $F_{[-i,i]} \in \tilde{U}_{m_0,i} \subset \text{Gr}^X(V_{[-i,i]})$ . Explicitly, given  $F$  a family  $F_{[-m,i]}$  is constructed as follows

$$F_{[-m,i]} := \frac{(F \cap z^{-m}V_+) + z^iV_+}{z^iV_+}, \quad i \geq m \geq m_0$$

Conversely, a family  $\{F_{m,i}\} \in U^m$  determines a subspace  $F$  by the expression

$$F := \bigcup_{m \geq m_0} \varprojlim_{i \geq m} (F_{m,i} \cap V_{[-m,i]})$$

In particular, it holds that  $F \cap z^{-m}V_+ = \varprojlim_{i \geq m} (F_{m,i} \cap V_{[-m,i]})$ .

### 3. GEOMETRIC QUOTIENT BY THE ACTION OF $\mathrm{Sl}(r, \mathbb{C})$

We will prove that the set of stable points of  $\mathrm{Gr}^X(V)$  admits a geometric quotient by the action of the group  $\mathrm{Sl}(r, \mathbb{C})$ . Recall [CMP] has proposed a natural notion of stability for points of the infinite Grassmannian  $\mathrm{Gr}^X(V)$  with respect to the action of the reductive group  $\mathrm{Sl}(r, \mathbb{C})$ . That proposal was based on the application of GIT to the finite Grassmannians,  $\mathrm{Gr}^X(V_{[-m,m]})$ . Then, finite and infinite Grassmannians were related with the help of diagram (2.1) since  $\mathrm{Sl}(r, \mathbb{C})$  acts on each term and all maps are equivariant. The following fundamental property was proved.

**Proposition 3.1** ([CMP], Proposition 3.6). *Let  $\tilde{U}_{m,i}^s$  denote the set of stable points of  $\tilde{U}_{m,i}$  w.r.t. the action of  $\mathrm{Sl}(r, \mathbb{C})$ .*

*It holds that  $\Phi_i^{-1}(\tilde{U}_{m,i}^s) \subseteq \tilde{U}_{m,i+1}^s$ . In particular, if  $\{F_{[-i,i]}\}_{i \geq m}$  are associated to  $F \in \mathrm{Gr}(V)$  as in equation (2.2) and  $F_{[-i_0,i_0]}$  is stable, then  $F_{[-i,i]}$  is stable for all  $i \geq i_0 \geq m$ .*

The definition is the following

**Definition 3.2** ([CMP], Definition 3.7). *Let  $F_{[-i,i]} \in U_{m,i} \subset \mathrm{Gr}(V_{[-i,i]})$  be those subspaces associated to a point  $F \in \mathrm{Gr}(V)$  by (2.2).*

*The point  $F$  is (semi)stable for the action of  $\mathrm{Sl}(r, \mathbb{C})$  if there exist  $m \in \mathbb{N}$  and  $i \geq m$  such that  $F_{[-i,i]}$  is (semi)stable.*

*We denote the set of the stable and semistable points of  $\mathrm{Gr}(V)$  by  $\mathrm{Gr}(V)^s$  and  $\mathrm{Gr}(V)^{ss}$ , respectively.*

The above proposition also implies the following

**Proposition 3.3.** *Let us denote by  $U_{m,m} := \tilde{U}_{m,m}^s$  and  $U_{m,i} := \Phi_{i-1}^{-1}(U_{m,i-1}) \subseteq \tilde{U}_{m,i}^s$  for each  $m \in \mathbb{N}$  and  $i > m$ .*

*It holds that  $\{\varprojlim_{i \geq m} U_{m,i} \mid m > 0\}$  is an increasing sequence of open subsets and that*

$$\mathrm{Gr}^X(V)^s = \bigcup_{m > 0} \varprojlim_{i \geq m} U_{m,i} \quad (3.1)$$

*Proof.* To begin with, note the following facts; firstly,  $\{(U_{m,i}, \Phi_i)\}_{i \geq m}$  is an inverse system for each  $m$ ; secondly, there is a diagram

$$\begin{array}{ccc} \varprojlim_{i \geq m} U_{m,i} \subset & \longrightarrow & (U^m)^s \\ \downarrow & & \downarrow \\ \varprojlim_{i \geq m+1} U_{m+1,i} \subset & \longrightarrow & (U^{m+1})^s \end{array}$$

and, finally, that  $\text{Gr}^X(V)^s = \bigcup_{m > 0} (U^m)^s$  (by (2.2)).

