The exact and near-exact distributions of the product of independent Beta random variables whose second parameter is rational

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Abstract

In this paper the exact distribution of the logarithm of the product of a given number of independent Beta random variables whose second parameter is rational is obtained under the form of a Generalized Integer Gamma distribution and near-exact distributions are obtained either as Generalized Near-Integer Gamma distributions or mixtures of these distributions. As particular cases of interest we have the exact and near-exact distributions of the generalized Wilks Lambda statistic.

Key words: Generalized Integer Gamma (GIG) distribution, Generalized Near-Integer Gamma distribution (GNIG) distribution of sum of independent Logbeta random variables, Wilks Lambda

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1. Introduction

The aim of this paper is to obtain the exact distribution of the logarithm of the product of a given number of independent Beta distributed random variables whose second parameter is rational, under the form of a Generalized Integer Gamma (GIG) distribution or near-exact distributions of such product under the form of a Generalized Near-Integer Gamma (GNIG) distribution or mixtures of these distributions. Results are obtained under fairly wide conditions. The only restrictions imposed are that the step in the first parameter has to be equal to the reciprocal of that same integer value, that is, if we write the second parameter in the Beta distributions as m/k, for $m, k \in I\!N$ (we will

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be using $I\!N$ to represent the set of positive integers, Q the set of rationals, with Q^+ representing the set of positive rationals, and $I\!R$ to represent the set of all reals, with $I\!R^+$ representing the set of positive reals), then the step in the first parameter of the Beta random variables involved in the product has to be equal to $1/k$, and in order to be able to obtain the exact distribution we need the number of such Beta distributed random variables to be a multiple of the integer in the denominator of the second parameter, say, nk, with $n \in I\!N$.

As a summary, we may say that the aim of this paper is to obtain the exact or near-exact distribution of statistics of the form

$$W' = \prod_{j=1}^{p} Y_j \qquad \text{and} \qquad W = -\log W' \tag{1.1}$$

where

$$Y_j \sim B\left(a - \frac{j}{k}, \frac{m}{k}\right) \qquad j = 1, \ldots, p$$

with $m, k \in I\!N$.

Since such distributions will assume the form of Generalized Integer Gamma (GIG) or Generalized Near-Integer Gamma (GNIG) distributions, we introduce now these distributions.

Let X be a random variable with a Gamma distribution with shape parameter r and rate parameter λ (we call this parameter 'rate', given the relation it has, for integer r, with the rate of a Poisson process). Then the pdf (probability density function) of X will be given by

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda x} x^{r-1}, \qquad (x > 0)$$

and we will represent this fact by

$$X \sim \Gamma(r, \lambda).$$

Let

$$X_i \sim \Gamma(r_i, \lambda_i) \qquad i = 1, \ldots, n$$

be n independent Gamma distributed random variables with $r_i \in I\!N$, for $i \in \{1, \ldots, n\}$ and all λ_i different. Then the distribution of the random variable

$$Z = \sum_{i=1}^{n} X_i$$

is what Coelho (1998, 2004) called a GIG distribution of depth n. The pdf and cdf (cumulative distribution function) of Z are respectively given by

$$f_Z(z) = K \sum_{j=1}^{n} P_j(z)\, e^{-\lambda_j z} \tag{1.2}$$

and

$$F_Z(z) \;=\; 1 - K \sum_{j=1}^{n} P_j^*(z)\, e^{-\lambda_j\, z} \tag{1.3}$$

where

$$K = \prod_{j=1}^{n} \lambda_j^{r_j} \;, \qquad P_j(z) = \sum_{k=1}^{r_j} c_{jk}\, z^{k-1} \tag{1.4}$$

and

$$P_j^*(z) = \sum_{k=1}^{r_j} c_{jk}\,(k-1)! \sum_{i=0}^{k-1} \frac{z^i}{i!\,\lambda_i^{k-i}}$$

with

$$c_{j,r_j} \;=\; \frac{1}{(r_j-1)!} \prod_{\substack{i=1\\ i\neq j}}^{n} (\lambda_i - \lambda_j)^{-r_i} \;, \qquad j=1,\ldots,n\,, \tag{1.5}$$

and

$$c_{j,r_j-k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(r_j-k+i-1)!}{(r_j-k-1)!}\, R(i,j,n)\, c_{j,r_j-(k-i)}\,, \quad \begin{array}{l}(k=1,\ldots,r_j-1)\\ (j=1,\ldots,n)\end{array} \tag{1.6}$$

where

$$R(i,j,n) = \sum_{\substack{k=1\\ k\neq j}}^{n} r_k\,(\lambda_j - \lambda_k)^{-i} \quad (i=1,\ldots,r_j-1)\,. \tag{1.7}$$

We will denote the fact that the random variable Z has a GIG distribution of depth n, with the above pdf and cdf by

$$Z \;\sim\; GIG(r_1,\ldots,r_n;\; \lambda_1,\ldots,\lambda_n)\,.$$

The GNIG distribution (Coelho, 2004) arises as the distribution of the sum of two independent random variables, one with a GIG distribution and the other with a Gamma distribution.

Let

$$Z_1 \;\sim\; GIG(r_1,\ldots,r_n;\lambda_1,\ldots,\lambda_n)$$

and

$$Z_2 \;\sim\; \Gamma(r,\lambda)$$

be two independent random variables. Let further $\lambda \neq \lambda_i$ $(i=1,\ldots,n)$ and $r \in I\!R^+ \backslash I\!N$. Then the random variable

$$Z = Z_1 + Z_2$$

has a GNIG distribution of depth $n+1$, with pdf

$$f_Z(z) = K \lambda^r \sum_{j=1}^{n} Q_j(z)\, e^{-\lambda_j z}$$

and cdf

$$F_Z(z) = \lambda^r \frac{z^r}{\Gamma(r+1)}\, {}_1F_1(r, r+1; -\lambda z) - K\lambda^r \sum_{j=1}^{n} Q_j^*(z)\, e^{-\lambda_j z}$$

where

$$Q_j(z) = \sum_{k=1}^{r_j} c_{jk} \frac{\Gamma(k)}{\Gamma(k+r)}\, z^{k+r-1}\, {}_1F_1\left(r, k+r; (\lambda_j - \lambda)z\right)$$

and

$$Q_j^*(z) = \sum_{k=1}^{r_j} \frac{c_{jk}}{\lambda_j^k}\, \Gamma(k) \sum_{i=0}^{k-1} \frac{z^{r+i}\, \lambda_j^i}{\Gamma(r+i+1)}\, {}_1F_1\left(r, r+i+1; (\lambda_j - \lambda)z\right)$$

with K and c_{jk} given respectively by (1.4) and (1.5) through (1.7) above, and

$$\begin{aligned} {}_1F_1(a, b; x) &= \frac{\Gamma(b)}{\Gamma(b-a)\,\Gamma(a)} \int_0^1 e^{xt}\, t^{a-1}(1-t)^{b-a-1}\, dt \\ &= \frac{\Gamma(b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)}{\Gamma(b+i)} \frac{x^i}{i!} \end{aligned}$$

being the Kummer confluent hypergeometric function.

We will denote the fact that the random variable Z has a GNIG distribution of depth $n+1$, with the above pdf and cdf by

$$Z \sim GNIG(r_1, \ldots, r_n, r;\ \lambda_1, \ldots, \lambda_n, \lambda)\,.$$

Since in section 3 we will be dealing with mixtures of these distributions we will now settle the notation for such mixtures. A mixture of k GNIG distributions, the j-th of which has weight θ_j and depth m_j, will be denoted by

$$MkGNIG\left(\theta_1; r_{11}, \ldots, r_{1m_1}; \lambda_{11}, \ldots, \lambda_{1m_1} | \ldots | \theta_k; r_{k1}, \ldots, r_{km_k}; \lambda_{k1}, \ldots, \lambda_{km_k}\right).$$

2. The exact distribution

The following Theorem gives the exact distribution of W or W' in (1.1) when $p = nk$, for some $n \in I\!N$, under a much manageable form.

