Semi-parametric second-order reduced-bias high quantile estimation

Frederico Caeiro
New University of Lisbon and CMA

and

M. Ivette Gomes
University of Lisbon, DEIO, CEAUL and FCUL

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Abstract. In many areas of application, like for instance *Climatology*, *Hydrology*, *Insurance*, *Finance* and *Statistical Quality Control*, a typical requirement is to estimate a high quantile of probability $1-p$, a value, high enough, so that the chance of an exceedance of that value is equal to p, small. The semi-parametric estimation of high quantiles depends not only on the estimation of the tail index γ, the primary parameter of extreme events, but also on the adequate estimation of a scale first order parameter, C. Recently, apart from new classes of reduced-bias estimators for $\gamma > 0$, new classes of the scale parameter C have been introduced in the literature. In all those classes, the second order parameters in the bias are estimated at a level k_1 of a larger order than that of the level k at which we compute the tail index estimators. The use of one of those classes of C-estimators in quantile estimation enables us to introduce new classes of high quantiles' estimators. The asymptotic distributional properties of the proposed classes of estimators are derived and the estimators are compared with alternative ones, not only asymptotically, but also for finite samples through Monte Carlo techniques. An application to the log-exchange rates of the Euro against the Sterling Pound is also provided.

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1 Introduction and preliminaries

A model F is said to be heavy-tailed if the tail function $\overline{F} := 1 - F \in RV_{-1/\gamma}$, $\gamma > 0$, where RV_α denotes the class of regularly varying functions with index of regular variation equal to α, i.e., non-negative measurable functions g such that, for all $x > 0$, $g(tx)/g(t) \to x^\alpha$, as $t \to \infty$ (Gnedenko, 1943). Let us denote $U(t) := F^{\leftarrow}(1 - 1/t) = \inf\{x : F(x) \geq 1 - 1/t\}$. Then, we may equivalently say that F is heavy-tailed if and only if $U \in RV_\gamma$ (de Haan, 1970), i.e.

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\gamma, \qquad \text{for any } x > 0. \tag{1.1}$$

For small values of p, we want to estimate χ_{1-p}, a value such that $F(\chi_{1-p}) = 1 - p$, a typical parameter in the most diversified areas of application, among which we mention climatology, hydrology, economics, insurance and finance. More specifically, we want to estimate

$$\chi_{1-p} = U(1/p), \qquad p = p_n \to 0, \quad np_n \to K \quad \text{as } n \to \infty, \quad K \in [0, 1], \tag{1.2}$$

and we shall assume to be working in Hall's class of models (Hall 1982; Hall and Welsh, 1985), where there exist $\gamma > 0$, $\rho < 0$, $C > 0$ and $\beta \neq 0$ such that

$$U(t) = Ct^\gamma(1 + \gamma\beta t^\rho/\rho + o(t^\rho)). \tag{1.3}$$

For some details in the paper we shall refer to a sub-class of Hall's class, such that

$$U(t) = Ct^\gamma(1 + \gamma\beta t^\rho/\rho + \beta' t^{2\rho} + o(t^{2\rho})), \tag{1.4}$$

i.e., relatively to Hall's class we merely make explicit a third order term $\beta' t^{2\rho}$, $\beta' \neq 0$. Such a class contains most of the heavy-tailed models important in applications, like the Fréchet, the Generalized Pareto and the Student's-t.

We are going to base inference on the largest k top order statistics (o.s.), and as usual in semi-parametric estimation of parameters of extreme events, we shall assume that k is an intermediate sequence of integers in $[1, n[$, i.e.,

$$k = k_n \to \infty, \qquad k/n \to 0, \qquad n \to \infty. \tag{1.5}$$

Since, from (1.2) and (1.3),

$$\chi_{1-p} = U(1/p) \sim Cp^{-\gamma}, \quad \text{as} \quad p \to 0,$$

an obvious estimator of χ_{1-p} is $\widehat{C}p^{-\hat{\gamma}}$, with $\widehat{C}$ and $\hat{\gamma}$ any consistent estimators of C and γ, respectively. Given a sample $(X_1, X_2, \ldots, X_n)$, let us denote $X_{i:n}$, $1 \le i \le n$, the set of associated ascending order statistics (o.s.). Denoting Y a standard Pareto model, i.e., a model such that $F_Y(y) = 1 - 1/y$, $y > 1$, the use of the universal uniform transformation enables us to write $X_{n-k:n} \stackrel{d}{=} U(Y_{n-k:n})$. Next, since $Y_{n-k:n} \stackrel{p}{\sim} (n/k)$ for intermediate k and (1.3) holds, we get $X_{n-k:n} \stackrel{p}{\sim} CY^{\gamma}_{n-k:n} \stackrel{p}{\sim} C(n/k)^{\gamma}$, as $n \longrightarrow \infty$. Consequently, an obvious estimator of C, proposed by Hall and Welsh (1985), is

$$C_{\hat{\gamma}}(k) := X_{n-k:n} \left(\frac{k}{n}\right)^{\hat{\gamma}} \tag{1.6}$$

and

$$Q^{(p)}_{\hat{\gamma}}(k) = \widehat{C}p^{-\hat{\gamma}} = X_{n-k:n} \left(\frac{k}{np}\right)^{\hat{\gamma}} \tag{1.7}$$

is the obvious quantile-estimator at the level p (Weissman, 1978).

For heavy tails, the classical tail index estimator, usually the one which is plugged in (1.7), for a semi-parametric quantile estimation, is the Hill estimator $\hat{\gamma} = \hat{\gamma}(k) =: H(k)$ (Hill, 1975), with the functional expression,

$$H(k) := \frac{1}{k}\sum_{i=1}^{k} V_{ik} = \frac{1}{k}\sum_{i=1}^{k} U_i, \tag{1.8}$$

where $V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}$, $1 \le i \le k < n$, are the log-excesses, and

$$U_i := i\left(\ln X_{n-i+1:n} - \ln X_{n-i:n}\right), \quad 1 \le i \le k < n, \tag{1.9}$$

are the scaled log-spacings. We thus get the so-called classical quantile estimator, based on the Hill tail index estimator H, with the obvious notation, $Q^{(p)}_H(k)$.

In order to derive the asymptotic non-degenerate behaviour of semi-parametric estimators of extreme events' parameters, we need more than the first order condition in (1.1). A typical condition for heavy-tailed models, which holds for the models in (1.3), with

$$A(t) = \gamma\,\beta\,t^{\rho}, \qquad \gamma > 0,\ \beta \ne 0,\ \rho < 0, \tag{1.10}$$

is the following:

$$\lim_{t\to\infty} \frac{\frac{U(tx)}{U(t)} - x^{\gamma}}{A(t)} = x^{\gamma}\frac{x^{\rho} - 1}{\rho}, \tag{1.11}$$

or equivalently,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \frac{x^\rho - 1}{\rho}, \tag{1.12}$$

for all $x > 0$, where A is a function of constant sign near infinity (positive or negative), and $\rho \le 0$ is the shape second order parameter.

Under the second order framework in (1.11) or in (1.12), and for intermediate k, i.e., whenever (1.5) holds, we may guarantee the asymptotic normality of the Hill estimator $H(k)$, for an adequate k. Indeed, we may write (de Haan and Peng, 1998),

$$H(k) \stackrel{d}{=} \gamma + \frac{\gamma}{\sqrt{k}} Z_k + \frac{A(n/k)}{1-\rho}(1 + o_p(1)), \tag{1.13}$$

with $Z_k = \sqrt{k}\left(\sum_{i=1}^k E_i/k - 1\right)$, and $\{E_i\}$ i.i.d. standard exponential r.v.'s. Consequently, if we choose k such that $\sqrt{k}\, A(n/k) \to \lambda \neq 0$, finite, as $n \to \infty$, $\sqrt{k}(H(k) - \gamma)$ is asymptotically normal, with variance equal to γ^2 and a non-null bias given by $\lambda/(1-\rho)$. Most of the times, this type of estimates exhibits a strong bias for moderate k and sample paths with very short stability regions around the target value γ. This has recently led researchers to consider the possibility of dealing with the bias term in an appropriate way, building new estimators, $\hat\gamma_{_R}(k)$ say, the so-called second order reduced-bias estimators discussed by Peng (1998), Beirlant *et al.* (1999), Feuerverger and Hall (1999), Gomes *et al.* (2000), among others. Then, for k intermediate, i.e., such that (1.5) holds, and under the second order framework in (1.12), we may write, with Z_k^R an asymptotically standard normal r.v.,

$$\hat\gamma_{_R}(k) \stackrel{d}{=} \gamma + \frac{\gamma\sigma_{_R}}{\sqrt{k}} Z_k^R + o_p(A(n/k)), \tag{1.14}$$

where $\sigma_{_R} > 0$, being A again the function in (1.12). Consequently, the sequence of r.v.'s, $\sqrt{k}(\hat\gamma_{_R}(k) - \gamma)$ is asymptotically normal with variance equal to $(\gamma\sigma_{_R})^2$ and a null mean value even when $\sqrt{k}\; A(n/k) \to \lambda \neq 0$, finite, as $n \to \infty$, possibly at expenses of an asymptotic variance $\gamma^2\sigma_{_R}^2 > \gamma^2$. Gomes and Figueiredo (2006) suggest the use, in (1.7), of reduced-bias tail index estimators, like the ones in Gomes and Martins (2001, 2002) and Gomes *et al.* (2004), all with $\sigma_{_R} > 1$ in (1.14), being then able to reduce also the dominant component of the classical quantile estimator's asymptotic bias.

