Adiabatic limit of the Zakharov-Rubenchik Equation

Filipe Oliveira
Centro de Matemática e Aplicações
FCT-Universidade Nova de Lisboa, Portugal
fso@fct.unl.pt

Abstract

The Zakharov-Rubenchik system

$$\begin{cases} i\partial_T B + \omega \partial_{XX} B - k(u - \frac{v}{2}\rho + q|B|^2)B = 0 \\ \epsilon \partial_T \rho + \partial_X(u - v\rho) = -k\partial_X |B|^2 \\ \epsilon \partial_T u + \partial_X(\beta\rho - vu) = \frac{k}{2} v \partial_X |B|^2 \end{cases}$$

appears in the context of Alfvén waves propagating in a magnetized plasma. This system "contains" the well-known Zakharov Equation and the Benney Equation for the interaction of high and low frequency waves. We prove the pointwise convergence of the magnetic field B to a solution of the Nonlinear Schrödinger Equation, in the adiabatic limit $\epsilon \to 0$.

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1 Introduction

In the presence of an external magnetic field, transverse oscillations of the magnetic field lines known as Alfvén waves can be observed in several magnetized plasmas. The dynamics of Alfvén waves are ruled by the so-called Magneto-Hydro-Dynamics Equations (MHD). By a multi-scale analysis and considering variations of the fields exclusively in the direction of propagation $(0x)$, Champeaux &al ([1]) derived from the MHD equations the following system for the evolution of wave trains of Alfvén waves with wave number k and frequency $\widetilde{\omega}$, in a frame travelling at the Alfvén-wave group velocity $v = 2\widetilde{\omega}^3 k^{-1}(k^2 + \widetilde{\omega}^2)^{-1}$:

$$\begin{cases} i\partial_T B + \omega \partial_{XX} B - k(u - \frac{1}{2} v\rho)B = 0 \\ \\ \epsilon \partial_T \widetilde{\rho} + \partial_X(\widetilde{u} - v\widetilde{\rho}) = 0 \\ \\ \epsilon \partial_T \widetilde{u} + \partial_X(\beta\widetilde{\rho} - v\widetilde{u} + \frac{1}{2}|B|^2) = 0. \end{cases} \tag{1}$$

Here, $B = B_y + iB_z$ denotes the transverse component of the magnetic field, $\widetilde{\rho}$ the density of mass, $\widetilde{u}$ the fluid speed in the direction of propagation, $\beta > 0$ the squared ratio of the sonic to the Alfvén speed and

$$\omega = \frac{\widetilde{\omega}^3}{2(\widetilde{\omega}^2 + k^2)} \left(\frac{1}{k^2} - 4\frac{k^2}{(k^2 + \widetilde{\omega}^2)^2} \right).$$

The "slow" variables (X, T) are given by the scaling $X = \epsilon(x - vt)$ and $T = \epsilon^2 t$, where ϵ is a small parameter. Here, the wave number k (assumed positive) and the frequency $\widetilde{\omega}$ are related by the dispersion relation

$$\tilde{\omega} = k \left(\sigma_1 \frac{k}{2R_i} + \sigma_2 \sqrt{1 + \frac{k^2}{4R_i}} \right),$$

where R_i is the non-dimensional ion-cyclotron frequency, $\sigma_1 = \pm 1$ (right/left-hand circularly polarization) and $\sigma_2 = \pm 1$ (forward/backward propagation).

This system becomes resonant when $\beta = v^2$. As mentioned in [1], far from this resonance, by setting

$$q = k + \frac{v}{4}\frac{kv-1}{\beta - v^2},\ u = \widetilde{u} - (k + \frac{v}{2}\frac{kv-1}{\beta - v^2})|B|^2 \quad \text{and } \rho = \widetilde{\rho} - \frac{kv-1}{2(\beta - v^2)}|B|^2,$$

system (1) can be transformed into the Zakharov-Rubenchik system

$$\begin{cases} i\partial_T B + \omega \partial_{XX} B - k(u - \frac{v}{2}\rho + q|B|^2)B = 0 & \text{(a)} \\ \epsilon \partial_T \rho + \partial_X (u - v\rho) = -k\partial_X |B|^2 & \text{(b)} \\ \epsilon \partial_T u + \partial_X (\beta\rho - vu) = \frac{k}{2} v \partial_X |B|^2. & \text{(c)} \end{cases} \qquad (2)$$

This system was introduced in [13] as a model for the interaction of high and low frequency waves. In [7], we proved that the Cauchy problem associated to (2) is globally well-posed for initial data (B_o, ρ_o, u_o) in $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and showed that for $\epsilon > 0$ small enough, (2) possesses solitary wave solutions which are orbitally stable. In [8], a study of the local well-posedness of a multi-dimensional version of (2) is made.

In the present report we are concerned with the following problem: by making ϵ tend to 0 (the so-called adiabatic limit), we get formally from $(2-b, c)$ that ρ and u become slaved to the magnetic field amplitude through

$$\rho = -\frac{kv}{2(\beta - v^2)}|B|^2 \quad \text{and} \quad u = -k\frac{\beta - v^2/2}{\beta - v^2}|B|^2.$$

Replacing in $(2-a)$, we find that the magnetic field B satisfies the cubic Nonlinear Schrödinger equation (NLS)

$$i\partial_T B + \omega \partial_{XX} B + \frac{kv}{4(\beta - v^2)} |B|^2 B = 0. \tag{3}$$

Here, we will justify rigourously this approximation: in the case where the conditions

$$\tilde{\omega} < 0, \;\; \beta - v^2 > 0 \text{ and } \omega > 0 \tag{4}$$

hold, we prove the pointwise convergence of the solutions $B^{(\epsilon)}$ of (2) to the solution B of (3).
Before stating more precisely our main theorem, we give a few notations:
For all $p \in [1; +\infty[$, we introduce the usual L^p spaces

$$L^p(\mathbb{R}) = \left\{ f \;\; ; \;\; \|f\|_{L^p} = (\int |f|^p)^{\frac{1}{p}} < +\infty \right\}.$$

We set $\Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$, i.e., for all $G(x) = (g_1(x), \ldots, g_n(x))$, where g_j is a tempered distribution,

$$\mathcal{F}(\Lambda G)(\xi) = \Big((1+\xi^2)^{\frac{1}{2}} \mathcal{F}(g_1)(\xi), \ldots, (1+\xi^2)^{\frac{1}{2}} \mathcal{F}(g_n)(\xi) \Big),$$

$\mathcal{F}$ denoting the Fourier transform.

