NATURALLY ORDERED REGULAR SEMIGROUPS WITH AN INVERSE MONOID TRANSVERSAL

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ABSTRACT. The notion of an inverse transversal of a regular semigroup is well-known. Here we investigate naturally ordered regular semigroups that have an inverse transversal. Such semigroups are necessarily locally inverse and the inverse transversal is a quasi-ideal. After considering various general properties that relate the imposed order to the natural order, we highlight the situation in which the inverse transversal is a monoid. The regularity of Green's relations is also characterised. Finally, we determine the structure of a naturally ordered regular semigroup with an inverse monoid transversal.

1. Introduction.

If S is a regular semigroup then an *inverse transversal* of S is an inverse subsemigroup T of S that contains a unique inverse of every $x \in S$. We let $T \cap V(x) = \{x^\circ\}$ and write T as $S^\circ = \{x^\circ \mid x \in S\}$. This concept has its roots in [8], the term itself being introduced in [6], where the structure of regular semigroups with a multiplicative inverse transversal was determined.

For a convenient summary of the basic properties of inverse transversals on a regular semigroup we refer the reader to [1] or to [2, Chapter 14]. For our purposes here we mention only the following notation and general properties:

(α) $(\forall x, y \in S) \quad (xy)^\circ = (x^\circ xy)^\circ x^\circ = y^\circ (xyy^\circ)^\circ = y^\circ (x^\circ xyy^\circ)^\circ x^\circ$.

(β) $(\forall x, y \in S) \quad (xy^\circ)^\circ = y^{\circ\circ} x^\circ, \ (x^\circ y)^\circ = y^\circ x^{\circ\circ}$.

(γ) Green's relations on S are given by

$$(x,y) \in \mathscr{L} \iff x^\circ x = y^\circ y, \quad (x,y) \in \mathscr{R} \iff xx^\circ = yy^\circ.$$

(δ) The subsets

$$\Lambda = \{x^\circ x \mid x \in S\}, \quad \mathrm{I} = \{xx^\circ \mid x \in S\}$$

are sub-bands of S with Λ right regular and I left regular.

(ε) $\mathrm{I}\Lambda = \{x \in S \mid x^\circ \in E(S^\circ)\}$ and $\Lambda \cap \mathrm{I} = E(S^\circ) = S^\circ \cap \mathrm{I}\Lambda$.

(ζ) The subsets

$$L = \{xx^\circ x^{\circ\circ} \mid x \in S\}, \quad R = \{x^{\circ\circ} x^\circ x \mid x \in S\}$$

are subsemigroups of S. Moreover, L is left inverse and a left ideal of S, R is right inverse and a right ideal of S, $L \cap R = S^\circ$, $E(L) = \mathrm{I}$, and $E(R) = \Lambda$.

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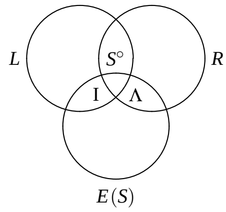
(η) The following Venn diagram provides a useful summary:



(ϑ) The inverse transversal S° is said to be a *quasi-ideal* if $S^\circ S S^\circ \subseteq S^\circ$ or, equivalently, $\Lambda \mathrm{I} \subseteq S^\circ$; *multiplicative* if $\Lambda\mathrm{I} \subseteq E(S^\circ)$; and *weakly multiplicative* if $(\Lambda\mathrm{I})^\circ \subseteq E(S^\circ)$ or, equivalently, $e^\circ \in E(S^\circ)$ whenever $e \in E(S)$.

(ι) S° is multiplicative if and only if it is both weakly multiplicative and a quasi-ideal.

(κ) If $x \in L$ or $y \in R$ then $(xy)^\circ = y^\circ x^\circ$.

(λ) S is orthodox if and only if $(\forall x, y \in S)\ \ (xy)^\circ = y^\circ x^\circ$.

Throughout what follows, $(S;\leqslant)$ *will denote an ordered regular semigroup with an inverse transversal* S°. Moreover, $\leqslant_n$ will denote the *natural order* on S, which is given by

$$(\forall x, y \in S) \quad x \leqslant_n y \iff \big(\exists e, f \in E(S)\big)\ \ x = ey = yf.$$

On the subset $E(S)$ of idempotents this reduces to

$$e \leqslant_n f \iff e = ef = fe.$$

On the inverse subsemigroup S° it reduces to

$$(\forall x, y \in S^\circ)\ \ x \leqslant_n y \iff \big(\exists e \in E(S^\circ)\big)\ \ x = ey \iff \big(\exists f \in E(S^\circ)\big)\ \ x = yf,$$

of which there are many variants, for example

$$(\forall x, y \in S^\circ) \quad x \leqslant_n y \iff x = xy^\circ x \iff xx^\circ = yx^\circ.$$

We say that $(S;\leqslant)$ is *naturally ordered* if $\leqslant$ extends the natural order $\leqslant_n$ on $E(S)$, in the sense that

$$\big(\forall e, f \in E(S)\big) \quad e \leqslant_n f \Rightarrow e \leqslant f.$$

For our purposes here, we recall the following particular results:

Theorem 1. [5] *Let $(S;\leqslant)$ be an ordered regular semigroup. If $e, f \in E(S)$ are such that $e \leqslant f$ then the products ef, fe, efe, fef are all idempotent. Moreover, if $(S;\leqslant)$ is naturally ordered then $e = efe$.* □

Theorem 2. [7] *A regular semigroup can be naturally ordered if and only if it is locally inverse.* □

Theorem 3. [3] *If S is a regular semigroup with an inverse transversal S° then S is locally inverse if and only if S° is a quasi-ideal.* □

The following result is contained in the proof of [4, Theorem 2].

Theorem 4. *The natural order $\leqslant_n$ on S has the following properties*:

(1) $(\forall x, y \in S) \quad x \leqslant_n y \Rightarrow x^\circ x \leqslant_n y^\circ y,\ \ xx^\circ \leqslant_n yy^\circ$;

(2) $\big(\forall e, f \in E(S)\big) \quad e \leqslant_n f \iff e^\circ e \leqslant_n f^\circ f,\ \ ee^\circ \leqslant_n ff^\circ$.

If, moreover, S is locally inverse then

(3) $\big(\forall e, f \in E(S)\big) \quad e \leqslant_n f \Rightarrow e^\circ \leqslant_n f^\circ$. □

2. Order properties.

In what follows, we shall assume that $(S;\leqslant)$ *is naturally ordered*. By Theorems 2 and 3 above the inverse transversal S° is then a quasi-ideal. Consequently, by (ι), S° is multiplicative if and only if it is weakly multiplicative. We begin by considering relationships between the imposed order $\leqslant$ and the natural order $\leqslant_n$ on S. We first note that

- *on* $E(S^\circ)$ *the orders* $\leqslant$ *and* $\leqslant_n$ *coincide*.

To see this, let $e,f \in E(S^\circ)$ be such that $e \leqslant_n f$. Then since S is naturally ordered we have $e \leqslant f$. Conversely, if $e \leqslant f$ then by Theorem 1 we have $e = efe$. Since idempotents in the inverse subsemigroup S° commute, it follows that $e = ef = fe$ and so $e \leqslant_n f$.