More recently, Gomes *et al.* (2004), Caeiro *et al.* (2005) and Gomes *et al.* (2005) consider new classes of tail index estimators, for which (1.14) holds with $\sigma_{_R} = 1$ at least for values k

such that $\sqrt{k}\, A(n/k) \to \lambda$, finite. These classes are dependent on $(\hat\beta, \hat\rho)$, an adequate consistent estimator of the vector of the second order parameters (β, ρ) in (1.10). The influence of these tail index estimators in quantile estimation has been studied by Gomes and Pestana (2005) and Beirlant *et al.* (2006).

Also recently, new estimators of C have been proposed (Caeiro, 2006), where, instead of $X_{n-k:n}$ alone, a spacing $X_{n-[\theta k]:n} - X_{n-k:n}$, $0 < \theta < 1$, is considered. More specifically, we may replace $C_{\hat\gamma}(k)$ in (1.6) by

$$\widetilde{C}_{\hat\gamma_R}(k;\theta) := \frac{X_{n-[\theta k]:n} - X_{n-k:n}}{\theta^{-\hat\gamma_R} - 1} \left(\frac{k}{n}\right)^{\hat\gamma_R}, \tag{1.15}$$

where $\theta \in]0,1[$ is a tuning parameter and $\hat\gamma_R \equiv \hat\gamma_R(k)$ is a second order reduced-bias extreme value index estimator. Similarly to the way developed by Caeiro *et al.* (2005) for the extreme value index estimation, Caeiro (2006) has worked out the main dominant component of the asymptotic bias of $\widetilde{C}_{\hat\gamma_R}(k;\theta)$. With the parametrization $A(t) = \gamma\,\beta\, t^\rho$, already given in (1.10), such a component is given by $C \times \mathcal{B}_\theta(\gamma, \rho, \beta)$, where

$$\mathcal{B}_\theta(\gamma, \rho, \beta) = \frac{\theta^{-(\gamma+\rho)} - 1}{\theta^{-\gamma} - 1} \times \frac{\gamma\,\beta\,(n/k)^\rho}{\rho}.$$

It is thus sensible to consider the semi-parametric C-estimator,

$$\overline{C}_{\hat\gamma_R}(k;\theta) := \frac{X_{n-[\theta k]:n} - X_{n-k:n}}{\theta^{-\hat\gamma_R} - 1} \left(\frac{k}{n}\right)^{\hat\gamma_R} \times (1 - \mathcal{B}_\theta(\hat\gamma_R, \hat\rho, \hat\beta)). \tag{1.16}$$

We shall here consider, for $\theta = 1/2$, the associated quantile estimator $\overline{Q}^{(p)}_{\hat\gamma_R}(k) \equiv \overline{Q}^{(p)}_{\hat\gamma_R,\hat\rho,\hat\beta}(k)$, with

$$\overline{Q}^{(p)}_{\hat\gamma_R,\hat\rho,\hat\beta}(k) := \frac{X_{n-[k/2]:n} - X_{n-k:n}}{2^{\hat\gamma_R} - 1} \left(\frac{k}{n\,p}\right)^{\hat\gamma_R} \times (1 - \mathcal{B}_{1/2}(\hat\gamma_R, \hat\rho, \hat\beta)). \tag{1.17}$$

Moreover, we shall restrict our attention to the second order reduced-bias extreme value index estimator estimator introduced in Caeiro *et al.* (2005),

$$\overline{H}(k) \equiv \overline{H}_{\hat\beta,\hat\rho}(k) := H(k)\left(1 - \tfrac{\hat\beta}{1-\hat\rho}\left(\tfrac{n}{k}\right)^{\hat\rho}\right) \tag{1.18}$$

for adequate consistent estimators $\hat\beta$ and $\hat\rho$ of the second order parameters β and ρ, respectively.

After a brief sketch on the estimation of the second order parameters, in Section 2, we provide, in Section 3, details on the reduced-bias estimators of γ and C, to be used for quantile estimation. Section 4 is devoted to the asymptotic behavior of quantile estimators and Section 5, to their finite sample properties. In Section 6, we study the robustness of the proposed class of quantile estimators to underlying heavy-tailed models that may be no longer supported by the developed theory. Finally, in Section 7, we provide an illustration, for data from the field of finance.

2 Estimation of second order parameters

The reduced-bias tail index estimator in (1.18) requires the estimation of the second order parameters ρ and β in (1.10). Such an estimation will now be briefly discussed.

2.1 Estimation of the shape second order parameter ρ

We shall consider here particular members of the class of estimators of the second order parameter ρ proposed by Fraga Alves *et al.* (2003). Such a class of estimators may be parameterized by a tuning real parameter $\tau \in \mathbb{R}$ (Caeiro and Gomes, 2004). These ρ-estimators depend on the statistics

$$T_n^{(\tau)}(k) := \begin{cases} \dfrac{\left(M_n^{(1)}(k)\right)^{\tau} - \left(M_n^{(2)}(k)/2\right)^{\tau/2}}{\left(M_n^{(2)}(k)/2\right)^{\tau/2} - \left(M_n^{(3)}(k)/6\right)^{\tau/3}}, & \text{if } \tau \neq 0 \\ \\ \dfrac{\ln\left(M_n^{(1)}(k)\right) - \frac{1}{2}\,\ln\left(M_n^{(2)}(k)/2\right)}{\frac{1}{2}\,\ln\left(M_n^{(2)}(k)/2\right) - \frac{1}{3}\ln\left(M_n^{(3)}(k)/6\right)}, & \text{if } \tau = 0 \end{cases},$$

which converge towards $3(1-\rho)/(3-\rho)$, independently of the tuning parameter τ, whenever the second order condition (1.12) holds and k is such that (1.5) holds and $\sqrt{k}\,A(n/k) \to \infty$, as $n \to \infty$. The ρ-estimators considered have the functional expression,

$$\hat{\rho}_n^{(\tau)}(k) := -\min\left(0\,,\, \frac{3(T_n^{(\tau)}(k)) - 1}{T_n^{(\tau)}(k) - 3}\right). \tag{2.1}$$

Remark 2.1. *Under adequate general conditions, and for an appropriate tuning parameter τ the ρ-estimators in* (2.1) *show highly stable sample paths as functions of k, the number of top o.s. used, for a wide range of large k-values.*

Remark 2.2. *The theoretical and simulated results in* Fraga Alves *et al.* (2003)*, together with the use of these estimators in different reduced-bias statistics, has led us to advise in practice the estimation of* ρ *through the estimator in* (2.1)*, computed at the value*

$$k_1 := \left[n^{0.995}\right], \tag{2.2}$$

not chosen in any optimal way, and the choice of the tuning parameter $\tau = 0$ *for the region* $\rho \in [-1, 0)$ *and* $\tau = 1$ *for the region* $\rho \in (-\infty, -1)$*. In the simulations of section 5 we have indeed done this. Anyway, we again advise practitioners not to choose blindly the value of* τ *in* (2.1)*. It is sensible to draw a few sample paths of* $\hat{\rho}_n^{(\tau)}(k)$*, as functions of* k*, electing the value of* τ *which provides higher stability for large* k*, by means of any stability criterion.*

2.2 Estimation of the scale second order parameter β

For the estimation of β we shall here consider the estimator developed in Gomes and Martins (2002), with the functional expression,

$$\widehat{\beta}_{\widehat{\rho}}(k) := \left(\frac{k}{n}\right)^{\widehat{\rho}} \frac{\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\widehat{\rho}}\right) N_n^{(1)}(k) - N_n^{(1-\widehat{\rho})}(k)}{\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\widehat{\rho}}\right) N_n^{(1-\widehat{\rho})}(k) - N_n^{(1-2\widehat{\rho})}(k)}, \tag{2.3}$$

where

$$N_n^{(\alpha)}(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{\alpha-1} U_i,$$

with U_i and $\hat{\rho} \equiv \hat{\rho}_n^{(\tau)}(k)$ defined in (1.9) and (2.1), respectively.

2.3 Asymptotic behaviour

In this paper, we intend to use the same level k_1 in (2.2) both for the estimation of ρ and β, through the estimators in (2.1) and (2.3), respectively, and we shall formalize, without proofs, the needed distributional properties of the second order paramers' estimators, essentially for the class of models in (1.4).

