Large deviations for analytical distributions on infinite dimensional spaces

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Abstract

Let θ be a Young function and N' the dual space of a complex nuclear Fréchet space N. In this paper we generalize Cramer's and Schilder's theorems to white noise measures in the dual space of the test space of entire functions on N' of θ-exponential growth.

Keywords: Cramer's theorem, Schilder's theorem , White noise distributions, Large deviations principle.

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1 Introduction

Among the principal results of large deviations, we cite Cramer's theorem which is proved in [4] for measures μ on $\mathbb{R}$ which are not singular to Lebesgue's measure. We cite also Schilder's theorem [16] which is the first example of a large deviations result for measures on a functions space, see also [5] and references therein. In this paper we study large deviations properties for white noise distributions. For white noise analysis theory see for example [8], [9], [11], [12], [13] and references therein.

Let $X' = \cup_{n=0}^{\infty} X_{-p}$ be the dual space of a real nuclear Fréchet space X. Consider μ a probability measure on X' such that μ is supported by some X_{-p}, $p \in \mathbb{N}^*$ and satisfies the following integrability condition: there exists $m > 0$ such that

$$\int_{X_{-p}} \exp(\theta(m|y|_{-p}))d\mu(y) < \infty, \tag{1}$$

where θ is a Young function (see [10]). First, for $n \geq 1$, let $X_i : \Omega \to X'$; $i \in \{1, 2, ..., n\}$ be a sequence of random variables on the probability space $(\Omega, B(\Omega), \mathcal{P})$. Next, let μ_n be

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the distribution of $\overline{S_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$, where $(X_1, X_2, ..., X_n)$ are n independent identically distributed (i.i.d) random variables with distribution μ. The main result of this paper is to prove Cramer's theorem for the measure μ, i.e., for all measurable subsets Γ in X',

$$-\inf_{\Gamma^\circ} \Lambda_\mu^* \leq \liminf_{n\to\infty} \frac{1}{n}\log(\mu_n(\Gamma)) \leq \limsup_{n\to\infty} \frac{1}{n}\log(\mu_n(\Gamma)) \leq -\inf_{\overline{\Gamma}} \Lambda_\mu^*, \tag{2}$$

where Λ_μ^* is the Legendre transform of the logarithmic moment generating function Λ_μ, i.e.,

$$\Lambda_\mu(\xi) = \log\left(\int_{X'} e^{\langle y,\xi\rangle} d\mu(y)\right), \forall\ \xi \in X, \tag{3}$$

and

$$\Lambda_\mu^*(\varphi) = \sup_{\xi\in X}\{\langle \xi, \varphi\rangle - \Lambda_\mu(\xi)\}, \forall\ \varphi \in X', \tag{4}$$

see Theorem 3.8, Section 3.

In Section 4, we consider the family $\{\mu_\varepsilon, \varepsilon > 0\}$, where μ_ε is the image measure of μ by the map g:

$$\begin{aligned} g : X' &\longrightarrow X' \\ \xi &\longmapsto \sqrt{\varepsilon}\xi. \end{aligned}$$

If the Laplace transform $\widehat{\mu}$ of μ satisfies the growth condition:

$$\exists p,\ m > 0;\ \widehat{\mu}(\xi) = e^{\theta^*(m|\xi|_p)},\ \xi \in X, \tag{5}$$

where θ^* is the Legendre transform of a Young function θ, we prove the upper bound condition for the family of measures $\{\mu_\varepsilon, \varepsilon > 0\}$, i.e., for all measurable subsets Γ in X',

$$\liminf_{\varepsilon\to 0} \varepsilon \log(\mu_\varepsilon(\Gamma)) \leq \limsup_{\varepsilon\to 0} \varepsilon \log(\mu_\varepsilon(\Gamma)) \leq -\inf_{\overline{\Gamma}}(\Lambda^*), \tag{6}$$

where Λ^* is the Legendre transform of $\Lambda(\xi) = \lim_{\varepsilon\to 0} \varepsilon \log\left(\int_{X'} e^{\langle y, \frac{\xi}{\varepsilon}\rangle} d\mu_\varepsilon(y)\right)$.

In Section 5, we apply the results of Theorems 3.8 and 4.3 to Gaussian and Poisson measures on particular spaces.

2 Notation and preliminaries

Let N be a complex nuclear Fréchet space, whose topology is defined by a family $\{|\cdot|_p, p \in \mathbb{N}\}$ of increasing hilbertian norms. We have the representation

$$N = \bigcap_{p\geq 0} N_p = \text{proj}\lim_{p\to\infty} N_p$$

where N_p is the completion of N with respect to the norm $|\cdot|_p$. Denote by N_{-p} the topological dual space of the space N_p. Then the dual N' of N can be written as

$$N' = \bigcup_{p\geq 0} N_{-p} = \text{ind}\lim_{p\to\infty} N_{-p}.$$

Let now $\theta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a Young function, i.e., θ is continuous, convex, strictly increasing and verifies $\theta(0) = 0$ and $\lim_{x\to\infty} \frac{\theta(x)}{x} = +\infty$. Denote by θ^* the Legendre transform of θ : $\theta^*(x) = \sup\{tx - \theta(t); t > 0\}$ for all $x \geq 0$, which is also a Young function. Given a complex Banach space $(B, \|.\|)$, let $H(B)$ be the space of entire functions on B, i.e., the space of continuous functions from B to $\mathbb{C}$, whose restriction to all affine lines of B are entire on $\mathbb{C}$. Let $Exp(B, \theta, m)$ denote the space of all entire functions on B with exponential growth of order θ and of finite type $m > 0$:

$$Exp(B, \theta, m) = \left\{ f \in H(B);\ \|f\|_{\theta,m} = \sup_{x\in B} |f(x)|e^{-\theta(m\|x\|)} < +\infty \right\}.$$