THEOREM 1: *Let $m, k, n \in I\!N$ and $a \in I\!R^+$, with $a > n$ and $m/k \in \mathbb{Q}^+ \backslash I\!N$, and let*

$$Y_j \sim B\left(a - \frac{j}{k}, \frac{m}{k}\right) \qquad j = 1, \ldots, nk$$

be nk independent Beta distributed random variables. Let further

$$W' = \prod_{j=1}^{nk} Y_j \qquad \textit{and} \qquad W = -\log W' \,. \tag{2.1}$$

Then the exact distribution of W is a GIG distribution of depth $m + k(n-1)$ with shape parameters

$$r_j = \begin{cases} h_j & j = 1, \dots, k \\ \\ h_j + r_{j-k} & j = k+1, \dots, m + k(n-1) \end{cases} \tag{2.2}$$

where

$$h_j = \begin{cases} 1 & j = 1, \dots, \min(nk, m) \\ 0 & j = 1 + \min(nk, m), \dots, \max(nk, m) \\ -1 & j = 1 + \max(nk, m), \dots, m + k(n-1) \end{cases} \tag{2.3}$$

or

$$h_j = (\#\ \textit{of elements in}\ \{nk, m\} \geq j) - 1 \,,$$

and rate parameters

$$a - n + \frac{j-1}{k} \qquad j = 1, \dots, m + k(n-1) \,,$$

that is,

$$W \sim GIG\Big(\underbrace{r_1, r_2, \dots, r_{m+k(n-1)}}_{m+k(n-1)} ; \underbrace{a-n, a-n+\tfrac{1}{k}, \dots, a-1+\tfrac{m-1}{k}}_{m+k(n-1)}\Big) \,. \tag{2.4}$$

Proof: Since we know that if

$$X \sim B(a, b)$$

then the h-th moment of X is

$$E\left(X^h\right) = \frac{\Gamma(a+b)}{\Gamma(a)} \frac{\Gamma(a+h)}{\Gamma(a+b+h)} \qquad (h > -a) \tag{2.5}$$

and since in (2.1) the Gamma functions are defined for any strictly complex h, and given the independence of the nk random variables Y_j in (1.4), using the Gauss multiplication formula for the Gamma function,

$$\Gamma(nz) = (2\pi)^{\frac{1}{2}(1-n)} \, n^{nz - \frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right)$$

and yet the fact that for $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{j=0}^{n-1}(a+j)$$

we may write the characteristic function of W as

$$\begin{aligned}
\Phi_W(t) &= E\left(e^{itW}\right) = E\left(e^{-it\,\log W'}\right) \\
&= E\left(W'^{-it}\right) = \prod_{j=1}^{nk} E\left(Y_j^{-it}\right) \\
&= \prod_{j=1}^{nk} \frac{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}\right)}{\Gamma\left(a-\frac{j}{k}\right)} \frac{\Gamma\left(a-\frac{j}{k}-it\right)}{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}-it\right)} \\
&= \prod_{j=1}^{n} \frac{\Gamma\left(k(a-j)+m\right)}{\Gamma\left(k(a-j)\right)} \frac{\Gamma\left(k(a-j)-kit\right)}{\Gamma\left(k(a-j)+m-kit\right)} \\
&= \prod_{j=1}^{n}\prod_{s=0}^{m-1} \left(k(a-j)+s\right)\,\left(k(a-j)+s-kit\right)^{-1} \\
&= \prod_{j=1}^{n}\prod_{s=0}^{m-1} \left(a-j+\frac{s}{k}\right)\,\left(a-j+\frac{s}{k}-it\right)^{-1} \\
&= \prod_{j=1}^{m+k(n-1)} \left(a-n+\frac{j-1}{k}\right)^{r_j}\left(a-n+\frac{j-1}{k}-it\right)^{-r_j},
\end{aligned}$$

that is the characteristic function of the sum of $m+k(n-1) = m-k+nk$ independent Gamma random variables with integer shape parameters r_j given by (2.2) and (2.3) and rate parameters

$$a-n+\frac{j-1}{k} \qquad j=1,\ldots,m+k(n-1)\,,$$

that is the GIG distribution in (2.4). ∎

The exact pdf and cdf of W are thus easily derived from (1.2) and (1.3), and from these, the exact pdf and cdf of $W' = e^{-W}$, are also easily obtained.

In the above Theorem, nk and m are interchangeable. This result is stated in the following Theorem.

THEOREM 2: *Let $m, k, n \in \mathbb{N}$ and $a \in \mathbb{R}^+$ with $a > n$ and let*

$$Y_j^* \sim B\left(a-n+\frac{m}{k}-\frac{j}{k}, n\right) \qquad j=1,\ldots,m$$

be m independent Beta distributed random variables. Let further

$$W_2 = \prod_{j=1}^{m} Y_j^* \qquad and \qquad W_2 = -\log W_2' \,.$$

Then W_2' and W_2 have, respectively, the same distribution as W' and W in (2.1) *in Theorem 1.*

Proof: Using the same technique and arguments used in the proof of Theorem 3, we may write

$$\begin{aligned}
\Phi_{W_2}(t) &= E\left(e^{itW_2}\right) \;=\; E\left(e^{-it\,\log W_2'}\right) \\
&= E\left(W_2'^{-it}\right) \;=\; \prod_{j=1}^{nk} E\left(Y_j^{*-it}\right) \\
&= \prod_{s=1}^{m} \frac{\Gamma\left(a-n+\frac{m}{k}-\frac{s}{k}+n\right)}{\Gamma\left(a-n+\frac{m}{k}-\frac{s}{k}\right)} \frac{\Gamma\left(a-n+\frac{m}{k}-\frac{s}{k}-it\right)}{\Gamma\left(a-n+\frac{m}{k}-\frac{s}{k}+n-it\right)} \\
&= \prod_{s=1}^{m} \frac{\Gamma\left(a-n+\frac{s-1}{k}+n\right)}{\Gamma\left(a-n+\frac{s-1}{k}\right)} \frac{\Gamma\left(a-n+\frac{s-1}{k}-it\right)}{\Gamma\left(a-n+\frac{s-1}{k}+n-it\right)} \\
&= \prod_{s=1}^{m}\prod_{j=0}^{n-1} \left(a-n+j+\frac{s-1}{k}\right)\left(a-n+j+\frac{s-1}{k}-it\right)^{-1} \\
&= \prod_{s=1}^{m}\prod_{j=1}^{n} \left(a-n-1+j+\frac{s-1}{k}\right)\left(a-n-1+j+\frac{s-1}{k}-it\right)^{-1} \\
&= \prod_{j=1}^{n}\prod_{s=0}^{m-1} \left(a-(n+1-j)+\frac{s}{k}\right)\left(a-(n+1-j)+\frac{s}{k}-it\right)^{-1} \\
&= \prod_{j=1}^{n}\prod_{s=0}^{m-1} \left(a-j+\frac{s}{k}\right)\left(a-j+\frac{s}{k}-it\right)^{-1} \\
&= \Phi_W(t)\,,
\end{aligned}$$

so that the distributions of W_2 and W are the same and thus so are the distributions of $W_2' = e^{-W_2}$ and $W' = e^{-W}$. ∎

Given the result in Theorem 2, the interchangeability of m and nk in Theorem 3 becomes more evident if we take in Theorem 1

$$a = a^* + n - \frac{m}{k} \qquad \text{with} \qquad a^* > \frac{m}{k}$$

so that then in Theorem 2 we have $a - n + \frac{m}{k} = a^*$, with $a^* > \frac{m}{k}$, so that we may write

$$Y_j^* \sim B\left(a^* - \frac{j}{k}, n\right) \qquad j = 1, \dots, m\,.$$

Yet under another equivalent view, the duality is in the fact that in Theorem 3 we have

$$Y_j \sim B\left(a^* + n - \frac{m}{k} - \frac{j}{k}, \frac{m}{k}\right) \qquad j = 1, \dots, nk$$

with $a^* > \frac{m}{k}$, while in Theorem 2 we have

$$Y_j^* \sim B\left(a + \frac{m}{k} - n - \frac{j}{k}, n\right) \qquad j = 1, \dots, m$$

with $a > n$.

