A general near-exact distribution theory for the most common likelihood ratio test statistics used in Multivariate Statistics

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Abstract

The aim of this paper is to show that while all the exact distributions of the most common likelihood ratio test (l.r.t.) statistics, that is, the ones used to test the independence of several sets of variables, the equality of several variance-covariance matrices, sphericity and the equality of several mean vectors, may be expressed as the distribution of the product of independent Beta random variables or the product of a given number of independent random variables whose logarithm has a Gamma distribution times a given number of independent Beta random variables, near-exact distributions for their logarithms may all be expressed as Generalized Near-Integer Gamma distributions or mixtures of these distributions, whose rate parameters associated with the integer shape parameters, for samples of size n, all have the form $(n-j)/n$ for $j=2,\ldots,p$, where for the first three statistics p is the number of variables involved, while for the fourth one it is the sum of the number of variables involved with the number of mean vectors being tested. What is interesting is that the similarities exhibited by these statistics are even more striking in terms of near-exact distributions than in terms of exact distributions. Moreover all the l.r.t. statistics that may be built as products of these basic statistics also inherit a similar structure for their near-exact distributions. To illustrate this fact, an application is made to the l.r.t. statistic to test the equality of several multivariate Normal distributions.

Key words: Wilks Lambda statistic, independence test, sphericity test, Generalized Integer Gamma distribution, Generalized Near-Integer Gamma distribution, mixtures.

1 Introduction

Although it is a rather well-known fact that the exact distributions of the l.r.t. (likelihood ratio test) statistics used in Multivariate Statistics (under normality assumptions) to test the independence of several sets of variables, the equality of several variance-covariance matrices, sphericity, and the equality of several mean vectors may be expressed as distributions of the product of independent Beta r.v.'s (random variables) raised to the power equal to one-half the sample-size (Anderson, 2003; Hsieh, 1979), we will present for each statistic our version of its exact distribution in the form of the distribution of the product of a given number, closely related with the number of variables being used, of independent r.v.'s whose logarithms have Gamma distributions with integer shape parameters and a given number, a function of the parity of the number of variables being tested, of independent Beta r.v.'s raised to the power equal to one-half the sample size.

This way of looking at the exact distribution of these statistics will enable us to show how in terms of near-exact distributions, the similarities among the distributions of these statistics are even more striking, and, based on previously published results (as well as other submitted for publication) (Coelho, 2004; Marques and Coelho, 2008; Coelho and Marques, 2007a, 2007b), to establish a common strategy concerning the development of near-exact distributions for these statistics, as well as a common formulation for such near-exact distributions.

If we think that the usual normality assumptions under which the l.r.t. statistics and their distributions are derived are too restrictive, we should bear in mind the results in Anderson et all. (1986), Anderson and Fang (1990) and Anderson (2003), which show that such statistics are still the l.r.t. statistics for the same hypotheses when the underlying distributions for the samples are multivariate elliptically contoured and that under the null hypotheses such statistics have the same distributions as in the normal case, this way making the results presented somewhat more comprehensive.

2 Exact distributions of the l.r.t. statistics

2.1 The l.r.t. statistic to test independence among m sets of variates

If we assume that

$$\underline{X} = [\underline{X}_1, \ldots, \underline{X}_k, \ldots, \underline{X}_m]' \sim N_p(\underline{\mu}, \Sigma)$$

where

$$\underline{\mu} = \left[\underline{\mu}_1, \ldots, \underline{\mu}_k, \ldots, \underline{\mu}_m\right]' \qquad \text{and} \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \cdots & \Sigma_{1k} & \cdots & \Sigma_{1m} \\ \vdots & \ddots & \vdots & & \vdots \\ \Sigma_{k1} & \cdots & \Sigma_{kk} & \cdots & \Sigma_{km} \\ \vdots & & \vdots & \ddots & \vdots \\ \Sigma_{m1} & \cdots & \Sigma_{mk} & \cdots & \Sigma_{mm} \end{bmatrix}$$

of course with

$$\underline{X}_k \sim N_{p_k}(\underline{\mu}_k, \Sigma_{kk}) \quad (k = 1, \ldots, m), \qquad \text{and} \qquad p = \sum_{k=1}^{m} p_k,$$

and we want to test the null hypothesis

$$\begin{aligned} H_{01} &: \Sigma = diag(\Sigma_{11}, \ldots, \Sigma_{kk}, \ldots, \Sigma_{mm}) \\ &\iff \Sigma_{ij} = 0_{p_i \times p_j}, \quad i \neq j;\ i, j \in \{1, \ldots, m\} \end{aligned} \tag{1}$$

that is the null hypothesis of independence among the m sets of variates $\underline{X}_k$ $(k = 1, \ldots, m)$, the l.r.t. statistic to test (1), based on a sample of size n from $\underline{X}$, is

$$\Lambda_1 = \left(\frac{|A|}{\prod_{k=1}^{m} |A_{kk}|} \right)^{n/2}, \tag{2}$$

where A is the m.l.e. (maximum likelihood estimator) of Σ and A_{kk} is the m.l.e. of Σ_{kk} $(k = 1, \ldots, m)$.

