NAVIER-STOKES EQUATION AND DIFFUSIONS ON THE GROUP OF HOMEOMORPHISMS OF THE TORUS

F. CIPRIANO AND A. B. CRUZEIRO

Abstract. A stochastic variational principle for the (two dimensional) Navier-Stokes equation is established. The velocity field can be considered as generalized velocity of a diffusion process with values on the volume preserving diffeomorphism group of the underlying manifold. Navier-Stokes equation is reinterpreted as a perturbed equation of geodesics for the L^2 norm. The method described here should hold as well in higher dimensions.

1. Introduction

V. Arnold (cf. [A] and [A-K]) discovered a beautiful relation between the Euler equation in hydrodynamics and the geometry of diffeomorphisms in $L^2(M)$ preserving the volume measure of the underlying manifold M. This equation coincides with the geodesic equation for the L^2 metric. In particular geometric properties like curvature reflect the dynamics of the Eulerian fluid. This discovery led in particular to new ideas in the study of geometry and topology of infinite dimensional manifolds.

On the other hand it is natural to think of Navier-Stokes equation as a "perturbation" of the Euler one, corresponding to a similar description, but of an underlying stochastic nature, the stochasticity being encoded in the Laplacian. This work explores this direction. We formulate solutions of Navier-Stokes as critical points of some regularized functional and show how they can be regarded as stochastically perturbed geodesics of Arnold's model. Our idea finds its roots in the works [N-Y-Z], [Y]. Other different stochastic variational formulations for Navier-Stokes can be found, for example in [I-F] or, more recently, in [Go] (c.f. also [C]). One could also benefit from a comparaison with the approach of [G].

Our approach relies upon the construction of diffusions on the (infinite dimensional) group of measure preserving homeomorphisms in the torus, a line of work which has recently been developed by [M], [F1] and others.

When M is a compact n-dimensional Riemannian manifold without boundary, we denote by $(\cdot,\cdot)$ the Riemannian metric and by μ the associated volume element. Let G^s, $s \geq 0$ be the infinite dimensional group of homeomorphisms on M which belong to H^s (the Sobolev space of order s), namely $g \in G^s$ if $g : M \to M$ is a bijection and g, $g^{-1} \in H^s$.

For $s > \frac{n}{2} + 1$ the group G^s is a C^∞ infinite dimensional Hilbert manifold (see [S]). For each $g \in G^s$, G^s is locally diffeomorphic to the Hilbert space

$$H^s_g(TM) = \{X \in H^s(M, TM) : \pi \circ X = g\}$$

where $\pi : TM \to M$ is the canonical projection.

For $g \in G^s$ denote by L_g and R_g respectively the right and left transformations on the group, $L_g : G^s \to G^s \quad (h \to g \circ h)$, $R_g : G^s \to G^s \quad (h \to h \circ g)$.

The adjoint transformation $Ad(g)$ is usually defined as follows:

$$\begin{aligned} Ad(g) : G^s &\to G^s \\ h &\to L_g R_{g^{-1}}(h) \end{aligned} \tag{1.1}$$

For an arbitrary Lie group G, the Lie algebra $\mathcal{G}$ is the space of left invariant vector fields on G, which can be identified with the tangent space at the identity. In the case of the group G^s, the Lie algebra $\mathcal{G}^s$ can be identified with the space of vector fields on M with Sobolev s-regularity.

We consider the volume preserving homeomorphism subgroup

$$G_V^s = \{g \in G^s : g_*\mu = \mu\}$$

For $g \in G_V^s$

$$H_g^s(TM) = \{X \in H^s(M, TM) : \pi \circ X = g \text{ and } \nabla \cdot X = 0\}$$

The Lie algebra $\mathcal{G}_V^s$ of the subgroup G_V^s corresponds to the space of divergence free vector fields on M which are in H^s.

We shall deal with the L^2 inner product defined on the Lie algebra $\mathcal{G}_V^s$ by

$$(X_g, Y_g)_{L^2} = \int_M (X_g(x), Y_g(x))_{g(x)} d\mu(x) \tag{1.2}$$

Since this inner product is right invariant it defines a metric on each tangent space $T_g(G_V^s)$. Therefore, G_V^s remains endowed with a Riemannian structure.

Arnold ([A]) has given a variational formulation for the Hydrodynamic Euler equation. More precisely, the motion of an ideal fluid (i.e., non viscous and incompressible) on M corresponds to a flow on G_V^0 which is critical for the energy functional

$$S[g] = \frac{1}{2}\int_0^T \|\dot{g}(t)\|_{L^2}^2 dt \tag{1.3}$$

Its velocity satisfies Euler equation $\dfrac{\partial u}{\partial t} + u \cdot \nabla u = \nabla p, \quad \nabla \cdot u = 0.$

In this work we consider the Navier-Stokes equation on the two dimensional torus $\mathbb{T}^2$ during the time interval $[0, T]$,

$$\begin{aligned} &\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u = \nabla p \\ &\nabla \cdot u = 0 \end{aligned} \tag{1.4}$$

This equation describes de motion of an incompressible fluid on $\mathbb{T}^2$ with viscosity $\nu > 0$. The vector field $u(t, \cdot)$ represents the velocity of the fluid and the function p the pressure.

Since the incompressibility condition $\nabla \cdot u = 0$ is intrinsically associated with the space $\mathcal{G}_V^s$, and the Brownian motion is closely related with the Laplacian operator, a stochastic approach of the above framework using G_V^s-valued processes seems to be natural when the viscosity is strictly positive.

We introduce a concept of solution of (the deterministic) Navier-Stokes equation as a the mean velocity of some stochastic flow.

First we define a generalization of the energy functional (1.3) for stochastic flows with values in G_V^0.

Let $(\Omega, \mathcal{F}_t, P)$ be a probability space endowed with an increasing filtration $\mathcal{F}_t$. Given a stochastic flow $g_\omega(t)$ with values in G_V^0 and adapted to the filtration, we generalize the above energy functional by considering:

$$S[g] = \frac{1}{2} E \int_0^T \| \lim_{\Delta t \to 0} \frac{1}{\Delta t} E^{\mathcal{F}_t}\big(g_\omega(t + \Delta t) - g_\omega(t)\big) \|_{L^2}^2 dt \tag{1.5}$$

where the increment $g_\omega(t + \Delta t) - g_\omega(t)$ is understood as the difference in the respective local coordinates and $E^{\mathcal{F}_t}$ is the conditional expectation with respect to the filtration. Notice that formally, when $\nu \to 0$, the functional (1.5) reduces to (1.3).