Now, let  $F \in \text{Gr}^X(V)$  be a stable point. Let  $\{F_{[-i,i]}\}$  be the subspaces associated to  $F$ . By Proposition 3.1 there exists  $m_0$  such that  $F_{[-i,i]} \in \widetilde{U}_{m,i}^s$  for all  $m$  and for all  $i > m_0$ . Then,  $\{F_{[-i,i]}\}$  defines a point of  $\varprojlim_{i \geq m} U_{m,i} \subseteq (U^m)^s$  for all  $m \geq m_0$ . And the conclusion follows.  $\square$

From Theorem 1.10 of [MF] we know that the open set of stable points of  $\text{Gr}^X(V_{[-m,m]})$  for the action of  $\text{Sl}(r, \mathbb{C})$  does admit a geometric quotient. Furthermore, the open subscheme  $U_{m,i}$ , which is acted by  $\text{Sl}(r, \mathbb{C})$  and whose points are stable, also admits a geometric quotient (see ‘‘Converse’’ 1.13 [MF]). We denote by  $p_{m,i} : U_{m,i} \rightarrow Y_{m,i}$  this quotient. The composition  $p_{m,i} \circ \Phi_i$  factors through  $p_{m,i+1}$ , that is

$$\begin{array}{ccc} U_{m,i+1} & \xrightarrow{\Phi_i} & U_{m,i} \\ p_{m,i+1} \downarrow & & \downarrow p_{m,i} \\ Y_{m,i+1} & \xrightarrow{\Theta_{m,i}} & Y_{m,i} \end{array} \quad (3.2)$$

We have an inverse system  $\{(Y_{m,i}, \Theta_{m,i})\}_{i \geq m}$  for each  $m > 0$ . Let  $Y^m := \varprojlim_{i \geq m} Y_{m,i}$  and let  $p^m$  the morphism induced by  $\{p_{m,i}\}$  between the inverse limits

$$p^m : \varprojlim_{i \geq m} U_{m,i} \longrightarrow Y^m = \varprojlim_{i \geq m} Y_{m,i}$$

Observe that the family  $\{U_{m,i}\}$  fits into a diagram similar to (2.1). Then, applying the properties of inverse limits, one gets a commutative diagram

$$\begin{array}{ccc} \varprojlim_{i \geq m} U_{m,i} \subset & \longrightarrow & \varprojlim_{i \geq m+1} U_{m+1,i} \\ p^m \downarrow & & \downarrow p^{m+1} \\ Y^m \subset & \xrightarrow{\gamma_m} & Y^{m+1} \end{array}$$

for each  $m$ . Since  $\gamma_m : Y^m \hookrightarrow Y^{m+1}$  are open immersions, it makes sense to consider the scheme  $Y$  defined by *recollement* of  $\{Y^m, \gamma_m\}$ ; or, in other words

$$Y := \bigcup_{m > 0} Y^m \quad (3.3)$$

Finally, note that there is a map induced by the  $p^m$ 's

$$p: (\mathrm{Gr}^X(V))^s = \bigcup_{m>0} \varprojlim_{i \geq m} U_{m,i} \longrightarrow Y$$

Now, it is straightforward to prove the main result of this section.

**Theorem 3.4.** *The scheme  $Y$  is the geometric quotient of  $\mathrm{Gr}^X(V)^s$  by the action of  $\mathrm{Sl}(r, \mathbb{C})$ .*

*Proof.* Recall that both  $\mathrm{Gr}^X(V)^s$  and  $Y$  are expressed by a *recollement* of open subschemes (see equations (3.1) and (3.3)). Further, it is straightforward to check that the claim will follow if we prove that  $Y^m$  is the geometric quotient of  $\varprojlim_{i \geq m} U_{m,i}$  for the action of  $\mathrm{Sl}(r, \mathbb{C})$  for all  $m$ .

In order to this fact we follow the items of Definition 0.6 of [MF].