Let also $\|f\|_{\theta,m,p} = \sup_{x\in N_p} |f(x)|e^{-\theta(m|x|_p)}$ for $f \in Exp(N_p, \theta, m)$. The intersection

$$\mathcal{F}_\theta(N') = \bigcap_{p\geq 0, m>0} Exp(N_{-p}, \theta, m),$$

equipped with the projective limit topology is called the space of entire functions on N' of θ-growth and minimal type. The union

$$\mathcal{G}_\theta(N) = \bigcup_{p\geq 0, m>0} Exp(N_p, \theta, m),$$

equipped with the inductive limit topology, is called the space of entire functions on N of θ-growth and (arbitrarily) finite type. Denote by $\mathcal{F}_\theta(N')^*$ the strong dual of the test functions space $\mathcal{F}_\theta(N')$. From the condition $\lim_{x\to\infty} \frac{\theta(x)}{x} = +\infty$, the exponential function defined as

$$\begin{aligned} e^\xi : N' &\longrightarrow \mathbb{C} \\ z &\longmapsto e^\xi(z) = e^{\langle z,\xi\rangle}, \end{aligned}$$

$\xi \in N$, belongs to $\mathcal{F}_\theta(N')$. For every $\phi \in \mathcal{F}_\theta(N')^*$ the Laplace transform of ϕ is defined by

$$\widehat{\phi}(\xi) = \mathcal{L}(\phi)(\xi) = \phi(e^\xi), \quad \xi \in N.$$

Theorem 2.1 *([6], Theorem 1). The Laplace transform of analytical functionals induces a topological isomorphism*

$$\mathcal{L} : \mathcal{F}_\theta(N')^* \longrightarrow \mathcal{G}_{\theta^*}(N). \tag{7}$$

As a consequence, $\phi \in \mathcal{F}_\theta(N')^*$ if and only if the Laplace transform of ϕ satisfies the growth condition

$$|\widehat{\phi}(\xi)| \leq C \exp(\theta^*(m|\xi|_p)), \quad \xi \in N, \tag{8}$$

for some $m > 0$ and $p \in \mathbb{N}^*$.

In the sequel we take $N = X + iX$, the complexification of a nuclear Fréchet space X. We denote by $\mathcal{F}_\theta(N')_+$ the cone of positive test functions, i.e., $f \in \mathcal{F}_\theta(N')_+$ if $f(y + i0) \geq 0$ for all y in the topological dual X' of X.

Definition 2.2 *The space* $\mathcal{F}_\theta(N')^*_+$ *of positive distributions is defined as the space of* $\phi \in \mathcal{F}_\theta(N')^*$ *such that* $\langle\langle \phi, f\rangle\rangle \geq 0$*; for all* $f \in \mathcal{F}_\theta(N')_+$.

We recall the following results on the representation of positive distributions:

Theorem 2.3 *[15] Let* $\phi \in \mathcal{F}_\theta(N')^*_+$. *There exists a unique Radon measure* μ_ϕ *on* X' *such that*

$$\langle\langle \phi, f\rangle\rangle = \phi(f) = \int_{X'} f(y+i0)d\mu_\phi(y) \ ; \ f \in \mathcal{F}_\theta(N').$$

Conversely, let μ *be a finite, positive Borel measure on* X'. *Then* μ *represents a positive distribution in* $\mathcal{F}_\theta(N')^*_+$ *if and only if* μ *is supported by some* X_{-p}, $p \in \mathbb{N}^*$ *and there exists* $m > 0$ *such that*

$$\int_{X_{-p}} \exp(\theta(m|y|_{-p}))d\mu(y) < \infty. \tag{9}$$

We recall from paper [14] (Theorem 2.1) the following tail estimates. Given $\xi \in X$ and $x \in \mathbb{R}$, let

$$A_{x,\xi} = \big\{y \in X' : \langle y, \xi\rangle > x\big\}$$

denote the half-plane in X' associated to ξ and x, then we obtain the tail estimate:

Theorem 2.4 *Let* $\phi \in \mathcal{F}_\theta(N')^*_+$ *defines a (positive) Radon measure* $\mu = \mu_\phi$ *on* X'. *For all* $\xi \in X$ *and* $x > 0$, *there exists* $m > 0$ *and* $p \in \mathbb{N}$ *such that*

$$\mu(A_{x,\xi}) \leq \|\hat{\phi}\|_{\theta,m,p} \exp\bigg(-\theta(\frac{x}{m|\xi|_p})\bigg). \tag{10}$$