For $s \in \mathbb{R}$, we introduce the Sobolev spaces $H^s(\mathbb{R}) = \{f \; ; \; \|f\|_s^2 = \|\Lambda^s f\|_{L^2}^2 < \infty\}$, and, setting $C_o^\infty(\mathbb{R})$ the space of smooth functions with compact support, we denote the local Sobolev spaces by

$$H^s_{loc}(\mathbb{R}) = \{ f \; ; \forall \theta \in C_o^\infty(\mathbb{R}) \; , \; \theta f \in H^s(\mathbb{R}) \}.$$

Finally, we denote by $[\Lambda^s, F]$ the commutator given by

$$[\Lambda^s, F] G = \Lambda^s(FG) - F\Lambda^s(G),$$

where $G = (g_1(x), \ldots, g_n(x))$ and $F = (f_{i,j}(x))$ is a n-square matrix.

We will prove the following result:

Theorem 1.1 *Assume conditions (4) hold. Let* $s > \frac{7}{2}$, $0 < \epsilon < 1$ *and*

$$(B_o, \rho_o, u_o) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}) \tag{5}$$

satisfying the compatibility conditions

$$u_o - v\rho_o = -k|B_o|^2 \quad \text{and} \quad \beta\rho_o - vu_o = k\frac{v}{2}|B_o|^2. \tag{6}$$

Then there exists $T > 0$ independent of ϵ such that the Zakharov-Rubenchik system (2) with initial data (5) possesses a unique solution

$$(B^{(\epsilon)}, \rho^{(\epsilon)}, u^{(\epsilon)}) \in \mathcal{C}^j([0;T]; H^{s+1-2j}(\mathbb{R}) \times H^{s-j}(\mathbb{R}) \times H^{s-j}(\mathbb{R})), \quad j = 0, 1.$$

Moreover, the following pointwise convergence holds:

$$B^{(\epsilon)} \to B \ \textit{in} \ \ C^{1-j}([0;T]; C^{2j}_{loc}), \tag{7}$$

where $B \in \mathcal{C}^j([0;T]; H^{s+1-2j})$ is the solution of the NLS equation (3) for initial data $B(0,x) = B_o(x)$.
Furthermore,

$$u^{(\epsilon)} - v\rho^{(\epsilon)} + k|B^{(\epsilon)}|^2 \to 0 \quad \textit{and} \quad \beta\rho^{(\epsilon)} - vu^{(\epsilon)} - k\frac{v}{2}|B^{(\epsilon)}|^2 \to 0 \quad \textit{in} \quad C_b([0;T] \times \mathbb{R}). \tag{8}$$

We will deal with the system

$$\begin{cases} iB_t + B_{xx} + a\phi B + ef\psi B + c|B|^2 B = 0 & \text{(a)} \\ \epsilon^2 \phi_{tt} + d\epsilon\phi_{tx} - \phi_{xx} = -a|B|^2_{xx} & \text{(b)} \\ \epsilon\psi_t + e\psi_x = f|B|^2_x, & \text{(c)} \end{cases} \tag{9}$$

where a, c, d, e, f are real constants. Indeed, the Zakharov-Rubenchik system (2) can be put in this form through linear transformations and space-time rescaling (see the Appendix), provided that conditions (4) hold. Hence we will prove the counterpart of Theorem 1.1 for system (9):

Theorem 1.2 *Let $s > \frac{7}{2}$ and $0 < \epsilon < 1$.*
Let

$$(B_o, \phi_o, \widehat{\phi}_o, \psi_o) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \times H^s(\mathbb{R}), \tag{10}$$

with $\widehat{\phi}_o = \theta_x$, $\theta \in H^s(\mathbb{R})$.
Then there exists $T > 0$ independent of ϵ such that the I.V.P. (9) with initial data

$$B(0,x) = B_o(x), \ \ \phi(0,x) = \phi_o(x), \ \ \psi(0,x) = \psi_o(x) \quad \textit{and} \ \phi_t(0,x) = \widehat{\phi}_o(0,x)$$

possesses a unique solution

$$(B^{(\epsilon)}, \phi^{(\epsilon)}, \psi^{(\epsilon)}) \in C^j([0;T]; H^{s+1-2j}(\mathbb{R}) \times H^{s-j}(\mathbb{R}) \times H^{s-j}(\mathbb{R})), \ \ j = 0, 1.$$

Furthermore, if $\phi_o = a|B_o|^2$ and $\psi_o = \dfrac{f}{e}|B_o|^2$,

$$B^{(\epsilon)} \to \hat{B} \ \textit{in} \ C^{1-j}([0;T]; C^{2j}_{loc}),$$

where $\hat{B} \in C^j([0;T]; H^{s+1-2j}(\mathbb{R}))$ is the solution to the NLS equation

$$i\hat{B}_t + \hat{B}_{xx} + (a^2 + f^2 + c)|\hat{B}|^2 \hat{B} = 0$$

for initial data $B(0,x) = B_o(x)$.
Also,

$$\phi^{(\epsilon)} - a|B^\epsilon|^2 \to 0 \quad \text{and } \psi^{(\epsilon)} - \frac{f}{e}|B^{(\epsilon)}|^2 \to 0 \qquad \text{in } C_b([0;T] \times \mathbb{R}).$$

Note that system (9) contains several well-known models:

On one hand, if $a = -1$, $c = d = e = f = 0$ and $\psi \equiv 0$, one gets the well-known Zakharov system ([12]):

$$\begin{cases} iB_t + B_{xx} - \phi B = 0 \\ \\ \epsilon^2 \phi_{tt} - (\phi + |B|^2)_{xx} = 0. \end{cases} \tag{11}$$

A result similar to Theorem 1.1 is obtained in [9] for this special case. In [3], the authors prove that for small initial data $(B_o, \phi_o, \widehat{\phi_o})$, the adiabatic limit of (11) is uniform in ϵ.