Referring to the variational calculus on path spaces of a Lie group, in the finite dimensional case, Üstünel [U], considers right as well as left derivatives. Such derivatives are defined by the multiplication on the right or on the left by a deterministic path of bounded variation. In that case the left product corresponds, at the level of paths, to shifts by an element in the Cameron-Martin subspace of the Wiener space. On the other hand, the right product corresponds to a rotation of the path in the Wiener space. Starting with a metric on the Lie algebra which is invariant for the *ad* transformation (the differential of $Ad(g)$ at the identity), both left and right variations are well defined from the measure theoretic point of view and Malliavin's calculus of variations can be used. Unfortunately, in our case, as refered in [A-K], an invariant metric for the $ad(g)$ transformation does not exist in G_V^0. We shall deal with G_V^0-valued stochastic flows and for each $g \in G_V^0$ a suitable "tangent space" of variations will be defined.

Let $g \in G_V^0$ and f be a C^2 function defined on M. The action of g on f is:

$$\big(g^* f\big)(\xi) = f(g(\xi))$$

In this work we shall consider $M = \mathbb{T}^2$, the two dimensional torus that we identify with $[0, 2\pi] \times [0, 2\pi]$, and denote by G_V^0 the space of volume preserving maps from $\mathbb{T}^2$ to $\mathbb{T}^2$. Our construction is based on the fundamental fact that the generator of our (infinite dimensional) process actually coincides with the finite-dimensional Laplacian when computed on functions of the torus (Theorem 2.2). It is not clear that this property can be generalized in three dimensions; nevertheless there is no conceptual reason preventing a priori the development of our construction in any dimension.

Let A_k, B_k, $k \in \mathbb{Z}^2$ be divergence free vector fields defined in local coordinates by

$$A_k = A_k^1 \partial_1 + A_k^2 \partial_2 \text{ with } A_k^1 = k_2 \cos(k \cdot \theta),\ A_k^2 = -k_1 \cos(k \cdot \theta)$$

and

$$B_k = B_k^1 \partial_1 + B_k^2 \partial_2 \text{ with } B_k^1 = k_2 \sin(k \cdot \theta),\ B_k^2 = -k_1 \sin(k \cdot \theta)$$

where $\theta = (\theta_1, \theta_2) \in T^2$ and $k \cdot \theta = k_1\theta_1 + k_2\theta_2$. Then $\big\{ \frac{A_k}{|k|}, \frac{B_k}{|k|} \ :\ k \neq 0 \big\}$ is a complete system of the space of divergence free vector fields u in $L^2(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} u(\theta) d\theta = 0$.

2. Brownian motion on the group of homeomorphisms

Let $x_k(t) = (x_k^1(t), x_k^2(t)),\ t \geq 0$, be a sequence of $\mathbb{R}^2$-valued independent standard Brownian motions defined on the probability space $(\Omega, \mathcal{F}_t, P)$.

We define a Brownian motion on the space of divergence free vector fields on the two dimensional torus by

$$dx(t) = \sum_{k \neq 0} \frac{1}{|k|^\beta} \Big(A_k dx_k^1(t) + B_k dx_k^2(t) \Big) \tag{2.1}$$

with $\beta \geq 3$. Using standard probabilistic techniques, this series can be shown to converge uniformly in $[0, T] \times \mathbb{T}^2$ a.e. and the stochastic process $x(t)$ belongs to H^α, for $0 < \alpha < \beta - 2$.

Let us consider the Stratonovich stochastic differential equation with respect to the filtration $\mathcal{F}_t$

$$\begin{cases} dg(t) = \big(\circ dx(t)\big)(g(t)) \\ g(0) = e, \end{cases} \tag{2.2}$$

where e denotes the identity of the group, or, more explicitly

$$\begin{aligned} dg^1(t) &= \sum_{k\neq 0} \frac{1}{|k|^\beta} \left[A_k^1(g(t)) \circ dx_k^1(t) + B_k^1(g(t)) \circ dx_k^2(t)\right] \\ dg^2(t) &= \sum_{k\neq 0} \frac{1}{|k|^\beta} \left[A_k^2(g(t)) \circ dx_k^1(t) + B_k^2(g(t)) \circ dx_k^2(t)\right] \end{aligned} \tag{2.3}$$

Lemma 2.1. *Equation (2.2) can be written in the Itô form as follows*

$$\begin{aligned} dg^1(t) &= \sum_{k\neq 0} \frac{1}{|k|^\beta} \left[A_k^1(g(t)) \cdot dx_k^1(t) + B_k^1(g(t)) \cdot dx_k^2(t)\right] \\ dg^2(t) &= \sum_{k\neq 0} \frac{1}{|k|^\beta} \left[A_k^2(g(t)) \cdot dx_k^1(t) + B_k^2(g(t)) \cdot dx_k^2(t)\right] \end{aligned} \tag{2.4}$$

i. e., the Itô contraction term vanishes.

Proof. We have:

$$\begin{aligned} d\big(A_k^1(g(t)\big) &= \big(\partial_1 A_k^1\big)(g(t)) \circ dg^1(t) + \big(\partial_2 A_k^1\big)(g(t)) \circ dg^2(t) \\ &= \sum_{m\neq 0} \frac{1}{|m|^\beta} \big(\partial_1 A_k^1\big)(g(t)) \left[A_m^1(g(t)) \circ dx_m^1(t) + B_m^1(g(t)) \circ dx_m^2(t)\right] \\ &+ \sum_{m\neq 0} \frac{1}{|m|^\beta} \big(\partial_2 A_k^1\big)(g(t)) \left[A_m^2(g(t)) \circ dx_m^1(t) + B_m^2(g(t)) \circ dx_m^2(t)\right] \end{aligned}$$

Since $dx_m^1 \cdot dx_k^1 = \delta_{mk} dt$ and $dx_m^2 \cdot dx_k^1 = 0$ we obtain

$$\begin{aligned} d\big(A_k^1(g(t)\big) \cdot dx_k^1(t) &= \frac{1}{|k|^\beta} \left[\big(\partial_1 A_k^1\big)(g(t)) A_k^1(g(t)) dt + \big(\partial_2 A_k^1\big)(g(t)) A_k^2(g(t)) dt\right] \\ &= \frac{1}{|k|^\beta} \big[-(k_2)^2 k_1 \sin(k\cdot\theta)\cos(k\cdot\theta) dt \\ &+ (k_2)^2 k_1 \sin(k\cdot\theta)\cos(k\cdot\theta) dt\big] \\ &= 0 \end{aligned}$$

All other Itô contractions can be shown to be zero in an analogous way. □

¿From the classical theory of stochastic flows (cf. [K]), for $\beta > 3$, the solution $g(t)$ of the equation (2.2) is well defined as a stochastic flow of diffeomorphisms.