3 The large deviations estimates for a sequence of measures

In the next, we consider $\phi \in \mathcal{F}_\theta(N')^*_+$ such that ϕ defines a (positive) Radon measure $\mu = \mu_\phi$ on X' (see Theorem 2.3). For $n \geq 1$, denote by μ_n the distributions on X' of $\overline{S_n} = \frac{1}{n}\sum_{i=1}^n X_i$, where $(X_1, X_2, ..., X_n)$ are n independent identically distributed (i.i.d) random variables with distribution μ. It is easy to see that, for all $n \geq 1$, $\xi \in X$

$$\widehat{\mu_n}(\xi) = \bigg(\widehat{\mu}(\frac{\xi}{n})\bigg)^n. \tag{11}$$

Proposition 3.1 *Let* $\mathcal{E}$ *be the collection of all non-empty, convex open sets* A *in* X'. *Then*

$$L_\mu(A) = -\lim_{n\mapsto\infty} \frac{1}{n}\log(\mu_n(A)) \in [0, \infty] \tag{12}$$

exists for every $A \in \mathcal{E}$.

The proof of Proposition 3.1 follows by combining the two following Lemmas.

Lemma 3.2 *For each convex subset $C \in B(X')$, where $B(X')$ denotes the Borel σ- filed of X'; the map $n \in \mathbb{N} \mapsto \mu_n(C)$ is super-multiplicative. In addition, if A is an open convex subset of X', then either $\mu_n(A) = 0$, for all $n \in \mathbb{N}$ or there exists an $n_0 \in \mathbb{N}$ such that $\mu_n(A) > 0$, for all $n \geq n_0$.*

PROOF:
Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d random variables on the probability space $(\Omega, B(\Omega), \mathbb{P})$ with distribution μ. For $n \geq 1$, we denote by μ_n the distribution of $\overline{S_n} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}S_n$. For all $n > m$, let $\overline{S_n^m} = \frac{1}{n-m}\sum_{i=m+1}^{n} X_i = \frac{1}{n-m}S_n^m$.

To prove the first assertion, observe that, by convexity

$$\{\omega; \overline{S}_{n+m}(\omega) \in C\} \supseteq \{\omega; \overline{S_n}(\omega) \in C\} \cap \{\omega; \overline{S}^m_{n+m}(\omega) \in C\}$$

therefore

$$\mu_{n+m}(C) \geq \mu_n(C)\mu_m(C).$$

We next turn to the second assertion. Let A be an open convex subset of X', such that $\mu_m(A) > 0$ for some $m \in \mathbb{N}$. For each $p \in \mathbb{N}$, let $A_p = A \cap X_{-p}$. So, there exists $q \in \mathbb{N}$, such that $\mu_m(A_q) > 0$. Since A_q is convex open subset of the Hilbert space $(X_{-q}, |\cdot|_{-q})$, we choose a convex compact subset K_q of A_q, such that $\mu_m(K_q) > 0$. Consider ρ_q such that $0 < 2\rho_q < \inf\{|x-y|_{-q}\ ;\ x \in K_q \text{ and } y \in A_q^c\}$. Take $G_q = \{\varphi \in X_{-q}; |\varphi - K_q|_{-q} < \rho_q\}$ and let $M_q = \sup_{\varphi\in K_q}\{|\varphi|_{-q}\}$. Then, for $n = sm + r$, where $0 \leq r < m$,

$$\begin{aligned}\mu_n(A_q) &\geq \mathbb{P}\Big(\{\omega; \tfrac{1}{n}S_{sm}(\omega) \in G_q \textit{ and } \tfrac{1}{n}|S_n^{sm}(\omega)|_{-q} < \rho_q\}\Big)\\ &\geq \mu_{sm}(K_q)\mathbb{P}\Big(\{\omega; |S_r(\omega)|_q < n\rho_q\}\Big),\end{aligned}$$

as long as $mM_q < n\rho_q$. Thus, if we choose n_0 so that $mM_q < n_0\rho_q$ and

$$\min_{1\leq r<m} \mathbb{P}\Big(\{\omega; |S_r(\omega)|_{-q} < n_0\rho_q\}\Big) \geq \frac{1}{2},$$

we obtain by the convexity of K_q

$$\mu_n(A_q) \geq \frac{1}{2}\Big(\mu_m(K_q)\Big)^{[\frac{n}{m}]} > 0,\ \textit{for all } n \geq n_0,$$

which implies

$$\mu_n(A) = \mu_n(\cup_{p\geq 0}A_p) = \lim_{p\mapsto\infty}\mu_n(A_p).$$

Therefore

$$\mu_n(A) > 0,\ \textit{for all } n \geq n_0.$$

■

Lemma 3.3 *Let $f : \mathbb{N} \longrightarrow [0, +\infty]$ be a sub-additive function and assume that there is an $n_0 \in \mathbb{N}$ such that $f(n) < \infty$ for all $n \geq n_0$. Then*

$$\lim_{n \mapsto \infty} \frac{f(n)}{n} = \inf_{n \geq n_0} \frac{f(n)}{n} \in [0, +\infty].$$

Definition 3.4 *For $\varphi \in X'$, $r > 0$, put*

$$I_\mu(\varphi) := \sup\{L_\mu(A) : \varphi \in A \in \mathcal{E}\}. \tag{13}$$

A function $I : X' \to [0, \infty]$ is said to be a rate function *if it is lower semi-continuous, and we will say that I is a* good rate function *if the set $\{q \in X' : I(q) \leq L\}$ is a compact subset of X' for all $L \geq 0$.*