- *The morphism  $p^m$  is  $\mathrm{Sl}(r, \mathbb{C})$ -equivariant.* It holds because the morphisms of the diagram (3.2) are  $\mathrm{Sl}(r, \mathbb{C})$ -equivariant.
- *$p^m$  is surjective.* We consider a sequence  $\{G_i\}_{i \geq m} \in Y^m$ . Since  $p_{m,i}$  is surjective, we may choose  $F_i \in U_{m,i}$  such that  $p_{m,i}(F_i) = G_i$ . Having in mind that the fiber of  $G_i$  is equal to its orbit and that  $\Phi_i(F_{i+1}), F_i \in p_{m,i}^{-1}(G_i)$ , there exists  $g_{i+1} \in \mathrm{Sl}(r, \mathbb{C})$  such that  $\Phi_i(g_{i+1}F_{i+1}) = F_i$ , for each  $i \geq m$ .

We now check that the sequence

$$\{F'_m := F_m, F'_{m+1} := g_{m+1}F_{m+1}, F'_{m+2} := g_{m+1}g_{m+2}F_{m+2}, \dots\}$$

is a preimage of  $\{G_i\}_{i \geq m}$  by  $p^m$ .

- *For any  $W \subseteq Y^m$ ,  $W$  is open if and only if  $(p^m)^{-1}(W)$  is open.*

Since  $p^m$  is a continuous map,  $(p^m)^{-1}(W)$  is open for all open subsets  $W \subseteq Y^m$ . Let us now show the converse. Let  $W \subseteq Y^m$  be a subset such that  $(p^m)^{-1}(W)$  is open. From Chap. I, §4, N 4 of [B], we know that any open subset of the inverse limit  $\varprojlim_{i \geq m} U_{m,i}$  is of the type  $\bigcup_{i \geq m} (j_i)^{-1}(X_i)$ , where  $j_i$  are the projections of the limit in each of its factors and  $X_i$  are open sets of  $U_{m,i}$ .

Note that  $X_i = p_{m,i}^{-1}(p_{m,i}(X_i))$  and, since  $p_{m,i}$  is a geometric quotient, it follows that  $p_{m,i}(X_i) \subseteq Y_{m,i}$  is open. Then, the surjectivity of  $p_m$  implies that

$$\begin{aligned} W &= p^m((p^m)^{-1}(W)) = p^m\left(\bigcup_{i \geq m} (j_i)^{-1}(X_i)\right) = \\ &= \bigcup_{i \geq m} p^m((j_i)^{-1}(X_i)) = \bigcup_{i \geq m} (j_i)^{-1}(p_{m,i}(X_i)) \end{aligned}$$

and we conclude that  $W \subseteq Y^m$  is open.

- *It holds that*

$$\mathrm{Im} \Gamma^m = \varprojlim_{i \geq m} U_{m,i} \times_{Y^m} \varprojlim_{i \geq m} U_{m,i}$$

where  $\Gamma^m$  is the morphism

$$\begin{aligned} \Gamma^m : \mathrm{Sl}(r, \mathbb{C}) \times \varprojlim_{i \geq m} U_{m,i} &\rightarrow \varprojlim_{i \geq m} U_{m,i} \times \varprojlim_{i \geq m} U_{m,i} \\ (g, F) &\mapsto (F, gF) \end{aligned}$$

The inclusion  $\subseteq$  is straightforward, so let us prove the reverse one  $\supseteq$ . Let us take an element

$$(F = \{F_i\}, G = \{G_i\}) \in \varprojlim_{i \geq m} U_{m,i} \times_{Y^m} \varprojlim_{i \geq m} U_{m,i}$$

therefore, it is verified that  $p_{m,i}(F_i) = p_{m,i}(G_i)$  for every  $i \geq m$ . By the properties of  $p_{m,i}$ , there exist  $g_i \in \mathrm{Sl}(r, \mathbb{C})$  for  $i \geq m$  such that  $F_i = g_i G_i$ . If we prove that  $g_i = g_j$  for all  $i, j \geq m$ , then we obtain  $F = gG$  for an element  $g \in \mathrm{Sl}(r, \mathbb{C})$  and, thus,  $(F, G) \in \mathrm{Im} \Gamma^m$  as it was to be shown.

So, let us check that all  $g_i$ 's are equal. Take  $i > j \geq m$  arbitrary, and observe that

$$\begin{aligned} g_{i_1} G_{i_1} &= F_{i_1} = \Phi_{i_1} \cdots \Phi_{i_2-1} F_{i_2} = \\ &= \Phi_{i_1} \cdots \Phi_{i_2-1} g_{i_2} G_{i_2} = g_{i_2} G_{i_1}. \end{aligned} \quad (3.4)$$

We conclude that  $g_{i_1}^{-1} g_{i_2}, g_{i_2}^{-1} g_{i_1} \in \mathrm{Stab}(G_{i_1})$  where  $\mathrm{Stab}$  is the stabilizer of a point for the action of a group.