For $\beta = 3$, we can follow [F1] to prove that the quadratic variation of the stochastic process $x(t)(\theta) - x(t)(\theta')$ can be estimated by $C|\theta-\theta'|^2 \log \frac{1}{|\theta-\theta'|}$ for $|\theta-\theta'|$ small enough. This estimate enables to prove the existence of the process $g(t)$.

More precisely, we have:

Theorem 2.1. *For $\beta = 3$, the solution $g(t)$ of the stochastic differential equation* (2.2) *exists and is a continuous process with values in the space of homeomorphisms on $\mathbb{T}^2$ preserving the volume measure.*

Proof. We follow the methodology of [F1]. We fix θ and denote by $g_i^n(t)(\theta)$ the solution of the following finite-dimensional s.d.e.:

$$\begin{aligned} d\gamma_1^n(t) &= \sum_{\substack{k\neq 0 \\ |k|\leq 2^n}} \frac{1}{|k|^3} \left[A_k^1(\gamma^n(t)) \cdot dx_k^1(t) + B_k^1(\gamma^n(t)) \cdot dx_k^2(t)\right] \\ d\gamma_2^n(t) &= \sum_{\substack{k\neq 0 \\ |k|\leq 2^n}} \frac{1}{|k|^3} \left[A_k^2(\gamma^n(t)) \cdot dx_k^1(t) + B_k^2(\gamma^n(t)) \cdot dx_k^2(t)\right] \end{aligned} \tag{2.5}$$

with initial condition $(\gamma_1^n(0), \gamma_2^n(0)) = (\theta_1, \theta_2)$. Denote $\eta_i(t) = \frac{\gamma_i^n(t) - \gamma_i^{n+1}(t)}{2^5}$; we have

$$d\eta_i(t)\cdot d\eta_i(t) = \frac{1}{2^{10}}\Bigg\{ \sum_{\substack{k\neq 0\\ |k|\le 2^n}} \frac{k_j^2}{|k|^6}\Big[\big(\cos(k\cdot\gamma^{n+1}(t)) - \cos(k\cdot\gamma^n(t))\big)^2$$
$$+ \big(\sin(k\cdot\gamma^{n+1}(t)) - \sin(k\cdot\gamma^n(t))\big)^2\Big]$$
$$+ \sum_{\substack{k\neq 0\\ 2^n+1\le |k|\le 2^{n+1}}} \frac{k_j^2}{|k|^6}\big[\cos^2(k\cdot\gamma^{n+1}(t)) + \sin^2(k\cdot\gamma^{n+1}(t))\big]\Bigg\}$$
$$\le \frac{1}{2^{10}}\Bigg\{ \sum_{\substack{k\neq 0\\ |k|\le 2^n}} \frac{k_j^2}{|k|^6} 4\sin^2\Bigg(k\cdot\frac{\gamma^{n+1}(t)-\gamma^n(t)}{2}\Bigg)$$
$$+ \sum_{\substack{k\in\mathbb{Z}^2\\ 2^n+1\le |k|\le 2^{n+1}}} \frac{k_j^2}{|k|^6}\Bigg\}$$

Now

$$\sum_{\substack{k\neq 0\\ |k|\le 2^n}} \frac{k_j^2}{|k|^6}\sin^2\Bigg(k\cdot\frac{\gamma^{n+1}(t)-\gamma^n(t)}{2}\Bigg) \le C|\eta(t)|^2\log\frac{1}{|\eta(t)|}$$

and

$$\sum_{\substack{k\neq 0\\ 2^n+1\le |k|\le 2^{n+1}}} \frac{k_j^2}{|k|^6} \le \sum_{\substack{k\neq 0\\ 2^n+1\le |k|\le 2^{n+1}}} \frac{1}{|k|^4} \le C2^{-n}$$

where $C = \sum_{|k|\ge 1}\frac{1}{|k|^3}$.

By Itô's formula, for $p\ge 1$,

$$d\eta_i^{2p}(t) = 2p\eta_i^{2p-1}(t)\cdot d\eta_i(t) + p(2p-1)\eta_i^{2p-2}(t)d\eta_i(t)\cdot d\eta_i(t)$$

we have

$$\mathbb{E}^{\mathcal{F}_t}\big(\eta_i^{2p}(t+\epsilon) - \eta_i^{2p}(t)\big) \le K_0 p\int_t^{t+\epsilon}\mathbb{E}^{\mathcal{F}_t}\Bigg(|\eta(s)|^{2p}\log\frac{1}{|\eta(s)|^{2p}}ds\Bigg) + K_p 2^{-n}\epsilon$$

Defining the function $\varphi(t) = \mathbb{E}(\eta_1^{2p}(t) + \eta_2^{2p}(t))$, we obtain

$$\varphi'(t) \le K_0 p\cdot\mathbb{E}\Bigg(|\eta(s)|^{2p}\log\frac{1}{|\eta(s)|^{2p}}\Bigg) + K_p 2^{-n}$$

therefore

$$\varphi(t) \le C_p 2^{-n\delta_p(t)}$$

where $\delta_p(t)$ is a constant verifying $\lim_{p\to+\infty}\delta_p(t) = 0$ and $\lim_{t\to 0^+}\delta_p(t) = 1$.

By the martingale maximal inequality,

$$\mathbb{E}\Bigg(\sup_{0\le t\le T}|g^n(t)(\theta) - g^{n+1}(t)(\theta)|^{2p}\Bigg) \le C_p 2^{-n\delta_p(T)}$$

Using Borel-Cantelli we deduce that

$$g(t)(\theta) = \lim_n g^n(t)(\theta) \quad \text{exists uniformly in } t\in[0,T]$$

Following [F1], one can show that g(t) satisfies equation (2.4) and that it is its unique solution.