3. Near-exact distributions

When we are not able to write p as nk for some $n \in I\!N$, then we are not able to obtain the exact distribution of W or W' under a manageable form. For these cases, the use of a near-exact distribution is a much adequate option.

The following Theorem gives near-exact distributions for W when $p \neq nk$, under the form of a GNIG distribution or mixtures of GNIG distributions.

Theorem 3: *Let $m, k \in I\!N$ and $a \in I\!R^+$, with $m/k \in Q^+ \backslash I\!N$, and let*

$$Y_j \sim B\left(a - \frac{j}{k}, \frac{m}{k}\right) \qquad j = 1, \dots, p$$

be p independent random variables, where $a > p/k$ and $p \neq n^* k$ for any $n^* \in I\!N$. Let further

$$W' = \prod_{j=1}^{p} Y_j\,, \qquad W = -\log W' \quad \textit{and} \quad n = \lfloor p/k \rfloor\,.$$

Then, near-exact distributions for W may be obatined under the form of a GNIG distribution of depth $m+k(n-1)+1$ or mixtures of 2 or 3 such distributions. More precisely, we may write

$$W \stackrel{ne}{\approx} GNIG\left(r_1, \dots, r_{m+k(n-1)}, r; \lambda_1, \dots, \lambda_{m+k(n-1)}, \lambda\right)\,, \tag{3.1}$$

$$W \stackrel{ne}{\approx} M2GNIG\left(\theta; r_1, \dots, r_{m+k(n-1)}, r^*; \lambda_1, \dots, \lambda_{m+k(n-1)}, \lambda_1^* \right| \\ \left. 1-\theta; r_1, \dots, r_{m+k(n-1)}, r^*; \lambda_1, \dots, \lambda_{m+k(n-1)}, \lambda_2^*\right)\,, \tag{3.2}$$

and

$$
\begin{aligned}
W \stackrel{ne}{\approx} M3GNIG \Big(& \theta_1; r_1, \ldots, r_{m+k(n-1)}, s; \lambda_1, \ldots, \lambda_{m+k(n-1)}, \nu_1 | \\
& \theta_2; r_1, \ldots, r_{m+k(n-1)}, s; \lambda_1, \ldots, \lambda_{m+k(n-1)}, \nu_2 | \\
& 1-\theta_1-\theta_2; r_1, \ldots, r_{m+k(n-1)}, s; \lambda_1, \ldots, \lambda_{m+k(n-1)}, \nu_3 \Big) ,
\end{aligned} \tag{3.3}
$$

where '$\stackrel{ne}{\approx}$' is to be read as 'is near-exactly distributed as', $0 < \theta, \theta_1, \theta_2 < 1$ and where for $j = 1, \ldots, m + k(n-1)$,

$$
r_j = \begin{cases} h_j & j = 1, \ldots, k \\ h_j + r_{j-k} & j = k+1, \ldots, m + k(n-1) \end{cases} \tag{3.4}
$$

where

$$
h_j = \begin{cases} 1 & j = 1, \ldots, \min(nk, m) \\ 0 & j = 1 + \min(nk, m), \ldots, \max(nk, m) \\ -1 & j = 1 + \max(nk, m), \ldots, m + k(n-1) \end{cases} \tag{3.5}
$$

or

$$
h_j = (\#\ \textit{of elements in}\ \{nk, m\} \geq j) - 1 ,
$$

$$
\lambda_j = a - \frac{p}{k} + \frac{j-1}{k} \qquad j = 1, \ldots, m + k(n-1) ,
$$

and r and λ in (3.1) defined in such a way that the two first moments of the exact and near-exact distribution coincide, that is,

$$
r = \frac{\mu_1^2}{\mu_2 - \mu_1^2} \qquad \text{and} \qquad \lambda = \frac{\mu_1}{\mu_2 - \mu_1^2} \tag{3.6}
$$

where

$$
\mu_h = \frac{d^h}{dt^h} \left(\prod_{j=1}^{p-nk} \frac{\Gamma\left(a - \frac{j}{k} + \frac{m}{k}\right)}{\Gamma\left(a - \frac{j}{k}\right)} \frac{\Gamma\left(a - \frac{j}{k} - it\right)}{\Gamma\left(a - \frac{j}{k} + \frac{m}{k} - it\right)} \right) \Bigg|_{t=0} \qquad (h = 1, 2) \tag{3.7}
$$

and θ, r^*, λ_1^* and λ_2^* in (3.2) are defined in such a way that

$$
\begin{aligned}
& \frac{d^h}{dt^h} \left(\prod_{j=1}^{p-nk} \frac{\Gamma\left(a - \frac{j}{k} + \frac{m}{k}\right)}{\Gamma\left(a - \frac{j}{k}\right)} \frac{\Gamma\left(a - \frac{j}{k} - it\right)}{\Gamma\left(a - \frac{j}{k} + \frac{m}{k} - it\right)} \right) \Bigg|_{t=0} \\
& \qquad = \theta \frac{\Gamma(r^* + h)}{\Gamma(r^*)} \lambda_1^{*-h} + (1-\theta) \frac{\Gamma(r^* + h)}{\Gamma(r^*)} \lambda_2^{*-h} \qquad (h = 1, \ldots, 4)
\end{aligned} \tag{3.8}
$$

and θ_1*,* θ_2*,* s*,* ν_1*,* ν_2 *and* ν_3 *in* (3.3) *are defined in such a way that*

$$\frac{d^h}{dt^h}\left(\prod_{j=1}^{p-nk}\frac{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}\right)}{\Gamma\left(a-\frac{j}{k}\right)}\,\frac{\Gamma\left(a-\frac{j}{k}-it\right)}{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}-it\right)}\right)\Bigg|_{t=0}$$
$$=\theta\,\frac{\Gamma(s{+}h)}{\Gamma(s)}\,\nu_1^{-h}+\theta_2\,\frac{\Gamma(s{+}h)}{\Gamma(s)}\,\nu_2^{-h}+(1{-}\theta_1{-}\theta_2)\,\frac{\Gamma(s{+}h)}{\Gamma(s)}\,\nu_3^{-h}$$
$$(h=1,\ldots,6). \tag{3.9}$$

Proof: Let $n{=}\lfloor p/k\rfloor$. Then, we may write the characteristic function of W as