From Coelho (1992, 2004), we may write the exact c.f. (characteristic function) of $W_1 = -\log \Lambda_1$, under H_{01} in (1), as

$$\begin{aligned} \Phi_{W_1}(t) &= \prod_{k=1}^{m-1} \prod_{j=1}^{p_k} \frac{\Gamma(a_{jk} + b_k)\, \Gamma\left(a_{jk} - \frac{n}{2}\mathrm{i}t\right)}{\Gamma(a_{jk})\, \Gamma\left(a_{jk} + b_k - \frac{n}{2}\mathrm{i}t\right)} \\ &= \underbrace{\prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{r_j} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-r_j}}_{\Phi_{1,W_1}(t)} \\ &\qquad \underbrace{\left(\frac{\Gamma\left(\frac{n-1}{2}\right)\, \Gamma\left(\frac{n-1}{2} - \frac{1}{2} - \frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-1}{2} - \frac{1}{2}\right)\, \Gamma\left(\frac{n-1}{2} - \frac{n}{2}\mathrm{i}t\right)} \right)^{k^*}}_{\Phi_{2,W_1}(t)} \end{aligned} \tag{3}$$

where $k^* = \left\lfloor \frac{\ell}{2} \right\rfloor$, with ℓ denoting the number of $\underline{X}_k$'s with an odd number of variates, and where, for $j = 1, \ldots, p_k$ and $k = 1, \ldots, m-1$,

$$a_{jk} = \frac{n - q_k - j}{2}\,, \qquad b_k = \frac{q_k}{2} \quad \text{with} \quad q_k = p_{k+1} + \ldots + p_m \tag{4}$$

and

$$r_j = \begin{cases} 0 & j = 2 \\ \displaystyle\sum_{k=1}^{m-1} r^*_{k,p_k+q_k+1-j} & j = 3, \ldots, p \end{cases} \tag{5}$$

where

$$r^*_{k,p_k+q_k+1-j} = \begin{cases} 0 & j = p_k + q_k + 1, \ldots, p \\ r_{k,p_k+q_k+1-j} & j = 3, \ldots, p_k + q_k \end{cases} \tag{6}$$

with

$$r_{k,j} = \begin{cases} h_{kj} & j = 1, 2 \\ r_{k,j-2} + h_{kj} & j = 3, \ldots, p_k + q_k - 2 \end{cases} \tag{7}$$

where, for $j = 1, \ldots, p_k + q_k - 2$,

$$h_{kj} = (\text{number of elements in } \{p_k, q_k\} \text{ greater or equal to } j) - 1\,. \tag{8}$$

The c.f. of W_1, in (3), is either the c.f. of the sum of $\sum_{k=1}^{m-1} p_k$ independent Logbeta r.v.'s with the parameters in (4), multiplied by $n/2$, or the c.f. of the sum of $p-2$ independent Gamma r.v.'s with rate parameters $\frac{n-j}{n}$ $(j = 3, \ldots, p)$ and shape parameters r_j given by (5) through (8), plus k^* independent Logbeta r.v.'s with parameters $n/2$ and $1/2$, multiplied by $n/2$. This shows that the distribution of Λ_1 in (2) is the same as the distribution of the product of $\sum_{k=1}^{m-1} p_k$ independent r.v.'s with Beta distributions, more precisely, the same distribution as the distribution of

$$\prod_{k=1}^{m-1} \prod_{j=1}^{p_k} (Y_{jk})^{n/2} \qquad \text{with} \qquad Y_{jk} \sim \mathit{Beta}\,(a_{jk}, b_k)$$

where a_{jk} and b_k are defined in (4) and where Y_{jk} are independent for $j = 1, \ldots, p_k$ and $k = 1, \ldots, m-1$, or alternatively that Λ_1 has, for $k^* = \left\lfloor \frac{\ell}{2} \right\rfloor$, where

ℓ is the number of $\underline{X}_k$'s with an odd number of variates, the same distribution as

$$(Y)^{\frac{n}{2}k^*} \prod_{j=3}^{p} e^{Z_j}\,, \qquad \text{where} \quad Y \sim Beta\left(\frac{n}{2}, \frac{1}{2}\right) \;\; \text{and} \;\; Z_j \sim \Gamma\left(\frac{n-j}{n}, r_j\right)\,,$$

with r_j given by (5) through (8), being the Z_j all independent $(j = 3, \ldots, p)$ and independent of Y.

If at most one of the m $\underline{X}_k$'s has an odd number of variates, Coelho (1998, 1999) has shown that in this case the exact c.f. of $W_1 = -\log \Lambda_1$ may be written, for a sample of size n, as

$$\Phi_{W_1}(t) = \prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{r_j} \left(\frac{n-j}{n} - \text{i}t\right)^{-r_j}$$

where

$$r_j = \begin{cases} 0 & j = 2 \\ h_{j-2} & j = 3, 4 \\ r_{j-2} + h_{j-2} & j = 5, \ldots, p \end{cases} \tag{9}$$

with

$$h_j = (\# \text{ of } p_k \;\, (k = 1, \ldots, m) \geq j) - 1\,, \qquad j = 1, \ldots, p-2\,, \tag{10}$$

what is the c.f. of the sum of $p-1$ independent Gamma r.v.'s with rate parameters $\frac{n-j}{n}$ and shape parameters r_j $(j = 2, \ldots, p)$, that is a GIG distribution of depth $p-1$ (see Appendix A) with those rate and shape parameters.

Alternatively, in this case, the exact c.f. of W_1 may be given by the second expression in (3), with the shape parameters r_j given by (5) through (8), and where for $k^* = 0$, $\Phi_2^*(t)$ vanishes.

2.2 The l.r.t. statistic to test the equality of several variance-covariance matrices

If we assume that

$$\underline{X}_k \sim N_p(\underline{\mu}_k, \Sigma_k)\,, \qquad k = 1, \ldots, q$$

and we want to test the null hypothesis

$$H_{02} : \Sigma_1 = \ldots = \Sigma_q \tag{11}$$

that is, the null hypothesis of equality of the q variance-covariance matrices. From Anderson (2003, sec. 10.4) and Coelho and Marques (2007) we may write the l.r.t. statistic to test H_{02} in (11), for samples of sizes n from each $\underline{X}_k$ $(k = 1, \ldots, q)$, as

$$\Lambda_2 = \left(q^{pq} \frac{\prod_{k=1}^{q} |A_k|}{|A|^q} \right)^{n/2} \tag{12}$$

where A_k is the m.l.e. of Σ_k $(k = 1, \ldots, q)$ and $A = A_1 + \ldots + A_q$.