The following estimate holds

$$\mathbb{E}\biggl(\sup_{0\le t\le T}|g(t)(\theta)-g(t)(\theta')|^{2p}\biggr)\le C_p|\theta-\theta'|^{2p\delta_p(T)}$$

Using this inequality, for fixed small T, we can apply Kolmogorov theorem to show that $g(t)(\cdot)$ is Hölder continuous. Following [F1] and [M], the flow property can be used to prove that the stochastic process $g(t)$ lives in the space of volume preserving homeomorphisms.

□

Definition 2.1. Let f be a function in C^2 defined on $\mathbb{T}^2$. On a functional $F(g)(\theta)=f(g(\theta))$, $\theta\in\mathbb{T}^2$, the infinitesimal generator of the process $g(t)$ is defined by

$$\mathcal{L}(F)(\theta)=\lim_{t\to 0}\frac{1}{t}E\biggl((g(t))^*f(\theta)-f(\theta)\biggr) \tag{2.6}$$

This infinitesimal generator corresponds to the usual Laplacian operator Δ on the torus, more precisely, we have:

Theorem 2.2. *Let $\mathcal{L}$ be the infinitesimal generator of the stochastic process $g(t)$. Then there exists a positive constant c such that, for $F(g)(\theta)=f(g(\theta))$,*

$$\mathcal{L}(F)=c\Delta f,\quad f\in C^2(\mathbb{T}^2)$$

Proof. Itô's formula reads

$$\begin{aligned} df(g(t)) &= \sum_{k\neq 0}\frac{1}{|k|^\beta}\bigl[\bigl(A_kf\bigr)(g(t))\cdot dx_k^1(t)+\bigl(B_kf\bigr)(g(t))\cdot dx_k^2(t)\bigr]\\ &+\frac{1}{2}\sum_{k\neq 0}\frac{1}{|k|^{2\beta}}\bigl[\bigl(A_k\bigl(A_kf\bigr)\bigr)(g(t))dt+\bigl(B_k\bigl(B_kf\bigr)\bigr)(g(t))dt\bigr]\end{aligned}$$

We have

$$\begin{aligned}\frac{1}{|k|^{2\beta}}\bigl(A_k\bigl(A_kf\bigr)\bigr)(\theta) &= \frac{1}{|k|^{2\beta}}\bigl[k_2\cos(k\cdot\theta)\partial_1\bigl(k_2\cos(k\cdot\theta)\partial_1 f-k_1\cos(k\cdot\theta)\partial_2 f\bigr)\\ &\quad -k_1\cos(k\cdot\theta)\partial_2\bigl(k_2\cos(k\cdot\theta)\partial_1 f-k_1\cos(k\cdot\theta)\partial_2 f\bigr)\bigr]\\ &= \frac{1}{|k|^{2\beta}}\bigl[(k_2)^2\cos^2(k\cdot\theta)\partial_1^2 f-2k_1k_2\cos^2(k\cdot\theta)\partial_1\partial_2 f\\ &\quad +(k_1)^2\cos^2(k\cdot\theta)\partial_2^2 f\bigr]\end{aligned}$$

and

$$\begin{aligned}\frac{1}{|k|^{2\beta}}\bigl(B_k\bigl(B_kf\bigr)\bigr)(\theta) &= \frac{1}{|k|^{2\beta}}\bigl[k_2\sin(k\cdot\theta)\partial_1\bigl(k_2\sin(k\cdot\theta)\partial_1 f-k_1\sin(k\cdot\theta)\partial_2 f\bigr)\\ &\quad -k_1\sin(k\cdot\theta)\partial_2\bigl(k_2\sin(k\cdot\theta)\partial_1 f-k_1\sin(k\cdot\theta)\partial_2 f\bigr)\bigr]\\ &= \frac{1}{|k|^{2\beta}}\bigl[(k_2)^2\sin^2(k\cdot\theta)\partial_1^2 f-2k_1k_2\sin^2(k\cdot\theta)\partial_1\partial_2 f\\ &\quad +(k_1)^2\sin^2(k\cdot\theta)\partial_2^2 f\bigr]\end{aligned}$$

Therefore the infinitesimal generator is given by

$$\frac{1}{2}\sum_{k\neq 0}\frac{1}{|k|^{2\beta}}\bigl[(k_2)^2\partial_1^2 f+(k_1)^2\partial_2^2 f-2k_1k_2\partial_1\partial_2 f\bigr]$$

Since

$$\sum_{k\neq 0}\frac{1}{|k|^{2\beta}}k_1k_2=0 \text{ and } \sum_{k\neq 0}\frac{1}{|k|^{2\beta}}(k_1)^2=\sum_{k\neq 0}\frac{1}{|k|^{2\beta}}(k_2)^2$$

the result follows from taking $c = \frac{1}{2}\sum_{k\neq 0}\frac{1}{|k|^{2\beta}}(k_1)^2$. □

3. **Stochastic differential equations on the group of homeomorphisms**

Let us consider $u : [0,T] \to \mathcal{G}_V^0$. For each t, $u(t)$ is a divergence free vector field on $\mathbb{T}^2$. We can associate to $u(t)$ the following $\mathcal{F}_t$ stochastic differential equation

$$\begin{aligned} dg_u(t) &= \left(u(t)dt + \sqrt{\frac{\nu}{c}} \circ dx(t)\right)(g_u(t)) \\ g_u(0) &= e \end{aligned} \tag{3.1}$$

where c denote the constant defined in theorem 2.2.

This equation can be written in local coordinates as follows:

$$\begin{aligned} dg_u^1(t) &= \left(u^1(t)dt + \sqrt{\frac{\nu}{c}} dx^1(t)\right)(g_u(t)) \\ dg_u^2(t) &= \left(u^2(t)dt + \sqrt{\frac{\nu}{c}} dx^2(t)\right)(g_u(t)) \\ g_u(0) &= e \end{aligned} \tag{3.2}$$

The method to solve this stochastic equation depends on the regularity of the underlying Brownian motion and of the drift. When $\beta = 3$ (the most irregular case) we can use Girsanov transformation as in [F2] with $u(t) \in H^2$. In this case, if u belong to $C([0,T];\mathcal{G}_V^2)$, one can show the existence of a stochastic process $g_u(t)$, solution of the s.d.e. (3.1), with values in the space of homeomorphisms of the torus preserving the Lebesgue measure.