Now we can give the following result:

Theorem 3.5 *The function I_μ define in (13) is a convex rate function on X' and $\{\mu_n; n \geq 1\}$ satisfies the weak large deviation principle with rate function I_μ, i.e.,*

$$\liminf_{n \to \infty} \frac{1}{n} \log(\mu_n(G)) \geq -\inf_G I_\mu, \ \textit{for all open } G \textit{ in } X', \tag{14}$$

and

$$\limsup_{n \to \infty} \frac{1}{n} \log(\mu_n(K)) \leq -\inf_K I_\mu, \ \textit{for all compact } K \textit{ in } X'. \tag{15}$$

Proof:
By definition I_μ is lower semi-continuous so I_μ is a rate function. Since I_μ is lower semi-continuous, to prove the convexity of I_μ it's sufficient to prove that for a given $q_1, q_2 \in X'$ and $q = \frac{1}{2}(q_1 + q_2)$ we have

$$I_\mu(q) \leq \frac{1}{2}\big(I_\mu(q_1) + I_\mu(q_2)\big). \tag{16}$$

Given an $A \in \mathcal{E}$ containing q, choose $A_i \in \mathcal{E}$, $i \in \{1, 2\}$ such that $q_i \in A_i$ and $\frac{1}{2}(A_1 + A_2) \subseteq A$. Then

$$\begin{aligned}
L_\mu(A) &= -\lim_{n \mapsto \infty} \tfrac{1}{2n} \log(\mu_{2n}(A)) \\
&\leq -\lim_{n \mapsto \infty} \tfrac{1}{2n} \log(\mathbb{P}(\{\omega; \overline{S_n}(\omega) \in A_1 \textit{ and } \overline{S_{2n}^n}(\omega) \in A_2\}) \\
&\leq \tfrac{1}{2}\big(-\lim_{n \mapsto \infty} \tfrac{1}{n} \log(\mu_n(A_1)) - \lim_{n \mapsto \infty} \tfrac{1}{n} \log(\mu_n(A_2))\big) \\
&\leq \tfrac{1}{2}\big(I_\mu(q_1) + I_\mu(q_2)\big)
\end{aligned}$$

and from this we conclude that $I_\mu(q) \leq \frac{1}{2}\big(I_\mu(q_1) + I_\mu(q_2)\big)$.

The inequality (14) is built into the definition of I_μ. Next, suppose that K is a compact subset of X' and let $\delta = \inf_K I_\mu$. Then, there is a finite cover $\{A_1, A_2, ..., A_{n_0}\} \subset \mathcal{E}$ of K such that $L_\mu(A_n) > \delta$ for each $1 \leq n \leq n_0$. Hence,

$$\limsup_{n\to\infty} \frac{1}{n}\log(\mu_n(K)) \quad \leq \quad \limsup_{n\to\infty} \frac{1}{n}\log(\max_{1\leq j\leq n_0} \mu_n(A_j)) \leq -\delta.$$

This proves the inequality (15) and therefore the weak large deviation principle holds. ■

Lemma 3.6 *For all $n \in \mathbb{N}$, there exists $m > 0$ and $p \in \mathbb{N}$ such that*

$$\mu_n(A_{x,\xi}) \leq C^n \exp\big(- n\theta(\frac{x}{m|\xi|_p})\big). \tag{17}$$

PROOF:
Using Theorem 2.1 and the condition (11), there exists $m > 0$ and $p \in \mathbb{N}$ such that for any $n \in \mathbb{N}$

$$|\widehat{\mu_n}(\xi)| \leq C^n \exp(n\theta^*(m|\frac{\xi}{n}|_p)), \quad \xi \in X.$$

For all $t \geq 0$ we have

$$\begin{aligned} \exp(tx)\mu_n(A_{x,\xi}) &= \int_{X'} \exp(tx) 1_{A_{x,\xi}} d\mu_n(y) \leq \int_{X'} \exp(t\langle y, \xi\rangle d\mu_n(y) \\ &\leq |\widehat{\mu_n}(t\xi)| \leq C^n \exp(n\theta^*(m|\tfrac{t\xi}{n}|_p)), \end{aligned}$$

hence

$$\mu_n(A_{x,\xi}) \leq C^n \exp(n\theta^*(m|\frac{t\xi}{n}|_p) - tx), \quad t \geq 0.$$

Minimizing in $t \geq 0$ we get

$$\mu_n(A_{x,\xi}) \leq C^n \exp(-n \sup_{t\geq 0}(\frac{t}{n}x - \theta^*(m\frac{t}{n}|\xi|_p)).$$

But by definition

$$\sup_{t\geq 0}(\frac{t}{n}x - \theta^*(m\frac{t}{n}|\xi|_p) = \theta^{**}(\frac{x}{m|\xi|_p}).$$

Since $\theta^{**} = \theta$, this implies

$$\mu_n(A_{x,\xi}) \leq C^n \exp\left(- n\theta(\frac{x}{m|\xi|_p})\right).$$

■

Lemma 3.7 *For each $L > 0$, $\xi \in X$, we define*

$$L_1(\xi) = \begin{cases} m|\xi|_p \theta^{-1}(L + \log(C)), & if\ \log(C) > 0, \\ m|\xi|_p \theta^{-1}(L), & if\ \log(C) < 0, \end{cases}$$

where $C = \|\hat{\phi}\|_{\theta,m,p}$. Then $K_L = \{y \in X'; |\langle y, \xi\rangle| \leq L_1(\xi),\ \forall \xi \in X\}$ is a compact of X', and we have

$$\lim_{n \mapsto \infty} \frac{1}{n} \log(\mu_n(K_L^C)) \leq -L \tag{18}$$

where K_L^C is the complementary of K_L in X'.