Proposition 2.1 (Fraga Alves et al., 2003)**.** *If the second order condition* (1.12) (*or equivalently,* (1.11)) *holds, with* $\rho \le 0$*,* k *is a sequence of intermediate integers, i.e.,* (1.5) *holds, and*

$\lim_{n\to\infty} \sqrt{k}\ A(n/k) = \infty$, *then* $\hat\rho_n^{(\tau)}(k)$ *in* (2.1) *converges in probability towards* ρ*, as* $n \to \infty$*. Moreover, and now for models in* (1.4), $\hat\rho_n^{(\tau)}(k) - \rho = o_p\left(1/\ln(n/k)\right)$ *for values* k *such that* $\sqrt{k}A^2(n/k) \to \lambda_A$*, finite and non-null, and for values* k *such that* $\sqrt{k}A^2(n/k) \to \infty$ *for some* $\epsilon > 0$ *and* $k = O(n^{1-\epsilon})$*.*

Proposition 2.2 (Gomes and Martins, 2002)**.** *If the second order condition* (1.12) *holds with* $A(t) = \gamma\ \beta\ t^\rho$, $\rho < 0$*, if* (1.5) *holds, and if* $\sqrt{k}A(n/k) \to \infty$*, then, with* $\hat\beta_\rho(k)$ *given in* (2.3)*,* $\hat\beta_\rho(k)$ *is asymptotically normal and converges in probability towards* β*, as* $n \to \infty$*.*

Proposition 2.3 (Gomes, de Haan and Rodrigues, 2005)**.** *Under the conditions in* Proposition 2.2*, with* $\hat\rho_n^{(\tau)}(k)$ *and* $\widehat\beta_{\widehat\rho}(k)$ *given in* (2.1) *and* (2.3)*, respectively, and* $\hat\rho = \hat\rho_n^{(\tau)}(k)$ *for any* τ *and* k*, such that* $\hat\rho - \rho = o_p(1/\ln n)$*, as* $n \to \infty$*,* $\widehat\beta_{\widehat\rho}(k)$ *is consistent for the estimation of* β*. Moreover,* $\hat\beta_{\hat\rho}(k) - \beta \overset{p}{\sim} -\beta \ln(n/k)(\hat\rho - \rho) = o_p(1)$*.*

Remark 2.3. *We shall denote generically* $\hat\rho$ *any of the estimators in* (2.1)*, computed at* k_1 *in* (2.2) *and* $\hat\beta$ *any estimator in* (2.3)*, also computed at the value* k_1*.*

3 Reduced-bias estimation of γ and C

3.1 The asymptotic behaviour of the reduced-bias tail index estimators

We now state the following:

Proposition 3.1 (Caeiro *et al.*, 2005)**.** *If the second order condition* (1.11) *holds, if* $k = k_n$ *is a sequence of intermediate positive integers, i.e.,* (1.5) *holds, and if* $\sqrt{k}A(n/k) \to \lambda$*, finite and non necessarily null, as* $n \to \infty$*, then*

$$\sqrt{k}\left(\overline{H}_{\beta,\rho}(k) - \gamma\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} Normal(0, \gamma^2).$$

This same limiting behaviour holds true if we replace $\overline{H}_{\beta,\rho}$ *by* $\overline{H}_{\hat\beta,\hat\rho}$*, provided that* $\hat\rho - \rho = o_p(1/\ln n)$*, and we choose* $\hat\beta := \hat\beta_{\hat\rho}(k_1)$*, with* k_1 *and* $\hat\beta_{\hat\rho}(k)$ *given in* (2.2) *and* (2.3)*, respectively. More specifically, and with* Z_k *an asymptotic standard normal r.v., we can then write*

$$\overline{H}_{\hat\beta,\hat\rho}(k) \overset{d}{=} \gamma + \frac{\gamma}{\sqrt{k}} Z_k + o_p(A(n/k)).$$

Remark 3.1. *Notice that, contrarily to what happens in Drees' class of functionals* (Drees, 1998), *where the minimal asymptotic variance of a reduced-bias tail index estimator is given by* $(\gamma(1-\rho)/\rho)^2$, *we have been here able to obtain a reduced-bias tail index estimator with an asymptotic variance equal to* γ^2, *the asymptotic variance of Hill's estimator, the maximum likelihood estimator of* γ *for a strict Pareto model.*

3.2 The asymptotic behaviour of the C-estimator

We first present the following result on the asymptotic behaviour of intermediate o.s.:

Proposition 3.2. *Under the second order framework in* (1.11) *and for* k *such that* (1.5) *holds*

$$\frac{X_{n-k:n}}{U(n/k)} \stackrel{d}{=} 1 + \frac{\gamma}{\sqrt{k}} B_k + o_p(A(n/k)), \tag{3.1}$$

where B_k *is asymptotically standard normal and*

$$Cov(B_i, B_j) = \frac{\sqrt{i\,j}(1 - j/n)}{j-1}, \quad i < j.$$

Proof. Since $X_{n-k:n} \stackrel{d}{=} U(Y_{n-k:n})$ and $Y_{n-k:n} \stackrel{p}{\sim} (n/k)$, where Y is a standard Pareto r.v., we can use the second order condition (1.11) with $t = n/k$ e $x = \frac{k}{n} Y_{n-k:n}$, and write

$$\frac{X_{n-k:n}}{U(n/k)} \stackrel{d}{=} \left(\frac{k}{n} Y_{n-k:n}\right)^{\gamma} \left[1 + \frac{\left(\frac{k}{n} Y_{n-k:n}\right)^{\rho} - 1}{\rho} A\left(Y_{n-k:n}\right)(1 + o(1))\right].$$

Then, since $x^{\alpha} = 1 + \alpha(x-1) + o(x-1),\ x \to 1$, we have

$$\frac{X_{n-k:n}}{U(n/k)} \stackrel{d}{=} 1 + \gamma\left(\tfrac{k}{n} Y_{n-k:n} - 1\right) + o_p(A(n/k)) = 1 + \frac{\gamma}{\sqrt{k}} B_k + o_p(A(n/k)),$$

where $B_k = \sqrt{k}\left(\frac{k}{n} Y_{n-k:n} - 1\right)$ is an asymptotic standard normal r.v. (Arnold *et al.*, 1992; Falk, 1989). Since $Cov\left(Y_{i:n}, Y_{j:n}\right) = n\ i/((n-i)(n-i-1)(n-j)),\ i < j$, the final part of the Proposition follows as well. □

Corollary 3.1. *Under the conditions of* Proposition 3.2, and for Hall's class of models in (1.3),

$$\frac{X_{n-k:n}}{(n/k)^{\gamma}} \stackrel{d}{=} C\left(1 + \frac{\gamma}{\sqrt{k}} B_k + \frac{A(n/k)}{\rho} + o_p(A(n/k))\right). \tag{3.2}$$

Proof. Since $\frac{X_{n-k:n}}{(n/k)^{\gamma}} = \frac{X_{n-k:n}}{U(n/k)} \times \frac{U(n/k)}{(n/k)^{\gamma}}$, the result in (3.2) follows from Proposition 3.2 and from equation (1.3). □

We may further state the following:

Proposition 3.3. *Let F be a model in Hall's class* (1.3). *If we consider the Hill estimator in* (1.8) *and plug it in* (1.6), *i.e., if we consider $C_H(k)$, the C-estimator proposed in* (1.6), *further assuming that $\sqrt{k}A(n/k) \to \lambda$, we have*

$$\frac{\sqrt{k}}{\ln n}\left(\frac{C_H(k)-C}{C}\right) \xrightarrow{d} N\left(\frac{-\lambda}{(1-\rho)(1-2\rho)}, \frac{\gamma^2}{(1-2\rho)^2}\right).$$

Proof. Since $C_H(k) = \frac{X_{n-k:n}}{(n/k)^\gamma}(k/n)^{H(k)-\gamma}$, using the result in Corollary 3.1 and applying the delta method to $(k/n)^{H(k)-\gamma}$, we have

$$\begin{aligned} C_H(k) &\stackrel{d}{=} C\left(1+\frac{\gamma}{\sqrt{k}}B_k + \frac{A(n/k)}{\rho}(1+o_p(1))\right)(1+\ln(k/n)(H(k)-\gamma)(1+o_p(1))) \\ &\stackrel{d}{=} C\left(1+\ln(k/n)(H(k)-\gamma)(1+o_p(1)) + \frac{\gamma}{\sqrt{k}}B_k + \frac{A(n/k)}{\rho}(1+o_p(1))\right), \end{aligned}$$

that is,

$$\frac{\sqrt{k}}{\ln n}\left(\frac{C_H(k)-C}{C}\right) \stackrel{d}{=} \left(\frac{\ln k}{\ln n}-1\right)\left(\gamma Z_k^{(1)} + \frac{\sqrt{k}A(n/k)}{1-\rho}\right) + \frac{\gamma}{\ln n}B_k + \frac{\sqrt{k}A(n/k)}{\rho \ln n}(1+o_p(1)).$$

As for values k such that $\sqrt{k}\, A(n/k) \to \lambda$, finite and non-null, we have $\ln k/\ln n \to -2\rho/(1-2\rho)$, as $n \to \infty$, the result follows. □