In the following result we show that the order $\leqslant$ also extends the natural order $\leqslant_n$ on the inverse transversal S°; and, as a consequence, the assignment $x \mapsto x^{\circ\circ}$ is always isotone.

Theorem 5. $(\forall x,y \in S) \quad x^\circ \leqslant_n y^\circ \;\Rightarrow\; x^\circ \leqslant y^\circ.$

Proof. Suppose that $x,y \in S$ are such that $x^\circ \leqslant_n y^\circ$. Then, by Theorem 4(1), in $E(S^\circ)$ we have $x^{\circ\circ}x^\circ \leqslant_n y^{\circ\circ}y^\circ$. Consequently, $x^{\circ\circ}x^\circ \leqslant y^{\circ\circ}y^\circ$ and so $y^\circ x^{\circ\circ}x^\circ \leqslant y^\circ$. But from $x^\circ \leqslant_n y^\circ$ we obtain $x^\circ x^{\circ\circ} = y^\circ x^{\circ\circ}$. It therefore follows that $x^\circ = y^\circ x^{\circ\circ}x^\circ \leqslant y^\circ$. □

Corollary. $(\forall x,y \in S) \quad x \leqslant y \;\Rightarrow\; x^{\circ\circ} \leqslant y^{\circ\circ}.$

Proof. If $x,y \in S$ are such that $x \leqslant y$ then, using the fact that S° is a quasi-ideal, we have

$$x^{\circ\circ} = x^{\circ\circ}x^\circ x x^\circ x^{\circ\circ} \leqslant x^{\circ\circ}x^\circ y x^\circ x^{\circ\circ} = x^{\circ\circ}x^\circ y^{\circ\circ}x^\circ x^{\circ\circ} \leqslant_n y^{\circ\circ}x^\circ x^{\circ\circ} \leqslant_n y^{\circ\circ}$$

whence, by Theorem 5, $x^{\circ\circ} \leqslant y^{\circ\circ}$. □

As for the corresponding assignment $x \mapsto x^\circ$, the following result shows that it is isotone on the subset $\mathrm{I}\Lambda$.

Theorem 6. $(\forall x,y \in \mathrm{I}\Lambda) \quad x \leqslant y \;\Rightarrow\; x^\circ \leqslant y^\circ.$

Proof. Let $x,y \in \mathrm{I}\Lambda$ be such that $x \leqslant y$. Then $x^\circ, y^\circ \in E(S^\circ)$ by (ε), and $x^\circ = x^\circ x x^\circ \leqslant x^\circ y x^\circ$. Since S° is a quasi-ideal we have $x^\circ y x^\circ = x^\circ y^{\circ\circ}x^\circ \in E(S^\circ)$. Thus, in $E(S^\circ)$ we have $x^\circ \leqslant x^\circ y^{\circ\circ}x^\circ$. It follows by Theorem 1 that $x^\circ = x^\circ y^{\circ\circ}x^\circ$ whence $x^\circ \leqslant_n y^\circ$. That $x^\circ \leqslant y^\circ$ now follows by Theorem 5. □

Precisely when $x \mapsto x^\circ$ is isotone on S is the substance of the following result.

Theorem 7. *The following statements are equivalent*:

(1) $(\forall x,y \in S) \quad x \leqslant y \;\Rightarrow\; x^\circ \leqslant y^\circ$;

(2) *the orders* $\leqslant$ *and* $\leqslant_n$ *coincide on* S°.

Proof. $(1) \Rightarrow (2)$: Suppose that (1) holds and let $x,y \in S$ be such that $x^\circ \leqslant y^\circ$. Then $x^{\circ\circ} \leqslant y^{\circ\circ}$ and consequently

$$y^{\circ\circ}x^\circ = y^{\circ\circ}x^\circ x^{\circ\circ}x^\circ \leqslant y^{\circ\circ}x^\circ \cdot y^{\circ\circ}x^\circ \leqslant y^{\circ\circ}y^\circ y^{\circ\circ}x^\circ = y^{\circ\circ}x^\circ$$

whence $y^{\circ\circ}x^\circ \in E(S^\circ)$. Then $y^{\circ\circ}x^\circ = y^{\circ\circ}x^\circ x^{\circ\circ}x^\circ \leqslant_n x^{\circ\circ}x^\circ$ and therefore, since S is naturally ordered, $y^{\circ\circ}x^\circ \leqslant x^{\circ\circ}x^\circ$. Since $x^{\circ\circ} \leqslant y^{\circ\circ}$ it follows from this that $y^{\circ\circ}x^\circ = x^{\circ\circ}x^\circ$ whence $x^\circ y^{\circ\circ}x^\circ = x^\circ$ and so $x^\circ \leqslant_n y^\circ$. Property (2) now follows by Theorem 5.

$(2) \Rightarrow (1)$: If $x \leqslant y$ then, by the Corollary to Theorem 5, $x^{\circ\circ} \leqslant y^{\circ\circ}$. If now (2) holds then $x^{\circ\circ} \leqslant_n y^{\circ\circ}$. Since on the ordered inverse semigroup $(S^\circ;\leqslant_n)$ the assignment $a \mapsto a^{-1} = a^\circ$ is isotone, we then have $x^\circ = x^{\circ\circ\circ} \leqslant_n y^{\circ\circ\circ} = y^\circ$ whence, by Theorem 5, $x^\circ \leqslant y^\circ$. □

We now add a basic property of naturally comparable idempotents that will be useful.

Theorem 8. *If* $e, f \in E(S)$ *are such that* $e \leqslant_n f$ *then* $e = fe^\circ e = ee^\circ f = fe^\circ f$.

Proof. Since $e = ef = fe$ we have $fe^\circ e \in E(S)$ with $fe^\circ e \leqslant_n f$, whence $fe^\circ e \leqslant f$. Since also $e \leqslant f$, it follows that $e = ee^\circ e \leqslant fe^\circ e = fe^\circ ee \leqslant fe = e$ and so $fe^\circ e = e$. Likewise we have $ee^\circ f = e$, and combining these we obtain $e = fe^\circ f$. $\square$

Using Theorem 8 we can now describe the situation in which the assignment $x \mapsto x^\circ$ is antitone.

Theorem 9. *The following statements are equivalent*:

(1) $(\forall x, y \in S) \quad x \leqslant y \Rightarrow y^\circ \leqslant x^\circ$;
(2) $\big(\forall e, f \in E(S)\big) \quad e \leqslant f \Rightarrow f^\circ \leqslant e^\circ$;
(3) *S is completely simple*;
(4) *the inverse transversal S° is a group.*

Proof. $(1) \Rightarrow (2)$: This is clear.

$(2) \Rightarrow (3)$: Let $e, f \in E(S)$ be such that $e \leqslant_n f$. Then $e \leqslant f$ and so, by the hypothesis and Theorem 8, $f = ff^\circ f \leqslant fe^\circ f = e$ whence $f = e$. Thus $\leqslant_n$ is equality on $E(S)$ and so S is completely simple.