Now, as  $G_i$  is a stable point for every  $i \geq m$ , we know that  $\mathrm{Stab}(G_i)$  is a finite set (Lemma 3.17 of [N]) and we easily have the inclusions

$$\mathrm{Stab}(G_m) \supseteq \mathrm{Stab}(G_{m+1}) \supseteq \cdots.$$

Therefore there exists  $i_0 \geq m$  such that  $\mathrm{Stab}(G_i) = \mathrm{Stab}(G_{i+1})$  for every  $i \geq i_0$ . For  $i < i_0$  we get  $g_i^{-1} g_{i_0} \in \mathrm{Stab}(G_i)$  by (3.4). For  $i \geq i_0$ , we obtain  $g_i^{-1} g_{i_0} \in \mathrm{Stab}(G_{i_0}) = \mathrm{Stab}(G_i)$  by (3.4) and by the equality of the stabilizers.

So for every  $i \geq m$ , we deduce

$$F_i = g_i G_i = g_i (g_i^{-1} g_{i_0} G_i) = g_{i_0} G_i.$$

We conclude that  $F = g_{i_0} G$  and therefore  $(F, G) \in \mathrm{Im} \Gamma^m$ .

- The morphism  $\mathcal{O}_{Y^m} \rightarrow (p^m)_* \mathcal{O}_{\varprojlim_{i \geq m} U_{m,i}}$  induces an isomorphism of  $\mathcal{O}_{Y^m}$  and the invariants of  $(p^m)_* \mathcal{O}_{\varprojlim_{i \geq m} U_{m,i}}$  under  $\mathrm{Sl}(r, \mathbb{C})$ . This is true because

$$\mathcal{O}_{Y^m} = \varinjlim_{i \geq m} \mathcal{O}_{Y_{m,i}} \simeq \varinjlim_{i \geq m} (\mathcal{O}_{U_{m,i}}^{\mathrm{Sl}(r, \mathbb{C})}) = \left( \mathcal{O}_{\varprojlim_{i \geq m} U_{m,i}} \right)^{\mathrm{Sl}(r, \mathbb{C})}$$

□

## 4. MODULI OF VECTOR BUNDLES WITH TRIVIALIZATION

In this section we will study the relation between the moduli spaces of vector bundles with finite trivialization and those with formal trivialization. Then, that relation will be interpreted in terms of Grassmannians. For the construction of the moduli spaces of vector bundles with finite trivialization in terms of the finite Grassmannian we follow [S, AM]. A triple  $(C, p, t_p)$  consisting of a irreducible non-singular projective curve over  $\mathbb{C}$ , a smooth point and an isomorphism of  $\mathbb{C}$ -algebras  $\hat{\mathcal{O}}_p \xrightarrow{\sim} \mathbb{C}[[z]]$  will be fixed from now on. Following [M], we know that there is a  $\mathbb{C}$ -scheme,  $\mathcal{M}_\infty(r, d)$ , whose set of rational points is given by

$$\mathcal{M}_\infty(r, d) := \left\{ \begin{array}{l} \text{pairs } (\mathcal{F}, \delta) \text{ s.t. } \mathcal{F} \text{ is a rank } r \text{ degree } d \text{ vector bundle} \\ \text{on } C \text{ and } \delta \text{ is an isomorphism } \hat{\mathcal{F}}_p \xrightarrow{\sim} \hat{\mathcal{O}}_p^{\oplus r} \end{array} \right\} / \sim$$

where we write  $(\mathcal{F}, \delta) \sim (\mathcal{F}', \delta')$  if and only if there exists an isomorphism of sheaves,  $f : \mathcal{F} \xrightarrow{\sim} \mathcal{F}'$  compatible with  $\delta$  and  $\delta'$ .

The Krichever map for  $\mathcal{M}_\infty(r, d)$  is the scheme homomorphism given by

$$\begin{aligned} \mathcal{K} : \mathcal{M}_\infty(r, d) &\longrightarrow \text{Gr}(V, V_+) \\ (\mathcal{F}, \delta) &\longmapsto (t_p \circ \delta) (H^0(C \setminus \{p\}, \mathcal{F})) \end{aligned}$$

with  $V := \mathbb{C}((z))^{\oplus r}$  and  $V_+ := \mathbb{C}[[z]]^{\oplus r}$ . Since this map is a closed immersion, the scheme  $\mathcal{M}_\infty(r, d)$  can be thought as a closed subscheme of  $\text{Gr}(V)$ .