Remark 3.2. *Asymptotically and for models in Hall's class, the minimum mean squared error of $H(k)$ and $C_H(k)$ is attained at the same level,*

$$k_0 = \left(\frac{(1-\rho)n^{-\rho}}{\beta\sqrt{-2\rho}}\right)^{2/(1-2\rho)}. \tag{3.3}$$

We shall now consider the r.v.'s $\widetilde{C}_\gamma$ and $\overline{C}_\gamma$, with $\widetilde{C}_{\hat\gamma_R}$ and $\widetilde{C}_{\hat\gamma_R}$ given in (1.15) and (1.16), respectively:

Theorem 3.1. *Under the second order framework in* (1.12), *for k values such that* (1.5) *holds and for models F in* (1.3),

$$\widetilde{C}_\gamma(k;\theta) \stackrel{d}{=} C\left(1+\frac{\gamma\sigma_{C,\theta}}{\sqrt{k}}Z_{k,\theta}^C + \frac{\theta^{-(\gamma+\rho)}-1}{\theta^{-\gamma}-1}\,\frac{A(n/k)}{\rho} + o_p(A(n/k))\right) \tag{3.4}$$

and

$$\overline{C}_\gamma(k;\theta) \stackrel{d}{=} C\left(1+\frac{\gamma\sigma_{C,\theta}}{\sqrt{k}}Z^C_{k,\theta}+o_p(A(n/k))\right) \tag{3.5}$$

where $0<\theta<1$,

$$\sigma^2_{C,\theta}=1+\left(\frac{1-\theta}{\theta}\right)\left(\frac{\theta^{-\gamma}}{\theta^{-\gamma}-1}\right)^2,$$

and with B_k *in* (3.1), $Z^C_{k,\theta}=\left(\theta^{-(\gamma+1/2)}B_{k\theta}-B_k\right)/\left(\sigma_{C,\theta}(\theta^{-\gamma}-1)\right)$ *is a sequence of asymptotically standard normal r.v.'s.*

Proof. Since in Hall's class of models, $A(n/k\theta)=\theta^{-\rho}A(n/k)$, and

$$\widetilde{C}_\gamma(k;\theta)=\frac{1}{\theta^{-\gamma}-1}\left(\frac{X_{n-k\theta:n}}{(\frac{n}{k\theta})^\gamma}\theta^{-\gamma}-\frac{X_{n-k:n}}{(\frac{n}{k})^\gamma}\right),$$

one can use Corolary 3.1 and (3.4) follows.

Consequently, due to the fact that $\overline{C}_\gamma(k;\theta)=\widetilde{C}_\gamma(k;\theta)(1-\frac{\theta^{-(\gamma+\rho-1)}}{\theta^{-\gamma}-1}\frac{A(n/k)}{\rho})$, (3.5) follows as well. □

The following Corollary is important in the sense that it shows that for some intermediate k-values, only $\overline{C}_\gamma(k;\theta)$ has an asymptotic null mean value and keeps the same asymptotic variance as $\widetilde{C}_\gamma(k;\theta)$.

Corollary 3.2. *Under the conditions in Theorem 3.1, and for intermediate* k *such that* $\sqrt{k}\,A(n/k)\to\lambda$,

$$\sqrt{k}\left(\frac{\widetilde{C}_\gamma(k;\theta)-C}{C}\right)\xrightarrow[n\to\infty]{d}N\left(\frac{\lambda(\theta^{-(\gamma+\rho)}-1)}{\rho(\theta^{-\gamma}-1)},\gamma^2\sigma^2_{C,\theta}\right)$$

and

$$\sqrt{k}\left(\frac{\overline{C}_\gamma(k;\theta)-C}{C}\right)\xrightarrow[n\to\infty]{d}N\left(0,\gamma^2\sigma^2_{C,\theta}\right).$$

In Figure 1, we picture $\gamma\sigma_{C,\theta}=\sqrt{Var_\infty(\sqrt{k}\left(\widetilde{C}_\gamma(k;\theta)-C\right)/C)}$ as a function of θ, for a few values of γ.

The value of θ minimizing this standard deviation depends on γ, but the value $\theta=1/2$ seems to be a good compromise. This is the reason why we have chosen the quantile estimator in (1.17). In the following Theorem we will consider $\theta=1/2$.

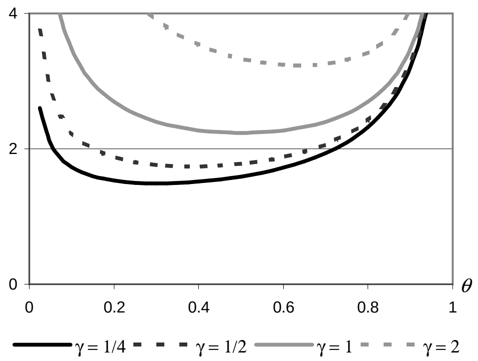

Figure 1: $\gamma\sigma_{\mathcal{C}}$ as function of θ, for some values of γ.

Theorem 3.2. *Under the conditions in Theorem 3.1, assume that* $\sqrt{k}\ A(n/k) \to \lambda$ *and* $\hat\gamma_{_R} \equiv \hat\gamma_{_R}(k)$ *is a second order reduced-bias extreme value index estimator, such that* (1.14) *holds. Then,*

$$\frac{\sqrt{k}}{\ln n}\left(\frac{\widetilde{C}_{\hat\gamma_R}(k) - C}{C}\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N\left(0, \left(\frac{\gamma\sigma_{_R}}{1-2\rho}\right)^2\right). \tag{3.6}$$

If we further consider second order parameters' estimators $\hat\rho$ *and* $\hat\beta$ *such that* $\hat\rho - \rho = o_p(1/\ln n)$ *and* $\hat\beta - \beta = o_p(1)$*, as* $n \to \infty$*,*

$$\frac{\sqrt{k}}{\ln n}\left(\frac{\overline{C}_{\hat\gamma_R}(k) - C}{C}\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} N\left(0, \left(\frac{\gamma\sigma_{_R}}{1-2\rho}\right)^2\right) \tag{3.7}$$

Proof. We can write

$$\widetilde{C}_{\hat\gamma_R}(k) = \widetilde{C}_{\gamma}(k)\ \frac{2^\gamma - 1}{2^{\hat\gamma_R} - 1}\ e^{-(\hat\gamma_R(k)-\gamma)\ln(n/k)}.$$

The use of the delta method enable us to write

$$\frac{1}{2^{\hat\gamma_R(k)} - 1} - \frac{1}{2^\gamma - 1} \overset{p}{\sim} -\frac{2^\gamma \ln 2}{(2^\gamma-1)^2}(\hat\gamma_R(k) - \gamma),$$

and $e^{(\hat\gamma_R(k)-\gamma)\ln(k/n)} - 1 \overset{p}{\sim} (\hat\gamma_{_R}(k) - \gamma)\ln(k/n)$. We now have

$$\widetilde{C}_{\hat\gamma_R}(k) \overset{d}{=} C\left\{1 + \left(\ln(k/n) - \frac{2^\gamma \ln 2}{2^\gamma - 1}\right)(\hat\gamma_R(k) - \gamma) + \frac{\gamma\sigma_{\mathcal{C}}}{\sqrt{k}} Z_k^{\mathcal{C}} + \frac{2^{\gamma+\rho}-1}{(2^\gamma-1)\rho} A(n/k)(1+o_p(1))\right\},$$

where

$$\sigma_C := \sigma_{C,1/2} = \sqrt{1 + \left(\frac{2^\gamma}{2^\gamma - 1}\right)^2}.$$

Therefore, since $\hat{\gamma}_R(k) - \gamma \stackrel{d}{=} \frac{\gamma\sigma_R}{\sqrt{k}} Z_k^R + o_p(A(n/k))$,

$$\frac{\sqrt{k}}{\ln n}\left(\frac{\widetilde{C}_{\hat{\gamma}_R}(k) - C}{C}\right) \stackrel{d}{=} \left(\frac{\ln k}{\ln n} - 1\right)\gamma\sigma_R Z_k^R + \frac{\gamma}{\ln n}\left(\sigma_{\mathcal{C}} Z_k^{\mathcal{C}} - \frac{2^\gamma \ln 2}{2^\gamma - 1} Z_k^R\right) + \frac{2^{\gamma+\rho}-1}{(2^\gamma-1)\rho}\frac{\sqrt{k}A(n/k)}{\ln n}(1+o_p(1)), \quad (3.8)$$

If we consider k-values, such that $\sqrt{k}\ A(n/k) \to \lambda$, non-null and finite, we have $\ln k/\ln n \to -2\rho/(1-2\rho)$, as $n \to \infty$ and the result in equation (3.6) follows.