From Anderson (2003) and Coelho and Marques (2007) we may write the c.f. of $W_2 = -\log \Lambda_2$, under H_{02} in (11), either as

$$\Phi_{W_2}(t) = \prod_{j=1}^{p} \prod_{k=1}^{q} \frac{\Gamma\Big(\frac{n}{2} + \frac{1-j}{2q} + \frac{k-1}{q}\Big)\, \Gamma\Big(\frac{n+1-j}{2} - \frac{n}{2}\mathrm{i}t\Big)}{\Gamma\Big(\frac{n}{2} + \frac{1-j}{2q} + \frac{k-1}{q} - \frac{n}{2}\mathrm{i}t\Big)\, \Gamma\Big(\frac{n+1-j}{2}\Big)}\,, \tag{13}$$

where we should note that for $j = k = 1$ the term in the product yields the value 1, or alternatively as

$$\begin{aligned}
\Phi_{W_2}(t) &= \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \frac{\Gamma(a_j + b_{jk})}{\Gamma(a_j + b_{jk} - nit)} \frac{\Gamma(a_j - nit)}{\Gamma(a_j)} \\
&\qquad \times \left(\prod_{k=1}^{q} \frac{\Gamma(a_p + b_{pk})}{\Gamma\left(a_p + b_{pk} - \frac{n}{2}it\right)} \frac{\Gamma\left(a_p - \frac{n}{2}it\right)}{\Gamma(a_p)} \right)^{p \perp\!\!\!\perp 2} \\
&= \underbrace{\prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{r_j} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-r_j}}_{\Phi_{1,W_2}(t)} \\
&\quad \underbrace{\times \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \frac{\Gamma(a_j + b_{jk})}{\Gamma(a_j + b^*_{jk})} \frac{\Gamma(a_j + b^*_{jk} - n\mathrm{i}t)}{\Gamma(a_j + b_{jk} - n\mathrm{i}t)} \times \left(\prod_{k=1}^{q} \frac{\Gamma(a_p + b_{pk})}{\Gamma(a_p + b^*_{pk})} \frac{\Gamma\Big(a_p + b^*_{pk} - \frac{n}{2}\mathrm{i}t\Big)}{\Gamma\Big(a_p + b_{pk} - \frac{n}{2}\mathrm{i}t\Big)} \right)^{p \perp\!\!\!\perp 2}}_{\Phi_{2,W_2}(t)}
\end{aligned} \tag{14}$$

for $p \perp\!\!\!\perp 2 = \left\lfloor \frac{p+1}{2} \right\rfloor - \left\lfloor \frac{p}{2} \right\rfloor$ being the remainder of the integer division of p by 2,

$$a_j = n - 2j \ , \qquad b_{jk} = 2j - 1 + \frac{k-2j}{q} \ , \tag{15}$$

$$a_p = \frac{n-p}{2} \ , \qquad b_{pk} = \frac{pq - q - p + 2k - 1}{2q} \ , \tag{16}$$

$$b^*_{jk} = \lfloor b_{jk} \rfloor \, , \qquad b^*_{pk} = \lfloor b_{pk} \rfloor \, , \tag{17}$$

and

$$r_j = \begin{cases} r^*_{j-1} & \text{for } j = 2, \dots, p, \\ & \text{except for } j = p - 2\alpha_1 \\ r^*_{j-1} + (p \perp\!\!\!\perp 2)(\alpha_2 - \alpha_1) & \\ \qquad \left(q - \frac{p-1}{2} + q \left\lfloor \frac{p}{2q} \right\rfloor \right) & \text{for } j = p - 2\alpha_1 \end{cases} \tag{18}$$

with

$$r^*_j = \begin{cases} \gamma_j & \text{for} \quad j = 1, \dots, \alpha + 1 \\ q \left(\left\lfloor \frac{p}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor \right) & \text{for} \quad j = \alpha + 2, \dots, \min(p - 2\alpha_1, p - 1) \\ & \text{and } j = 2 + p - 2\alpha_1, \dots, 2\left\lfloor \frac{p}{2} \right\rfloor - 1, \ \text{step } 2 \\ q \left(\left\lfloor \frac{p+1}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor \right) & \text{for} \quad j = 1 + p - 2\alpha_1, \dots, p - 1, \ \text{step } 2 \, , \end{cases} \tag{19}$$

and

$$\alpha = \left\lfloor \frac{p-1}{q} \right\rfloor , \qquad \alpha_1 = \left\lfloor \frac{q-1}{q} \frac{p-1}{2} \right\rfloor , \qquad \alpha_2 = \left\lfloor \frac{q-1}{q} \frac{p+1}{2} \right\rfloor , \tag{20}$$

where,

$$\gamma_j = \left\lfloor \frac{q}{2} \right\rfloor \left((j-1)q - 2\left((q+1) \perp\!\!\!\perp 2\right) \left\lfloor \frac{j}{2} \right\rfloor \right) + \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q + j \perp\!\!\!\perp 2}{2} \right\rfloor \qquad \text{for } j = 1, \dots, \alpha \tag{21}$$

and

$$\gamma_{\alpha+1} = -\left(\left\lfloor\frac{p}{2}\right\rfloor - \alpha\left\lfloor\frac{q}{2}\right\rfloor\right)^2 + q\left(\left\lfloor\frac{p}{2}\right\rfloor - \left\lfloor\frac{\alpha+1}{2}\right\rfloor\right) \\ +(q \perp\!\!\!\perp 2)\left(\alpha\left\lfloor\frac{p}{2}\right\rfloor + \frac{\alpha \perp\!\!\!\perp 2}{4} - \frac{\alpha^2}{4} - \alpha^2\left\lfloor\frac{q}{2}\right\rfloor\right), \tag{22}$$