$(3) \Rightarrow (4)$: This is a well-known result of Saito (see [9] or [1]).

$(4) \Rightarrow (1)$: If $x, y \in S$ are such that $x \leqslant y$ then by the Corollary to Theorem 5 we have $x^{\circ\circ} \leqslant y^{\circ\circ}$. Since S° is a group, it follows that $y^\circ \leqslant x^\circ$. $\square$

Example 1. For each $x \in \mathbb{R}$ with $x \geqslant 0$ let

$$Q_x = \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix}, \begin{bmatrix} x & 0 \\ x & 0 \end{bmatrix}, \begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

Under matrix multiplication, the set $Q = \bigcup\limits_{x \geqslant 0} Q_x$ is a semigroup which is regular since, for every $X \in Q_x$ $(x \neq 0)$, we have $\begin{bmatrix} x^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in V(X)$. The subset

$$Q^\circ = \left\{ \begin{bmatrix} x^{-1} & 0 \\ 0 & 0 \end{bmatrix} \;\middle|\; x > 0 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\},$$

which is a group with zero, is then an inverse transversal of Q.

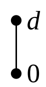
The set of idempotents of Q is

$$E(Q) = \left\{ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \equiv \{a, b, c, d, 0\},$$

and the natural order on $E(Q)$ is given by

$$E \leqslant_n F \iff E = 0 \text{ or } 0 \neq E = F.$$

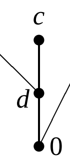
In particular, $\big(E(Q^\circ); \leqslant_n\big)$ consists of the 2-element chain



Consider now the component-wise order $\leqslant$ on Q, this being defined for $X = [x_{ij}]$ and $Y = [y_{ij}]$ in Q by

$$X \leqslant Y \iff (\forall i, j)\ \ x_{ij} \leqslant y_{ij}.$$

As can readily be verified, $(Q;\leqslant)$ is an ordered semigroup and the restriction of $\leqslant$ to $E(Q)$ gives the Hasse diagram



Since $\leqslant$ extends $\leqslant_n$ on $E(Q)$ we see that Q is naturally ordered. Moreover, it follows by Theorem 9 that the subsemigroup $Q\setminus\{0\}$ is completely simple.

In view of Theorem 9 it is natural to investigate the more general situation in which the inverse transversal S° is a monoid. This produces some properties that are similar to those that hold in the case where S has a biggest idempotent (see [5]).

Theorem 10. *The following statements are equivalent*:

(1) *the inverse transversal* S° *is a monoid*;
(2) $(S^\circ;\leqslant)$ *has a biggest idempotent*;
(3) $\big(\exists \xi \in E(S^\circ)\big)\quad S^\circ = \xi S \xi$;
(4) $\big(\exists \xi \in E(S^\circ)\big)(\forall x \in S)\quad x^{\circ\circ} = \xi x \xi$;
(5) $\big(\exists \xi \in E(S^\circ)\big)\quad L = S\xi,\ R = \xi S$;
(6) L [*resp.* R] *is an idempotent-generated principal left* [*resp. right*] *ideal.*

Proof. $(1) \Rightarrow (2)$: If (1) holds then for every $e \in E(S^\circ)$ we have $e \leqslant_n 1_{S^\circ}$ whence $e \leqslant 1_{S^\circ}$.

$(2) \Rightarrow (1)$: Suppose that (2) holds and let $\xi = \max E(S^\circ)$. For every $x^\circ \in S^\circ$ we have $x^\circ x^{\circ\circ} \leqslant \xi$ whence $x^\circ x^{\circ\circ} \leqslant_n \xi$ since $\leqslant$ coincides with $\leqslant_n$ on $E(S^\circ)$. Consequently,

$$\xi x^\circ = \xi x^\circ x^{\circ\circ} x^\circ = x^\circ x^{\circ\circ} x^\circ = x^\circ,$$

and similarly $x^\circ \xi = x^\circ$. Hence $\xi = 1_{S^\circ}$.

$(1) \Rightarrow (3)$: If (1) holds then we have

$$1_{S^\circ} S 1_{S^\circ} \subseteq S^\circ S S^\circ \subseteq S^\circ = 1_{S^\circ} S^\circ 1_{S^\circ} \subseteq 1_{S^\circ} S 1_{S^\circ}$$

whence $S^\circ = \xi S \xi$ where $\xi = 1_{S^\circ}$.

$(3) \Rightarrow (4)$: If (3) holds then for every $x \in S$ we have $x^{\circ\circ} = \xi x^{\circ\circ} \xi = \xi x \xi$.

$(4) \Rightarrow (1)$: If (4) holds then for every $x \in S$ we have

$$x^\circ = (\xi x \xi)^\circ = (\xi^\circ x \xi^\circ)^\circ = \xi^{\circ\circ} x^\circ \xi^{\circ\circ} = \xi x^\circ \xi$$

whence $\xi x^\circ = x^\circ = x^\circ \xi$ and we have (1).

$(4) \Rightarrow (5)$: If (4) holds then for every $x \in S$ we have $x x^\circ x^{\circ\circ} = x x^\circ \xi x \xi \in S\xi$ whence $L \subseteq S\xi$; and similarly $R \subseteq \xi S$. But, as in the above, by (4) we have $x^\circ = \xi x^\circ \xi$ whence it follows that

$$x\xi (x\xi)^\circ (x\xi)^{\circ\circ} = x\xi x^\circ x^{\circ\circ} \xi = x\xi x^\circ \xi x \xi = x x^\circ x \xi = x\xi$$

whence $S\xi \subseteq L$. Similarly, $\xi S \subseteq R$ and we have (5).

$(5) \Rightarrow (3)$: If (5) holds then $S^\circ = L \cap R = S\xi \cap \xi S = \xi S \xi$.

$(5) \Rightarrow (6)$: This is clear.

(6) $\Rightarrow$ (5): Suppose that $L = Sp$ and $R = qS$ where p and q are idempotents. Then $p \in E(L) = \mathrm{I}$ and so $p^\circ = p^{\circ\circ} \in E(S^\circ)$. Likewise, $q \in E(R) = \Lambda$ and $q^\circ = q^{\circ\circ} \in E(S^\circ)$. Consequently, $L = Sp = Sp^\circ p = Sp^\circ p^{\circ\circ} = Sp^\circ$ and $R = qS = qq^\circ S = q^{\circ\circ}q^\circ S = q^\circ S$. Now since $p^\circ \in E(S^\circ) \subseteq R = q^\circ S$ we have $p^\circ = q^\circ p^\circ$; and since $q^\circ \in E(S^\circ) \subseteq L = Sp^\circ$ we have $q^\circ = q^\circ p^\circ$. Consequently $p^\circ = q^\circ$ and (5) follows. □

Theorem 11. *If S° is a monoid with identity element ξ then the following statements are equivalent*:

(1) *the assignment $x \mapsto x^{\circ\circ}$ is a morphism*;
(2) $(\forall x, y \in S)\ \ (xy)^\circ = (x\xi y)^\circ$;
(3) *S is orthodox.*

Proof. (1) $\Rightarrow$ (2): If (1) holds then by Theorem 10 we have $(xy)^{\circ\circ} = x^{\circ\circ}y^{\circ\circ} = \xi x\xi y\xi = (x\xi y)^{\circ\circ}$. It follows that $(xy)^\circ = (x\xi y)^\circ$.