Now, since $\overline{C}_{\hat{\gamma}_R}(k) = \widetilde{C}_{\hat{\gamma}_R}(k) \times \left(1 - \frac{(2^{(\hat{\gamma}_R+\hat{\rho})}-1)}{2^{\hat{\gamma}_R}-1}\frac{\hat{\gamma}_R\hat{\beta}\,(n/k)^{\hat{\rho}}}{\hat{\rho}}\right)$, the use of the delta method enable us to write,

$$\left(\frac{2^{\hat{\gamma}_R+\hat{\rho}}-1}{2^{\hat{\gamma}_R}-1}\right)\frac{\hat{\gamma}_R\hat{\beta}}{\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}} = \frac{2^{\gamma+\rho}-1}{2^\gamma-1}\frac{A(n/k)}{\rho}\times \\ \times\left[1+\left(1-\frac{\gamma 2^\gamma(2^\rho-1)}{(2^\gamma-1)(2^{\gamma+\rho}-1)}\right)\frac{\hat{\gamma}_R-\gamma}{\gamma}+\frac{\hat{\beta}-\beta}{\beta}+(\hat{\rho}-\rho)\ln(n/k)\right].$$

As $\hat{\gamma}_R - \gamma = o_p(1)$, $(\hat{\rho}-\rho)\ln(n/k) = o_p(1)$ and $\hat{\beta} - \beta = o_p(1)$, we have

$$\left(\frac{2^{\hat{\gamma}_R+\hat{\rho}}-1}{2^{\hat{\gamma}_R}-1}\right)\frac{\hat{\gamma}_R\hat{\beta}}{\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}} = \frac{2^{\gamma+\rho}-1}{2^\gamma-1}\frac{A(n/k)}{\rho}(1+o_p(1)),$$

and

$$\overline{C}_{\hat{\gamma}_R}(k) \stackrel{d}{=} C\left\{1+\left(\ln(k/n) - \frac{2^\gamma\ln 2}{2^\gamma-1}\right)(\hat{\gamma}_R(k)-\gamma) + \frac{\gamma\sigma_{\mathcal{C}}}{\sqrt{k}}Z_k^{\mathcal{C}} + \frac{2^{\gamma+\rho}-1}{(2^\gamma-1)\rho}o_p(A(n/k))\right\}.$$

Therefore, we can write,

$$\frac{\sqrt{k}}{\ln n}\left(\frac{\overline{C}_{\hat{\gamma}_R}(k) - C}{C}\right) \stackrel{d}{=} \left(\frac{\ln k}{\ln n} - 1\right)\gamma\sigma_R Z_k^R + \frac{\gamma}{\ln n}\left(\sigma_{\mathcal{C}} Z_k^{\mathcal{C}} - \frac{2^\gamma \ln 2}{2^\gamma - 1} Z_k^R\right) + o_p\left(\frac{\sqrt{k}A(n/k)}{\ln n}\right),$$

and the asymptotic result in (3.7) follows as well. □

Remark 3.3. *Although both $\widetilde{C}_{\hat{\gamma}_R}(k)$ and $\overline{C}_{\hat{\gamma}_R}(k)$ have the same limit distribution, that result is achieved very slowy. In (3.8), the term $\frac{2^{\gamma+\rho}-1}{(2^\gamma-1)\rho}\frac{\sqrt{k}A(n/k)}{\ln n}(1+o_p(1))$ may change the bias. Also, the term $\frac{\gamma}{\ln n}\left(\sigma_{\mathcal{C}} Z_k^{\mathcal{C}} - \frac{2^\gamma \ln 2}{2^\gamma-1}Z_k^R\right)$ may increase the asymptotic variance of both estimators.*

4 The asymptotic behaviour of reduced-bias quantile estimators

Details on semi-parametric estimation of extremely high quantiles for a general extreme value index $\gamma \in \mathbb{R}$ may be found in de Haan and Rootzén (1993) and more recently in Ferreira *et al.* (2003). Matthys and Beirlant (2003), Gomes and Figueiredo (2006), Mathys *et al.* (2004), Gomes and Pestana (2005) and Beirlant *et al.* (2006) deal with heavy tails and reduced-bias quantile estimation.

Since we will work only with the asympotic unbiased extreme value estimator $\hat{\gamma}_R \equiv \overline{H}$ in (1.18), we shall next consider and study the high quantile estimator,

$$\overline{Q}^{(p)}_{\overline{H}}(k) := \frac{X_{n-[k/2]:n} - X_{n-k:n}}{2^{\overline{H}(k)} - 1} \left(\frac{k}{n\,p}\right)^{\overline{H}(k)} \times (1 - B_{1/2}(\overline{H}(k), \hat{\rho}, \hat{\beta})). \tag{4.1}$$

We may state the following results:

Theorem 4.1. *Under the second order framework in* (1.12) *with* $A(t) = \gamma\beta\, t^{\rho}$, *for intermediate* k, *i.e.,* k *such that* (1.5) *holds, whenever* $\ln(np)/\sqrt{k} \to 0$, *and* $\sqrt{k}\ A(n/k) \to \lambda$, *as* $n \to \infty$,

$$\frac{\sqrt{k}}{\ln(\frac{k}{np})} \left(\frac{Q^{(p)}_{H}(k)}{\chi_{1-p}} - 1\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} Normal\left(\frac{\lambda}{1-\rho}, \gamma^2\right). \tag{4.2}$$

Moreover, for $\hat{\rho}$ *and* $\hat{\beta}$ *introduced in* Remark 2.3*, such that* $\hat{\rho} - \rho = o_p(1/\ln n)$*, as* $n \to \infty$,

$$\frac{\sqrt{k}}{\ln(\frac{k}{np})} \left(\frac{\overline{Q}^{(p)}_{\overline{H}}(k)}{\chi_{1-p}} - 1\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} Normal\left(0, \gamma^2\right). \tag{4.3}$$

Proof. Since $\chi_{1-p} = U(1/p)$, and with $\hat{\gamma} = \hat{\gamma}(k)$ any tail index estimator, one can write

$$Q^{(p)}_{\hat{\gamma}}(k) - \chi_{1-p} = X_{n-k:n}\left((k/np)^{\hat{\gamma}}(k) - \frac{U(1/p)}{X_{n-k:n}}\right).$$

Using the second order condition (1.11) and the result in Proposition 3.2,

$$\frac{U(1/p)}{X_{n-k:n}} = \frac{U(\frac{n}{k} \times \frac{k}{np})}{U(\frac{n}{k})} \times \frac{U(\frac{n}{k})}{X_{n-k:n}} \overset{d}{=} (\tfrac{k}{np})^{\gamma}\left(1 + \frac{(\frac{k}{np})^{\rho}-1}{\rho} A(n/k)(1 + o(1))\right)$$
$$\times\left(1 - \frac{\gamma}{\sqrt{k}} B_k + o_p(A(n/k))\right).$$

As $\frac{(\frac{k}{np})^{\rho}-1}{\rho} \to -1/\rho$, we have

$$\frac{U(1/p)}{X_{n-k:n}} \stackrel{d}{=} (\tfrac{k}{np})^{\gamma}\left(1-\frac{\gamma}{\sqrt{k}}B_k-\frac{A(n/k)}{\rho}(1+o_p(1))\right).$$

The delta method enable us to write $c^{\hat{\gamma}} \stackrel{d}{=} c^{\gamma}(1+\ln c(\hat{\gamma}-\gamma)(1+o_p(1)))$, for any $c>1$. Therefore, also using the result from Corollary 3.1, we get

$$Q_{\hat{\gamma}}^{(p)}(k)-\chi_{1-p} \stackrel{d}{=} C(\tfrac{1}{p})^{\gamma}\left[\ln(\tfrac{k}{np})(\hat{\gamma}(k)-\gamma)+\frac{\gamma}{\sqrt{k}}B_k+\frac{A(n/k)}{\rho}(1+o(1))\right], \tag{4.4}$$

that is, since $\chi_{1-p} \sim Cp^{-\gamma}$,

$$\frac{\sqrt{k}}{\ln(\frac{k}{np})}\left(\frac{Q_{\hat{\gamma}}^{(p)}(k)}{\chi_{1-p}}-1\right) \stackrel{d}{=} \sqrt{k}(\hat{\gamma}(k)-\gamma)+\frac{\gamma B_k}{\ln(\frac{k}{np})}+\frac{\sqrt{k}A(n/k)}{\rho\ln(\frac{k}{np})}(1+o(1)).$$

Since we consider $\sqrt{k}\ A(n/k) \to \lambda$, finite, as $n\to\infty$, then $\ln(\frac{k}{np}) \to \infty$ and the asymptotic result in (4.2) follows from (1.13).