On the other hand, taking $\phi \equiv 0$ and $a = 0$, (9) becomes the Benney Equation

$$\begin{cases} iB_t + B_{xx} + ef\psi B + c|B|^2 B = 0 \\ \\ \epsilon\psi_t + e\psi_x = f|B|^2_x, \end{cases} \tag{12}$$

which has been extensively studied by many authors: see for instance [6], [10] and [11] for the well-posedness and [5] for a study of the stability of the solitary wave type solutions in the case where $c = 0$. To the best of our knowledge, there is no reference in the literature to the adiabatic limit of the Benney equation.

The rest of this paper is organized as follows: in Section 2 we show that (9) can be put in the form of a quasilinear symmetric hyperbolic system. This will allow us to use the general Friedrich's theory for symmetric hyperbolic systems and derive in section 3 via a fixed-point technique a life span T independent of ϵ. In section 4, we complete the proof of Theorem 1.2 and finally, in the Appendix, we explain why Theorem 1.2 implies Theorem 1.1.

2 Obtaining a perturbed hyperbolic system

We show here that an hyperbolic system can be obtained from (9). The strategy is to treat the problem of the adiabatic limit of the Zakharov-Rubenchik system in the frame of the pioneer work of S. Klainerman and A. Majda ([4]) for the limiting behaviour of quasilinear hyperbolic systems as certain coeficients become infinite. A similar method was also introduced in [9].

We start from a solution (B, ϕ, ψ), corresponding to initial data given by (10), and set

$$F = \phi - a|B|^2, \quad G = -e\psi + f|B|^2, \tag{13}$$

and

$$V_x = -\epsilon \phi_t. \tag{14}$$

Note that, from (9-b),

$$\partial_t(\phi_t) = [-\frac{a}{\epsilon^2}|B|_x^2 + \frac{1}{\epsilon^2}\phi_x - \frac{d}{\epsilon}\phi_t]_x,$$

and $\phi_t(0,.) = \hat{\phi}_o = \theta_x$, hence ϕ_t is an x-derivative for all times.

From (9):

$$V_t + \frac{d}{\epsilon}V_x + \frac{1}{\epsilon}F_x = 0. \tag{15}$$

Differentiating (13) with respect to t,

$$F_t + \frac{1}{\epsilon}V_x + a|B|_t^2 = 0 \tag{16}$$

and

$$G_t + \frac{e}{\epsilon}G_x - f|B|_t^2 = 0. \tag{17}$$

From (9-a),

$$iB_t + B_{xx} + (aF - fG + (a^2 + f^2 + c)|B|^2)B = 0. \tag{18}$$

We now set

$$\sqrt{2}B = \alpha + i\beta \text{ and } \sqrt{2}B_x = \gamma + i\delta.$$

From (18),

$$\alpha_t + \beta(aF - fG + \frac{1}{2}(a^2 + f^2 + c)(\alpha^2 + \beta^2)) = -\beta_{xx}, \tag{19}$$

$$\beta_t - \alpha(aF - fG + \frac{1}{2}(a^2 + f^2 + c)(\alpha^2 + \beta^2)) = \alpha_{xx}. \tag{20}$$

Differentiating (18) with respect to x:

$$iB_{xt} + B_{xxx} + (aF - fG + (a^2 + f^2 + c)|B|^2)B_x \\ + (aF_x - fG_x + (a^2 + f^2 + c)(\overline{B}B_x + \overline{B}_x B))B = 0. \tag{21}$$

Taking the real and the imaginary part,

$$\gamma_t + (aF - fG + \frac{1}{2}(a^2 + f^2 + c)(\alpha^2 + \beta^2))\delta \\ + (aF_x - fG_x + (a^2 + f^2 + c)(\alpha\gamma + \beta\delta))\beta = -\delta_{xx}, \tag{22}$$

$$\delta_t - (aF - fG + \frac{1}{2}(a^2 + f^2 + c)(\alpha^2 + \beta^2))\gamma \\ - (aF_x - fG_x + (a^2 + f^2 + c)(\alpha\gamma + \beta\delta))\alpha = \gamma_{xx}. \tag{23}$$

Multiplying (18) by $\overline{B}$ and taking the imaginary part yields

$$|B|_t^2 = \frac{1}{i}(B\overline{B}_{xx} - B_{xx}\overline{B}) = \beta\gamma_x - \alpha\delta_x. \tag{24}$$

Replacing in (16) and (17):

$$F_t + \frac{1}{\epsilon}V_x + a(\beta\gamma_x - \alpha\delta_x) = 0, \tag{25}$$

and

$$G_t + \frac{e}{\epsilon}G_x - f(\beta\gamma_x - \alpha\delta_x) = 0. \tag{26}$$