(2) $\Rightarrow$ (3): Since $x\xi \in L$ we have, by (κ), $(x\xi y)^\circ = y^\circ(x\xi)^\circ = y^\circ\xi x^\circ = y^\circ x^\circ$. Then (2) and (λ) give S is orthodox.

(3) $\Rightarrow$ (1): If (3) holds then, by (λ), $(xy)^{\circ\circ} = (y^\circ x^\circ)^\circ = x^{\circ\circ}y^{\circ\circ}$ and (1) follows. □

3. The regularity of $\mathscr{L}$ and $\mathscr{R}$.

Guided by Theorem 4(2) and property (γ), we now consider the following concept.

Definition. We shall say that $\mathscr{L}$ [resp. $\mathscr{R}$] is *weakly regular* if

$$\big(\forall e, f \in E(S)\big) \quad e \leqslant f \Rightarrow e^\circ e \leqslant f^\circ f \ \ [\text{resp. } ee^\circ \leqslant ff^\circ].$$

The weak regularity of $\mathscr{L}$ and $\mathscr{R}$ can be characterised in the following way, which should be viewed in comparison with Theorem 4(3).

Theorem 12. *The following statements are equivalent*:

(1) *$\mathscr{L}$ and $\mathscr{R}$ are weakly regular*;
(2) $\big(\forall e, f \in E(S)\big) \quad e \leqslant f \ \Rightarrow\ e^\circ \leqslant f^\circ$.

Proof. (1) $\Rightarrow$ (2): Suppose that $e, f \in E(S)$ are such that $e \leqslant f$. Then if (1) holds we have $e^\circ e \leqslant f^\circ f$ and $ee^\circ \leqslant ff^\circ$ whence $e^\circ = e^\circ e \cdot ee^\circ \leqslant f^\circ f \cdot ff^\circ = f^\circ$.

(2) $\Rightarrow$ (1): This is immediate. □

Corollary. *If S° is (weakly) multiplicative then $\mathscr{L}$ and $\mathscr{R}$ are weakly regular.*

Proof. Since S° is weakly multiplicative, for every $e \in E(S)$ we have $e^\circ = e^{\circ\circ} \in E(S^\circ)$. Thus, if $e \leqslant f$ then, by the Corollary to Theorem 5, $e^\circ = e^{\circ\circ} \leqslant f^{\circ\circ} = f^\circ$. It follows that $\mathscr{L}$ and $\mathscr{R}$ are weakly regular. □

Remark. Note that in Example 1 we have

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \in E(Q) \quad \text{but} \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^\circ = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \notin E(Q^\circ)$$

and so Q° is not weakly multiplicative. However, by applying Theorem 12(2) to $E(Q)$, we see that both $\mathscr{L}$ and $\mathscr{R}$ are weakly regular. The converse of the above Corollary therefore fails in general.

Definition. We shall say that $\mathscr{L}$ [resp. $\mathscr{R}$] is *regular* if

$$(\forall x, y \in S) \quad x \leqslant y \Rightarrow x^\circ x \leqslant y^\circ y \text{ [resp. } xx^\circ \leqslant yy^\circ].$$

Clearly, 'regular' implies 'weakly regular'. In order to characterise the regularity of $\mathscr{L}$ and $\mathscr{R}$ we require the following concepts.

Definition. An equivalence relation Θ on an ordered set A is said to satisfy the *link property* if

$$a \,\Theta\, x \leqslant b \;\Rightarrow\; (\exists y \in A) \;\; a \leqslant y \,\Theta\, b;$$

and the *dual link property* if

$$a \leqslant x \,\Theta\, b \;\Rightarrow\; (\exists y \in A) \;\; a \,\Theta\, y \leqslant b.$$

Theorem 13. *If* $\mathscr{L}$ [resp. $\mathscr{R}$] *is regular then* $\mathscr{L}$ [resp. $\mathscr{R}$] *satisfies the dual link property.*

Proof. If $a \leqslant x \,\mathscr{L}\, b$ then since $\mathscr{L}$ is regular we have $a^\circ a \leqslant x^\circ x = b^\circ b$ and so $ba^\circ a \leqslant b$. Now, by Theorem 1 and property (δ), we have $a^\circ a = a^\circ ab^\circ ba^\circ a = b^\circ ba^\circ a$ and so

$$a^\circ a = (a^\circ a)^\circ a^\circ a = (b^\circ ba^\circ a)^\circ b^\circ ba^\circ a = (ba^\circ a)^\circ ba^\circ a.$$

Hence $a \,\mathscr{L}\, ba^\circ a \leqslant b$.

Similarly, when regular, $\mathscr{R}$ satisfies the dual link property. $\square$

As the following example shows, even when S° is multiplicative $\mathscr{L}$ can be weakly regular and satisfy the dual link property but fail to be regular

Example 2. Let $k > 1$ be a fixed integer. For every $n \in \mathbb{Z}$ let n_k be the biggest multiple of k that is less than or equal to n. On the cartesian ordered set $S = \mathbb{Z} \times -\mathbb{N} \times \mathbb{Z}$ define a multiplication by

$$(x, -p, m)(y, -q, n) = (\min\{x, y\}, -p, m_k + n).$$

Then S is an ordered semigroup in which the idempotents are $(x, -p, m)$ where $m_k = 0$, i.e. where $0 \leqslant m \leqslant k-1$. It is readily observed that S is naturally ordered, and that for every $(x, -p, m) \in S$ we have $(x, 0, -m_k + k - 1) \in V(x, -p, m)$ so S is regular. Moreover, with $(x, -p, m)^\circ = (x, 0, -m_k + k - 1)$, the set $S^\circ = \{z^\circ \mid z \in S\}$ is an inverse transversal of S which is a quasi-ideal. Since the product of two idempotents is idempotent we see that S is orthodox. Hence S° is also weakly multiplicative [3] and so is multiplicative.

Consider now the relation $\mathscr{L}$ which, by the Corollary to Theorem 12, is weakly regular. Since

$$(x, -p, m)^\circ (x, -p, m) = (x, 0, m - m_k)$$

we see that $\mathscr{L}$ is not regular. To see that $\mathscr{L}$ satisfies the dual link property, suppose that $(x, -p, m) \leqslant (z, -r, t) \,\mathscr{L}\, (y, -q, n)$. Then $x \leqslant z = y$. Let $a = n_k - k + m - m_k$. Then $a_k = n_k - k$ and so $a - a_k = m - m_k$. Also, $a \leqslant a_k + k - 1 = n_k - 1 < n_k \leqslant n$. Hence $(x, -p, m) \,\mathscr{L}\, (x, -q, a) \leqslant (y, -q, n)$.