Notice next that we can write $\overline{Q}_{\overline{H}}^{(p)}(k) = \overline{C}_{\overline{H}}(k)\ p^{-\overline{H}}$. Using $\chi_{1-p} \sim Cp^{-\gamma}$ and the same type of results used before for $Q_{\hat{\gamma}}^{(p)}(k)$, one can write,

$$\overline{Q}_{\overline{H}}^{(p)}(k) \stackrel{d}{=} \chi_{1-p}\left\{1+\left(\ln(\tfrac{k}{np})-\frac{2^{\gamma}\ln 2}{2^{\gamma}-1}\right)(\overline{H}(k)-\gamma)+\frac{\gamma\sigma_{\mathcal{C}}}{\sqrt{k}}Z_k^{\mathcal{C}}+\frac{2^{\gamma+\rho}-1}{(2^{\gamma}-1)\rho}o_p(A(n/k))\right\}.$$

Therefore, since $\overline{H}(k)-\gamma \stackrel{d}{=} \frac{\gamma}{\sqrt{k}}Z_k+o_p(A(n/k))$,

$$\frac{\sqrt{k}}{\ln(\frac{k}{np})}\left(\frac{\overline{Q}_{\overline{H}}^{(p)}(k)-\chi_{1-p}}{\chi_{1-p}}\right) \stackrel{d}{=} \gamma Z_k+\frac{\gamma}{\ln(\frac{k}{np})}\left(\sigma_{\mathcal{C}}Z_k^{\mathcal{C}}-\frac{2^{\gamma}\ln 2}{2^{\gamma}-1}Z_k\right)+o_p\left(\frac{\sqrt{k}A(n/k)}{\ln(\frac{k}{np})}\right),$$

and (4.3) follows. □

Remark 4.1. *Notice that, in equation* (4.3)*, we have a mean value equal to 0, even if* $\sqrt{k}A(n/k) \to \lambda \neq 0$*, as* $n\to\infty$*.*

Remark 4.2. *Since* $\ln(\frac{k}{np})$ *goes to infinity very slowly, we can state a better distributional representation, for moderate* k *and* n (pre-asymptotic case)*:*

$$\frac{\sqrt{k}}{\ln(\frac{k}{np})}\left(\frac{Q_{H}^{(p)}(k)}{\chi_{1-p}}-1\right) \stackrel{d}{\approx} Normal\left(\frac{\lambda}{1-\rho}\left(1+\frac{1-\rho}{\rho\ln(\frac{k}{np})}\right),\gamma^2\left(1+\frac{1}{\ln^2(\frac{k}{np})}\right)\right),$$

$$\frac{\sqrt{k}}{\ln(\frac{k}{np})}\left(\frac{\overline{Q}^{(p)}_{H}(k)}{\chi_{1-p}}-1\right)\overset{d}{\approx} Normal\left(0,\gamma^2\left(\left\{1-\frac{s_1(\gamma)}{\ln(\frac{k}{np})}\right\}^2+\frac{1+s_2(\gamma)}{\ln^2(\frac{k}{np})}\right)\right),$$

with $s_1(\gamma)=\frac{2^\gamma\ln 2}{(2^\gamma-1)}$ *and* $s_2(\gamma)=\left(\frac{2^\gamma}{2^\gamma-1}\right)^2$*. Notice that* $s_1(\gamma)\to\infty$ *and* $s_2(\gamma)\to\infty$ *as* $\gamma\to 0$*.*

5 Finite sample behavior — a simulation study

We compare the finite sample behaviour of the proposed high quantile estimator $\overline{Q}^{(p)}_{H}(k)$ with the one of the classical estimator $Q^{(p)}_{H}(k)$ for $p=1/n$ and $p=1/(n\ln n)$. We will also consider the high quantile estimator $Q^{(p)}_{\overline{H}}(k)$ based on the classical C estimador and the reduced-bias γ-estimator, to see whether the bias reduction in the C estimator is relevant in practice.

Notice that when $p=p_n=1/n$ we will have $np_n\to 1$ and when $p=1/(n\ln n)$ we get $np_n\to 0$, as assumed in equation (1.2). The results are based on 10000 samples from the *Fréchet* model, with d.f.,

$$F(x)=1-\exp\left(-x^{-1/\gamma}\right),\quad \gamma>0,\quad x>0.$$

We assume that the second order parameters ρ and β are unknown and they are estimated through (2.1) and (2.3), respectively, both computed at the level $k_1=[n^{0.995}]$, in (2.2). In Figures 2 and 3 we show, for $p=1/n$, the simulated patterns of the mean value, $E[\cdot]$, and the mean squared error, $MSE[\cdot]$ of $Q^{(p)}_{H}(k)$, $Q^{(p)}_{\overline{H}}(k)$ and $\overline{Q}^{(p)}_{H}(k)$. In Figures 4 and 5 we show the same patterns, but now for $p=1/(n\ln n)$.

Remark 5.1. *The computation of both second order parameters' estimators, at the high value* $k_1=[n^{0.995}]$*, enables us to work with high quantiles' estimators with a mean squared error smaller than the mean squared error of the classical estimator* $Q^{(p)}_{H}(k)$*, for small to moderate values of* k *when* $\gamma=0.25$*, or for all* k *values when* $\gamma=0.5$*. The Figures show us that for almost every level* k*, fixed,* $MSE[\overline{Q}^{(p)}_{H}(k)]<MSE[Q^{(p)}_{\overline{H}}(k)]<MSE[Q^{(p)}_{H}(k)]$*. The reduction of the mean squared error is a direct consequence of the bias reduction. So, as expected,* $\overline{Q}^{(p)}_{H}(k)$ *exhibits highly stable sample paths around the true value* χ_{1-p}*. Only for small values of* γ*, like the value* $\gamma=0.25$ *and for small* k *we have* $MSE[\overline{Q}^{(p)}_{H}(k)]>MSE[Q^{(p)}_{H}(k)]$*. This may be explained by the pre-asymptotic behavior provided in* Remark 4.2.

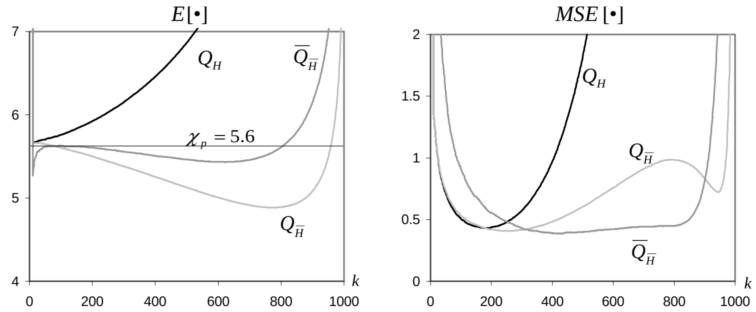

Figure 2: Underlying Fréchet parent with $\gamma = 0.25$ and $p = 1/n$.

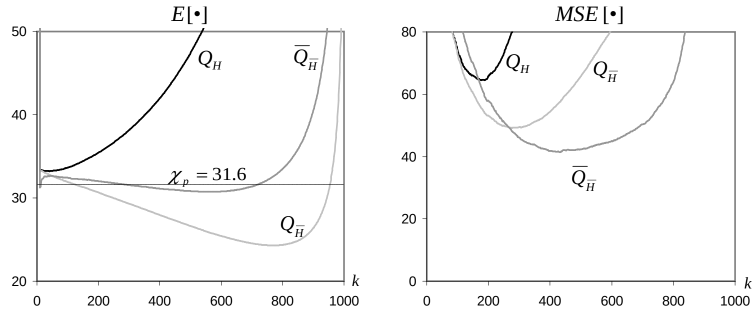

Figure 3: Underlying Fréchet parent with $\gamma = 0.5$ and $p = 1/n$.

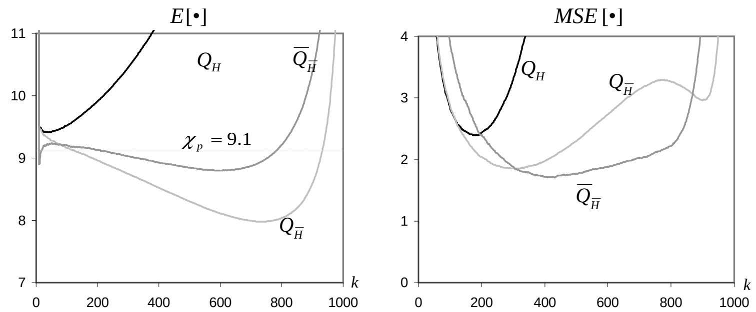

Figure 4: Underlying Fréchet parent with $\gamma = 0.25$ and $p = 1/(n \ln n)$.

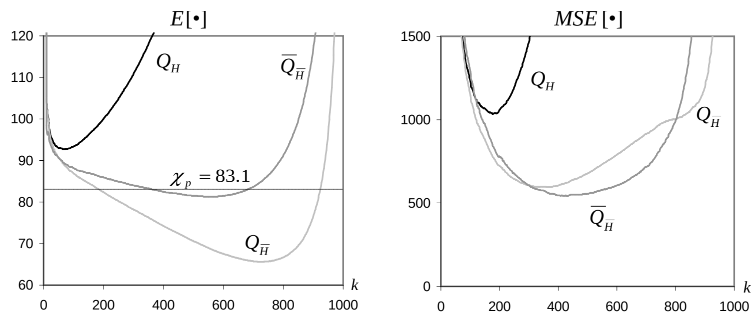

Figure 5: Underlying Fréchet parent with $\gamma = 0.5$ and $p = 1/(n \ln n)$.