Finally, setting $Y = (V, F, G, \alpha, \beta, \gamma, \delta)$, from (15),(19),(20),(22),(23),(25) and (26), we obtain the perturbed quasilinear hyperbolic system

$$Y_t + \left(\frac{1}{\epsilon}M + N(Y)\right)Y_x + R(Y) + AY_{xx} = 0, \tag{27}$$

where

$$M = \begin{pmatrix} d & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; N(Y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a\beta & -a\alpha \\ 0 & 0 & 0 & 0 & 0 & -f\beta & f\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a\beta & -f\beta & 0 & 0 & 0 & 0 \\ 0 & -a\alpha & f\alpha & 0 & 0 & 0 & 0 \end{pmatrix}$$

are symmetric matrixes,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

is antisymmetric, and the nonlinear part $R(Y)$ reads

$$R(Y) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \beta(aF - fG + \frac{1}{2}(a^2 + f^2 + c)(\alpha^2 + \beta^2)) \\ -\alpha(aF - fG + \frac{1}{2}(a^2 + f^2 + c)(\alpha^2 + \beta^2)) \\ (aF - fG + \frac{1}{2}(a^2 + f^2 + c)(\alpha^2 + \beta^2))\delta + (a^2 + f^2 + c)(\alpha\gamma + \beta\delta)\beta \\ -(aF - fG + \frac{1}{2}(a^2 + f^2 + c)(\alpha^2 + \beta^2))\gamma - (a^2 + f^2 + c)(\alpha\gamma + \beta\delta)\alpha \end{pmatrix}.$$

3 Uniform estimates

In this section, we prove the following result for the system (27):

Lemma 3.1 *Let* $s > \frac{3}{2}$, $0 < \epsilon < 1$ *and*

$$Y_o = (V_o, F_o, G_o, \alpha_o, \beta_o, \gamma_o, \delta_o) \in H^s(\mathbb{R})^7. \tag{28}$$

There exists a life-span $T > 0$ *independent of* ϵ *such that the I.V.P. (27) with initial data (28) possesses a unique solution* $Y^{(\epsilon)} \in C^j([0;T]; H^{s-2j}(\mathbb{R})^7)$.

Furthermore, for all $t \leq T$, $\|Y(t)\|_s \leq C(T)$, *where* $C(T)$ *is a positive constant independent of* ϵ.

We begin by considering, for a fixed function W, the linear equation

$$Y_t + \left(\frac{1}{\epsilon}M + N(W)\right) Y_x + AY_{xx} = 0, \tag{29}$$

where M, $N(W)$ and A were computed in Section 2. The solutions to (29) are given by the following lemma:

Lemma 3.2 *Let* $s > \frac{3}{2}$ *and* $W \in \mathcal{C}(\mathbb{R}, H^s(\mathbb{R})^7)$. *There exists an evolution family* $\{U(t,\tau)\}_{t \geq \tau \geq 0}$ *acting on* $H^s(\mathbb{R})^7$:

$$\begin{array}{llll} U(t,\tau): & H^s(\mathbb{R})^7 & \to & H^s(\mathbb{R})^7 \\ & f & \to & U(t,\tau)f \end{array}$$

which generates the solution $Y(x,t) = U(t,\tau)f \in C(\mathbb{R}_+, H^s(\mathbb{R})^7) \cap C^1(\mathbb{R}_+, H^{s-2}(\mathbb{R})^7)$ *of the I.V.P. (29) for initial data* $Y(x,\tau) = f(x)$.

Moreover, for $0 \leq \tau \leq t \leq T$ *and for every* $f \in H^s(\mathbb{R})^7$,

$$\|U(t,\tau)f\|_s \leq e^{CT \sup_{0\leq t\leq T} \|W(t)\|_s^2} \|f\|_s, \quad t \in [0;T], \tag{30}$$

where C *is a positive constant independent of* ϵ.

Proof of Lemma 3.2:
We derive here an a priori energy estimate. Applying Λ^s to (29) and taking the inner product with $\Lambda^s Y$ yields

$$\langle \Lambda^s Y_t, \Lambda^s Y\rangle + \frac{1}{\epsilon}\langle M\Lambda^s Y_x, \Lambda^s Y\rangle + \langle \Lambda^s(N(W(t))Y_x), \Lambda^s Y\rangle + \langle A\Lambda^s Y_{xx}, \Lambda^s Y\rangle = 0. \tag{31}$$

Since M is a symmetric constant matrix, $\langle M\Lambda^s Y_x, \Lambda^s Y\rangle = \frac{1}{2}\langle M\Lambda^s Y, \Lambda^s Y\rangle_x$. Moreover,

$$\langle \Lambda^s(N(W(t))Y_x), \Lambda^s Y\rangle = \langle N(W(t))\Lambda^s Y_x, \Lambda^s Y\rangle + \langle [\Lambda^s, N(W(t))]Y_x, \Lambda^s Y\rangle.$$

Since $N(W(t))$ is symmetric,

$$\langle N(W(t))\Lambda^s Y_x, \Lambda^s Y\rangle = \frac{1}{2}\langle N(W(t))\Lambda^s Y, \Lambda^s Y\rangle_x - \frac{1}{2}\langle N(W(t))_x\Lambda^s Y, \Lambda^s Y\rangle.$$

Also, integrating (31) over $\mathbb{R}$:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^s Y\|_o^2 &= \frac{1}{2}\int\langle N(W(t))_x\Lambda^s Y, \Lambda^s Y\rangle dx - \int\langle A\Lambda^s Y_{xx}, \Lambda^s Y\rangle dx \\
&+ \int\langle[\Lambda^s, N(W(t))]Y_x, \Lambda^s Y\rangle dx \\
&\leq \|N(W(t))_x\|_\infty\|\Lambda^s Y\|_o^2 - \int\langle A\Lambda^s Y_x, \Lambda^s Y_x\rangle dx \\
&+ \|[\Lambda^s, N(W(t))]Y_x\|_{L^2}\|\Lambda^s Y\|_{L^2}.
\end{aligned}$$

Since A is antisymmetric, $\displaystyle\int\langle A\Lambda^s Y_x, \Lambda^s Y_x\rangle dx = 0.$