$$\begin{aligned}
\Phi_W(t) &= \prod_{j=1}^{p}\frac{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}\right)}{\Gamma\left(a-\frac{j}{k}\right)}\,\frac{\Gamma\left(a-\frac{j}{k}-it\right)}{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}-it\right)}\\
&= \prod_{j=1}^{p-nk}\frac{\Gamma\left(a{-}\frac{j}{k}{+}\frac{m}{k}\right)}{\Gamma\left(a{-}\frac{j}{k}\right)}\,\frac{\Gamma\left(a{-}\frac{j}{k}{-}it\right)}{\Gamma\left(a{-}\frac{j}{k}{+}\frac{m}{k}{-}it\right)}\\
&\qquad\qquad \prod_{j=p-nk+1}^{p}\frac{\Gamma\left(a{-}\frac{j}{k}{+}\frac{m}{k}\right)}{\Gamma\left(a{-}\frac{j}{k}\right)}\,\frac{\Gamma\left(a{-}\frac{j}{k}{-}it\right)}{\Gamma\left(a{-}\frac{j}{k}{+}\frac{m}{k}{-}it\right)}\\
&= \prod_{j=1}^{p-nk}\frac{\Gamma\left(a{-}\frac{j}{k}{+}\frac{m}{k}\right)}{\Gamma\left(a{-}\frac{j}{k}\right)}\,\frac{\Gamma\left(a{-}\frac{j}{k}{-}it\right)}{\Gamma\left(a{-}\frac{j}{k}{+}\frac{m}{k}{-}it\right)}\\
&\qquad\qquad \prod_{j=1}^{nk}\frac{\Gamma\left(a{-}\frac{j+p-nk}{k}{+}\frac{m}{k}\right)}{\Gamma\left(a{-}\frac{j+p-nk}{k}\right)}\,\frac{\Gamma\left(a{-}\frac{j+p-nk}{k}{-}it\right)}{\Gamma\left(a{-}\frac{j+p-nk}{k}{+}\frac{m}{k}{-}it\right)}\\
&= \prod_{j=1}^{p-nk}\frac{\Gamma\left(a{-}\frac{j}{k}{+}\frac{m}{k}\right)}{\Gamma\left(a{-}\frac{j}{k}\right)}\,\frac{\Gamma\left(a{-}\frac{j}{k}{-}it\right)}{\Gamma\left(a{-}\frac{j}{k}{+}\frac{m}{k}{-}it\right)}\\
&\qquad\qquad \prod_{j=1}^{nk}\frac{\Gamma\left(a{+}n{-}\frac{p}{k}{-}\frac{j}{k}{+}\frac{m}{k}\right)}{\Gamma\left(a{+}n{-}\frac{p}{k}{-}\frac{j}{k}\right)}\,\frac{\Gamma\left(a{+}n{-}\frac{p}{k}{-}\frac{j}{k}{-}it\right)}{\Gamma\left(a{+}n{-}\frac{p}{k}{-}\frac{j}{k}{+}\frac{m}{k}{-}it\right)}\\
&= \underbrace{\prod_{j=1}^{p-nk}\frac{\Gamma\left(a{-}\frac{j}{k}{+}\frac{m}{k}\right)}{\Gamma\left(a{-}\frac{j}{k}\right)}\,\frac{\Gamma\left(a{-}\frac{j}{k}{-}it\right)}{\Gamma\left(a{-}\frac{j}{k}{+}\frac{m}{k}{-}it\right)}}_{\Phi_1(t)}\\
&\qquad\qquad \underbrace{\prod_{j=1}^{m+k(n-1)}\left(a{-}\frac{p}{k}{+}\frac{j-1}{k}\right)^{r_j}\left(a{-}\frac{p}{k}{+}\frac{j-1}{k}{-}it\right)^{-r_j}}_{\Phi_2(t)}
\end{aligned} \tag{3.10}$$

with r_j ($j=1,\ldots,m{+}k(n{-}1)$) given by (3.1) and (3.2), and where in passing from

$$\Phi_2(t)=\prod_{j=1}^{nk}\frac{\Gamma\left(a+n-\frac{p}{k}-\frac{j}{k}+\frac{m}{k}\right)}{\Gamma\left(a+n-\frac{p}{k}-\frac{j}{k}\right)}\,\frac{\Gamma\left(a+n-\frac{p}{k}-\frac{j}{k}-it\right)}{\Gamma\left(a+n-\frac{p}{k}-\frac{j}{k}+\frac{m}{k}-it\right)}$$

to

$$\Phi_2(t)=\prod_{j=1}^{m+k(n-1)}\left(a-\frac{p}{k}+\frac{j-1}{k}\right)^{r_j}\left(a-\frac{p}{k}+\frac{j-1}{k}-it\right)^{-r_j}$$

we used the result coming from the proof of Theorem 1, when handling the characteristic function of W, replacing a by $a+n-p/k$.

Then in (3.10) we may approximate $\Phi_1(t)$ in several different ways, as for example by

$$\Phi_1^*(t)=\lambda^r(\lambda-it)^{-r} \tag{3.11}$$

in such a way that

$$\frac{d^h}{dt^h}\,\Phi_1(t)=\frac{d^h}{dt^h}\,\Phi_1^*(t)\qquad\text{for}\qquad h=1,2$$

yielding λ and r given by (3.6) and (3.7) in the Theorem statement, or by

$$\Phi_1^{**}(t)=\theta\,\lambda_1^{*r^*}(\lambda_1^*-it)^{-r^*}+(1-\theta)\lambda_2^{*r^*}(\lambda_2^*-it)^{-r^*} \tag{3.12}$$

in such a way that

$$\frac{d^h}{dt^h}\,\Phi_1(t)=\frac{d^h}{dt^h}\,\Phi_1^{**}(t)\qquad\text{for}\qquad h=1,\dots,4$$

yielding θ, λ_1^*, λ_2^* and r^* given by (3.8) in the Theorem statement, or yet by

$$\Phi_1^{***}(t)=\theta_1\,\nu_1^s(\nu_1-it)^{-s}+\theta_2\,\nu_2^s(\nu_2-it)^{-s}+(1-\theta_1-\theta_2)\nu_3^s(\nu_3-it)^{-s} \tag{3.13}$$

in such a way that

$$\frac{d^h}{dt^h}\,\Phi_1(t)=\frac{d^h}{dt^h}\,\Phi_1^{***}(t)\qquad\text{for}\qquad h=1,\dots,6$$

yielding θ_1, θ_2, ν_1, ν_2, ν_3 and s given by (3.9) in the Theorem statement.

Then if we approximate $\Phi_W(t)$ by keeping $\Phi_2(t)$ unchanged and replacing $\Phi_1(t)$ by the characteristic functions in (3.11), (3.12) or (3.13), we will get near-exact approximations of $\Phi_W(t)$ which are respectively the GNIG distribution, the mixture of two GNIG distributions or the mixture of three GNIG distributions respectively in (3.1), (3.2) and (3.3) in the statement of the Theorem, with these distributions yielding respectively the first two, four and six moments equal to the exact ones. ∎

We should stress that the expressions in Theorem 3 will work and give the correct result for any value of $n=\lfloor p/k\rfloor\geq 0$, since at first sight this may be not completely clear for $n=1$ and $n=0$.

We will define, as it is usual, that a product that is empty of terms, that is, a product which upper limit is smaller than its lower limit, is equal to 1. Given the definition of $n(=\lfloor p/k \rfloor)$ and given that we consider $p \neq nk$, we will always have $p-nk \geq 1$, so that the product that defines $\Phi_1(t)$ is never empty, while the product that defines $\Phi_2(t)$ is never empty for $n \geq 1$. However, for $n=1$, there may be some questions about the definition of the shape parameters r_j given by (3.4) and (3.5), since we may either have $m>k$ or $m<k$. If $n=1$ and $m>k$, there is clearly no problem with the definitions in (3.4) and (3.5), with the upper limit of the index j for r_j being m, both in (3.4) and (3.10). If $m<k$, (3.4) will give

$$h_j = \begin{cases} 1 & j=1,\dots,m \\ 0 & j=m+1,\dots,k \end{cases}$$

and (3.5)

$$r_j = h_j \qquad j=1,\dots,k$$

with only the first m of them, all equal to 1, being used in $\Phi_2(t)$ in (3.10).

For $n=0$, $\Phi_2(t)$ doesn't exist, or rather, it is equal to 1 and the whole $\Phi_W(t)$, in this case reduced to $\Phi_1(t)$, is asymptotically approximated by $\Phi_1^*(t)$, $\Phi_1^{**}(t)$ or $\Phi_1^{***}(t)$. For $n=0$, if $m<k$, $\Phi_2(t)$ will clearly reduce to 1 since in this case the upper limit of the product that defines $\Phi_2(t)$ will be $m-k<0$ and thus this product reduces to 1. However, if $n=0$ and $m>k$, the product defining $\Phi_2(t)$ will have its upper limit equal to $m-k>0$ and it may seem that in this case $\Phi_2(t)$ would not reduce to 1. But indeed, in this case, all the shape parameters r_j in $\Phi_2(t)$ will be equal to zero since from (3.5) we will have

$$h_j = 0\,, \qquad j=1,\dots,,m-k$$

and from (3.4),

$$r_j = h_j = 0\,, \qquad j=1,\dots,m-k\,.$$

Another issue related with Theorem 3 is that one may question why was it that in the proof of Theorem 3 the decomposition of the characteristic function of W was not done in a much simpler way, like

$$\begin{aligned}\Phi_W(t) &= \prod_{j=1}^{p} \frac{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}\right)}{\Gamma\left(a-\frac{j}{k}\right)} \frac{\Gamma\left(a-\frac{j}{k}-it\right)}{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}-it\right)} \\ &= \prod_{j=1}^{nk} \frac{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}\right)}{\Gamma\left(a-\frac{j}{k}\right)} \frac{\Gamma\left(a-\frac{j}{k}-it\right)}{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}-it\right)} \\ &\qquad\qquad \prod_{j=nk+1}^{p} \frac{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}\right)}{\Gamma\left(a-\frac{j}{k}\right)} \frac{\Gamma\left(a-\frac{j}{k}-it\right)}{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}-it\right)}\end{aligned}$$