which shows that the exact distribution of Λ_2 in (12) may either be seen, from (13), as the distribution of the $n/2$ power of the product of $pq-1$ independent Beta r.v.'s, more precisely the distribution of

$$\prod_{j=1}^{p}\prod_{k=1}^{q}(Y_{kj})^{n/2} \qquad \text{where} \qquad Y_{kj} \sim Beta\left(\frac{n+1-j}{2}, \frac{j(q-1)-q-1+2k}{2q}\right)$$

and where the Y_{kj} are independent for $k = 1, \ldots, q$ and $j = 1, \ldots, p$ and for $j = k = 1$ yield a degenerate r.v. with the value 1, or alternatively, based on the first expression in (14), as the distribution of

$$\left(\prod_{j=1}^{\lfloor p/2\rfloor}\prod_{k=1}^{q}(Y_{jk})^{n/2}\right) \times \left(\prod_{k=1}^{q}(Y_k^*)^{n/2}\right)^{p \perp\!\!\!\perp 2}$$

where

$$Y_{jk} \sim Beta(a_j, b_{jk}) \qquad \text{and} \qquad Y_k^* \sim Beta(a_p, b_{pk})$$

for a_j, b_{jk}, a_p and b_{pk} given by (15) and (16), with all the r.v.'s involved being independent, that is, the Y_{jk} independent for $j = 1, \ldots, \lfloor p/2\rfloor$ and $k = 1, \ldots, q$ and independent from Z_k ($k = 1, \ldots, q$), which are also independent among themselves, and where the r.v.'s Z_k are not present if p is even; or yet, based on the second expression of (14), that the distribution of Λ_2 may be seen as the distribution of

$$\left\{\prod_{j=2}^{p} e^{Z_j}\right\}\left\{\prod_{j=1}^{\lfloor p/2\rfloor}\prod_{k=1}^{q}(Y_{jk})^{n}\right\}\left\{\prod_{k=1}^{q}(Y_k^*)^{n/2}\right\}^{p \perp\!\!\!\perp 2}$$

where

$$Z_j \sim \Gamma\left(\frac{n-j}{n}, r_j\right), \qquad Y_{jk} \sim Beta\left(a_j + b_{jk}, b_{jk} - b_{jk}^*\right)$$

and

$$Y_k^* \sim Beta(a_p + b_{pk}, b_{pk} - b_{pk}^*)$$

for r_j given by (18) through (22), a_j, b_{jk}, a_p, a_{pk}, b^*_{jk} and b^*_{pk} are given by (15) through (17) and where the r.v.'s Z_j, Y_{jk} and Y^*_k are all independent and where the r.v.'s Y^*_k only exist if p is odd.

2.3 The l.r.t. statistic for sphericity

If we assume that

$$\underline{X} \sim N_p(\underline{\mu}, \Sigma)$$

and we want to test

$$H_{03} : \Sigma = \sigma^2 I_p \quad (\sigma^2 \text{ unspecified}), \tag{23}$$

based on a sample of size n from $\underline{X}$, the l.r.t. statistic is (Mauchly, 1940; Anderson, 2003)

$$\Lambda_3 = \left(\frac{|A|}{\left(tr\,\frac{1}{p}\,A\right)^p} \right)^{n/2} \tag{24}$$

where A is the m.l.e. of Σ.

From Mauchly (1940), Anderson (2003), Marques and Coelho (2008) and Coelho and Marques (2008), the c.f. of $W_3 = -\log \Lambda_3$ may be written as

$$\begin{aligned}\Phi_{W_3}(t) &= \prod_{j=2}^{p} \frac{\Gamma\left(\frac{n-1}{2} + \frac{j-1}{p}\right)\,\Gamma\left(\frac{n-j}{2} - \frac{n}{2}\text{i}t\right)}{\Gamma\left(\frac{n-1}{2} + \frac{j-1}{p} - \frac{n}{2}\text{i}t\right)\,\Gamma\left(\frac{n-j}{2}\right)} \\ &= \underbrace{\prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{r^*_j} \left(\frac{n-j}{n} - \text{i}t\right)^{-r^*_j}}_{\Phi_{1,W_3}(t)} \\ &\qquad\qquad \times \underbrace{\prod_{j=2}^{p} \frac{\Gamma(a_j + b^*_j + c_j)}{\Gamma(a_j + b^*_j)}\, \frac{\Gamma\left(a_j + b^*_j - \frac{n}{2}\text{i}t\right)}{\Gamma\left(a_j + b^*_j + c_j - \frac{n}{2}\text{i}t\right)}}_{\Phi_{2,W_3}(t)}\end{aligned}$$

$$
= \underbrace{\prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{r_j} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-r_j}}_{\Phi_{1,W_3}(t)}
$$

$$
\underbrace{\times \left\{ \prod_{j=1}^{p-k^*} \frac{\Gamma\left(\frac{n-1}{2}+\frac{j-1}{p}\right)\,\Gamma\left(\frac{n-1}{2}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-1}{2}\right)\,\Gamma\left(\frac{n-1}{2}+\frac{j-1}{p}-\frac{n}{2}\mathrm{i}t\right)} \right\}
\times \left\{ \prod_{j=p-k^*+1}^{p} \frac{\Gamma\left(\frac{n-1}{2}+\frac{j-1}{p}\right)\,\Gamma\left(\frac{n}{2}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n}{2}\right)\,\Gamma\left(\frac{n-1}{2}+\frac{j-1}{p}-\frac{n}{2}\mathrm{i}t\right)} \right\}}_{\Phi_{2,W_3}(t)},
$$