Definition. We shall say that $\mathscr{L}$ is *lower* Λ*-stable* if

$$(\forall x \in S) \quad x \leqslant l \in \Lambda \;\Rightarrow\; x^\circ x \leqslant l;$$

and *upper* Λ*-stable* if

$$(\forall x \in S) \quad x \geqslant l \in \Lambda \;\Rightarrow\; x^\circ x \geqslant l.$$

Dually, we define $\mathscr{R}$ to be lower and upper I-stable.

Theorem 14. *The following statements are equivalent*:

(1) *$\mathscr{L}$ is regular*;
(2) *$\mathscr{L}$ satisfies the dual link property and is lower Λ-stable*;
(3) *$\mathscr{L}$ is upper Λ-stable.*

Proof. $(1) \Rightarrow (2)$: If $\mathscr{L}$ is regular then by Theorem 13 it satisfies the dual link property. Moreover, if $x \leqslant l \in \Lambda$ then $x^\circ x \leqslant l^\circ l = l$ and so $\mathscr{L}$ is also lower Λ-stable.

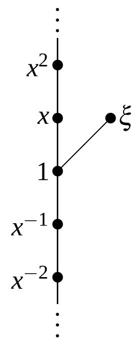
$(2) \Rightarrow (1)$: Let $x, y \in S$ be such that $x \leqslant y$. Then from $x \leqslant y \mathscr{L} y^\circ y$ and the fact that $\mathscr{L}$ satisfies the dual link property we have the existence of $z \in S$ such that $x \mathscr{L} z \leqslant y^\circ y$. Then $x^\circ x = z^\circ z$ and, since $\mathscr{L}$ is lower Λ-stable, $z^\circ z \leqslant y^\circ y$. Hence $\mathscr{L}$ is regular.

$(1) \Rightarrow (3)$: This is clear.

$(3) \Rightarrow (1)$: Observe first that $(x^\circ y)^\circ x^\circ y = y^\circ (x^\circ y y^\circ)^\circ x^\circ y \leqslant_n y^\circ y$ and therefore, since S is naturally ordered, $(x^\circ y)^\circ x^\circ y \leqslant y^\circ y$. Suppose now that $x \leqslant y$. Then $x^\circ x \leqslant x^\circ y$ and so, by (3), $x^\circ x \leqslant (x^\circ y)^\circ x^\circ y \leqslant y^\circ y$ whence we have (1). $\square$

We now pass to a consideration of the link property. That this is not in general satisfied by $\mathscr{L}$ or $\mathscr{R}$ is shown by the following example.

Example 3. Let G be an ordered group and let $x \in G$ be such that $x > 1$. Let $H = \langle x \rangle$ be the subgroup generated by x. Add to H a new identity element ξ together with the single comparability $1 < \xi$. Then $\xi^2 = \xi$ and $\xi y = y = y\xi$ for all $y \in H$, and we obtain the ordered inverse semigroup S with Hasse diagram



Clearly, S is naturally ordered and $S^\circ = S$. Moreover, since the $\mathscr{L}$-classes (and the $\mathscr{R}$-classes) of S are H and $\{\xi\}$, we see that $\mathscr{L}$ is regular. Now $x \mathscr{L} x^{-1} < \xi$ but the existence of $y \in S$ such that $x \leqslant y \mathscr{L} \xi$ forces $y = \xi$ and gives the contradiction $x \leqslant \xi$. Hence $\mathscr{L}$ does not satisfy the link property.

A situation in which $\mathscr{L}$ [resp. $\mathscr{R}$] satisfies the link property is the following.

Theorem 15. *If $\mathscr{L}$* [resp. *$\mathscr{R}$*] *is regular and if S is upper directed then $\mathscr{L}$* [resp. *$\mathscr{R}$*] *satisfies the link property.*

Proof. Suppose that $a \mathscr{L} x \leqslant b$ and let $k \in S$ be such that $a, b \leqslant k$. Then $a^\circ a = x^\circ x \leqslant b^\circ b$ whence $a = aa^\circ a \leqslant kb^\circ b$. Also, $b^\circ b \leqslant k^\circ k$ which, by Theorem 1 and property (δ), gives $b^\circ b = k^\circ k b^\circ b$ whence

$$b^\circ b = (b^\circ b)^\circ b^\circ b = (k^\circ k b^\circ b)^\circ k^\circ k b^\circ b = (kb^\circ b)^\circ k b^\circ b.$$

Hence $a \leqslant kb^\circ b \mathscr{L} b$. A similar proof holds for $\mathscr{R}$. $\square$

4. A structure theorem.

Our objective now is to obtain the structure of a naturally ordered regular semigroup with an inverse monoid transversal. This we can deduce from the algebraic structure of a locally inverse semigroup with such a transversal, which we now proceed to describe. For this purpose, we recall that if S is a regular semigroup then $\xi \in S$ is said to be a *weak middle unit* if, for all $x \in S$ and all $x' \in V(x)$, $x\xi x' = xx'$ and $x'\xi x = x'x$. If ξ is a weak middle unit then ξ is necessarily idempotent; for $\xi = \xi\xi'\xi = \xi\xi\xi'.\xi = \xi.\xi\xi'\xi = \xi\xi$.

Theorem 16. *Let A be a regular semigroup with a weak middle unit ξ. Then the following statements are equivalent*:

(1) *A is locally inverse*;
(2) *the regular subsemigroup $\xi A\xi$ is inverse.*

Proof. $(1) \Rightarrow (2)$: This clear.

$(2) \Rightarrow (1)$: Suppose that $\xi A\xi$ is inverse. If $x \in A$ and $x' \in V(x) \cap A$ then $x = xx'x = x\xi x'\xi x$ and $\xi x'\xi x\xi x'\xi = \xi x'\xi$, and so $\xi x'\xi \in V(x) \cap \xi A\xi$ whence $V(x) \cap \xi A\xi \neq \emptyset$. If now $y \in V(x) \cap \xi A\xi$ then $y = yxy = y\xi x\xi y$, and $x = xyx$ gives $\xi x\xi = \xi x\xi y\xi x\xi$, whence $y \in V(\xi x\xi) \cap \xi A\xi$ and consequently $y = (\xi x\xi)^{-1}$. Hence $V(x) \cap \xi A\xi = \{(\xi x\xi)^{-1}\}$ and so $\xi A\xi$ is an inverse transversal of A. Since $\xi A\xi \cdot A \cdot \xi A\xi \subseteq \xi A\xi$ we see that $\xi A\xi$ is a quasi-ideal. Then, by Theorem 3, A is locally inverse. □

The structure of regular semigroups with various kinds of inverse transversal has been determined by Saito [10]. This is immensely complicated, even in the case where the semigroup is locally inverse, i.e. the inverse transversal is a quasi-ideal. The particular situation in which the semigroup is locally inverse and the inverse transversal is a monoid can be treated more simply as follows.