$$= \underbrace{\prod_{j=1}^{m+k(n-1)} \left(a-n+\frac{j-1}{k}\right)^{r_j} \left(a-n+\frac{j-1}{k}-it\right)^{-r_j}}_{\tilde{\Phi}_2(t)}$$

$$\underbrace{\prod_{j=nk+1}^{p} \frac{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}\right)}{\Gamma\left(a-\frac{j}{k}\right)} \frac{\Gamma\left(a-\frac{j}{k}-it\right)}{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}-it\right)}}_{\tilde{\Phi}_1(t)} \tag{3.14}$$

with r_j ($j = 1, \ldots, m + k(n-1)$) given by (2.2) and (2.3), allowing a direct use of the result obtained in the proof of Theorem 1, when passing from

$$\tilde{\Phi}_2(t) = \prod_{j=1}^{nk} \frac{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}\right)}{\Gamma\left(a-\frac{j}{k}\right)} \frac{\Gamma\left(a-\frac{j}{k}-it\right)}{\Gamma\left(a-\frac{j}{k}+\frac{m}{k}-it\right)}$$

to

$$\tilde{\Phi}_2(t) = \prod_{j=1}^{m+k(n-1)} \left(a-n+\frac{j-1}{k}\right)^{r_j} \left(a-n+\frac{j-1}{k}-it\right)^{-r_j},$$

and approximating then $\Phi_W(t)$ by keeping $\tilde{\Phi}_2(t)$ unchanged and replacing $\tilde{\Phi}_1(t)$ by the characteristic functions in (3.11), (3.12) or (3.13), now with the parameters respectively defined in such a way that

$$\left.\frac{d^h}{dt^h}\tilde{\Phi}_1(t)\right|_{t=0} = \frac{\Gamma(r+h)}{\Gamma(r)}\lambda^{-h} \qquad (h=1,2)$$

$$\left.\frac{d^h}{dt^h}\tilde{\Phi}_1(t)\right|_{t=0} = \theta\,\frac{\Gamma(r^*+h)}{\Gamma(r^*)}\lambda_1^{*-h} + (1-\theta)\,\frac{\Gamma(r^*+h)}{\Gamma(r^*)}\lambda_2^{*-h} \qquad (h=1,\ldots,4)$$

or

$$\left.\frac{d^h}{dt^h}\tilde{\Phi}_1(t)\right|_{t=0} = \theta\,\frac{\Gamma(s+h)}{\Gamma(s)}\nu_1^{-h} + \theta_2\,\frac{\Gamma(s+h)}{\Gamma(s)}\nu_2^{-h} + (1-\theta_1-\theta_2)\,\frac{\Gamma(s+h)}{\Gamma(s)}\nu_3^{-h} \qquad (h=1,\ldots,6),$$

obtaining this way near-exact approximations for $\Phi_W(t)$ which would be similar to the GNIG distribution, the mixture of two GNIG distributions or the mixture of three GNIG distributions respectively in (3.1), (3.2) and (3.3) in the statement of the Theorem, with

$$\lambda_j = a - n + \frac{j-1}{k} \qquad j = 1, \ldots, m + k(n-1),$$

still yielding respectively the first two, four and six moments equal to the exact ones? Well, the answer is: because the way the decomposition of the characteristic function of W was done in the proof of Theorem 3 yields much better approximations, as it may be confirmed by observing the results in the next section. But not only this. Also, while the near-exact distributions obtained in Theorem 3 display a marked asymptotic behaviour for increasing values of p which yielding the same values for $p-nk$ show higher values of n, these latter distributions would have a worse performance for these increasing values of p, as it may be observed by analysing the results obtained in the numerical studies in the next section. This is due to the fact that in either case $\Phi_1(t)$ is the characteristic function of either a Logbeta random variable or the sum of independent Logbeta random variables and these distributions are much better approximated by Gamma random variables or mixtures of Gamma random variables when ther first parameter in the Logbeta random variables has higher values.

In the next section we will carry out a numerical study in order to evaluate the performance of the near-exact approximations proposed.

4. Numerical Studies

A rather extensive numerical study was carried out for different values of each of the parameters in the distributions, a, m, k and p, in order to better assess the proximity between the exact and the near-exact distributions. All cases considered are cases for which we do not have the exact distribution given by Theorem 1, that is, cases for which $p/k \in \mathbb{Q}^+\backslash\mathbb{N}$. All studies were carried out for the distribution of the statistic W in (1.1).

In order to evaluate the proximity between the exact and the near-exact distributions, and given that the exact pdf or cdf is not known, two measures based on the characteristic function are used. These measures are

$$\Delta_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\Phi_W(t) - \Phi^*(t)|\, dt \quad \text{and} \quad \Delta_2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\frac{\Phi_W(t) - \Phi^*(t)|}{t}\right| dt$$

where $\Phi_W(t)$ represents the exact characteristic function of W and $\Phi^*(t)$ the characteristic function corresponding to the near-exact distribution under study. Both Δ_1 and Δ_2 are directly derived from the inversion formulas respectively for the pdf and the cdf, with

$$\max_{w>0} |F_W(w) - F^*(w)| \leq \Delta_2 \qquad \text{and} \qquad \max_{w>0} |f_W(w) - f^*(w)| \leq \Delta_1\,,$$

where $F_W(w)$ and $f_W(w)$ represent respectively the exact cdf and pdf of W evaluated at $w > 0$ and $F^*(w)$ and $f^*(w)$ represent respectively the near-exact cdf and pdf of W corresponding to the characteristic function $\Phi^*(t)$. The measure Δ_2 was already used by Grilo & Coelho (2007) and it may also be directly derived from the Berry-Esseen bound (Grilo & Coelho, 2007), while the measure Δ_1 is a slight modification of a measure used by the same authors, in order to enable us to obtain a bound on the absolute value of the difference between the exact and near-exact pdf's.

In all tables we denote by $GNIG$, $M2GNIG$ and $M3GNIG$ the three near-exact distributions with similar acronyms in Theorem 3 and by $GNIG*$, $M2GNIG*$ and $M3GNIG*$ the three 'corresponding' near-exact distributions obtained by following the procedure outlined after Theorem 3, by replacing $\tilde{\Phi}_1(t)$ in $\Phi_W(t)$ respectively by $\Phi_1^*(t)$, $\Phi_1^{**}(t)$ and $\Phi_1^{***}(t)$ in (3.11), (3.12) and (3.13).

Table 1 – Values of Δ_2 for $m = 5$ and several values of a, k and p.