where $k^* = \lfloor p/2 \rfloor$ and, for $j = 2, \ldots, p$,

$$
a_j = \frac{n-j}{2}, \quad b_j^* = \left\lfloor \frac{j-1}{p} + \frac{j-1}{2} \right\rfloor = \begin{cases} \frac{j-1}{2} & j \text{ odd} \\ \frac{j-2}{2} & j-1 < \frac{p}{2},\ j \text{ even} \\ \frac{j}{2} & j-1 \geq \frac{p}{2},\ j \text{ even}, \end{cases} \tag{25}
$$

$$
c_j = \begin{cases} \frac{j-1}{p} & j \text{ odd} \\ \frac{j-1}{p} + \frac{1}{2} & j-1 < \frac{p}{2},\ j \text{ even} \\ \frac{j-1}{p} - \frac{1}{2} & j-1 \geq \frac{p}{2},\ j \text{ even}, \end{cases} \tag{26}
$$

$$
r_j^* = \begin{cases} \left\lfloor \frac{p-j}{2} + 1 \right\rfloor & j = 3, \ldots p \\ \left\lfloor \frac{p}{4} + \frac{1}{2} \right\rfloor & j = 2,\ p \text{ even} \\ \left\lfloor \frac{p}{4} \right\rfloor & j = 2,\ p \text{ odd}. \end{cases} \tag{27}
$$

and

$$
r_j = \left\lfloor \frac{p-j+2}{2} \right\rfloor \qquad j = 2, \ldots, p, \tag{28}
$$

what shows that the distribution of Λ_3 is the same as the distribution of the power $n/2$ of the product of $p-1$ independent Beta r.v.'s, more precisely, the distribution of

$$
\prod_{j=2}^{p} (Y_j)^{n/2} \qquad \text{where} \qquad Y_j \sim Beta\left(\frac{n+1-j}{2}, \frac{j-1}{p} + \frac{j-1}{2}\right)
$$

and where the Y_j are independent for $j = 2, \ldots, p$, or, alternatively, the same distribution as

$$\left\{ \prod_{j=2}^{p} e^{Z_j} \right\} \left\{ \prod_{j=2}^{p} (Y_j)^{n/2} \right\}$$

where

$$Z_j \sim \Gamma\left(\frac{n-j}{n}, r_j^*\right) \qquad \text{and} \qquad Y_j \sim Beta(a_j + b_j^*, c_j)$$

for r_j^* given by (27) and a_j, b_j^* and c_j given by (25) and (26), and where all the Z_j and Y_j r.v.'s are independent, or yet alternatively the same distribution as the distribution of

$$\left\{ \prod_{j=2}^{p} e^{Z_j} \right\} \left\{ \prod_{j=1}^{p-k^*} (Y_j)^{n/2} \right\} \left\{ \prod_{j=p-k^++1}^{p} \left(Y_j^*\right)^{n/2} \right\}$$

where

$$Z_j \sim \Gamma\left(\frac{n-j}{n}, r_j\right) \,, \quad Y_j \sim Beta\left(\frac{n-1}{2}, \frac{j-1}{p}\right)$$

and

$$Y_j^* \sim Beta\left(\frac{n}{2}, \frac{j-1}{p} - \frac{1}{2}\right)$$

for r_j given by (28) and where all the Z_j, Y_j and Y_j^* r.v.'s are independent.

2.4 The l.r.t. statistic to test the equality of several mean vectors

Let us suppose we have q independent samples from the q multivariate Normal distributions $N_p(\underline{\mu}_j, \Sigma)$ $(j = 1, \ldots, q)$, the sample from the j-th population being of size n_j. Then the l.r.t. to test the null hypothesis

$$H_{04} : \underline{\mu}_1 = \ldots = \underline{\mu}_q \tag{29}$$

may be written as (Kshirsagar, 1972, Ch. 10, sec. 10.1)

$$\Lambda_4 = \left(\frac{|G|}{|G+H|}\right)^{n/2} \tag{30}$$

where $n = n_1 + \ldots + n_q$ and G and H are two $p\times p$ Wishart matrices which under H_{04} in (29) are independent, both with parameter matrix Σ and respectively $n-q$ and $q-1$ degrees of freedom.

The distribution of Λ_4 under H_0 in (29) is thus the same as the distribution of Λ_1 in subsection 2.1 for $m = 2$, taking $p_1 = p$ and $p_2 = q-1$. Thus, we have the exact distribution for $W_4 = -\log\Lambda_4$ for Λ_4 in (30) above either for even p or odd q, given under the form of a GIG distribution of depth $p+q-1$, with the c.f. of $W_4 = -\log\Lambda_4$ being in this case given by

$$\Phi_{W_4}(t) = \prod_{j=2}^{p+q-1} \left(\frac{n-j}{n}\right)^{r_j} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-r_j} \qquad (31)$$

where

$$r_j = \begin{cases} 0 & j = 2 \\ h_{j-2} & j = 3,4 \\ r_{j-2} + h_{j-2} & j = 5, \ldots, p+q-1 \end{cases} \qquad (32)$$

with

$$h_j = (\#\text{ of elem.'s in } \{p, q-1\} \geq j) - 1\,, \qquad j = 1, \ldots, p+q-3 \qquad (33)$$

or, equivalently

$$h_j = \begin{cases} 1 & j = 1, \ldots, \min(p, q-1) \\ 0 & j = 1 + \min(p, q-1), \ldots, \max(p, q-1) \\ -1 & j = 1 + \max(p, q-1), \ldots, p+q-3\,, \end{cases} \qquad (34)$$

what shows that in this case the exact distribution of Λ_4 may be seen as being the same as the distribution of

$$\prod_{j=2}^{p+q-1} e^{Y_j} \quad \text{where} \quad Y_j \sim \Gamma\left(r_j, \frac{n-j}{n}\right) \quad \text{are independent r.v.'s.} \qquad (35)$$

We may note that in (31) we may write the product through $p+q$ instead of $p+q-1$, considering then the r_j defined as in (32), where in the last line j runs from 5 through $p+q$, as long as the h_j are defined as in (33) or (34) with j running through $p+q$. This would yield indeed $r_{p+q} = 0$.