In Tables 1 and 2 we present, for sample sizes $n = 100, 500, 1000, 2000, 10000, 20000$ and 50000, and $p = 1/n$, the simulated optimal sample fractions, k_0/n, mean values and mean squared errors of the different estimators under study, at their optimal levels k_0. Tables 3 and 4 are equivalent to Tables 1 and 2, but now with $p = 1/(n \ln n)$.

Remark 5.2. *The tables show us that* $\overline{Q}_H^{(p)}(k)$ *has almost generally the smallest minimum mean squared error, although we need to go slightly deeper in the sample. Also, for every sample size,* $E\big[\overline{Q}_H^{(p)}(k_0)\big]$ *is the mean value closest to the true quantile* χ_{1-p}.

6 Sensitivity to the model F: a simulation study

In this section we will study the sensivity of the new semi-parametric estimators to changes of the underlying heavy-tailed model F, when we consider models that may be even no longer supported by the developed theory, i.e., models that do not belong to Hall's class of models in (1.3). In practice, F is an unknown model, and it is thus important to know not only the way these estimators react to changes of F, but also to detect whether their eficiency decrease drastically. We have only considered the Log-gamma model, a model that does not belong to Hall's class of models in (1.3), but it is under the second order framework in (1.12), with $\rho = 0$. This model has the following tail function

$$
\begin{aligned}
1 - F(x) &= \frac{1}{\gamma^m \Gamma(m)} \int_x^\infty (\ln u)^{m-1} u^{-1/\gamma - 1} du \\
&= \frac{1}{\gamma^{m-1}\Gamma(m)} x^{-1/\gamma} (\ln x)^{m-1} \sum_{k=0}^{\infty} \left(\frac{\gamma}{\ln x}\right)^k \frac{\Gamma(m)}{\Gamma(m-k)} \\
&= \frac{x^{-1/\gamma}(\ln x)^{m-1}}{\gamma^{m-1}\Gamma(m)} \left[1 + \gamma \frac{m-1}{\ln x} + \gamma^2 \frac{(m-1)(m-2)}{(\ln x)^2}(1 + o(1))\right],
\end{aligned}
$$

for $\gamma > 0$, $m > 0$ and $x > 1$. We have here chosen $m = 2$ and for this value the d.f. is:

$$
1 - F(x) = x^{-1/\gamma}\left(1 + \frac{\ln x}{\gamma}\right), \quad \gamma > 0, \quad x > 1.
$$

Although we have considered the same values of n, as in the previous section, we have decided to show results only for $n = 1000$ ($p = 1/n$ and $p = 1/(n \ln n)$). The simulation results

Table 1: Simulated optimal sample fraction, mean value and MSE (at the optimal level) of the estimators under study for $p = 1/n$ and the Fréchet model with $\gamma = 0.25$.

n	100	500	1000	2000	5000	10000	20000	50000
$k_0(Q_H^{(p)})/n$	0.330	0.225	0.181	0.140	0.113	0.089	0.075	0.053
$k_0(Q_{\overline{H}}^{(p)})/n$	0.348	0.271	0.259	0.237	0.209	0.191	0.167	0.155
$k_0(\overline{Q}_{\overline{H}}^{(p)})/n$	0.711	0.486	0.419	0.462	0.400	0.414	0.396	0.343
χ_{1-p}	3.158	4.728	5.623	6.687	8.409	10.000	11.892	14.954
$\mathbb{E}(Q_H^{(p)})/\chi_{1-p}$	1.056	1.053	1.047	1.040	1.036	1.030	1.027	1.021
$\mathbb{E}(Q_{\overline{H}}^{(p)})/\chi_{1-p}$	0.935	0.961	0.966	0.973	0.980	0.985	0.989	0.992
$\mathbb{E}(\overline{Q}_{\overline{H}}^{(p)})/\chi_{1-p}$	0.972	0.976	0.981	0.987	0.990	0.994	0.996	0.997
$MSE(Q_H^{(p)})$	**0.344**	0.411	0.426	0.444	0.451	0.446	0.436	0.447
$MSE(Q_{\overline{H}}^{(p)})$	0.350	0.412	0.403	0.385	0.348	0.305	0.263	0.216
$MSE(\overline{Q}_{\overline{H}}^{(p)})$	0.369	**0.396**	**0.379**	**0.347**	**0.303**	**0.259**	**0.218**	**0.173**

Table 2: Simulated optimal sample fraction, mean value and MSE (at the optimal level) of the estimators under study for $p = 1/n$ and the Fréchet model with $\gamma = 0.5$.

n	100	500	1000	2000	5000	10000	20000	50000
$k_0(Q_H^{(p)})/n$	0.322	0.212	0.173	0.132	0.106	0.087	0.072	0.052
$k_0(Q_{\overline{H}}^{(p)})/n$	0.440	0.317	0.284	0.255	0.220	0.200	0.177	0.157
$k_0(\overline{Q}_{\overline{H}}^{(p)})/n$	0.636	0.477	0.435	0.453	0.401	0.374	0.418	0.336
χ_{1-p}	9.975	22.349	31.615	44.716	70.707	99.997	141.420	223.606
$\mathbb{E}(Q_H^{(p)})/\chi_{1-p}$	1.142	1.117	1.102	1.084	1.073	1.063	1.055	1.044
$\mathbb{E}(Q_{\overline{H}}^{(p)})/\chi_{1-p}$	0.858	0.920	0.933	0.948	0.961	0.969	0.977	0.985
$\mathbb{E}(\overline{Q}_{\overline{H}}^{(p)})/\chi_{1-p}$	0.981	0.978	0.981	0.985	0.984	0.988	0.992	0.994
$MSE(Q_H^{(p)})$	18.239	44.686	63.589	90.649	142.123	194.614	265.661	423.613
$MSE(Q_{\overline{H}}^{(p)})$	13.378	35.430	48.605	65.646	94.611	117.926	144.961	189.758
$MSE(\overline{Q}_{\overline{H}}^{(p)})$	**12.146**	**29.868**	**40.804**	**54.103**	**75.052**	**91.864**	**109.890**	**139.540**

Table 3: Simulated optimal sample fraction, mean value and MSE (at the optimal level) of the estimators under study for $p = 1/(n \ln n)$ and the Fréchet model with $\gamma = 0.25$.

n	100	500	1000	2000	5000	10000	20000	50000
$k_0(Q_H^{(p)})/n$	0.331	0.221	0.178	0.134	0.108	0.087	0.074	0.052
$k_0(Q_{\overline{H}}^{(p)})/n$	0.456	0.339	0.300	0.262	0.242	0.224	0.196	0.181
$k_0(\overline{Q}_{\overline{H}}^{(p)})/n$	0.638	0.437	0.411	0.427	0.390	0.371	0.401	0.317
χ_{1-p}	4.631	7.466	9.117	11.104	14.365	17.421	21.096	27.121
$\mathbb{E}(Q_H^{(p)})/\chi_{1-p}$	1.113	1.089	1.076	1.060	1.052	1.044	1.039	1.030
$\mathbb{E}(Q_{\overline{H}}^{(p)})/\chi_{1-p}$	0.906	0.948	0.959	0.971	0.977	0.982	0.988	0.992
$\mathbb{E}(\overline{Q}_{\overline{H}}^{(p)})/\chi_{1-p}$	0.948	0.975	0.980	0.984	0.986	0.994	0.995	0.996
$MSE(Q_H^{(p)})$	1.843	2.254	2.361	2.478	2.556	2.548	2.527	2.622
$MSE(Q_{\overline{H}}^{(p)})$	1.433	1.841	1.831	1.782	1.644	1.462	1.277	1.068
$MSE(\overline{Q}_{\overline{H}}^{(p)})$	**1.376**	**1.702**	**1.683**	**1.608**	**1.447**	**1.261**	**1.082**	**0.876**

Table 4: Simulated optimal sample fraction, mean value and MSE (at the optimal level) of the estimators under study for $p = 1/(n \ln n)$ and the Fréchet model with $\gamma = 0.5$.

n	100	500	1000	2000	5000	10000	20000	50000
$k_0(Q_H^{(p)})/n$	0.323	0.215	0.174	0.128	0.104	0.085	0.069	0.051
$k_0(Q_{\overline{H}}^{(p)})/n$	0.540	0.410	0.337	0.302	0.261	0.236	0.205	0.190
$k_0(\overline{Q}_{\overline{H}}^{(p)})/n$	0.572	0.462	0.426	0.453	0.412	0.412	0.364	0.355
χ_{1-p}	21.448	55.739	83.110	123.294	206.362	303.485	445.050	735.519
$\mathbb{E}(Q_H^{(p)})/\chi_{1-p}$	1.307	1.215	1.175	1.135	1.111	1.093	1.078	1.061
$\mathbb{E}(Q_{\overline{H}}^{(p)})/\chi_{1-p}$	0.836	0.893	0.923	0.938	0.953	0.963	0.974	0.981
$\mathbb{E}(\overline{Q}_{\overline{H}}^{(p)})/\chi_{1-p}$	0.994	0.989	0.988	0.985	0.985	0.983	0.990	0.993
$MSE([Q_H^{(p)})$	266.23	690.34	1016.44	1496.94	2465.70	3509.42	4990.13	8363.82
$MSE(Q_{\overline{H}}^{(p)})$	123.03	399.01	586.29	849.08	1305.39	1710.00	2212.28	3076.93
$MSE(\overline{Q}_{\overline{H}}^{(p)})$	**122.99**	**369.55**	**531.61**	**746.43**	**1102.15**	**1406.77**	**1773.83**	**2393.19**

are presented in Figures 6, 7. These Figures are parallel to Figures 2, 3, 4 and 5, but for the above mentioned model.