In [S] (see also [AM]) trivializations of finite order have been considered. It has been shown that there exist  $\mathbb{C}$ -schemes  $\mathcal{M}_{m,i}^s(r, d)$ , for each pair  $(i, m)$  with  $i \geq m > m_0 := 2g(r+1)$  (where  $g$  is the genus of  $C$ ), whose set of rational points is given by

$$\mathcal{M}_{m,i}^s(r, d) := \left\{ \begin{array}{l} (\mathcal{F}, \delta_i) \text{ s.t. } \mathcal{F} \text{ is a rank } r \text{ degree } d \text{ stable v. b.} \\ \text{on } C, \delta_i \text{ is a surjection } \mathcal{F} \rightarrow (\mathcal{O}_C/\mathcal{O}_C(-ip))^{\oplus r} \\ \text{and } H^0(C, \mathcal{F}(-mp)) = 0, H^1(C, \mathcal{F}(mp)) = 0 \end{array} \right\} / \sim$$

The equivalence relation is analogous to the previous one.

The Krichever map for  $\mathcal{M}_{m,i}^s(r, d)$  is the scheme homomorphism given by

$$\begin{aligned} \mathcal{K}_{m,i} : \mathcal{M}_{m,i}^s(r, d) &\hookrightarrow \text{Gr}^\chi(V_{[-i,i]}) \\ (\mathcal{F}, \delta_i) &\mapsto (t_p \circ \delta_i) (H^0(C \setminus \{p\}, \mathcal{F}(ip))) \end{aligned} \tag{4.1}$$

with  $\chi = d + r(1 - g)$  and  $V_{[-i,i]} = z^{-i}V_+/z^iV_+$  (see [AM], Corollary 2.1).

Let us write down maps relating these spaces. Firstly, note that  $\mathcal{M}_{m+1,i}^s(r, d) \subset \mathcal{M}_{m,i}^s(r, d)$  is an open subscheme for each  $i \geq m+1$ .

Let us now define an affine and surjective map

$$\Phi_i : \mathcal{M}_{m,i+1}^s(r, d) \twoheadrightarrow \mathcal{M}_{m,i}^s(r, d)$$



which maps  $(\mathcal{F}, \delta_{i+1})$  to  $(\mathcal{F}, \delta_i)$  where  $\delta_i$  is given by

$$\mathcal{F} \xrightarrow{\delta_{i+1}} (\mathcal{O}_C/\mathcal{O}_C(-(i+1)p))^{\oplus r} \longrightarrow (\mathcal{O}_C/\mathcal{O}_C(-ip))^{\oplus r}$$

Finally, we introduce the rational map

$$\begin{aligned} \mathcal{M}_\infty(r, d) & \dashrightarrow \mathcal{M}_{m,i}^s(r, d) \\ (\mathcal{F}, \delta) & \mapsto (\mathcal{F}, \delta_i) \end{aligned} \quad (4.2)$$

whose domain of definition is the open subscheme consisting of those pairs  $(\mathcal{F}, \delta)$  such that  $\mathcal{F}$  is stable and  $H^0(C, \mathcal{F}(-m)) = H^1(C, \mathcal{F}(m)) = 0$ . Here  $\delta_i$  is constructed from  $\delta$ , since giving an isomorphism  $\delta : \mathcal{F} \xrightarrow{\sim} \hat{\mathcal{O}}_{C,p}^{\oplus r}$  is equivalent to giving a compatible family of surjections  $\{\delta_i\}$ .