If p is odd and q is even, then the exact c.f. of W_4 is given by

$$\begin{aligned}\Phi_{W_4}(t) &= \prod_{j=1}^{p} \frac{\Gamma\left(\frac{n-j}{2}\right)\Gamma\left(\frac{n-q+1-j}{2}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-j}{2}-\frac{n}{2}\mathrm{i}t\right)\Gamma\left(\frac{n-q+1-j}{2}\right)} \\ &= \underbrace{\prod_{j=1}^{p+q-1}\left(\frac{n-j}{n}\right)^{r_j}\left(\frac{n-j}{n}-\mathrm{i}t\right)^{-r_j}}_{\Phi_{1,W_4}(t)} \times \underbrace{\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-1}{2}-\frac{n}{2}\mathrm{i}t\right)\Gamma\left(\frac{n-2}{2}\right)}}_{\Phi_{2,W_4}(t)}\end{aligned}$$

for r_j defined as above by (32) and (33) or (34), what shows that in this case the distribution of W_4 may be seen as the same as the distribution of the sum of p independent Logbeta r.v.'s with parameters

$$\frac{n-q+1-j}{2} \quad \text{and} \quad \frac{q-1}{2} \quad (j=2,\ldots,p) \tag{36}$$

multiplied by $n/2$, or as the distribution of the sum of a GIG distribution of depth $p+q-1$ with rate parameters $\frac{n-j}{n}$ and shape parameters r_j ($j = 2,\ldots,p$) given by (32) and (33) or (34) with an independent Logbeta r.v. with parameters

$$\frac{n-1}{2} \quad \text{and} \quad \frac{1}{2}\,, \tag{37}$$

and that thus the distribution of Λ_4 is in this case the same as the distribution of

$$\left\{\prod_{j=1}^{p+q-1} e^{Z_j}\right\} \times Y$$

where

$$Z_j \sim \Gamma\left(\frac{n-j}{n}, r_j\right) \quad \text{and} \quad Y \sim Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

for r_j given by (32) and (33) and where the $p+q-2$ r.v.'s Z_j and Y are all independent.

2.5 *About the distributions of the modified l.r.t. statistics and about the l.r.t. statistics for elliptically contoured distributions*

This subsection is devoted to two brief notes, one concerning the distribution of the modified l.r.t. statistics and the other concerning the l.r.t. statistics and their distributions when sampling from multivariate elliptically contoured distributions.

In the previous subsections our choice was to consider the unmodified l.r.t. statistics. However, it is known that the modified l.r.t. statistics yield unbiased tests. In every case presented in the previous subsections, if we consider Λ as denoting the unmodified l.r.t. statistic, the modified l.r.t. statistic will be denoted by

$$\Lambda^* = \Lambda^{(n-1)/n}$$

so that if $W^* = -\log \Lambda^*$, the c.f. of W^* will be given by

$$\Phi_{W^*}(t) = \Phi_W\left(\frac{n-1}{n}\,t\right)$$

so that in each case if we want to devise the exact distribution of W^* or Λ^* we only have to consider $\Phi_{2,W}\left(\frac{n-1}{n}\,t\right)$ instead of $\Phi_{2,W}(t)$ and $\Phi_{1,W}\left(\frac{n-1}{n}\,t\right)$ instead of $\Phi_{1,W}(t)$, where we may note that $\Phi_{1,W}\left(\frac{n-1}{n}\,t\right)$ may be written as

$$\Phi_{1,W}\left(\frac{n-1}{n}\,t\right) = \prod_{j=2}^{p^*}\left(\frac{n-j}{n-1}\right)^{r_j}\left(\frac{n-j}{n-1}-\mathrm{i}t\right)^{r_j},$$

that is, instead of considering the Gamma r.v.'s considered in the previous subsections, multiplied by $\frac{n-1}{n}$, we may alternatively consider Gamma r.v.'s with rate parameters equal to $\frac{n-j}{n-1}$ and the same shape parameters that were considered in each of the previous subsections.

As shown by Anderson et all (1986), Anderson and Fang (1990) and Anderson (2003, Sec.'s 8.11, 9.11, 10.11), if we consider sampling from multivariate elliptically contoured or left-spherical distributions, instead of sampling from the multivariate Normal distribution, not only will the l.r.t. statistics to test the null hypotheses discussed in the previous subsections remain the same as for the Normal case but also their distributions under the null hypothesis will remain unchanged. Because of this, the results presented in the present paper may be applied without change in these more general settings.

3 Near-exact distributions for the l.r.t. statistics

3.1 Preamble

Like the asymptotic distributions, near-exact distributions for a given statistic may usually be obtained under several different forms. The near-exact distributions we will focus our attention on in this paper are near-exact distributions that are derived from a factorization of the exact c.f. of the logarithm of the statistic under study, $W_k = -\log \Lambda_k$ (where $k \in \{1, \dots, 4\}$ denotes the l.r.t. statistic in subsection k of section 2) of the form

$$\Phi_{W_k}(t) = \Phi_{1,W_k}(t)\, \Phi_{2,W_k}(t)\,, \tag{38}$$

where $\Phi_{1,W_k}(t)$ is the part of $\Phi_{W_k}(t)$ that will be left unchanged and $\Phi_{2,W_k}(t)$ is the part of $\Phi_{W_k}(t)$ that will be replaced by an asymptotic result, which is intended to be asymptotic both in terms of sample size and also in terms of the overall number of variables involved, in the sense that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left| \frac{\Phi_{W_k}(t) - \Phi_{W_k}(t; n, p)}{t} \right| dt = 0$$

and

$$\lim_{p\to\infty} \int_{-\infty}^{\infty} \left| \frac{\Phi_{W_k}(t) - \Phi_{W_k}(t; n, p)}{t} \right| dt = 0$$

where $\Phi_{W_k}(t)$ represents the exact c.f. of the negative logarithm of the l.r.t. statistic and $\Phi_{W_k}(t; n, p)$ represents the near-exact c.f. of the same statistic, seen as a function of n (the sample size) and p (the overall number of variables involved). In (38), $\Phi_{1,W_k}(t)$ is the c.f. of a sum of independent Logbeta random variables (multiplied by $n/2$, for samples of size n) which may also be expressed as a c.f. of the sum of independent Gamma random variables, all with integer shape parameters, while $\Phi_{2,W_k}(t)$ is the c.f. of another sum of independent Logbeta random variables (also multiplied by $n/2$) which is not possible to express as a sum of independent Gamma random variables with integer shape parameters.