Theorem 17. *Let L and R be regular semigroups with a common weak middle unit ξ and a common inverse submonoid $T = \xi L\xi = \xi R\xi$. Define a mapping $R \times L \to T$ by $(a,x) \mapsto a \circ x$ with the following properties*:

(1) $(\forall a,b \in R)(\forall y,z \in L) \quad a(a\circ y)b \circ z = [(a\circ y)b\xi]^{-1}(a\circ y)b\xi(b\circ z)$;
(2) $(\forall a,b \in R)(\forall y,z \in L) \quad a \circ y(b\circ z)z = (a\circ y)\xi y(b\circ z)[\xi y(b\circ z)]^{-1}$;
(3) $(\forall a \in R)(\forall x \in L)(\forall t \in T) \quad a\circ t = (a\circ\xi)(\xi\circ t), \quad t\circ x = (t\circ\xi)(\xi\circ x)$;
(4) $(\forall a \in R)(\forall x \in L) \quad a(a\circ\xi) = \xi a\xi, \quad (\xi\circ x)x = \xi x\xi$;
(5) $(\forall a \in R)(\forall a' \in V(a)\cap R) \quad (a\circ\xi)\xi a'\xi = \xi a'\xi$,
$(\forall x \in L)(\forall x' \in V(x)\cap L) \quad \xi x'\xi(\xi\circ x) = \xi x'\xi$.

On the set $L \,|\!\times\!|_\xi\, R = \{(x,a) \in L\times R \mid \xi x = a\xi\}$ define a multiplication by

$$(x,a)(y,b) = \big(x(a\circ y)y,\; a(a\circ y)b\big).$$

Then $L \,|\!\times\!|_\xi\, R$ is a locally inverse semigroup that has an inverse monoid transversal.

Moreover, every such semigroup is obtained in this way. More precisely, if S is a locally inverse semigroup with an inverse transversal S° that is a monoid with identity ξ then $S^\circ = \xi S\xi$, ξ is a weak middle unit of both $S\xi$ and ξS, the mapping $\xi S \times S\xi \to S^\circ = \xi S\xi$ given by $(a,x) \mapsto a^\circ axx^\circ$ satisfies properties (1) to (5) above, and there is a semigroup isomorphism

$$S \simeq S\xi \,|\!\times\!|_\xi\, \xi S.$$

Proof. For $x,y \in L$ and $a \in R$ we have $x(a\circ y)y \in LTL \subseteq L$; and for all $a,b \in R$ and $y \in L$ we have $a(a\circ y)b \in RTR \subseteq R$. Moreover, if $(x,a),(y,b) \in L \,|\!\times\!|_\xi\, R$ then

$$\xi x(a\circ y)y = a\xi(a\circ y)y = a(a\circ y)y = a(a\circ y)\xi y = a(a\circ y)b\xi.$$

Consequently, the multiplication on $L \,|\!\times\!|_\xi\, R$ is well-defined.

We proceed in the following stages.

(i) *$L \,|\!\times\!|_\xi\, R$ is a semigroup.*

From the multiplication, we see that in $L \,|\!\times\!|_\xi\, R$ the first component of $[(x,a)(y,b)](z,c)$ is $x(a\circ y)y[a(a\circ y)b \circ z]z$ which by (1) is $x(a\circ y)y[(a\circ y)b\xi]^{-1}(a\circ y)b\xi(b\circ z)z$. Using twice the fact that $b\xi = \xi y$ and ξ is an identity of T, we see that this reduces to $x(a\circ y)y(b\circ z)z$. On the other hand, the first component of $(x,a)[(y,b)(z,c)]$ is $x[a \circ y(b\circ z)z]y(b\circ z)z$ which by (2) is $x(a\circ y)\xi y(b\circ z)[\xi y(b\circ z)]^{-1}\xi y(b\circ z)z$ which also reduces to $x(a\circ y)y(b\circ z)z$. Thus the first components of the products are the same; and similarly, so are the second components. Thus, with the above multiplication, $L \,|\!\times\!|_\xi\, R$ is a semigroup.

(ii) *$L \,|\!\times\!|_\xi\, R$ is regular.*

Observe first that, as in Theorem 16, if $x \in L$ and $x' \in V(x)\cap L$ then $\xi x'\xi = (\xi x\xi)^{-1}$. Likewise, if $a \in R$ and $a' \in V(a)\cap R$ then $\xi a'\xi = (\xi a\xi)^{-1}$. Suppose now that $(x,a) \in L \,|\!\times\!|_\xi\, R$. Then $\xi x = a\xi$ gives $\xi x\xi = \xi a\xi$ and so, for $x' \in V(x)\cap L$ and $a' \in V(a)\cap R$, we can define $\beta = \xi x'\xi = \xi a'\xi \in T$.

Then the first component of the product $(x,a)(\beta,\beta)(x,a)$ is, as in (i),

$$\begin{aligned} x(a\circ\beta)\beta(\beta\circ x)x &= x(a\circ\beta)\beta(\beta\circ\xi)(\xi\circ x)x && \text{by (3)}\\ &= x(a\circ\beta)\xi\beta\xi\xi x\xi && \text{by (4)}\\ &= x(a\circ\xi)(\xi\circ\beta)\beta a\xi && \text{by (3)}\\ &= x(a\circ\xi)\xi\beta\xi a\xi && \text{by (4)}\\ &= x(a\circ\xi)\xi a'\xi x\\ &= x\xi a'\xi x && \text{by (5)}\\ &= x\xi x'\xi x\\ &= x. \end{aligned}$$

In a similar way we can see that the second component of the product is a. Thus we see that $(x,a)(\beta,\beta)(x,a) = (x,a)$ and so $L \,|\!\times\!|_\xi\, R$ is regular.

(iii) *$\widehat{T} = \operatorname{diag}(T\times T)$ is an inverse submonoid of $L \,|\!\times\!|_\xi\, R$.*

Clearly, $\widehat{T}$ is is a subsemigroup of $L \,|\!\times\!|_\xi\, R$, and is regular since if $(\gamma,\gamma) \in \widehat{T}$ then, by (ii),

$$\big(\forall\gamma' \in V(\gamma)\cap T\big) \quad (\gamma,\gamma)(\xi\gamma'\xi,\xi\gamma'\xi)(\gamma,\gamma) = (\gamma,\gamma).$$

If now (γ,γ) is an idempotent in $\widehat{T}$ then by (3) and (4) we see that $\gamma = \gamma(\gamma\circ\gamma)\gamma = \gamma(\gamma\circ\xi)(\xi\circ\gamma)\gamma = \gamma\gamma$, so that γ is idempotent in T. Furthermore, if $(\gamma,\gamma),(\delta,\delta) \in \widehat{T}$ are idempotents then

$$\begin{aligned} (\gamma,\gamma)(\delta,\delta) &= \big(\gamma(\gamma\circ\delta)\delta, \gamma(\gamma\circ\delta)\delta\big)\\ &= \big(\gamma(\gamma\circ\xi)(\xi\circ\delta)\delta, \gamma(\gamma\circ\xi)(\xi\circ\delta)\delta\big) && \text{by (3)}\\ &= (\xi\gamma\xi\delta\xi, \xi\gamma\xi\delta\xi) && \text{by (4)}\\ &= (\gamma\delta,\gamma\delta). \end{aligned}$$

Now since γ, δ are idempotents in T they commute, and therefore it follows that so do (γ,γ) and (δ,δ) in $\widehat{T}$. Hence $\widehat{T}$ is an inverse semigroup.