Furthermore, $\|[\Lambda^s, N(W(t))]Y_x\|_{L^2} \leq \|N(W(t))\|_s\|Y_x\|_{s-1}$, and since $s-1\frac{1}{2}$, by the Sobolev injection $H^{s-1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$,

$$\|N(W(t))_x\|_\infty \leq C\|N(W(t))_x\|_{s-1} \leq C\|N(W(t))\|_s.$$

Finally, since $s > \frac{3}{2}$, $H^s(\mathbb{R})$ is an algebra: for $f, g \in H^s$, $fg \in H^s$ and $\|fg\|_s \leq C\|f\|_s\|g\|_s$. Hence, since N is quadratic in W, $\|N(W(t))\|_s \leq C\|W(t)\|_s^2$, Therefore, we obtain the a priori estimate

$$\frac{d}{dt}\|Y(t)\|_s^2 \leq C(t)\|Y(t)\|_s^2, \quad \text{where } C(t) = C\|W(t)\|_s^2 \tag{32}$$

It is well-known from the general theory of semigroups for non-autonomous linear equations that (32) implies the existence of an evolution family $\{U(t,\tau)\}_{t\geq\tau\geq 0}$ as described in Lemma 3.2 (see for instance [2]. ■

Proof of Lemma 3.1:
For $Y_o \in H^s$, $s > \frac{3}{2}$ and $T > 0$ we consider the Banach space $X(T) = C([0;T]; H^s(\mathbb{R}^7))$ endowed with the norm $\|W\|_X = \sup_{0\leq t\leq T}\|W(t)\|_s$ and

$$E(T) = \{W \in X \ ; \ \|W\|_X \leq 2K\},$$

where K is such that $\|Y_o\|_s \leq K$.
Moreover, we consider the map

$$\begin{aligned}
\Psi \quad : \quad E(T) &\to C([0;T], H^s(\mathbb{R}^7)) \\
W &\to U(t,0)Y_o + \int_0^t U(t,\tau)R(W(\tau))d\tau,
\end{aligned}$$

We claim that for T small enough, Ψ is a contraction of the X-ball $E(T)$.
Indeed, for all $t \le T$,

$$\|\Psi(W(t))\|_s \le \|U(t,0)Y_o\|_s + \int_0^t \|U(t,\tau)R(W(s))\|_s ds.$$

By (30), $\|U(t,0)Y_o\|_s \le e^{CT\|W\|_X^2}\|Y_o\|_s \le Ke^{4CTK^2}$.

Also,

$$\begin{aligned}\int_0^t \|U(t,\tau)R(W(\tau))\|_s d\tau &\le \int_0^T e^{CT\|W\|_X^2}\|R(W(\tau))\|_s d\tau\\ &\le Te^{4CTK^2}\sup_{0\le\tau\le T}\|R(W(\tau))\|\\ &\le CTe^{4CTK^2}(\|W\|_X^2+\|W\|_X^3+\|W\|_X^4).\end{aligned}$$

Finally,

$$\|\Psi(W)\|_X \le e^{4CTK^2}\left(K+TC(K^2+K^3+K^4)\right) \le 2K$$

for T small enough.

By the exact same computations, one can easily prove that Ψ is a contraction of $E(T)$.

Hence, there exists a unique fixed point $Y \in L^\infty([0,T]; H^s(\mathbb{R})^7)$ of Ψ:

$$\forall t \in [0;T],\ \ Y(t) = U(t,0)Y_o + \int_0^t t,\tau)R(Y(\tau))d\tau. \tag{33}$$

Now, observing that the right-hand-side of (33) is in $C([0;T];H^s(\mathbb{R})^7) \cap C^1([0;T];H^{s-2}(\mathbb{R})^7)$, we get the announced regularity

$$Y \in C([0;T], H^s(\mathbb{R})^7) \cap C^1([0;T], H^{s-2}(\mathbb{R})^7)$$

and Y solves the I.V.P. (27) for initial data Y_o in the time interval $[0;T]$. ■

By transposing this result for equation (9), we get the first part of Theorem 1.2:

Proposition 3.3 *Let $s > \frac{3}{2}$ and $0 < \epsilon < 1$.*
Let

$$(B_o, \phi_o, \widehat{\phi}_o, \psi_o) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \times H^s(\mathbb{R}), \tag{34}$$

with $\widehat{\phi} = \theta_x$, $\theta \in H^s(\mathbb{R})$.
Then there exists a life-span $T > 0$ independent of ϵ such that the I.V.P. (9) with initial data (34) possesses a unique solution

$$(B^{(\epsilon)}, \phi^{(\epsilon)}, \psi^{(\epsilon)}) \in C^j([0;T]; H^{s+1-2j}(\mathbb{R}) \times H^{s-j}(\mathbb{R}) \times H^{s-j}(\mathbb{R}), \quad j \in \{0;1\}. \tag{35}$$

Proof:
For initial data (34) we set

$$Y_o = (V_o, F_o, G_o, \alpha_o, \beta_o, \gamma_o, \delta_o)$$

$$:= \left(-\epsilon\theta, \phi_o - |B_o|^2, -e\psi_o + f|B_o|^2, \frac{1}{\sqrt{2}} Re(B_o), \frac{1}{\sqrt{2}} Im(B_o), \frac{1}{\sqrt{2}} Re(B_{ox}), \frac{1}{\sqrt{2}} Im(B_{ox}) \right).$$

Note that Y_o depends on ϵ but $\|Y_o\|_s$ is estimated independently of ϵ when this parameter is small: there exists $K > 0$ depending exclusively on the initial data (34) such that $\|Y_o\|_s \leq K$.
We consider the solution $Y = (V, F, G, \alpha, \beta, \gamma, \delta)$ of (27). We set

$$B := \frac{1}{\sqrt{2}}(\alpha + i\beta).$$

After differentiating (19) and (20) with respect to x, we obtain from these equation and from (22), (23) an evolution equation for $z(t) = (\alpha_x - \gamma) + i(\beta_x - \delta)$. Multiplying by $\overline{z(t)}$ and integrating, we obtain an energy estimate which yields

$$\|z(t)\|_{L^2} \leq C(T)\|z(0)\|_{L^2}.$$

Since $z(0) = 0$, we get $B_x = \frac{1}{\sqrt{2}}(\gamma + i\delta)$. Setting $\phi := F + a|B|^2$ and $\psi := \frac{f}{e}|B|^2 - \frac{1}{e}G$, it is straightforward that (B, ϕ, ψ) satisfies (9) with the adequate initial data. ■

4 Proof of the Main Theorem

Let $s > \frac{7}{2}$.