PROOF:
The proof of this Lemma is similar to the proof of the Lemma 3.6. In fact, first we can prove the inequality

$$\mu_n(K_L^C) \leq 2C^n \exp(-n\theta(\frac{L_1(\xi)}{m|\xi|_p}))$$

which implies the inequality (18). ■

Theorem 3.8 *Let $\phi \in \mathcal{F}_\theta(N')_+^*$ and let μ be the associated measure. Then I_μ defined by (13) is a good rate function and the family $\{\mu_n, n \geq 1\}$ satisfies the full large deviation principle with rate function I_μ, i.e.,*

$$-\inf_{\Gamma^\circ} \Lambda_\mu^* \leq \liminf_{n\to\infty} \frac{1}{n} \log(\mu_n(\Gamma)) \leq \limsup_{n\to\infty} \frac{1}{n} \log(\mu_n(\Gamma)) \leq -\inf_{\overline{\Gamma}} \Lambda_\mu^*. \tag{19}$$

Moreover the logarithmic moment generating function Λ_μ defined by (3) is the Legendre transform of I_μ, i.e.,

$$\Lambda_\mu(\xi) = \sup_{y \in X'} \{\langle y, \xi\rangle - I_\mu(y)\},\ \xi \in X$$

and

$$I_\mu(\varphi) = \sup_{\xi \in X} \{\langle \varphi, \xi\rangle - \Lambda_\mu(\xi)\},\ \varphi \in X'.$$

PROOF:
First note that

$$\inf_{K_L^C} I_\mu \geq -\liminf_{n\to\infty} \frac{1}{n} \log(\mu_n(K_L^C)) \geq L;$$

and so $\{q : I_\mu(q) \leq L\} \subseteq K_{L+1}$. Since I_μ is lower semi-continuous, this proves that I_μ is a good rate function.

To prove that the family $\{\mu_n, n \geq 1\}$ satisfies the full large deviation principle with rate function I_μ, it is sufficient to prove

1. (Upper Bound) for all closed subsets F of X'

$$\limsup_{n\to\infty} \frac{1}{n}\log(\mu_n(F)) \leq -\inf_{y\in F} I_\mu(y) \tag{20}$$

2. (Lower Bound) for all open sets G of X'

$$\liminf_{n\to\infty} \frac{1}{n}\log(\mu_n(G)) \geq -\inf_{y\in G} I_\mu(y) \tag{21}$$

Let F be a closed subset of X', let $\ell = \inf_F I_\mu$, and for $L > 0$, set $F_L = F\bigcap K_L$, where K_L is the compact set produced in the Lemma 3.7. Then

$$\mu_n(F) \leq \mu_n(F_L) + \mu_n(K_L^C)$$

and so by the inequality (15) in Theorem 3.5 and Lemma 3.7, we have

$$\limsup_{n\to\infty} \frac{1}{n}\log(\mu_n(F)) \leq -min(\ell, L).$$

After letting $L \longrightarrow \infty$, we obtain the inequality (20). Finally, the inequality (21) follows from Theorem 3.5. ■

4 The large deviations estimates for a family of measures

Let μ be a measure such that there exist $p, m > 0$; $\widehat{\mu}(\xi) = e^{\theta^*(m|\xi|_p)}$. So the logarithmic moment generating function is given by

$$\Lambda_\mu(\xi) := \log(\widehat{\mu}(\xi)) = \theta^*(m|\xi|_p), \forall\ \xi \in X,$$

and let Λ^*_μ be the Legendre transform of Λ_μ:

$$\Lambda^*_\mu(\varphi) = \sup_{\xi\in X}\{\langle \xi, \varphi\rangle - \Lambda_\mu(\xi)\}, \forall\ \varphi \in X'.$$

We denote by μ_ε the distribution of $\xi \mapsto \sqrt{\varepsilon}\xi$ under μ. The logarithmic moment generating function for the measure μ_ε is given by

$$\Lambda_{\mu_\varepsilon}(\xi) := \log(\widehat{\mu_\varepsilon}(\xi)) = \theta^*(m\sqrt{\varepsilon}|\xi|_p), \forall\ \xi \in X.$$

Put

$$\Lambda(\xi) = \lim_{\varepsilon\to 0} \varepsilon \Lambda_{\mu_\varepsilon}(\frac{\xi}{\varepsilon}). \tag{22}$$

From now we suppose the additional condition:

$$\lim_{x\to+\infty} \frac{\theta^*(x)}{x^2} < \infty \tag{23}$$

which implies that for all $\xi \in X$, $\Lambda(\xi) < \infty$.