Using (4) and the fact that ξ is an identity for T, it is readily seen that (ξ,ξ) is an identity element for $\widehat{T}$.

(iv) *$\widehat{T}$ is an inverse transversal of $L\;|\!\times\!|_\xi\; R$.*

First we observe that if $(x,a) \in L\;|\!\times\!|_\xi\; R$ then, with β as in (ii),

$$(\beta,\beta) \in V(x,a) \cap \widehat{T}.$$

In fact, we have seen in (ii) that $(x,a)(\beta,\beta)(x,a) = (x,a)$. Consider now the product $(\beta,\beta)(x,a)(\beta,\beta)$. The first component of this is, as in (i),

$$\begin{aligned}
\beta(\beta\circ x)x(a\circ\beta)\beta &= \beta(\beta\circ\xi)(\xi\circ x)x(a\circ\xi)(\xi\circ\beta)\beta \qquad \text{by (3)}\\
&= \xi\beta\xi\xi x\xi(a\circ\xi)\xi\beta\xi \qquad \text{by (4)}\\
&= \beta\xi x\xi(a\circ\xi)\beta\\
&= \beta\xi a\xi(a\circ\xi)\beta \qquad \text{since } \xi x\xi = \xi a\xi\\
&= \beta a(a\circ\xi)\beta\\
&= \beta\xi a\xi\beta \qquad \text{by (4)}\\
&= \beta a\beta\\
&= \xi a'\xi a\xi a'\xi\\
&= \xi a'\xi\\
&= \beta.
\end{aligned}$$

Similarly, the second component is also β. Thus $(\beta,\beta)(x,a)(\beta,\beta) = (\beta,\beta)$ and therefore $V(x,a)\cap\widehat{T}$ contains (β,β).

Suppose now that $(\gamma,\gamma) \in V(x,a)\cap\widehat{T}$. On the one hand $(\gamma,\gamma) = (\gamma,\gamma)(x,a)(\gamma,\gamma)$ gives, as in the above, $\gamma = \gamma\xi a\xi\gamma = \gamma\xi x\xi\gamma = \gamma x\gamma$. On the other hand, $(x,a) = (x,a)(\gamma,\gamma)(x,a)$ gives, as in (ii), $x = x(a\circ\xi)\gamma x$ whence

$$x\gamma x = x(a\circ\xi)\gamma x\gamma x = x(a\circ\xi)\gamma x = x.$$

It follows that $\gamma \in V(x)\cap T$ whence, as in Theorem 16, $\gamma \in V(\xi x\xi)\cap T$ and therefore $\gamma = (\xi x\xi)^{-1} = \beta$.

It follows from the above that $V(x,a)\cap\widehat{T} = \{(\beta,\beta)\}$ whence $\widehat{T}$ is an inverse transversal of $L\;|\!\times\!|_\xi\; R$.

(v) *$\widehat{T}$ is a quasi-ideal.*

If $(\gamma,\gamma),(\delta,\delta) \in \widehat{T}$ and $(x,a) \in L\;|\!\times\!|_\xi\; R$ then the product $(\gamma,\gamma)(x,a)(\delta,\delta)$ is

$$\big(\gamma(\gamma\circ x)x[\gamma(\gamma\circ x)a \,\circ\, \delta]\delta,\ \gamma(\gamma\circ x)a[\gamma(\gamma\circ x)a \,\circ\, \delta]\delta\big)$$

which belongs to $\widehat{T}$ since $\xi x\xi = \xi a\xi$. Thus $\widehat{T}(L\;|\!\times\!|_\xi\; R)\widehat{T} \subseteq \widehat{T}$ and hence the inverse transversal $\widehat{T}$ of $L\;|\!\times\!|_\xi\; R$ is a quasi-ideal.

In summary, from the above and Theorem 3 we have that $L\;|\!\times\!|_\xi\; R$ is a locally inverse semigroup with an inverse monoid transversal.

To show that every such semigroup is obtained in this way, let S be a locally inverse semigroup with an inverse monoid transversal S°. Let the identity element of S° be ξ. By Theorem 10(3) we have $S^\circ = \xi S\xi$. Moreover, by Theorem 10(5) and property (ζ) we see that $L = S\xi$ is a left inverse semigroup with right identity ξ, and $R = \xi S$ is a right inverse semigroup with left identity ξ. Thus ξ is a weak middle unit of both $S\xi$ and ξS. Moreover, $\xi L\xi = \xi S\xi = \xi R\xi$. We are therefore in the initial conditions of the first part. Consider therefore the mapping $\xi S \times S\xi \to S^\circ = \xi S\xi$ given by

$$(a,x) \mapsto a \circ x = a^\circ axx^\circ.$$

To see that this satisfies property (1) above, observe that

$$\begin{aligned}
a(a\circ y)b \circ z &= ayy^\circ b \circ z \\
&= (ayy^\circ b)^\circ ayy^\circ bzz^\circ \\
&= b^\circ(ayy^\circ)^\circ ayy^\circ bb^\circ bzz^\circ \qquad \text{by } (\kappa) \text{ since } b \in R \\
&= (\xi b\xi)^\circ(a^\circ ayy^\circ)^\circ a^\circ ayy^\circ b\xi b^\circ bzz^\circ \qquad \text{by } (\alpha) \\
&= (a^\circ ayy^\circ \xi b\xi)^\circ a^\circ ayy^\circ b\xi b^\circ bzz^\circ \qquad \text{by } (\kappa) \\
&= (a^\circ ayy^\circ b\xi)^\circ a^\circ ayy^\circ b\xi b^\circ bzz^\circ \\
&= [(a\circ y)b\xi]^{-1}(a\circ y)b\xi(b\circ z).
\end{aligned}$$

It is readily verified that (2), (3), (4), (5) also hold.

Consider now the mapping $\vartheta : S \to S\xi \,|\!\times\!|_\xi\, \xi S$ given by $\vartheta(x) = (x\xi,\, \xi x)$. For all $x,y \in S$ we have

$$\begin{aligned}
\vartheta(x)\vartheta(y) &= (x\xi,\xi x)(y\xi,\xi y) \\
&= \big(x\xi(\xi x\circ y\xi)y\xi,\, \xi x(\xi x\circ y\xi)\xi y\big) \\
&= (x\xi x^\circ xyy^\circ y\xi,\, \xi xx^\circ xyy^\circ \xi y) \\
&= (xy\xi,\, \xi xy) \\
&= \vartheta(xy)
\end{aligned}$$

and so ϑ is a morphism.

If now $\vartheta(x) = \vartheta(y)$ then $x\xi = y\xi$ and $\xi x = \xi y$ whence $x^{\circ\circ} = \xi x\xi = \xi y\xi = y^{\circ\circ}$ and then $x^\circ = y^\circ$. Consequently, $x = xx^\circ x = x\xi x^\circ \xi x = y\xi y^\circ \xi y = yy^\circ y = y$ and so ϑ is injective.