a	m	k	p	$GNIG*$ $GNIG$	$M2GNIG*$ $M2GNIG$	$M3GNIG*$ $M3GNIG$
7.3	5	3	16	1.945×10^{-4} 8.311×10^{-7}	4.067×10^{-7} 1.496×10^{-11}	2.492×10^{-9} 2.357×10^{-15}
12.3	5	3	16	9.189×10^{-6} 7.647×10^{-7}	6.337×10^{-10} 7.243×10^{-12}	3.649×10^{-13} 2.324×10^{-15}
7.3	5	3	17	5.475×10^{-4} 1.594×10^{-6}	5.039×10^{-6} 1.744×10^{-10}	3.912×10^{-8} 2.970×10^{-14}
12.3	5	3	17	2.115×10^{-5} 1.657×10^{-6}	9.992×10^{-9} 1.152×10^{-10}	6.180×10^{-12} 1.302×10^{-14}
7.3	5	3	19	8.542×10^{-4} 2.516×10^{-7}	9.783×10^{-6} 2.245×10^{-12}	2.259×10^{-8} 1.815×10^{-16}
12.3	5	3	19	1.167×10^{-5} 4.719×10^{-7}	1.107×10^{-9} 3.308×10^{-12}	3.146×10^{-13} 7.947×10^{-16}
7.3	5	3	20	3.446×10^{-3} 3.683×10^{-7}	1.530×10^{-4} 1.762×10^{-11}	6.768×10^{-6} 1.442×10^{-15}
12.3	5	3	20	2.756×10^{-5} 1.026×10^{-6}	1.727×10^{-8} 5.283×10^{-11}	1.343×10^{-11} 4.490×10^{-15}
7.3	5	3	31	— —	— —	— —
12.3	5	3	31	1.553×10^{-4} 4.797×10^{-8}	2.868×10^{-7} 7.985×10^{-14}	1.560×10^{-9} 4.830×10^{-18}
7.3	5	6	31	2.161×10^{-5} 7.709×10^{-8}	1.883×10^{-6} 1.056×10^{-9}	1.174×10^{-7} 1.525×10^{-11}
12.3	5	6	31	8.977×10^{-7} 7.100×10^{-8}	5.796×10^{-8} 1.818×10^{-9}	3.534×10^{-9} 4.796×10^{-11}
7.3	5	6	32	3.052×10^{-5} 9.980×10^{-8}	2.482×10^{-6} 1.326×10^{-9}	1.127×10^{-7} 1.930×10^{-11}
12.3	5	6	32	1.270×10^{-6} 9.807×10^{-8}	8.419×10^{-8} 2.553×10^{-9}	5.455×10^{-9} 7.109×10^{-11}
7.3	5	6	35	2.125×10^{-4} 2.861×10^{-7}	7.021×10^{-7} 1.184×10^{-11}	9.463×10^{-9} 9.629×10^{-16}
12.3	5	6	35	5.058×10^{-6} 3.498×10^{-7}	1.081×10^{-9} 1.074×10^{-11}	3.960×10^{-13} 1.008×10^{-15}

In Tables 1 through 5 we may see the values of Δ_2 for a number of different combinations of values for the parameters a, m, k and p and in Appendix A, in Tables A.1 through A.5 the values of Δ_1 for the same combinations of parameters.

Table 2 – Values of Δ_2 for $m = 7$, $k = 3$, and several values of a and p.

a	m	k	p	$GNIG*$ $GNIG$	$M2GNIG*$ $M2GNIG$	$M3GNIG*$ $M3GNIG$
7.3	7	3	16	$3.283{\times}10^{-4}$ $1.698{\times}10^{-6}$	$2.216{\times}10^{-6}$ $2.345{\times}10^{-10}$	$1.715{\times}10^{-8}$ $3.504{\times}10^{-14}$
12.3	7	3	16	$1.775{\times}10^{-5}$ $1.536{\times}10^{-6}$	$9.750{\times}10^{-9}$ $1.334{\times}10^{-10}$	$5.073{\times}10^{-12}$ $1.120{\times}10^{-14}$
7.3	7	3	17	$8.109{\times}10^{-4}$ $2.976{\times}10^{-6}$	$1.274{\times}10^{-5}$ $7.285{\times}10^{-10}$	$2.240{\times}10^{-7}$ $2.216{\times}10^{-13}$
12.3	7	3	17	$3.683{\times}10^{-5}$ $3.016{\times}10^{-6}$	$3.980{\times}10^{-8}$ $4.899{\times}10^{-10}$	$4.657{\times}10^{-11}$ $9.175{\times}10^{-14}$
7.3	7	3	19	$1.341{\times}10^{-3}$ $5.446{\times}10^{-7}$	$2.959{\times}10^{-5}$ $3.787{\times}10^{-11}$	$7.469{\times}10^{-7}$ $2.998{\times}10^{-15}$
12.3	7	3	19	$2.221{\times}10^{-5}$ $9.550{\times}10^{-7}$	$1.550{\times}10^{-8}$ $6.126{\times}10^{-11}$	$1.050{\times}10^{-11}$ $3.845{\times}10^{-15}$
7.3	7	3	31	— —	— —	— —
12.3	7	3	31	$2.585{\times}10^{-4}$ $1.043{\times}10^{-7}$	$1.515{\times}10^{-6}$ $1.625{\times}10^{-12}$	$1.029{\times}10^{-8}$ $2.602{\times}10^{-17}$

We may see that, as it was indeed expected, the conclusions drawn when comparing distributions using the values of the measure Δ_2 in Tables 1 through 5 or the values of the measure Δ_1 in Tables A.1 through A.5 are exactly the same, since the relations among the values of both measures are similar.

In Tables 1 and 2, and also in Tables A.1 and A.2, the cases with $a = 7.3$, $k = 3$ and $p = 31$ are displayed only for the sake of completeness and also to alert us, that as stated in Theorems 1 and 3, we need to have $a > p/k$.

All starred versions of the distributions (see note before Table 1) display higher values of the measures, indicating a less good proximity to the exact distribution. Indeed all the near-exact distributions obtained in Theorem 3 show a very good behaviour, with the mixtures, which also equate more of the first exact moments showing a much better performance, mainly the mixture of three GNIG distributions, which shows and outstanding performance for all cases.

We should stress that for $\lfloor p/k \rfloor = 0$ the starred and the non-starred versions of the near-exact distributions coincide (see section 3, Table 4 and Table A.4), since in these cases the part of the characteristic function of W that is usually left untouched is actually lacking.

We may also see how increases in the value of the parameter a lead to a slight worsening (higher values) of the measures for the non-starred versions of the near-exact distributions and to an improvement on the values of the measures (lower values) for the starred versions, indicating that the former do not display an asymptotic behaviour with respect to this parameter. On the limit, that is, for very high values of a (see Tables 5 and A.5), the corresponding starred and non-starred versions of the near-exact distributions converge to the same value of the measures, nevertheless with the non-starred versions always with better (smaller) values of the measures.

Table 3 – Values of Δ_2 for $m = 7$, $k = 6$, and several values of a and p.

a	m	k	p	$GNIG*$ $GNIG$	$M2GNIG*$ $M2GNIG$	$M3GNIG*$ $M3GNIG$
7.3	7	6	16	$3.196{\times}10^{-5}$ $6.530{\times}10^{-6}$	$1.780{\times}10^{-8}$ $1.097{\times}10^{-9}$	$1.159{\times}10^{-11}$ $4.331{\times}10^{-13}$
12.3	7	6	16	$6.561{\times}10^{-6}$ $2.783{\times}10^{-6}$	$7.635{\times}10^{-10}$ $1.720{\times}10^{-10}$	$2.850{\times}10^{-13}$ $3.955{\times}10^{-14}$
7.3	7	6	17	$5.037{\times}10^{-4}$ $9.943{\times}10^{-4}$	$5.226{\times}10^{-8}$ $3.130{\times}10^{-9}$	$5.843{\times}10^{-11}$ $1.258{\times}10^{-12}$
12.3	7	6	17	$1.008{\times}10^{-5}$ $4.232{\times}10^{-6}$	$2.267{\times}10^{-9}$ $5.036{\times}10^{-10}$	$6.570{\times}10^{-13}$ $8.240{\times}10^{-14}$
7.3	7	6	31	$1.890{\times}10^{-5}$ $7.465{\times}10^{-8}$	$2.653{\times}10^{-8}$ $3.426{\times}10^{-12}$	$1.543{\times}10^{-9}$ $7.976{\times}10^{-14}$
12.3	7	6	31	$8.573{\times}10^{-7}$ $6.832{\times}10^{-8}$	$1.671{\times}10^{-10}$ $2.098{\times}10^{-12}$	$1.727{\times}10^{-11}$ $9.277{\times}10^{-14}$
7.3	7	6	35	$3.919{\times}10^{-4}$ $6.726{\times}10^{-7}$	$2.990{\times}10^{-6}$ $4.023{\times}10^{-11}$	$2.210{\times}10^{-8}$ $3.406{\times}10^{-15}$
12.3	7	6	35	$1.148{\times}10^{-5}$ $8.096{\times}10^{-7}$	$3.713{\times}10^{-9}$ $3.454{\times}10^{-11}$	$1.472{\times}10^{-12}$ $2.149{\times}10^{-15}$
7.3	7	6	37	$8.196{\times}10^{-5}$ $2.324{\times}10^{-8}$	$7.286{\times}10^{-8}$ $5.181{\times}10^{-13}$	$4.436{\times}10^{-8}$ $5.926{\times}10^{-15}$
12.3	7	6	37	$1.089{\times}10^{-6}$ $4.184{\times}10^{-8}$	$2.651{\times}10^{-10}$ $9.396{\times}10^{-13}$	$2.515{\times}10^{-11}$ $2.985{\times}10^{-14}$
7.3	7	6	41	$3.872{\times}10^{-3}$ $1.089{\times}10^{-7}$	$1.893{\times}10^{-4}$ $2.256{\times}10^{-12}$	$1.061{\times}10^{-5}$ $7.343{\times}10^{-17}$
12.3	7	6	41	$1.537{\times}10^{-5}$ $4.981{\times}10^{-7}$	$6.711{\times}10^{-9}$ $1.558{\times}10^{-11}$	$3.445{\times}10^{-12}$ $7.188{\times}10^{-16}$