Summing up, we have the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{M}_{m_0, m_0+3}^s & \xrightarrow{\Phi_{m_0+2}} & \mathcal{M}_{m_0, m_0+2}^s & \xrightarrow{\Phi_{m_0+1}} & \mathcal{M}_{m_0, m_0+1}^s & \xrightarrow{\Phi_{m_0}} & \mathcal{M}_{m_0, m_0}^s \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathcal{M}_{m_0+1, m_0+3}^s & \xrightarrow{\Phi_{m_0+2}} & \mathcal{M}_{m_0+1, m_0+2}^s & \xrightarrow{\Phi_{m_0+1}} & \mathcal{M}_{m_0+1, m_0+1}^s & & \\ & & \downarrow & & \downarrow & & & & \\ \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & & & & \end{array}$$

From these arguments one deduces the following

**Theorem 4.1.** *There is an identification*

$$\{(\mathcal{F}, \delta) \in \mathcal{M}_\infty(r, d) \text{ s.t. } \mathcal{F} \text{ is stable}\} = \bigcup_{m \geq m_0} \varprojlim_{i \geq m} \mathcal{M}_{m,i}^s(r, d)$$

The following proposition unveils the relation between this result and Theorem 2.1.

**Proposition 4.2.** *Let  $m \geq m_0$ . The diagram*

$$\begin{array}{ccc} \varprojlim_{i \geq m} \mathcal{M}_{m,i}^s(r, d) & \longrightarrow & U^m = \varprojlim_{i \geq m} U_{m,i} \\ \downarrow & & \downarrow \\ \{(\mathcal{F}, \delta) \in \mathcal{M}_\infty(r, d) \text{ s.t. } \mathcal{F} \text{ is stable}\} & \xrightarrow{\mathcal{K}} & \text{Gr}^X(V) \end{array}$$

*is cartesian and the four maps are  $\text{Sl}(r, \mathbb{C})$ -equivariant.*

*Proof.* Given  $(\mathcal{F}, \delta) \in \mathcal{M}_\infty(r, d)$  such that  $\mathcal{F}$  is stable, we know from Theorem 2.7 of [M] (see also [O]) that the formal trivialization  $\delta$  induces canonical isomorphisms

$$\begin{aligned} F \cap z^m V_+ & \simeq H^0(C, \mathcal{F}(-m)) \\ \frac{V}{F + z^{-m} V_+} & \simeq H^1(C, \mathcal{F}(m)) \end{aligned}$$

for every integer  $m$ . Furthermore, the stability of  $\mathcal{F}$  implies the stability of  $\mathcal{K}((\mathcal{F}, \delta)) \in \text{Gr}^X(V)$  w.r.t. the action of  $\text{Sl}(r, \mathbb{C})$  (see [CMP], §3.2).

Then, the Krichever map (4.1) takes values in  $U_{m,i} \subseteq \text{Gr}^X(V_{[-i,i]})$ ; taking inverse limits one gets the arrow on the top row. Now, it is easy to check that

$$\mathcal{K}(\mathcal{F}, \delta) \in U^m \iff H^0(C, \mathcal{F}(-m)) = H^1(C, \mathcal{F}(m)) = 0$$

and the conclusion follows.  $\square$

**Remark 4.3.** *As a consequence of these results and those of §2, it holds that*

$$\mathcal{K}((\mathcal{F}, \delta)) = \bigcup_{m \geq m_0} \varprojlim_{i \geq m} (\mathcal{K}_{m,i}(\mathcal{F}, \delta_i) \cap V_{[-m,i]})$$

**Remark 4.4.** *Let us write down the condition that  $(\mathcal{F}, \delta), (\mathcal{F}', \delta') \in \mathcal{M}_\infty(r, d)$  have the same image in the quotient. Since their images under  $p$  are equal, there exists  $g \in \text{Sl}(r, \mathbb{C})$  such that  $\mathcal{K}((\mathcal{F}, \delta))$  and  $g(\mathcal{K}((\mathcal{F}', \delta')))$  coincide or, what is tantamount, there is an isomorphism  $\bar{g}: \mathcal{F} \xrightarrow{\sim} \mathcal{F}'$  making commutative the following diagram*

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \hat{\mathcal{F}}_p & \xrightarrow[\sim]{\delta} & \hat{\mathcal{O}}_p^{\oplus r} \\ \bar{g} \downarrow \wr & & & & \downarrow g \\ \mathcal{F}' & \longrightarrow & \hat{\mathcal{F}}'_p & \xrightarrow[\sim]{\delta'} & \hat{\mathcal{O}}_p^{\oplus r} \end{array}$$

**Theorem 4.5.** *Let  $(\mathcal{F}, \delta) \in \mathcal{M}_\infty(r, d)$ . Then,  $(\mathcal{F}, \delta)$  is not stable (as a point of  $\text{Gr}(V)$ ) if and only if there exists a vector subbundle  $\mathcal{G} \subset \mathcal{F}$  with a formal trivialization,  $\gamma: \hat{\mathcal{G}}_p \xrightarrow{\sim} \hat{\mathcal{O}}_p^{\oplus l}$  with  $l < r$ , and  $g \in \text{Sl}(r, \mathbb{C})$  such that the following two conditions hold*

- $\mu(\mathcal{G}) \geq \mu(\mathcal{F})$  where  $\mu$  is the slope of the bundle; and,
- $\delta|_{\hat{\mathcal{G}}_p} = g \circ \gamma$ .