Let $(B^{(\epsilon)}, \phi^{(\epsilon)}, \psi^{(\epsilon)}) \in C([0;T]; H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ the solution of (9) corresponding to initial data (34) such that

$$\phi_o = a|B_o|^2 \text{ and } \psi_o = \frac{f}{e}|B_o|^2. \tag{36}$$

Let $\hat{B} \in C^1([0;T]; H^{s+1}(\mathbb{R})$ the solution of the Cubic Nonlinear Schrödinger Equation

$$i\hat{B}_t + \hat{B}_{xx} + (a^2 + f^2 + c)|\hat{B}|^2\hat{B} = 0 \tag{37}$$

for initial data $\hat{B}(0, x) = B_o(x)$.

We finish here the proof of Theorem 1.2 by showing that

$$B^{(\epsilon)} \to \hat{B} \text{ in } C^{1-j}([0;T], C^{2j}) \tag{38}$$

and

$$\phi^{(\epsilon)} - a|B^{(\epsilon)}|^2 \to 0 \quad \text{and } -e\psi^{(\epsilon)} + f|B^{(\epsilon)}|^2 \to 0 \quad \text{ in } C([0;T] \times \mathbb{R}). \tag{39}$$

Once again we will use the new variables:

Proposition 4.1 *Let $s > \frac{7}{2}$ and*

$$Y^{(\epsilon)} = (F^{(\epsilon)}, V^{(\epsilon)}, G^{(\epsilon)}, \alpha^{(\epsilon)}, \beta^{(\epsilon)}, \gamma^{(\epsilon)}, \delta^{(\epsilon)}) \in C([0;T]; H^s(\mathbb{R})^7)$$

the solution of (27) given by Lemma 3.1, with $F_o \equiv G_o \equiv 0$.

Then

$$F^{(\epsilon)} \to 0 \ , \ \ G^{(\epsilon)} \to 0 \quad in\ C([0;T] \times \mathbb{R})$$

and

$$(\alpha^{(\epsilon)}, \beta^{(\epsilon)}, \gamma^{(\epsilon)}, \delta^{(\epsilon)}) \to (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}) \quad in\ C^{1-j}([0;T], C^{2j}_{loc}),\ j \in \{0,1\},$$

where $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ satisfies $\hat{\gamma} = \alpha_x$, $\hat{\delta} = \beta_x$ and

$$\left\{ \begin{array}{rcl} \hat{\alpha}_t + \frac{1}{2}\hat{\beta}(a^2 + f^2 + c)(\hat{\alpha}^2 + \hat{\beta}^2) & = & -\hat{\beta}_{xx} \\ \\ \hat{\beta}_t - \frac{1}{2}\hat{\alpha}(a^2 + f^2 + c)(\hat{\alpha}^2 + \hat{\beta}^2) & = & \hat{\alpha}_{xx} \end{array} \right. \tag{40}$$

Proof:
We differentiate (27) with respect to t. Since $Y_t^{(\epsilon)} \in H^{s-2}(\mathbb{R})$ and $s - 2 > \frac{3}{2}$, we obtain exacly by the energy method used in Section 2 that for all $t \leq T$,

$$\|Y_t^{(\epsilon)}(t, \epsilon)\|_{s-2} \leq C, \quad C \text{ independent of } \epsilon.$$

Note that the condition $F_o \equiv G_o \equiv 0$, is used to get, at $t = 0$, $Y_t^{(\epsilon)}(0) = Y_t(0)$ independent of ϵ.
From (16), we get that for all $t \leq T$, $\|\frac{1}{\epsilon} V_x^{(\epsilon)}(t)\|_{s-2} \leq C$, and yet, by (15),

$$\|F_x^{(\epsilon)}(t)\|_{s-2} \leq C\epsilon.$$

By the Gagliardo-Nirenberg inequality,

$$\|F^{(\epsilon)}(t)\|_{L^\infty} \leq C\|D^k F^{(\epsilon)}(t)\|_o^\lambda \|F^{(\epsilon)}(t)\|_o^{1-\lambda}, \tag{41}$$

provided that $\lambda = \frac{1}{2k}$. Taking $\lambda = \frac{1}{2(s-1)}$,

$$\|F^{(\epsilon)}(t)\|_{L^\infty} \leq C\|D^{s-1} F^{(\epsilon)}(t)\|_o^{\frac{1}{2(s-1)}} \|F^{(\epsilon)}(t)\|_o^{1-\frac{1}{2(s-1)}} \leq C\epsilon^{\frac{1}{2(s-1)}}.$$

Hence

$$F^{(\epsilon)} \to 0 \quad \text{in } C([0;T] \times \mathbb{R}).$$

Also, from (17),

$$\frac{1}{\epsilon}\|G_x^{(\epsilon)}(t)\|_{s-2} \leq C,$$

and by the same computations

$$G^{(\epsilon)} \to 0 \quad \text{in } C([0;T] \times \mathbb{R}).$$

Since $\Omega_\epsilon = (\alpha^{(\epsilon)}, \beta^{(\epsilon)}, \gamma^{(\epsilon)}, \delta^{(\epsilon)})$ is bounded in $C([0;T];H^s) \cap C^1([0;T];H^{s-2})$, any subsequence of Ω_ϵ has a subsequence (still denoted Ω_ϵ) such that

$$\Omega_\epsilon \to (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}) \ \text{ weak* in } L^\infty([0;T];H^s),$$

$$\partial_t\Omega_\epsilon \to \partial_t(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}) \ \text{ weak* in } L^\infty([0;T];H^{s-2}),$$

and, by the Lions-Aubin compacity lemma,

$$\Omega_\epsilon \to (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}) \quad \text{in } C([0;T], H^{s-\delta}_{loc}),\ \delta > 0. \tag{42}$$