In the case where only L^2 regularity is available, we prove the following

Theorem 3.1. *Let u belong to the space $L^2([0,T];\mathcal{G}_V^0)$; then there exists a stochastic process $g_u(t)$, weak solution of the s.d.e.* (3.1)*, with values in G_V^0.*

Proof. Since $u \in H_V^0$, we can write

$$u(t,\theta) = \sum_{k\neq 0}\left[u_k^1(t)\frac{A_k(\theta)}{|k|} + u_k^2(t)\frac{B_k(\theta)}{|k|}\right] \tag{3.3}$$

with $\sum_{k\neq 0}|u_k^1|^2 + |u_k^2|^2 < +\infty$.

Let us consider the following smooth aproximation u^n of the vector field u,

$$u^n(t,\theta) = \sum_{\substack{k\neq 0\\ |k|\leq n}}\left[u_k^1(t)\frac{A_k(\theta)}{|k|} + u_k^2(t)\frac{B_k(\theta)}{|k|}\right] \tag{3.4}$$

and a smooth finite dimensional aproximation $x^n(t)$ of the Brownian motion $x(t)$ defined in (2.1)

$$dx^n(t,\theta) = \sum_{\substack{k\neq 0\\ |k|\leq n}}\frac{1}{|k|^\beta}\left(A_k(\theta)dx_k^1(t) + B_k(\theta)dx_k^2(t)\right) \tag{3.5}$$

Then there exists a smooth global stochastic flow, strong solution of the stochastic differential equation (3.1) with u replaced by u^n and $x(t)$ replaced by $x^n(t)$. More precisely, there exists $g^n(t)(\theta)$ such that

$$g^n(t)(\theta) = \theta + \int_0^t u^n(\tau, g^n(\tau)(\theta))d\tau + \sqrt{\frac{\nu}{c}}\int_0^t dx^n(\tau, g^n(\tau)(\theta)) \tag{3.6}$$

We consider the sequence of measures ν^n, defined on the space $C([0,T];G_V^0)$ as the laws of g^n. We prove that this sequence is tight. It is sufficient to show that:

$$\lim_{R\to\infty}\sup_n \nu^n(\|y(0)\|_{L^2} > R) = 0 \tag{3.7}$$

$$\lim_{\delta\to 0}\sup_n \nu^n\left(\max_{\substack{|t-s|\le\delta\\ s,t\in[0,T]}} \|y(t)-y(s)\|_{L^2} \ge \rho\right) = 0, \quad \forall \rho > 0 \tag{3.8}$$

Since $g^n(0)(\theta) = \theta$, condition (3.7) is clearly satisfied. Concerning condition (3.8), we have

$$\nu^n\left(\max_{\substack{|t-s|\le\delta\\ s,t\in[0,T]}} \|y(t)-y(s)\|_{L^2} \ge \rho\right) \le \frac{1}{\rho}E\left(\max_{\substack{|t-s|\le\delta\\ s,t\in[0,T]}} \|g^n(t)-g^n(s)\|_{L^2}\right)$$
$$\le \frac{1}{\rho}E\left(\max_{\substack{|t-s|\le\delta\\ s,t\in[0,T]}} \left\|\int_s^t u^n(\tau, g^n(\tau)(\theta))d\tau\right\|_{L^2}\right) + E\left(\max_{\substack{|t-s|\le\delta\\ s,t\in[0,T]}} \left\|\int_s^t dx^n(\tau, g^n(\tau)(\theta))\right\|_{L^2}\right)$$

Using the invariance with respect to Lebesgue measure of the flow $g^n(t)(\cdot)$, we obtain

$$E\left(\max_{\substack{|t-s|\le\delta\\ s,t\in[0,T]}} \left\|\int_s^t u^n(\tau, g^n(\tau)(\theta))d\tau\right\|_{L^2}\right) \le \sqrt{\delta}\|u\|_{L^2([0,T];\mathbb{T}^2)}$$

Also by the invariance of Lebesgue measure and the Hölder continuity of the Brownian motion, we have

$$E\left(\max_{\substack{|t-s|\le\delta\\ s,t\in[0,T]}} \left\|\int_s^t dx^n(\tau, g^n(\tau)(\theta))\right\|_{L^2}\right) = E\left(\max_{\substack{|t-s|\le\delta\\ s,t\in[0,T]}} \left\|\int_s^t dx^n(\tau, \theta)\right\|_{L^2}\right) \le C\delta^{2\alpha}$$

with $0 < \alpha \le 1$. Condition (3.8) follows.

The space L^2 endowed with the weak topology is a relatively compact space and the σ-algebras generated by the Borelian sets defined by the weak and the strong topologies are the same. Therefore, there exists a subsequence ν_{n_k} of ν_n and a measure ν such that $\nu_{n_k} \to \nu$, with respect to the weak topology on the space of measures on $C([0,T];L^2)$. To simplify the notation, we still denote such subsequence by ν_n. By Skorohod's theorem, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a family of stochastic processes $\tilde{g}^n_\omega(t)$, $g_\omega(t)$, $\omega \in \Omega$ with laws, respectively, ν_n and ν such that for a.e. ω, $\tilde{g}^n_\omega(\cdot) \to g_\omega(\cdot)$ in the space $C([0,T];L^2)$.

For any continuous function f defined on $\mathbb{T}^2$ and a.e. $\omega \in \Omega$, we have

$$\int_{\mathbb{T}^2} f(g^n_\omega(t)(\theta))d\theta = \int_{\mathbb{T}^2} f(\theta)d\theta$$

The left hand side integral can be identified in law with

$$\int_{\mathbb{T}^2} f(\tilde{g}^n_\omega(t)(\theta))d\theta$$

For a.e. ω, the convergence of $\tilde{g}^n$ to g implies the existence of a subsequence $\tilde{g}^{n_j}_\omega(t)(\theta)$ of $\tilde{g}^n_\omega(t)(\theta)$ such that $\tilde{g}^{n_j}_\omega(t)(\theta) \to g_\omega(t)(\theta)$ for a.e. θ. Using Lebesgue's theorem

$$\int_{\mathbb{T}^2} f(\tilde{g}^{n_j}_\omega(t)(\theta))d\theta \to \int_{\mathbb{T}^2} f(g_\omega(t)(\theta))d\theta$$

which proves that g lives in G_V^0.