*Proof.* Let  $F = \mathcal{K}((\mathcal{F}, \delta))$ . Recall that Theorem 3.11 of [CMP] states that  $F \in \text{Gr}(k[[z]]^{\oplus r})$  is not stable w.r.t. the action of  $\text{Sl}(r, \mathbb{C})$  if and only if there exist  $l < r$  and  $g \in \text{Sl}(r, \mathbb{C})$  such that  $\frac{1}{l}\chi(F \cap gV^l) \geq \frac{1}{r}\chi(F)$  where  $V^l$  denotes the subspace  $k((z))^{\oplus l} \oplus 0 \oplus \dots \oplus 0 \subseteq V$ .

Let  $(\mathcal{F}, \delta)$  be a non stable point and let  $l$  and  $g$  be as above. Then, by Proposition 4.2 of [CMP] (see also Proposition 1 of [O]), the subspace  $F \cap gV^l$  lies on  $\text{Gr}(k((z))^{\oplus l})$  and gives rise a vector bundle on  $C$ ,  $\mathcal{G}$  endowed with a formal trivialization  $\gamma$  satisfying the conditions of the statement.

The converse can be proved similarly.  $\square$

## REFERENCES

- [AM] Álvarez-Vázquez, A., Muñoz Porras, J. M., *Canonical Immersions of Moduli schemes of vector bundles into Grassmannians*. Aportaciones Mat. Investig., **13**, 87–95, Soc. Mat. Mexicana, México, 1998.
- [AMP] Álvarez Vázquez, A., Muñoz Porras, J. M., Plaza Martín, F. J., *The algebraic formalism of soliton equations over arbitrary base fields*. Aportaciones Mat. Investig., **13**, 3–40, Soc. Mat. Mexicana, México, 1998.

- [B] Bourbaki, N., *Elements of Mathematics: General Topology Part 1*. Addison-Wesley, Reading, Mass., 1966.
- [CMP] Casimiro, A. C., Muñoz Porras, J. M., Plaza Martín, F. J., *Stability on the Sato Grassmannian. Applications to the moduli of vector bundles.*, Preprint, Departamento de Matemática, FCT, UNL, No. **3**, 2007.
- [EGA-II] Grothendieck, A., *Éléments de géométrie algébrique. Chapitre II. Étude globale élémentaire de quelques classes de morphismes.*, Inst. Hautes Études Sci. Publ. Math., No. **8**, 1961.
- [M] Mulase, M., *Category of vector bundles on algebraic curves and infinite dimensional grassmannians*. Internat. J. Math. **1**(3) (1990), 293–342.
- [MF] Fogarty, J., Mumford, D., *Geometric invariant theory*. Springer-Verlag, Berlin-Heidelberg-New York, 1982.
- [N] Newstead, P. E., *Introduction to Moduli problems and orbit spaces*. Tata Institute of Fundamental Research Lectures on Mathematical Physics **51**, Tata Institute of Fundamental Research, Bombay, 1978.
- [O] Osipov, D. V., *The infinite-dimensional Sato Grassmannian and coherent sheaves of rank 2 on curves*. Sb. Math. **194**(11-12) (2003), 1665–1678.
- [PS] Pressley, A., Segal, G., *Loop Groups*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986.
- [S] Seshadri, C. S., *Fibrés vectoriels sur les courbes algébriques*. Astérisque **96**, Société Mathématique de France, Paris, 1982.
- [SS] Sato, M., Sato, Y., *Soliton equations as dynamical systems on an infinite dimensional grassmannian manifold*. Nonlinear partial differential equations in applied science (Tokyo, 1982), 259–271, North-Holland Math. Stud. **81**, North-Holland, Amsterdam, 1983.
- [SW] Segal, G., Wilson G., *Loop groups and equations of KdV type*. Inst. Hautes Études Sci. Publ. Math. **61** (1985), 5–65.

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