For a given $\xi \in X'$, let $B_q(\xi, r) = \{y \in X'; |\xi - y|_{-q} < r\}$.

Lemma 4.1 *Let $\xi \in X'$ and $q \in \mathbb{N}$ be given. Then for each $\delta > 0$ there exists an $r > 0$ such that*

$$\mu_\varepsilon(\overline{B}_q(\xi, r)) \leq \exp(-\frac{\Lambda^*(\xi) - \delta}{\varepsilon})$$

for all $\varepsilon > 0$. In particular, for all compact $K \subset X'$, we have

$$\limsup_{\varepsilon \mapsto 0} \varepsilon log(\mu_\varepsilon(K) \leq -\inf_K \Lambda^*. \tag{24}$$

PROOF:

$$\begin{aligned}
\mu_\varepsilon(\overline{B}_q(\xi, r)) &= \mu(\overline{B}_q(\tfrac{\xi}{\sqrt{\varepsilon}}, \tfrac{r}{\sqrt{\varepsilon}})) \\
&\leq \sup\nolimits_{y \in \overline{B}_q(\frac{\xi}{\sqrt{\varepsilon}}, \frac{r}{\sqrt{\varepsilon}})} \exp(-\langle \tfrac{y}{\sqrt{\varepsilon}}, \varphi \rangle) \int_{\overline{B}_q(\frac{\xi}{\sqrt{\varepsilon}}, \frac{r}{\sqrt{\varepsilon}}))} \exp(\langle \frac{y}{\sqrt{\varepsilon}}, \varphi \rangle) d\mu(y), \forall\ \varphi \in X, \\
&\leq \exp\Big(-\tfrac{1}{\varepsilon}\Big(\langle \xi, \varphi \rangle - r|\varphi|_q - \varepsilon\theta^*(\tfrac{m|\varphi|_p}{\sqrt{\varepsilon}})\Big)\Big), \forall\ \varphi \in X, \\
&\leq \exp\Big(-\tfrac{1}{\varepsilon}\Big(\langle \xi, \varphi \rangle - r|\varphi|_q - \varepsilon\Lambda_{\mu_\varepsilon}(\tfrac{\varphi}{\varepsilon})\Big)\Big), \forall\ \varphi \in X.
\end{aligned}$$

Since $\Lambda(\varphi) = \lim_{\varepsilon \to 0} \varepsilon \Lambda_{\mu_\varepsilon}(\frac{\varphi}{\varepsilon})$, so for all $\delta > 0$, there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$, $\varepsilon\Lambda_{\mu_\varepsilon}(\frac{\varphi}{\varepsilon}) \leq \frac{\delta}{3} + \Lambda(\varphi)$.

We choose $\varphi \in X$, such that $\langle \xi, \varphi \rangle - \Lambda(\varphi) \geq \Lambda^*(\xi) - \frac{\delta}{3}$ and $r \leq \frac{\delta}{3(1+|\varphi|_q)}$, then we have

$$\mu_\varepsilon(\overline{B}_q(\xi, r)) \leq \exp(-\frac{\Lambda^*(\xi) - \delta}{\varepsilon}).$$

To prove the inequality (24) denote by $\ell = \inf_K \Lambda^*$. Since K is a compact of X', we choose $\xi_1, \xi_2, ..., \xi_n \in K$, $r_1, r_2, ..., r_n \in \mathbb{R}_+^*$ and $p_1, p_2, ..., p_n \in \mathbb{N}$ such that $K \subset \bigcup_{k=1}^n B_{p_k}(\xi_k, r_k)$ and

$$\log(\mu_\varepsilon(\overline{B}_{p_k}(\xi_k, r_k)) \leq -\big(\ell - \delta\big).$$

Then we have

$$\limsup_{\varepsilon \to 0} \varepsilon \log(\mu_\varepsilon(K)) \leq -\big(\ell - \delta\big).$$

Finally, considering $\delta \searrow 0$, we obtain

$$\limsup_{\varepsilon \to 0} \varepsilon \log(\mu_\varepsilon(K)) \leq -\inf_K \Lambda^*.$$

Lemma 4.2 *For each $L > 0$, $K_L = \big\{y \in X'; \theta(m|y|_{-p}) \leq L\big\}$ is a compact of X' and*

$$\limsup_{\varepsilon \mapsto 0} \varepsilon log(\mu_\varepsilon(K_L^C) \leq -L. \tag{25}$$

PROOF:
For all $\varepsilon > 0$

$$\begin{aligned}\mu_\varepsilon(K_L^C) &\leq \mu\{y \in X'; \theta(m|y|_{-p}) > \tfrac{L}{\varepsilon}\} \\ &\leq \exp(-\tfrac{L}{\varepsilon}) \int_{X_{-p}} \exp(\theta(m|y|_{-p}))d\mu(y).\end{aligned}$$

By the integrability condition (9) this surely leads to (25). ■

Lemmas 4.1 and 4.2 lead to the following results

Theorem 4.3 *Let F be a closed subset of X'. Then we have*

$$\limsup_{\varepsilon\to 0} \varepsilon \log(\mu_\varepsilon(F)) \leq -\inf_F(\Lambda^*). \tag{26}$$

Moreover for every measurable subsets Γ of X'

$$\liminf_{\varepsilon\to 0} \varepsilon \log(\mu_\varepsilon(\Gamma)) \leq \limsup_{\varepsilon\to 0} \varepsilon \log(\mu_\varepsilon(\Gamma)) \leq -\inf_{\overline{\Gamma}}(\Lambda^*).$$

5 Examples

In this Section, we apply the results of Theorems 3.8 and 4.3, respectively, to Gaussian and Poisson measures.