By, (19), (20), (22) and (23), $(\alpha_t^{(\epsilon)}, \beta_t^{(\epsilon)}, \gamma_t^{(\epsilon)}, \delta_t^{(\epsilon)})$ converges to some limit in $C([0;T], H^{s-2-\delta}_{loc})$, therefore

$$(\alpha_t^{(\epsilon)}, \beta_t^{(\epsilon)}, \gamma_t^{(\epsilon)}, \delta_t^{(\epsilon)}) \to (\hat{\alpha}_t, \hat{\beta}_t, \hat{\gamma}_t, \hat{\delta}_t) \quad \text{in } C([0;T], H^{s-2-\delta}_{loc}).$$

Hence,

$$\begin{cases} \hat{\alpha}_t + \frac{1}{2}\hat{\beta}(a^2+f^2+c)(\hat{\alpha}^2+\hat{\beta}^2) + \hat{\beta}_{xx} = 0 \\ \hat{\beta}_t - \frac{1}{2}\hat{\alpha}(a^2+f^2+c)(\hat{\alpha}^2+\hat{\beta}^2) - \hat{\alpha}_{xx} = 0 \\ \hat{\gamma}_t + \frac{1}{2}(a^2+f^2+c)(\hat{\alpha}^2+\hat{\beta}^2)\hat{\delta} + (a^2+f^2+c)(\alpha\gamma+\beta\delta)\beta + \hat{\delta}_{xx} = 0 \\ \hat{\delta}_t - \frac{1}{2}(a^2+f^2+c)(\hat{\alpha}^2+\hat{\beta}^2)\hat{\gamma} - (a^2+f^2+c)(\hat{\alpha}\hat{\gamma}+\hat{\beta}\hat{\delta})\hat{\alpha} - \hat{\gamma}_{xx} = 0. \end{cases}$$

This system possesses a unique solution in the functional space considered (this can be obtained by a simple energy estimate), yet the limit (42) takes place without the need of taking a subsequence. Finally, by the Sobolev imbedding

$$\|f\|_{C^k} \hookrightarrow C\|f\|_{H^s}$$

provided that $s > k + \frac{1}{2}$, the convergence $(\alpha^{(\epsilon)}, \beta^{(\epsilon)}, \gamma^{(\epsilon)}, \delta^{(\epsilon)}) \to (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ takes place in $C^{1-j}([0;T], C^{2j})$.

The end of the proof of Theorem 1.2 is straightforward, one only has to consider Proposition 4.1 with respect to the variables (B, ϕ, ψ). ■

5 Appendix

We show in this Appendix that for $\tilde{\omega} < 0$, $\beta - v^2 > 0$ and $\omega > 0$,the Zakharov-Rubenchik system (2) can be put in the form

$$\begin{cases} iB_t + B_{xx} + a\phi B + ef\psi B + c|B|^2 B = 0 & \text{(a)} \\ \epsilon^2\phi_{tt} + d\epsilon\phi_{tx} - \phi_{xx} = -a|B|^2_{xx} & \text{(b)} \\ \epsilon\psi_t + e\psi_x = f|B|^2_x & \text{(c),} \end{cases} \tag{43}$$

where a, c, d, e, f are real constants.
Note that for $\tilde{\omega} < 0$, the group speed $v = 2\tilde{\omega}^3 k^{-1}(k^2 + \tilde{\omega}^2)^{-1}$ is negative: this correponds to backward propagation.

Differentiating $(2-b, c)$ in respect to T,

$$\epsilon^2 \rho_{TT} - 2v\epsilon\rho_{TX} + (v^2 - \beta)\rho_{XX} = -k\epsilon|B|^2_{XT} + \frac{1}{2}kv|B|^2_{XX} \tag{44}$$

and

$$\epsilon^2 u_{TT} - 2v\epsilon u_{TX} + (v^2 - \beta)u_{XX} = \frac{1}{2}\epsilon kv|B|^2_{XT} + k(\beta - \frac{v^2}{2})|B|^2_{XX}. \tag{45}$$

From equation (44) and (45), setting

$$\widetilde{\phi} = u + \frac{v}{2}\rho,$$

$$\epsilon^2 \widetilde{\phi}_{TT} - 2v\epsilon\widetilde{\phi}_{TX} + (v^2 - \beta)\widetilde{\phi}_{XX} = k(\beta - \frac{v^2}{4})|B|^2_{XX}. \tag{46}$$

Also, setting $\widetilde{\psi} = \sqrt{\beta}\rho + u$, equations (2-b),(2-c) yields

$$\epsilon\widetilde{\psi}_T + (\sqrt{\beta} - v)\widetilde{\psi}_X = -k(\sqrt{\beta} - \frac{v}{2})|B|^2_X. \tag{47}$$

Finally, since $u = \dfrac{1}{\sqrt{\beta} - \frac{v}{2}}(\sqrt{\beta}\widetilde{\phi} - \dfrac{v}{2}\widetilde{\psi})$ and $\rho = \dfrac{1}{\sqrt{\beta} - \frac{v}{2}}(-\widetilde{\phi} + \widetilde{\psi})$,
(note that $v = 2\widetilde{\omega}^3 k^{-1}(k^2 + \widetilde{\omega}^2)^{-1} 0$ hence $\sqrt{\beta} - \frac{v}{2} \neq 0$) we get the equivalent system:

$$\begin{cases} iB_T + \omega B_{XX} - \dfrac{k}{\beta^{\frac{1}{2}} - \frac{v}{2}}\left((\sqrt{\beta} + \dfrac{v}{2})\widetilde{\phi} - v\widetilde{\psi} + q|B|^2\right)B = 0 & \text{(a)} \\ \epsilon^2 \widetilde{\phi}_{TT} - 2v\epsilon\widetilde{\phi}_{TX} + (v^2 - \beta)\widetilde{\phi}_{XX} = k(\beta - \dfrac{v^2}{4})|B|^2_{XX} & \text{(b)} \\ \epsilon\widetilde{\psi}_T + (\sqrt{\beta} - v)\widetilde{\psi}_X = -k(\sqrt{\beta} - \dfrac{v}{2})|B|^2_X & \text{(c).} \end{cases} \tag{48}$$