We now prove that the stochastic process $g_\omega(t)(\theta)$ is a weak solution of the s.d.e. (3.1). The process $g^n(t)$ can be identified in law with the solution of equation (3.6); we have:

$$\begin{aligned}
&E\int_{\mathbb{T}^2}\left|\int_0^t u^n(\tau,\tilde{g}^n(\tau)(\theta))d\tau-\int_0^t u(\tau,g(\tau)(\theta))d\tau\right|^2\\
&\le E\int_{\mathbb{T}^2}\left|\int_0^t u^n(\tau,\tilde{g}^n(\tau)(\theta))d\tau-\int_0^t u(\tau,\tilde{g}^n(\tau)(\theta))d\tau\right|^2\\
&+E\int_{\mathbb{T}^2}\left|\int_0^t u(\tau,\tilde{g}^n(\tau)(\theta))d\tau-\int_0^t u(\tau,g(\tau)(\theta))d\tau\right|^2\\
&=I_1^n+I_2^n
\end{aligned}$$

By the invariance of the Lebesgue measure and the fact that $u^n\to u$ in $L^2([0,T];\mathbb{T}^2)$ the integral I_1^n converges to zero as $n\to\infty$. Applying Lusin's theorem to the vector field u on $[0,T]\times\mathbb{T}^2$, and considering, for a.e. ω, a subsequence of $\tilde{g}^n_\omega(t)(\theta)$ that converges to $g_\omega(t)(\theta)$ uniformly in t and θ, there exists a subsequence of the integral I_2^n that converges to zero. Concerning the stochastic integral, for a.e. ω, we have:

$$\begin{aligned}
&\int_{\mathbb{T}^2}\left|\int_0^t dx^n(\tau,\tilde{g}^n(\tau)(\theta))-\int_0^t dx(\tau,g(\tau)(\theta))\right|^2\\
&\le\int_{\mathbb{T}^2}\left|\int_0^t dx^n(\tau,\tilde{g}^n(\tau)(\theta))-\int_0^t dx(\tau,\tilde{g}^n(\tau)(\theta))\right|^2\\
&+\int_{\mathbb{T}^2}\left|\int_0^t dx(\tau,\tilde{g}^n(\tau)(\theta))-\int_0^t dx(\tau,g(\tau)(\theta))\right|^2\\
&=I_1^n(\omega)+I_2^n(\omega)
\end{aligned}$$

Using again the invariance with respect to Lebesgue measure

$$\begin{aligned}
&\int_{\mathbb{T}^2}\left|\int_0^t dx^n(\tau,\theta)-dx(\tau,\theta)\right|^2 d\theta=\int_{\mathbb{T}^2}\left|x^n(t,\theta)-x(t,\theta)\right|^2 d\theta\\
&\le C\sup_{t\in[0,T],\theta\in\mathbb{T}^2}|x^n(t,\theta)-x(t,\theta)|
\end{aligned}$$

which converges to zero as $n\to\infty$. Since for a.e. ω, $x_\omega(t,\theta)$ are continuous random variables on $[0,T]\times\mathbb{T}^2$, there exists a subsequence of the integral

$$\int_{\mathbb{T}^2}\left|x(\tau,\tilde{g}^n(\tau)(\theta))-x(\tau,g(\tau)(\theta))\right|^2 d\theta$$

that converges to zero.

Therefore, taking the limit in $L^2(\Omega\times\mathbb{T}^2)$ of the integral equation (3.6), we conclude that $g(\tau)(\theta)$ satisfies the s.d.e. (3.1).

□

Corollary 3.1. *Suppose that u satisfy the hypothesis of Theorem 3.1 and let $g_u(t)$ the solution of equation* (3.1). *Then the infinitesimal generator of this process, when computed at functionals of the form $F(g)(\theta)=f(g(\theta))$, is given by*

$$\mathcal{L}^u F=u\cdot\nabla f+\nu\Delta f,\quad\forall f\in C^2(\mathbb{T}^2)\tag{3.9}$$

Proof. The proof is analogous to the proof of theorem 2.2. □

4. **The variational principle**

Given a G^0_V-valued stochastic process $g(t)$ we define the action functional

$$S[g] = \frac{1}{2} E \int_0^T \|Dg(t)\|^2_{L^2} dt - \frac{1}{2} E \|Dg(T)\|^2_{L^2} \tag{4.1}$$

where D is the generalized derivative, defined for smooth functionals F by

$$DF(g(t), t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} E^{\mathcal{F}_t} \Big(F\big(g(t+\epsilon), t+\epsilon\big) - F\big(g(t), t\big) \Big) \tag{4.2}$$

Given $v \in C^1\big([0,T]; \mathcal{G}^\infty_V\big)$ with $v(0, \cdot) = 0$, consider the following ordinary differential equation

$$\begin{aligned} \frac{de_t(v)}{dt} &= \dot{v}(t, e_t(v)) \\ e_0(v) &= e \end{aligned} \tag{4.3}$$

where $\dot{v} = \dfrac{dv}{dt}$. Since v is divergence free, $e_\cdot(v)$ is a G^∞_V- valued deterministic path.

Let us denote by $\mathcal{P}$ the set of continuous G^0_V-valued $\mathcal{F}_t$-semimartingales $g(t)$ such that $g(0) = e$.

The left product $e_t(v) \circ g_u(t)$ of the diffusion $g_u(t)$ by an arbitrary deterministic path $e_\cdot(v)$ is well defined on $\mathcal{P}$. So, we define the derivative of a functional on $\mathcal{P}$ at a process $g \in \mathcal{P}$ using the left product by an arbitrary element of the following set

$$\mathcal{H} = \big\{ e_t(v) : v \in C^1\big([0,T]; \mathcal{G}^\infty_V\big) \text{ and } v(0, \cdot) = 0 \big\}$$

This set can be considered as the "tangent" space to $\mathcal{P}$, appropriate to a calculus of variation for (4.1). A small perturbation of $g \in \mathcal{P}$ in the direction $h(t) = e_t(v) \in \mathcal{H}$ will correspond to the product $e_t(\epsilon v) \circ g_u(t)$ where $e_t(\epsilon v)$ is the solution of (4.3) associated with the perturbation ϵv of v.