5.1 Gaussian measures

Let $X = S(\mathbb{R})$ be the Schwartz space of real-valued rapidly decreasing functions on $\mathbb{R}$ and X' the corresponding dual space, i.e., $X' = S'(\mathbb{R})$ the Schwartz distributions space. Then we have the Gel'fand triple

$$S(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}, dx) \hookrightarrow S'(\mathbb{R}).$$

Using Bochner-Minlos theorem [7], there exists a unique measure γ on $S'(\mathbb{R})$ such that

$$\int_{S'(\mathbb{R})} e^{i\langle y,\xi\rangle} d\gamma(y) = e^{-\frac{1}{2}\sigma^2|\xi|_0^2} = e^{-\frac{1}{2}\sigma^2(\xi,\xi)},\ \xi \in S(\mathbb{R}),$$

where $\langle y, \xi\rangle$ denote the dual paring between $S'(\mathbb{R})$ and $S(\mathbb{R})$ which coincide with the inner product (y, ξ) on $L^2(\mathbb{R}, dx)$ if $y \in L^2(\mathbb{R}, dx)$. Hence the Gaussian measure γ on $S'(\mathbb{R})$ with variance σ^2 belongs to $\mathcal{F}_\theta(S'_\mathbb{C}(\mathbb{R}))^*$, where $S'_\mathbb{C}(\mathbb{R}) = S'(\mathbb{R}) + iS'(\mathbb{R})$ and $\theta(t) = \frac{1}{2}\sigma^2 t^2$. Theorems (3.8) and (4.3) allow us to get the following estimates.

Proposition 5.1 *For a given $\xi \in X$ and $a \in \mathbb{R}_+$, let $A_{a,\xi} = \{y \in X'; a < \langle y, \xi\rangle\}$. Then there exists $p > 0$ such that:*

$$\liminf_{\varepsilon\to 0} \varepsilon \log(\gamma_\varepsilon(A_{a,\xi})) \leq \limsup_{\varepsilon\to 0} \varepsilon \log(\gamma_\varepsilon(A_{a,\xi})) \leq -\frac{1}{2\sigma^2|\xi|_p^2}a^2,$$

$$-\inf_{y\in A_{a,\xi}} \sup_{\lambda\in S(\mathbb{R})} \frac{1}{2}\left(\frac{\langle y, \lambda\rangle}{|\lambda|_0}\right)^2 \leq \liminf_{n\to\infty} \frac{1}{n}\log(\gamma_n(A_{a,\xi})) \leq \limsup_{n\to\infty} \frac{1}{n}\log(\gamma_n(A_{a,\xi})) \leq -\frac{1}{2\sigma^2|\xi|_p^2}a^2.$$

Remark 5.2 *1.* For the Gaussian measure, if we take $\epsilon = \frac{1}{n}$, $n > 0$, we recover the Schilder's theorem [16], i.e., for every measurable subset Γ of $S'(\mathbb{R})$, we have

$$-\inf_{\Gamma^0}(\Lambda_\gamma^*) \leq \liminf_{\epsilon\to 0} \epsilon \log(\gamma_\epsilon(\Gamma)) \leq \limsup_{\epsilon\to 0} \epsilon \log(\gamma_\epsilon(\Gamma)) \leq -\inf_{\overline{\Gamma}}(\Lambda_\gamma^*). \qquad (27)$$

The estimate given by (27) is proved in [2], see also [1].

2. In the particular case where γ_ϵ is the gaussian measure with mean 0 and variance ϵ on $\mathbb{R}$, the large deviation principle given in (27) becomes the following equality:

$$\lim_{\epsilon\to 0} \epsilon \log(\gamma_\epsilon(A_{\xi,a})) = -\frac{a^2}{2\xi^2}.$$

5.2 Poisson measure

5.2.1 Poisson measure on $\mathbb{R}$

The Poisson measure π_λ on $X = \mathbb{R}$, with intensity $\lambda > 0$, defined by

$$\widehat{\pi_\lambda}(t) = \int_{\mathbb{R}} e^{ty} d\pi_\lambda(y) = e^{-\theta(t)}, \; t \in \mathbb{R}$$

belongs to $\mathcal{F}_\theta(\mathbb{R})^*$, where $\theta(t) = \lambda(e^t - 1)$. Using Theorems 3.8 and 4.3 we obtain that for all measurable subsets Γ of $\mathbb{R}$:

$$-\inf_{\Gamma^\circ} \Lambda_\mu^* \leq \liminf_{n\to\infty} \frac{1}{n} \log(\mu_n(\Gamma)) \leq \limsup_{n\to\infty} \frac{1}{n} \log(\mu_n(\Gamma)) \leq -\inf_{\overline{\Gamma}} \Lambda_\mu^*,$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon \log(\mu_\varepsilon(\Gamma)) \leq \limsup_{\varepsilon\to 0} \varepsilon \log(\mu_\varepsilon(\Gamma)) \leq -\inf_{\overline{\Gamma}}(\Lambda^*).$$

In particular, if $\Gamma = A_{a,\xi} = \{y \in \mathbb{R}; y\xi > a\}$, $a > 0$ and $\xi > 0$, then

$$\lim_{n\mapsto\infty} \frac{1}{n} \log((\pi_\lambda)_n) = \theta^*(\frac{a}{\xi}),$$

where $\theta^*(x) = -x + \lambda + x\log(\frac{x}{\lambda})$.