To see that ϑ is also surjective, let $(x\xi,\xi y) \in S\xi \,|\!\times\!|_\xi\, \xi S$. Then $\xi x\xi = \xi y\xi$, whence $x^\circ = y^\circ$. Now let $t = xx^\circ y$. Then $t\xi = xx^\circ \xi y\xi = xx^\circ \xi x\xi = x\xi$ and $\xi t = \xi x\xi x^\circ y = \xi y\xi y^\circ y = \xi y$. Thus $\vartheta(t) = (t\xi,\xi t) = (x\xi,\xi y)$.

It follows from the above that $S \simeq S\xi \,|\!\times\!|_\xi\, \xi S$. □

Corollary. *The inverse transversal in question is multiplicative if and only if*

$$\big(\forall l \in E(\xi R)\big)\big(\forall i \in E(L\xi)\big) \quad l \circ i \in E(T).$$

Proof. As in the proof of Theorem 16, T is an inverse transversal of R with $a^\circ = \xi a'\xi$ for every $a' \in V(A) \cap R$. Consequently,

$$\Lambda(R) = \{a^\circ a \mid a \in R\} = \{\xi a'a \mid a \in R,\, a' \in V(a) \cap R\}.$$

Since ξ is a weak middle unit, it is clear that $\Lambda(R) \subseteq E(\xi R)$. Conversely, if $e \in E(\xi R)$ then $e = \xi e = \xi ee'e = \xi e^\star e$ where $e^\star = ee' \in V(e)$. Hence $\Lambda(R) = E(\xi R)$. Similarly

$$\mathrm{I}(L) = \{xx'\xi \mid x \in L,\, x' \in V(x) \cap L\} = E(L\xi).$$

Suppose now that $(x,a) \in L \,|\!\times\!|_\xi\, R$. Then, as in (iv) above, $(x,a)^\circ = (\beta,\beta)$ where $\beta = \xi a'\xi = \xi x'\xi$. Consequently, in $L \,|\!\times\!|_\xi\, R$ we have

$$\begin{aligned}
(x,a)^\circ(x,a) &= (\beta,\beta)(x,a)\\
&= \big(\beta(\beta\circ x)x,\ \beta(\beta\circ x)a\big)\\
&= \big(\beta(\beta\circ\xi)(\xi\circ x)x,\ \beta(\beta\circ\xi)(\xi\circ x)a\big) \quad \text{by (3)}\\
&= \big(\xi\beta\xi.\xi x\xi,\ \xi\beta\xi(\xi\circ x)a\big) \quad \text{by (4)}\\
&= \big(\xi x'\xi.\xi x\xi,\ \xi x'\xi(\xi\circ x)a\big)\\
&= (\xi x'x\xi,\ \xi x'\xi a) \quad \text{by (5)}\\
&= (\xi x'x\xi,\ \xi a'a) \quad \text{since } \xi x'\xi = \xi a'\xi.
\end{aligned}$$

Similarly, $(y,b)(y,b)^\circ = (yy'\xi,\ \xi bb'\xi)$. It follows that if $l \in \Lambda(L \,|\!\times\!|_\xi\, R)$ and $i \in \mathrm{I}(L \,|\!\times\!|_\xi\, R)$ then li is of the form

$$\begin{aligned}
li &= (\xi x'x\xi, \xi a'a)(yy'\xi, \xi bb'\xi)\\
&= \big(\xi x'x\xi(\xi a'a \circ yy'\xi)yy'\xi,\ \xi a'a(\xi a'a \circ yy'\xi)\xi bb'\xi\big).
\end{aligned}$$

Now if the stated condition holds then we have $\xi a'a \circ yy'\xi \in E(T)$. This, together with $\xi x'x\xi = \xi a'a\xi \in E(T)$ and $\xi yy'\xi = \xi bb'\xi \in E(T)$, gives $li = (\gamma,\gamma) \in \widehat{T}$ where $\gamma \in E(T)$. Then since $(\gamma,\gamma)^2 = (\gamma^2,\gamma^2) = (\gamma,\gamma)$ we see that $li \in E(\widehat{T})$, whence the inverse transversal $\widehat{T}$ is multiplicative.

Conversely, if S is a locally inverse semigroup with an inverse monoid transversal $S^\circ = \xi S\xi$ that is multiplicative then for $l \in \Lambda = E(\xi S)$ and $i \in \mathrm{I} = E(S\xi)$ we have $l \circ i = l^\circ l i i^\circ = li \in E(S^\circ)$ whence the condition holds. □

We can apply the above theorem to obtain a structure theorem for naturally ordered regular semigroups with an inverse monoid transversal.

Theorem 18. *Let S be a naturally ordered regular semigroup with an inverse transversal S° that is a monoid with identity element ξ. Let $S\xi \,|\!\times\!|_\xi\, \xi S$ consist of the subset of the cartesian ordered set $S\xi \times \xi S$ given by*

$$S\xi \,|\!\times\!|_\xi\, \xi S = \{(x\xi, \xi x) \mid x \in S\}$$

together with the multiplication

$$(x\xi, \xi x)(y\xi, \xi y) = (xy\xi, \xi xy).$$

Then $S\xi \,|\!\times\!|_\xi\, \xi S$ is an ordered regular semigroup. Moreover, if either $\mathscr{L}$ or $\mathscr{R}$ is regular on S then there is an ordered semigroup isomorphism

$$S \simeq S\xi \,|\!\times\!|_\xi\, \xi S.$$

Proof. Since S is locally inverse by Theorem 2, it follows immediately from Theorem 10(5) and Theorem 17 that there is an algebraic isomorphism $\vartheta : S \to S\xi \,|\!\times\!|_\xi\, \xi S$ given by $\vartheta(x) = (x\xi, \xi x)$. Suppose now that, for example, $\mathscr{R}$ is regular on S. If $\vartheta(x) \leqslant \vartheta(y)$ then $x\xi \leqslant y\xi$ and $\xi x \leqslant \xi y$, whence $x = xx^\circ x = x\xi(x\xi)^\circ\xi x \leqslant y\xi(y\xi)^\circ\xi y = yy^\circ y = y$. Thus we see that $x \leqslant y \iff \vartheta(x) \leqslant \vartheta(y)$ and so the isomorphism ϑ is also an order isomorphism. The same is true if $\mathscr{L}$ is regular. □

Example 4. With the above notation, in Example 1 the identity element of the inverse transversal Q° is $\xi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then, by Theorem 18, we have $Q \simeq Q\xi \;|\!\times\!|_\xi\; \xi Q$ where

$$Q\xi = \left\{ \begin{bmatrix} x & 0 \\ x & 0 \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \mid x > 0 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\};$$

$$\xi Q = \left\{ \begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \mid x > 0 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

Since Λ consists of the chain $0 < d < c$ it is readily verified that $\mathscr{L}$ is upper Λ-stable. It then follows from Theorem 14 that $\mathscr{L}$ is regular. Consequently the above semigroup isomorphism is also an order isomorphism.

Finally, we note that if S is a naturally ordered regular semigroup that contains a biggest idempotent ξ then S has a quasi-ideal inverse monoid transversal, namely $\xi S \xi$ (see, for example, [2, Theorem 13.16]). Theorem 18 therefore applies in this case. More particularly, it applies in the case of a naturally ordered regular Dubreil-Jacotin semigroup.

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