Definition 4.1. Let J be a functional defined on $\mathcal{P}$ taking values in $\mathbb{R}$. We define its left and right derivatives in the direction of $h(\cdot) = e_\cdot(v) \in \mathcal{H}$ at a process $g \in \mathcal{P}$, respectively, by:

$$(D_L)_h J[g] = \frac{d}{d\epsilon} J[e_\cdot(\epsilon v) \circ g(\cdot)]\Big|_{\epsilon=0},$$

$$(D_R)_h J[g] = \frac{d}{d\epsilon} J[g(\cdot) \circ e_\cdot(\epsilon v)]\Big|_{\epsilon=0}$$

A process $g \in \mathcal{P}$ will be called a critical point of the functional J if

$$(D_L)_h J[g] = (D_R)_h J[g] = 0, \, \forall h \in \mathcal{H},$$

Theorem 4.1. *Let $u \in L^2([0,T]; \mathcal{G}^0_V)$ and $g_u(t) \in C([0,T]; G^0_V)$ be a weak solution of equation (3.1). The stochastic process $g_u(t)$ is a critical point of the energy functional S defined in* (4.1) *if and only if the vector field $u(t)$ verifies the Navier-Stokes equation:*

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u = \nu \Delta u + \nabla p \\ \nabla \cdot u = 0 \\ u(T, \theta) = u_T(\theta) \end{cases} \tag{4.4}$$

Proof. Since the functional S is right invariant, the right derivative is not relevant.

Let $\epsilon > 0$ and $h(t) = e_t(v)$ an arbitrary element in the "tangent" space $\mathcal{H}$. We have

$$e_t(\epsilon v) = e + \epsilon \int_0^t \dot{v}(s, e_s(\epsilon v)) ds \tag{4.5}$$

Since $e_t(0)(\theta) = \theta$ for all t,

$$\frac{d}{d\epsilon} e_t(\epsilon v)\big|_{\epsilon=0} = \int_0^t \dot{v}_s(\theta) ds = v_t(\theta) \tag{4.6}$$

therefore

$$\frac{d}{d\epsilon}\big(e_t(\epsilon v) \circ g_u(t)\big)\big|_{\epsilon=0} = v_t \circ g_u(t) \tag{4.7}$$

and

$$\begin{aligned} \frac{d}{d\epsilon} S[e.(\epsilon v) \circ g_u(\cdot)]\bigg|_{\epsilon=0} =& E\int_0^T \big(Dg_u(t), D(v(t, g_u(t)))\big)_{L^2} dt \\ & - \int_{\mathbb{T}^2} \big(u_T(\theta) v(T,\theta)\big) d\theta \end{aligned} \tag{4.8}$$

Using Itô's formula

$$\begin{aligned} d\big(Dg_u(t), v(t, g_u(t))\big)_{L^2} =& \big(dDg_u(t), v(t, g_u(t))\big)_{L^2} + \big(Dg_u(t), dv(t, g_u(t))\big)_{L^2} \\ &+ \big(dDg_u(t), dv(t, g_u(t))\big)_{L^2} \end{aligned}$$

and the expression of the contraction term,

$$\big(dDg_u(t), dv(t, g_u(t))\big)_{L^2} = 2\nu \int_{\mathbb{T}^2} \sum_{i,j=1}^{2} \frac{\partial v^i}{\partial \theta_j}(t, g_u(t)) \frac{\partial u^i}{\partial \theta_j}(t, g_u(t)) d\theta, \tag{4.9}$$

we deduce that

$$\begin{aligned} & E\int_0^T \big(Dg_u(t), D(v(t, g_u(t)))\big)_{L^2} dt = E\big(Dg_u(T), v(T, g_u(T))\big)_{L^2} \\ & - E\int_0^T \big(DDg_u(t), v(t, g_u(t))\big)_{L^2} dt - E\int_0^T \big(dDg_u(t), dv(t, g_u(t))\big)_{L^2} dt \\ & = \int_{\mathbb{T}^2} u(T,\theta) v(T,\theta) d\theta - \int_0^T \int_{\mathbb{T}^2} \left(\frac{\partial u}{\partial t} + (u \cdot \nabla u) + \nu \Delta u\right)(t,\theta) v(t,\theta) d\theta dt \\ & - 2\nu \int_{\mathbb{T}^2} \sum_{i,j=1}^{2} \frac{\partial v^i}{\partial \theta_j}(t,\theta) \frac{\partial u^i}{\partial \theta_j}(t,\theta) d\theta dt \\ & = \int_{\mathbb{T}^2} u(T,\theta) v(T,\theta) d\theta - \int_0^T \int_{\mathbb{T}^2} \left(\frac{\partial u}{\partial t} + (u \cdot \nabla u) - \nu \Delta u\right)(t,\theta) v(t,\theta) d\theta dt \end{aligned}$$

Therefore

$$\frac{d}{d\epsilon} S[e.(\epsilon v) \circ g_u(\cdot)]\bigg|_{\epsilon=0} = 0, \quad \forall v \in \mathcal{H}$$

is equivalent to

$$\int_0^T \int_{\mathbb{T}^2} \left(\frac{\partial u}{\partial t} + (u \cdot \nabla u) - \nu \Delta u\right)(t,\theta) v(t,\theta) d\theta dt = \int_{\mathbb{T}^2} \Big(u(T,\theta) - u_T(\theta)\Big) v(T,\theta) d\theta,$$

which corresponds to the weak formulation of the Navier-Stokes equation (4.4), since v is arbitrary in $C^1\big([0,T]; \mathcal{G}_V^\infty\big)$. □

5. **Existence of the critical diffusion**

In this paragraph, we consider the action functional S defined in (4.1) on the set $\mathcal{P}$ of semimartingales. The next theorem proves the existence of a process in $\mathcal{P}$ that is a minimum of the functional.