This way the near-exact distributions we will be dealing with in this paper will have c.f.'s of the form

$$\Phi_{1,W_k}(t)\, \Phi^*_{2,W_k}(t)\,, \tag{39}$$

where $\Phi_{1,W_k}(t)$ is the same as in (38) and will be written under the form of the c.f. of a GIG distribution of a given depth, that is, the c.f. of the sum of a given number of independent Gamma r.v.'s, all with integer shape parameters, while $\Phi^*_{2,W_k}(t)$ may be either the c.f. of a single Gamma distribution or of a mixture of two or three Gamma distributions, depending on the number of exact moments we want to match. The c.f. $\Phi^*_{2,W_k}(t)$ will indeed have, accordingly, the same 2, 4 or 6 first derivatives (with respect to t at $t=0$) as the part of the exact c.f. of the statistic under study that will be replaced, that is, $\Phi_{2,W_k}(t)$. In other words, we will have

$$\frac{d^j}{dt^j}\,\Phi^*_{2,W_k}(t)\bigg|_{t=0} = \frac{d^j}{dt^j}\,\Phi_{2,W_k}(t)\bigg|_{t=0}\,, \qquad j=1,\dots,h \tag{40}$$

for $h=2,4$ or 6, according to the case of $\Phi^*_{2,W_k}(t)$ being the c.f. of a single Gamma distribution, or the c.f. of a mixture of 2 or 3 Gamma distributions with the same rate parameter, that is,

$$\Phi^*_{2,W_k}(t) = \sum_{k=1}^{h/2} p_k\,\lambda^{s_k}\,(\lambda - \mathrm{i}t)^{-s_k}\,, \tag{41}$$

with

$$p_{h/2} = 1 - \sum_{k=1}^{h/2-1} p_k\,. \tag{42}$$

While if $\Phi^*_{2,W_k}(t)$ is the c.f. of a single Gamma distribution, equating the two first derivatives of $\Phi_{2,W_k}(t)$ at $t=0$, there is a simple analytical solution for the problem of equating moments, with the rate and shape parameters of $\Phi^*_{2,W_k}(t)$ being given by

$$\lambda = \frac{m_1}{m_2 - m_1^2} \qquad \text{and} \qquad s_1 = \frac{m_1}{m_2 - m_1^2}\,,$$

where

$$m_1 = \frac{1}{i}\,\frac{d}{dt}\,\Phi_{2,W_k}(t)\bigg|_{t=0} \qquad \text{and} \qquad m_2 = -\,\frac{d^2}{dt^2}\,\Phi_{2,W_k}(t)\bigg|_{t=0}\,,$$

if $\Phi^*_{2,W_k}(t)$ is the c.f. of a mixture of two Gamma distributions it is possible to prove (through quite long and tedious calculations) that there is always one unique analytic real solution, or rather, a pair of conjugate real solutions, with the values for the two shape parameters and corresponding weights interchanged, and if $\Phi^*_{2,W_k}(t)$ is the c.f. of a mixture of three Gamma distributions, it is believed that there is also always one only real solution with all positive parameters, or rather, a six-tuple of conjugate solutions, although this is not easy to prove analytically. Anyway, for the cases where $\Phi^*_{2,W_k}(t)$ is the c.f. of a mixture of 2 or 3 Gamma distributions (respectively for $h = 4$ or $h = 6$) we advocate the numerical solution of the system of equations (40).

As already remarked in Marques & Coelho (2008) and Coelho & Marques (2007), the replacement of $\Phi_{2,W_k}(t)$ by $\Phi^*_{2,W_k}(t)$, that is, the replacement of a sum of independent Logbeta random variables (multiplied by a constant) by a single Gamma distribution or a mixture of 2 or 3 Gamma distributions, matching the first 2, 4 or 6 exact moments is a much adequate decision, since, as it is shown in Coelho et al. (2006), a single Logbeta distribution may be represented either under the form of an infinite mixture of Exponential or GIG distributions, and thus a sum of independent Logbeta random variables may thus be represented under the form of an infinite mixture of sums of Exponential or GIG distributions, which are themselves GIG distributions, while the GIG distribution may itself be seen as a mixture of Gamma distributions (Coelho, 2007).

$\Phi_{1,W_k}(t)$ in (39) will have, for samples of size n, the form

$$\Phi_{1,W_k}(t) = \prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{r_{j,k}} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-r_{j,k}} \tag{43}$$

where p is the overall number of variables involved and the $r_{j,k}$ are the integer shape parameters of the Gamma r.v.'s involved and will be obtained under specific forms, according to the l.r.t. statistic under study. This amounts to be able to write the near-exact c.f. of the logarithm of the l.r.t. statistics under consideration in the form

$$\Phi^*_{2,W_k}(t) \times \prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{r_{j,k}} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-r_{j,k}}, \tag{44}$$

where $\Phi^*_{2,W_k}(t)$ is either the c.f. of a Gamma distribution or the c.f. of a mixture of 2 or 3 Gamma distributions, being thus the near-exact distributions obtained in this way, correspondingly a GNIG distribution of depth at most $p+1$ (see Appendix A) or a mixture of two or three GNIG distributions of depth at most $p+1$, which have very manageable expressions, allowing this way for an easy computation of very accurate near-exact quantiles.