But when looking at the values of the measures for fixed values of a, m and k and different values of p, we may see how the non-starred versions of the near-exact distributions show a very good asymptotic behaviour for increasing values of p, opposite to the starred versions. More precisely, the non-starred versions of the near-exact distributions display much lower values for the measures when for fixed values of a, m, k and $p/k - \lfloor p/k \rfloor$ the value of $\lfloor p/k \rfloor$ increases.

When a, m, k and $\lfloor p/k \rfloor$ remain fixed and $p/k - \lfloor p/k \rfloor$ increases, then the behaviour of the near-exact distributions, although consistent for different values of the parameter a, it seems to be different for different values of p and also for the other parameters. All the near-exact distributions, both the starred and non-starred versions seem to display worse (higher) values of both measures when we go from $p/k - \lfloor p/k \rfloor = 1$ to $p/k - \lfloor p/k \rfloor = 2$. However, when for example for $m = 5$ and $k = 6$ we go from $p = 31$ to $p = 35$ both measures worsen for the single GNIG near-exact distribution but both measures improve for the near-exact distributions based on mixtures. But, when we take a similar jump on the value of p, now for $m = 7$ and $k = 6$, considering now $p = 37$ and $p = 41$, the measures worsen for all the near-exact distributions except for the non-starred version of the mixture of 3 GNIG distributions, which shows a sharp improvement in its measures values.

Table 4 – Values of Δ_2 for $a = 7.3$, $p = 16$ and different values of m and higher vales of k, yielding values of $n = 1$ or $n = 0$.

a	m	k	p	$GNIG*$ $GNIG$	$M2GNIG*$ $M2GNIG$	$M3GNIG*$ $M3GNIG$
7.3	7	9	16	1.072×10^{-5} 5.125×10^{-6}	6.948×10^{-9} 2.124×10^{-9}	1.982×10^{-11} 4.042×10^{-12}
7.3	7	19	16	1.789×10^{-5} 1.789×10^{-5}	4.746×10^{-6} 4.746×10^{-6}	1.523×10^{-6} 1.523×10^{-6}
7.3	27	9	16	1.611×10^{-4} 8.128×10^{-5}	7.639×10^{-7} 2.317×10^{-7}	3.837×10^{-9} 6.890×10^{-10}
7.3	27	19	16	8.259×10^{-5} 8.259×10^{-5}	1.456×10^{-7} 1.456×10^{-7}	2.430×10^{-10} 2.430×10^{-10}

Table 5 – Values of Δ_2 for $m = 7$, $k = 3$, $p = 16$ and higher values of a.

a	m	k	p	$GNIG*$ $GNIG$	$M2GNIG*$ $M2GNIG$	$M3GNIG*$ $M3GNIG$
123	7	3	16	3.871×10^{-8} 3.169×10^{-8}	6.171×10^{-14} 4.340×10^{-14}	1.229×10^{-15} 5.612×10^{-17}
223	7	3	16	1.107×10^{-8} 9.926×10^{-9}	5.106×10^{-15} 4.211×10^{-15}	9.783×10^{-17} 5.588×10^{-17}
523	7	3	16	1.928×10^{-9} 1.840×10^{-9}	1.095×10^{-15} 4.202×10^{-16}	1.603×10^{-16} 1.522×10^{-16}

For smaller values of m all distributions show a decrease in the value of their measures of proximity, what shows an improvement of their proximity to the exact distribution. For smaller values of k it seems that the starred versions of the near-exact distributions display a worse behaviour than for higher values of k, while the non-starred versions show the opposite behaviour. Only for $\lfloor p/k \rfloor = 0$, in which case the starred and non-starred versions coincide, the distributions based on mixtures improve their performance for higher values of k.

5. Final Remarks

All the near-exact distributions developed show a very good behaviour. As expected, the ones that equate more moments exhibiting a better behaviour and also the versions built in Theorem 3 with a better performance than the corresponding versions specified after Theorem 3. The mixture of 3 GNIG distributions developed in Theorem 3, which equates six moments has the best performance among all the near-exact distributions and, when compared with the mixture of 2 GNIG distributions, seems to be well worth the extra effort of determining two more parameters whenever some extra precision is reuired, showing that going beyond the mitic number of four moments may be worth the effort.

A particular case of interest of the above results arises for $k=2$, as the exact distribution of the generalized Wilks Λ statistic (Wilks, 1932, 1935) used to test the independence of severeal sets of normally distributed variables, when at most one of them has an odd number of variables (Coelho, 1998, 1999).

Appendix A

Table 1 – Values of Δ_1 for $m = 5$ and several values of a, k and p.

a	m	k	p	$\frac{GNIG*}{GNIG}$	$\frac{M2GNIG*}{M2GNIG}$	$\frac{M3GNIG*}{M3GNIG}$
7.3	5	3	16	2.413×10^{-4}	6.515×10^{-7}	4.711×10^{-9}
				1.140×10^{-6}	2.859×10^{-11}	5.705×10^{-15}
12.3	5	3	16	2.754×10^{-5}	2.494×10^{-9}	1.740×10^{-12}
				2.386×10^{-6}	3.066×10^{-11}	1.202×10^{-14}
7.3	5	3	17	6.051×10^{-4}	7.226×10^{-6}	6.575×10^{-8}
				1.957×10^{-6}	3.002×10^{-10}	6.422×10^{-14}
12.3	5	3	17	5.996×10^{-5}	3.731×10^{-8}	2.741×10^{-11}
				4.885×10^{-6}	4.611×10^{-10}	6.353×10^{-14}
7.3	5	3	19	6.893×10^{-4}	1.016×10^{-5}	2.769×10^{-8}
				2.373×10^{-7}	3.031×10^{-12}	3.162×10^{-16}
12.3	5	3	19	2.950×10^{-5}	3.665×10^{-9}	1.303×10^{-12}
				1.247×10^{-6}	1.188×10^{-11}	3.496×10^{-15}
7.3	5	3	20	2.241×10^{-3}	1.282×10^{-4}	6.637×10^{-6}
				2.912×10^{-7}	2.039×10^{-11}	2.170×10^{-15}
12.3	5	3	20	6.600×10^{-5}	5.435×10^{-8}	5.013×10^{-11}
				2.567×10^{-6}	1.798×10^{-10}	1.868×10^{-14}
7.3	5	3	31	—	—	—
				—	—	—
12.3	5	3	31	1.760×10^{-4}	4.193×10^{-7}	2.686×10^{-9}
				5.961×10^{-8}	1.377×10^{-13}	1.039×10^{-17}
7.3	5	6	31	2.611×10^{-5}	3.171×10^{-6}	2.448×10^{-7}
				1.031×10^{-7}	2.002×10^{-9}	3.684×10^{-11}
12.3	5	6	31	2.673×10^{-6}	2.382×10^{-7}	1.790×10^{-8}
				2.205×10^{-7}	7.853×10^{-9}	2.588×10^{-10}
7.3	5	6	32	3.498×10^{-5}	3.976×10^{-6}	2.269×10^{-7}
				1.262×10^{-7}	2.380×10^{-9}	4.418×10^{-11}
12.3	5	6	32	3.676×10^{-6}	3.362×10^{-7}	2.683×10^{-8}
				2.958×10^{-7}	1.070×10^{-8}	3.719×10^{-10}
7.3	5	6	35	1.996×10^{-4}	8.573×10^{-7}	1.367×10^{-8}
				3.007×10^{-7}	1.756×10^{-11}	1.814×10^{-15}
12.3	5	6	35	1.345×10^{-5}	3.789×10^{-9}	1.664×10^{-12}
				9.677×10^{-7}	4.037×10^{-11}	4.629×10^{-15}