Now, rescaling the space and time variables: $T = \dfrac{\omega t}{\beta - v^2}$, $X = \dfrac{x}{\sqrt{\beta - v^2}}$, (48) yields

$$\begin{cases} i\dfrac{\omega}{\beta - v^2}B_t + \dfrac{\omega}{\beta - v^2}B_{xx} - \dfrac{k}{\sqrt{\beta} - \frac{v}{2}}\left((\sqrt{\beta} + \dfrac{v}{2})\widetilde{\phi} - v\widetilde{\psi} + q|B|^2\right)B = 0 \\ (\dfrac{\omega\epsilon}{\beta - v^2})^2\widetilde{\phi}_{tt} - 2\dfrac{v}{\sqrt{\beta - v^2}}\dfrac{\omega\epsilon}{\beta - v^2}\widetilde{\phi}_{tx} - \widetilde{\phi}_{xx} = k\dfrac{\beta - \frac{v^2}{4}}{\beta - v^2}|B|^2_{xx} \\ \dfrac{\omega\epsilon}{\beta - v^2}\widetilde{\psi}_t + \dfrac{\sqrt{\beta} - v}{\sqrt{\beta - v^2}}\widetilde{\psi}_x = -k\dfrac{\sqrt{\beta} - \frac{v}{2}}{\sqrt{\beta - v^2}}|B|^2_x. \end{cases} \tag{49}$$

Setting $\alpha := \dfrac{\omega}{\beta - v^2}$ and $\epsilon' := \alpha\epsilon$,

$$\begin{cases} iB_t + B_{xx} - \dfrac{k}{\alpha(\sqrt{\beta}-\frac{v}{2})}\left((\sqrt{\beta}+\dfrac{v}{2})\widetilde{\phi} - v\widetilde{\psi} + q|B|^2\right)B = 0 \\ \\ {\epsilon'}^2\widetilde{\phi}_{tt} - 2\dfrac{v}{\sqrt{\beta - v^2}}\epsilon'\widetilde{\phi}_{tx} - \widetilde{\phi}_{xx} = k\dfrac{\beta - \frac{v^2}{4}}{\beta - v^2}|B|^2_{xx} \\ \\ \epsilon'\widetilde{\psi}_t + \dfrac{\sqrt{\beta}-v}{\sqrt{\beta - v^2}}\widetilde{\psi}_x = -k\dfrac{\sqrt{\beta}-\frac{v}{2}}{\sqrt{\beta - v^2}}|B|^2_x. \end{cases} \tag{50}$$

Finally, setting $\phi := \dfrac{1}{A_1}\widetilde{\phi}$ and $\psi := \dfrac{1}{A_2}\widetilde{\psi}$,

$$\begin{cases} iB_t + B_{xx} - \dfrac{k}{\alpha(\sqrt{\beta}-\frac{v}{2})}\left((\sqrt{\beta}+\dfrac{v}{2})A_1\phi - vA_2\psi + q|B|^2\right)B = 0 \\ \\ {\epsilon'}^2\phi_{tt} - 2\dfrac{v}{\sqrt{\beta - v^2}}\epsilon'\phi_{tx} - \phi_{xx} = k\dfrac{\beta - \frac{v^2}{4}}{A_1(\beta - v^2)}|B|^2_{xx} \\ \\ \epsilon'\psi_t + \dfrac{\sqrt{\beta}-v}{\sqrt{\beta - v^2}}\psi_x = -k\dfrac{\sqrt{\beta}-\frac{v}{2}}{A_2\sqrt{\beta - v^2}}|B|^2_x. \end{cases} \tag{51}$$

Choosing $A_1 = \sqrt{\omega}\dfrac{\sqrt{\beta}-\frac{v}{2}}{\beta - v^2}$ and $A_2 = \dfrac{\sqrt{\beta}-\frac{v}{2}}{\beta - v^2}\sqrt{\dfrac{\omega(\sqrt{\beta}-v)}{-v}}$,
we get the announced result, with

$$a = -\frac{k}{\sqrt{\omega}}(\sqrt{\beta}+\frac{v}{2}),\ \ c = \frac{-qk}{\alpha(\sqrt{\beta}-\frac{v}{2})},\ \ d = -2\frac{v}{\sqrt{\beta - v^2}},$$

$$e = \frac{\sqrt{\beta}-v}{\sqrt{\beta - v^2}} \ \text{ and } f = -k\sqrt{\frac{-v(\sqrt{\beta}+v)}{\omega}}. \qquad \blacksquare$$

It now becomes clear that Theorem 1.2 implies Theorem 1.1: for a given (B_o, ρ_o, u_o), the conclusions in Theorem 1.1 are obtained by setting

$$\phi_o(x) := \frac{1}{A_1}(u_o(X) + \frac{v}{2}\rho_o(X)) \quad \text{and} \quad \psi_o(x) := \frac{1}{A_2}(\sqrt{\beta}\rho_o(X) + u_o(X))$$

in Theorem 1.2.For the third initial data $\phi_t(0,x) = \widehat{\phi}_o(x) = \phi_t(0,x)$, we choose

$$\widehat{\phi}_o := \frac{1}{\epsilon}\left(\frac{1}{A_1}([\frac{k}{2}v|B_o|^2 - (\beta\rho_o - vu_o)] + \frac{v}{2}[-k|B_o|^2 - (u_o - v\rho_o)]\right)_X = 0$$

(i.e. $\theta := 0$) in view of (6).

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