Theorem 5.1. *The action functional S defined in* (4.1) *has a minimum $g(t)$ on the subset of $\mathcal{P}$ with fixed final energy $\|Dg(T)\|_{L^2}$.*

Proof. The functional is bounded below, let α be its infimum. We consider $g^n(t)$ a minimizing sequence of the functional S. This means that $S[g^n(\cdot)] \to \alpha$ as $n \to \infty$. Let us denote by u_n the drift of $g^n(t)$. The sequence $2S[g^n(\cdot)] = \|u_n\|_{L^2([0,T];\mathbb{T}^2)} - \|u_T\|_{L^2}$ is bounded, therefore there exists a subsequence u_{n_j} of u_n that converges with respect to the weak topology, more precisely there exists $u \in L^2([0,T];\mathbb{T}^2)$ such that

$$u_{n_j} \to u, \quad \text{weakly in } \ L^2([0,T];\mathbb{T}^2)$$

We obtain $\int_0^T \int_{\mathbb{T}^2} u(t,\theta)\nabla f(\theta) dt d\theta = 0$, for all regular function f, which implies that $u(t,\cdot) \in \mathcal{G}_V^0$ a.e. in t. The limit function u satisfies the assumptions of Theorem 3.1. Then we can construct a stochastic process $g_u(t)$ in $\mathcal{P}$ as solution of the stochastic differential equation (3.1). Since

$$S\big[g_u(\cdot)\big] \le S\big[g^{n_j}(\cdot)\big], \quad \forall j$$

we deduce that $S\big[g_u(\cdot)\big] = \alpha$ and $g_u(t)$ is a minimum. □

We may define a solution of the Navier-Stokes equation as the drift (ou mean velocity) of the critical process obtained in the last theorem. We have,

Corollary 5.1. *The mean generalized velocity of the minimum of the action functional in Theorem 5.1 is a solution of the Navier-Stokes equation* (4.4). *Moreover, the generalized kinetic energy $\|Dg_u(t)\|^2_{L^2}$ of such minimum $g_u(t)$ is a $\mathcal{F}_t$-supermartingale.*

Proof. By construction $g_u(t)$, the minimum of the action functional obtained in Theorem 5.1, satisfies the stochastic differential equation (3.1). Let us consider an arbitrary deterministic path $h(t) = e_t(\epsilon v) \in \mathcal{H}$ with $v(T) = 0$. The final energy of all variations $e_t(\epsilon v) \circ g_u(t)$ coincide with the final energy of the minimum $\|Dg_u(T)\|^2_{L^2}$. Therefore $(D_L)_h J[g] = 0$ and, according to theorem 4.1, $u(t,\theta)$ satisfies Navier-Stokes equation.

We prove that

$$E^{\mathcal{F}_s}\big(\|Dg_u(t)\|^2_{L^2}\big) \le \|Dg_u(s)\|^2_{L^2}, \quad 0 \le s \le t \le T.$$

Using Itô's formula for the functional $g_u(t)(\theta)$, we have:

$$d\|Dg_u(t)\|^2_{L^2} = 2(dDg_u(t), Dg_u(t))_{L^2} + (dDg_u(t), dDg_u(t))_{L^2}$$

Considering the expression of the stochastic contraction term (4.9), we derive

$$\begin{aligned} E^{\mathcal{F}_s}\big(\|Dg_u(t)\|^2_{L^2}\big) - \|Dg_u(s)\|^2_{L^2} &= 2E\int_s^t (DDg_u(\tau), Dg_u(\tau))_{L^2} \\ &\quad + 2\nu\|\nabla u(\tau, g_u(\tau))\|^2_{L^2} \\ &= 2\int_s^t \int_{\mathbb{T}^2} \Big(\frac{\partial u}{\partial t} + u\cdot\nabla u\Big)(\tau,\theta) u(\tau,\theta) d\theta d\tau \\ &= -2\nu \int_s^t \int_{\mathbb{T}^2} |\nabla u(\tau,\theta)|^2 d\theta d\tau \end{aligned}$$

□

Since we have associated stochastic processes to solutions of Navier-Stokes equation, a detailed study of these processes will determine various properties of Navier-Stokes flows. On the other hand, the existence of a variational principle should lead naturally to a study of the symmetries of our action functional, their probabilistic interpretation and geometrical implications for the same flow.

Acknowledgements

The authors are grateful to Prof. J. C. Zambrini for his suggestions and very helpful discussions.

We acknowledge the support of FCT, projects POCTI/0208/2003 and POCI / MAT /55977 / 2004.

References

[A] V.I. Arnold: Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications a l'hidrodynamique des fluides parfaits. *Ann. Inst. Fourier* **16**, 316–361 (1966)

[A-K] V.I. Arnold, B.A. Khesin: *Topological Methods in Hydrodynamics*. Springer-Verlag, New York, 1998

[C] P. Constantin: An Eulerian-Lagrangian approach to the Navier-Stokes equations. *Comm. Math. Phys.* **216(3)**, 663–686 (2001)

[F1] S. Fang: Canonical Brownian motion on the diffeomorphism group of the circle. *J. Funct. Anal.* **196**, 162–179 (2002)

[F2] S. Fang: Solving s.d.e.'s on Home(S^1). *J. Funct. Anal.* **216**, 22–46 (2004)

[G] K. Gawedzki: Simple models of turbulent transport. *XIV Intern. Congress on Mathem. Physics*, World Scientific (2005)

[Go] D. A. Gomes: A variational formulation for the Navier-Stokes equation. *Comm. Math. Phys.* **257(1)**, 227–234 (2005)

[I-F] A. Inoue and T. Funaki: A new derivation of the Navier-Stokes equation. *Comm. Math. Phys.* **65**, 83–90 (1979)

[K] H. Kunita: *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, 1990

[M] P. Malliavin: The canonic diffusion above the diffeomorphism group of the circle, *C. R. Acad. Sci. Paris*, t. **329**, Série I, p. 325–329, (1999)

[N-Y-Z] T. Nakagomi, K. Yasue and J. C. Zambrini: Stochastic variational derivations of the Navier-Stokes equation. *Letters in Math. Phys.* **160**, 337–365 (1981)

[S] S. Shkoller: Geometry and curvature of diffeomorphism group with H^1 metric and mean hydrodynamics. *J. Funct. Anal.* (1998)

[U] A. S. Üstünel: Stochastic analysis on Lie groups. *Stoch. Anal. and Rel. Topics* **VI**, *Progr. Probab.* **42**, Birkhäuser Boston, 1998

[Y] K. Yasue: A variational principle for the Navier-Stokes equation. *J. Funct. Anal.* **51(2)**, 133–141, (1983)

GFM e Dep. de Matemática FCT-UNL, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

E-mail address: cipriano@cii.fc.ul.pt

GFM e Dep. de Matemática IST-TUL, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

E-mail address: abcruz@math.ist.utl.pt