Table 2 – Values of Δ_1 for $m = 7$, $k = 3$, and several values of a and p.

a	m	k	p	$GNIG*$ $GNIG$	$M2GNIG*$ $M2GNIG$	$M3GNIG*$ $M3GNIG$
7.3	7	3	16	$3.668{\times}10^{-4}$	$3.236{\times}10^{-6}$	$2.952{\times}10^{-8}$
				$2.044{\times}10^{-6}$	$3.891{\times}10^{-10}$	$7.193{\times}10^{-14}$
12.3	7	3	16	$4.602{\times}10^{-5}$	$3.339{\times}10^{-8}$	$2.065{\times}10^{-11}$
				$4.099{\times}10^{-6}$	$4.810{\times}10^{-10}$	$4.905{\times}10^{-14}$
7.3	7	3	17	$8.112{\times}10^{-4}$	$1.663{\times}10^{-5}$	$3.452{\times}10^{-7}$
				$3.221{\times}10^{-6}$	$1.091{\times}10^{-9}$	$4.121{\times}10^{-13}$
12.3	7	3	17	$9.045{\times}10^{-5}$	$1.291{\times}10^{-7}$	$1.797{\times}10^{-10}$
				$7.616{\times}10^{-6}$	$1.671{\times}10^{-9}$	$3.801{\times}10^{-13}$
7.3	7	3	19	$9.980{\times}10^{-4}$	$2.858{\times}10^{-5}$	$8.486{\times}10^{-7}$
				$4.576{\times}10^{-7}$	$4.479{\times}10^{-11}$	$4.466{\times}10^{-15}$
12.3	7	3	19	$4.884{\times}10^{-5}$	$4.492{\times}10^{-8}$	$3.614{\times}10^{-11}$
				$2.168{\times}10^{-6}$	$1.881{\times}10^{-10}$	$1.436{\times}10^{-14}$
7.3	7	3	31	—	—	—
				—	—	—
12.3	7	3	31	$2.636{\times}10^{-4}$	$2.016{\times}10^{-6}$	$1.612{\times}10^{-8}$
				$1.139{\times}10^{-7}$	$2.434{\times}10^{-12}$	$4.807{\times}10^{-17}$

Table 3 – Values of Δ_1 for $m = 7$, $k = 6$, and several values of a and p.

a	m	k	p	$GNIG*$ $GNIG$	$M2GNIG*$ $M2GNIG$	$M3GNIG*$ $M3GNIG$
7.3	7	6	16	$7.242{\times}10^{-5}$	$5.322{\times}10^{-8}$	$4.175{\times}10^{-11}$
				$1.534{\times}10^{-5}$	$3.491{\times}10^{-9}$	$1.686{\times}10^{-12}$
12.3	7	6	16	$2.797{\times}10^{-5}$	$4.319{\times}10^{-9}$	$1.932{\times}10^{-12}$
				$1.209{\times}10^{-5}$	$1.006{\times}10^{-9}$	$2.802{\times}10^{-13}$
7.3	7	6	17	$1.084{\times}10^{-4}$	$1.486{\times}10^{-7}$	$1.980{\times}10^{-10}$
				$2.215{\times}10^{-5}$	$9.454{\times}10^{-9}$	$4.625{\times}10^{-12}$
12.3	7	6	17	$4.120{\times}10^{-5}$	$1.230{\times}10^{-8}$	$4.260{\times}10^{-12}$
				$1.761{\times}10^{-5}$	$2.820{\times}10^{-9}$	$5.579{\times}10^{-13}$
7.3	7	6	31	$2.001{\times}10^{-5}$	$3.717{\times}10^{-8}$	$2.745{\times}10^{-9}$
				$8.501{\times}10^{-8}$	$5.379{\times}10^{-12}$	$1.570{\times}10^{-13}$
12.3	7	6	31	$2.176{\times}10^{-6}$	$5.602{\times}10^{-10}$	$7.149{\times}10^{-11}$
				$1.787{\times}10^{-7}$	$7.420{\times}10^{-12}$	$4.060{\times}10^{-13}$
7.3	7	6	35	$3.260{\times}10^{-4}$	$3.245{\times}10^{-6}$	$2.823{\times}10^{-8}$
				$6.058{\times}10^{-7}$	$5.031{\times}10^{-11}$	$5.314{\times}10^{-15}$
12.3	7	6	35	$2.607{\times}10^{-5}$	$1.114{\times}10^{-8}$	$5.254{\times}10^{-12}$
				$1.891{\times}10^{-6}$	$1.090{\times}10^{-10}$	$8.240{\times}10^{-15}$
7.3	7	6	37	$5.756{\times}10^{-5}$	$7.307{\times}10^{-8}$	$5.027{\times}10^{-8}$
				$1.817{\times}10^{-8}$	$5.678{\times}10^{-13}$	$8.226{\times}10^{-15}$
12.3	7	6	37	$2.331{\times}10^{-6}$	$7.487{\times}10^{-10}$	$8.738{\times}10^{-11}$
				$9.260{\times}10^{-8}$	$2.814{\times}10^{-12}$	$1.103{\times}10^{-13}$
7.3	7	6	41	$1.695{\times}10^{-3}$	$1.072{\times}10^{-4}$	$7.067{\times}10^{-6}$
				$5.642{\times}10^{-8}$	$1.706{\times}10^{-12}$	$7.219{\times}10^{-17}$
12.3	7	6	41	$2.950{\times}10^{-5}$	$1.699{\times}10^{-8}$	$1.036{\times}10^{-11}$
				$9.871{\times}10^{-7}$	$4.179{\times}10^{-11}$	$2.347{\times}10^{-15}$

Table 4 – Values of Δ_1 for $a = 7.3$, $p = 16$ and different values of m and higher vales of k, yielding values of $n = 1$ or $n = 0$.

a	m	k	p	$\frac{GNIG*}{GNIG}$	$\frac{M2GNIG*}{M2GNIG}$	$\frac{M3GNIG*}{M3GNIG}$
7.3	7	9	16	3.293×10^{-5}	2.851×10^{-8}	9.815×10^{-11}
				1.614×10^{-5}	9.060×10^{-9}	2.103×10^{-11}
7.3	7	19	16	9.803×10^{-5}	4.075×10^{-5}	1.813×10^{-5}
				9.803×10^{-5}	4.075×10^{-5}	1.813×10^{-5}
7.3	27	9	16	2.722×10^{-4}	1.719×10^{-6}	1.035×10^{-8}
				1.382×10^{-4}	5.276×10^{-7}	1.917×10^{-9}
7.3	27	19	16	2.053×10^{-4}	4.834×10^{-7}	9.692×10^{-10}
				2.053×10^{-4}	4.834×10^{-7}	9.692×10^{-10}

Table 5 – Values of Δ_1 for $m = 7$, $k = 3$, $p = 16$ and higher values of a.

a	m	k	p	$\frac{GNIG*}{GNIG}$	$\frac{M2GNIG*}{M2GNIG}$	$\frac{M3GNIG*}{M3GNIG}$
123	7	3	16	1.260×10^{-6}	2.679×10^{-12}	5.872×10^{-14}
				1.034×10^{-6}	1.892×10^{-12}	2.617×10^{-15}
223	7	3	16	6.594×10^{-7}	4.055×10^{-13}	4.266×10^{-15}
				5.917×10^{-7}	3.352×10^{-13}	4.047×10^{-15}
523	7	3	16	2.704×10^{-7}	3.069×10^{-14}	3.000×10^{-14}
				2.583×10^{-7}	2.705×10^{-14}	2.627×10^{-14}

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