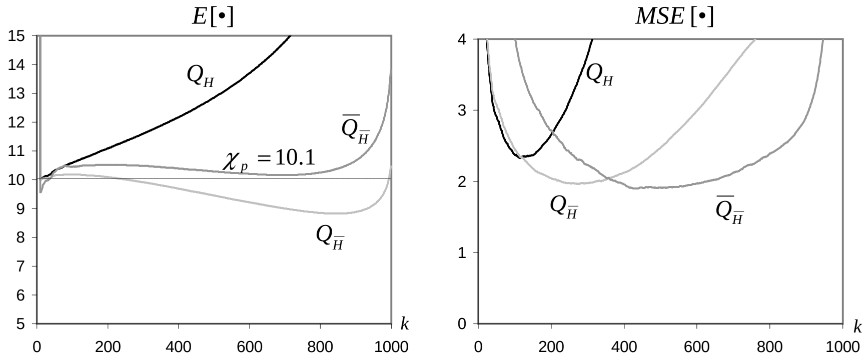

Figure 6: Underlying Log-gamma parent with $\gamma = 0.25$ and $p = 1/n$.

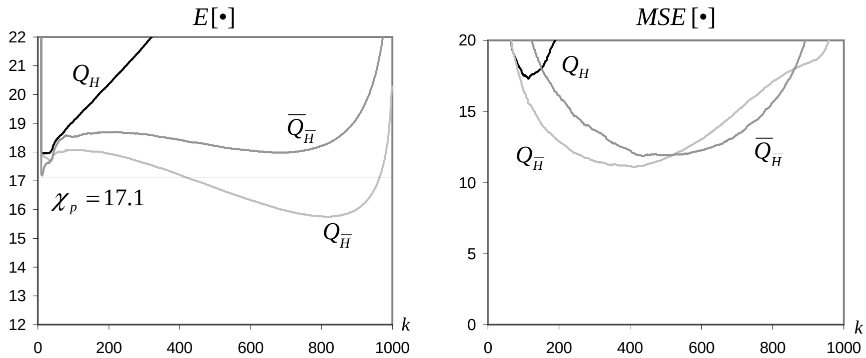

Figure 7: Underlying Log-gamma parent with $\gamma = 0.25$ and $p = 1/n \ln n$.

For this model, outside Hall's class, $Q_{\overline{H}}$ behaves slightly better than $\overline{Q}_{\overline{H}}$ but not a long way from it, both better than Q_H.

7 Application to Financial Data

We shall finally consider an illustration of the performance of the above mentioned estimators, reporting results associated to the Euro-UK Pound exchange rates from January 2, 2004 until December 29, 2006, which correspond to a sample of size $n = 771$. This data has been collected by the European System of Central Banks, and was obtained from http://www.bportugal.pt/.

The Value at Risk (VaR) is a common risk measure, defined as a large quantile of the log-returns, i.e., of $L_t = \ln(X_{t+1}/X_t)$, $1 \le t \le n-1$, assumed to be stationary and weakly dependent.

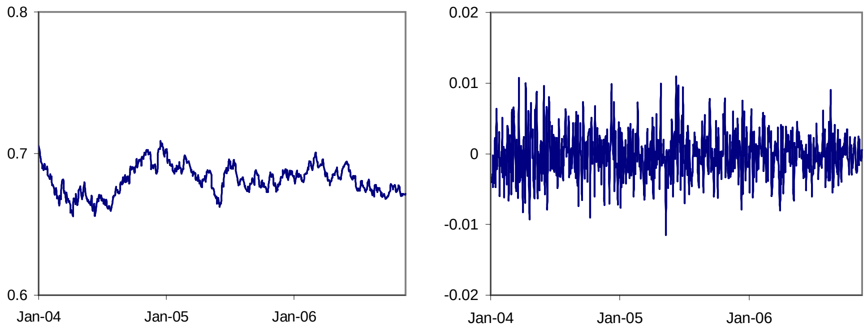

Figure 8: Daily exchange rates (left) and daily log-returns (right) on Euro vs UK pound exchange rate.

Working with the $n^- = 384$ negative log-returns, we show in Figure 9 the sample paths of the most common estimators of the second order parameters ρ and β.

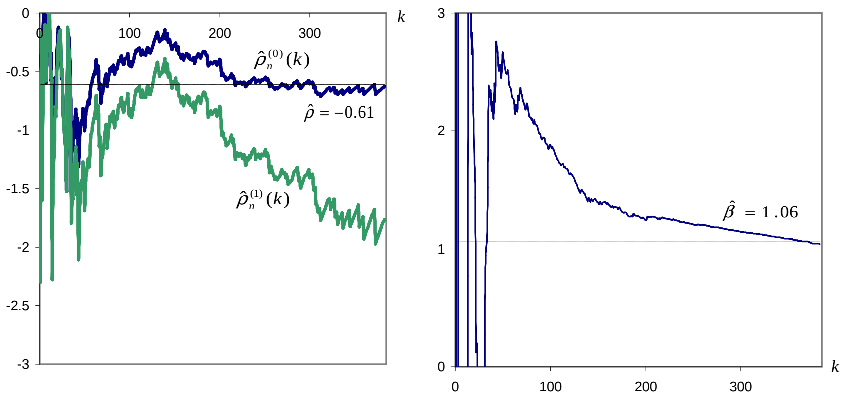

Figure 9: Estimates of the second-order parameter ρ (*left*) and β (*right*) for the Daily Log-Returns on the Euro vs UK Pound exchange rate.

The sample paths of the ρ-estimates associated to $\tau = 0$ and $\tau = 1$ lead us to choose, on the basis of any stability criterion for large values of k, the estimate associated to $\tau = 0$. From the experience we have with this class of estimates, this means that $|\rho| \le 1$ and the tuning parameter $\tau = 0$ is then advisable. We have got $\hat\rho = -0.61$. The use of $\hat\beta$ in (2.3), computed at the level k_1 in (2.2), i.e., at $k_1 = (n^-)^{0.995} = 372$, leads then us to the estimate $\hat\beta = 1.06$

The sample paths of the classical Hill estimator in (1.8), the second order reduced-bias tail index estimator $\overline{H}$ in (1.18) and the associated Var-estimators in (1.7) and (4.1), respectively, for $p = 0.001$, are pictured in Figure 10. For the Hill estimator, as we know how to estimate the second order parameters ρ and β, we can estimate the optimal sample fraction, in (3.3), and the extreme value index. We get $\hat{k}_0^H = 24$ and $H(24) = 0.16$. Since we do not have yet the possibility of adaptively estimate the optimal sample fraction associated to any second order reduced-bias estimator, the estimate pictured, $\hat{\gamma} = 0.24$, is the median of the $\overline{H}(k)$ estimates for k between k_0^H and $5 \times k_0^H$. A similar technique led us to the quantile estimate $\chi_{0.001} = 0.0197$, as pictured in Figure 10 (right).

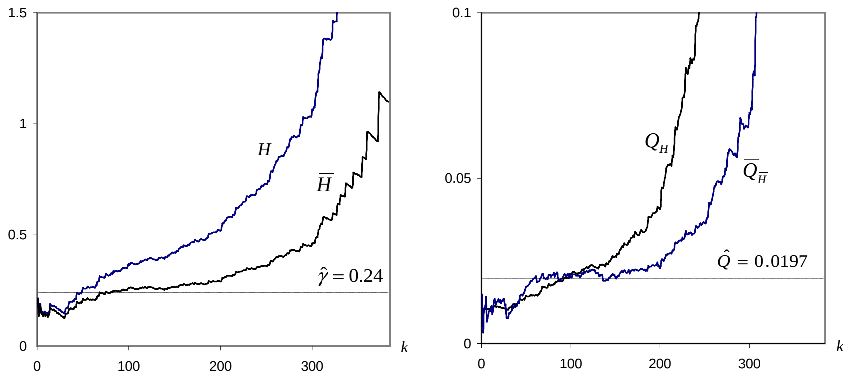

Figure 10: Estimates of the first-order parameter γ (*left*) and of the high quantile $\chi_{0.001}$ (*right*).

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