5.2.2 Poisson measure on $D'(\mathbb{R})$

Consider the measure space $(\mathbb{R}, B(\mathbb{R}), \sigma)$ where σ is a non-atomic σ-finite measure. Bellow we consider $X = D(\mathbb{R})$ the space of C^∞- functions on $\mathbb{R}$ with compact support and $X' = D'(\mathbb{R})$ the corresponding topological dual space. Then we have the Gel'fand triple

$$D(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}, \sigma) \hookrightarrow D'(\mathbb{R}).$$

On $D'(\mathbb{R})$ we fixed the σ-algebra $C_\sigma(D'(\mathbb{R}))$ generated by the cylinder sets

$$\big\{\omega \in D'(\mathbb{R});\; (\langle\omega, \varphi_1\rangle, ..., \langle\omega, \varphi_n\rangle) \in B\big\},\; \varphi_i \in D(\mathbb{R}),\; B \in B(\mathbb{R}^n),\; n \in \mathbb{N}.$$

By the Bochner-Minlos Theorem, there exists a unique measure π_σ on $D'(\mathbb{R})$ such that the Laplace transform of π_σ is given by

$$\int_{D'(\mathbb{R})} e^{(\langle f,\omega\rangle)} d\pi_\sigma(\omega) = \exp\left(\int_{\mathbb{R}} (e^{f(x)} - 1) d\sigma(x)\right), \; f \in D(\mathbb{R}), \tag{28}$$

π_σ is the Poisson measure with intensity measure σ on $D'(\mathbb{R})$.

If σ is a finite measure on $\mathbb{R}$, the Poisson measure π_σ on $D'(\mathbb{R})$ belongs to $\mathcal{F}_\theta(D'_{\mathbb{C}}(\mathbb{R}))^*$, where $D'_{\mathbb{C}}(\mathbb{R}) = D'(\mathbb{R}) + iD'(\mathbb{R})$ and $\theta(t) = \lambda(e^t - 1)$ with $\lambda = \pi_\sigma(D(\mathbb{R}))$. Then Theorems (3.8) and (4.3) which correspond, repectively, to the Cramer's and Schilder's theorems can be applied to this Poisson measure π_σ.

Remark 5.3 The configuration space $\Gamma = \Gamma_{\mathbb{R}}$ over $\mathbb{R}$ is defined as the set of all locally subsets of $\mathbb{R}$:

$$\Gamma := \left\{\gamma \subset \mathbb{R};\ |\gamma \cap K| < \infty \quad \text{for every compact } K \subset \mathbb{R}\right\},$$

here $|A|$ denote the cardinality of the set A.

Denote by $\mathcal{M}_+(\mathbb{R})$ (resp. $\mathcal{M}_{\mathbb{N}}(\mathbb{R})$) the set of all positive (resp. positive integer-valued) Radon measures on $B(\mathbb{R})$. For $\wedge \subset \mathbb{R}$ we define $\Gamma_\wedge := \left\{\gamma \in \Gamma_{\mathbb{R}}\ ;\ \gamma \cap \wedge^c = \emptyset\right\}$ where $\wedge^c$ is the complementary of $\wedge$ in $\mathbb{R}$. Then we can identify any $\gamma \in \Gamma_{\mathbb{R}}$ with the positive integer-valued Radon measure by the following map

$$\gamma \in \Gamma \to \sum_{x\in\gamma} \delta_x \in \mathcal{M}_{\mathbb{N}}(\mathbb{R}) \subset \mathcal{M}_+(\mathbb{R}), \tag{29}$$

where δ_x is the Dirac measure at x and by convention $\sum_{x\in\emptyset} \delta_x$ is the zero measure. The space $\mathcal{M}_+(\mathbb{R})$ is endowed with the vague topology, i.e., the weakest topology on $\Gamma_{\mathbb{R}}$ such that all maps

$$\Gamma_{\mathbb{R}} \ni \gamma \longmapsto \langle f, \gamma\rangle := \int_{\mathbb{R}} f(x)\gamma(dx) = \sum_{x\in\gamma} f(x), \; f \in C_0(\mathbb{R})$$

are continuous, where $C_0(\mathbb{R})$ is the set of all real-valued continuous functions on $\mathbb{R}$ with compact support. Then the space Γ can be endowed with the enduced vague topology.

Let $B(\Gamma)$ denote the corresponding Borel σ-algebra. The restriction of the Poisson measure π_σ defined by (28) to the space $(\Gamma, B(\Gamma))$ is a probability measure, i. e., $\pi_\sigma(\Gamma) = 1$. This implies that the Poisson measure π_σ on the configuration space Γ satisfies both the Cramer's and Schilder's Theorems.

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