3.2 Expressions for the near-exact p.d.f.'s and c.d.f.'s of the l.r.t. statistics

Expressions for the near-exact p.d.f.'s and c.d.f.'s of the l.r.t. statistics Λ_k in section 2 (where k denotes the subsection k of section 2 where Λ_k is studied) may thus be obtained, using the notation in Appendix A, under the form

$$f_{\Lambda_k}(\ell) = \sum_{\nu=1}^{h/2} p_\nu \, f^{\,GNIG}\left(-\log \ell | r_{i_k}, \dots, r_{f_k}, s_\nu; \frac{n-i_k}{n}, \dots, \frac{n-f_k}{n}, \lambda; d_k\right) \frac{1}{\ell}\,,$$

and

$$F_{\Lambda_k}(\ell) = 1 - \sum_{\nu=1}^{h/2} p_\nu \, F^{\,GNIG}\left(-\log \ell | r_{i_k}, \dots, r_{f_k}, s_\nu; \frac{n-i_k}{n}, \dots, \frac{n-f_k}{n}, \lambda; d_k\right),$$

where the weights p_ν are subject to the relation (42) and where

$$i_k = \begin{cases} 3 & k=1,4 \\ 2 & k=2,3 \end{cases}, \qquad f_k = \begin{cases} p & k=1,2,3 \\ p+q-1 & k=4\,, \end{cases} \tag{45}$$

being

$$d_k = f_k - i_k + 2 \tag{46}$$

and

$$\begin{array}{ll} r_3, \dots, r_p & \text{given by (5)-(8) for } k=1 \\ r_2, \dots, r_p & \text{given by (18)-(22) for } k=2 \\ r_2, \dots, r_p & \text{given by (27) or (28) for } k=3 \\ r_3, \dots, r_{p+q-1} & \text{given by (32)-(33) for } k=4. \end{array} \tag{47}$$

We should briefly note here that, as referred in the previous section, there are situations in which we may have the exact distribution of some Λ_k given under the form of a GIG distribution (Coelho, 1998, 1999), with p.d.f.'s and c.d.f.'s respectively given by

$$f_{\Lambda_k}(\ell) = f^{\,GIG}\left(-\log \ell | r_{i_k}, \dots, r_{f_k}; \frac{n-i_k}{n}, \dots, \frac{n-f_k}{n}; d_k - 1\right) \frac{1}{\ell}$$

and

$$F_{\Lambda_k}(\ell) = 1 - F^{GIG}\left(-\log\,\ell | r_{i_k}, \ldots, r_{f_k}; \frac{n-i_k}{n}, \ldots, \frac{n-f_k}{n}; d_k - 1\right)$$

with i_k, f_k and d_k given by (45) and (46) above and $r_{i_k}, \ldots, r_{f_k}$ given by (47) above, for $k = 1$ when there is at most one set of variables with an odd number of variables and for $k = 4$ when p is even or q is odd.

4 The l.r.t. statistic to test the equality of several p-multivariate Normal distributions

Let us suppose we want to test if q multivariate Normal distributions $N_p(\underline{\mu}_j, \Sigma_j)$ $(j = 1, \ldots, q)$ are equal, that is, we want to test the null hypothesis

$$H_{05} : \underline{\mu}_1 = \ldots = \underline{\mu}_q,\ \Sigma_1 = \ldots = \Sigma_q\,, \tag{48}$$

based on q independent samples, each of size n_j $(j = 1, \ldots, q)$.

We will decompose, as it is indeed usual, the above null hypothesis as

$$H_{05} :\ H_{04|02} \,\mathrm{o}\, H_{02}$$

where 'o' is to be read as 'after' or 'composed with' and where

$$\begin{aligned} &H_{02} : \Sigma_1 = \ldots = \Sigma_q \\ &H_{04|02} : \underline{\mu}_1 = \ldots = \underline{\mu}_q \ \text{ given that } \ \Sigma_1 = \ldots = \Sigma_q (= \Sigma) \end{aligned} \tag{49}$$

and use the induced factorization on the test statistic to obtain near-exact distributions for the overall test statistic. Now the overall l.r.t. statistic to test H_{05} in (48) may be written as

$$\Lambda_5 = \Lambda_2\,\Lambda_4$$

where Λ_2 is the l.r.t. statistic treated in subsection 2.2 and used here to test H_{02} in (49) and Λ_4 is the l.r.t. statistic in subsection 2.4, to test $H_{04|02}$ in (49), that is the same statistic treated in subsection 2.4, and where Λ_4 and Λ_2 are independent under H_0 in (48).

If we take $n_j = n$ for all $j = 1, \ldots, q$, we may then write, for the case of even p, taking $\alpha \in I\!N$, the exact c.f. of $W_5 = -\log \Lambda_5$ as

$$\Phi_{W_5}(t) = \left.\begin{array}{l} \displaystyle\prod_{\substack{j=2\\ j\neq\alpha q}}^{p+q-1} \left(\frac{n-j/q}{n}\right)^{r_j} \left(\frac{n-j/q}{n} - \mathrm{i}t\right)^{-r_j} \\ \displaystyle\times \prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{u_j} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-u_j} \end{array}\right\} \Phi_{1,W_5}(t)$$
$$\left.\times \prod_{j=1}^{p/2} \prod_{k=1}^{q} \frac{\Gamma\left(a_j + b_{jk}\right) \Gamma\left(a_j + b^*_{jk} - n\mathrm{i}t\right)}{\Gamma\left(a_j + b^*_{jk}\right) \Gamma\left(a_j + b_{jk} - n\mathrm{i}t\right)} \right\} \Phi_{2,W_5}(t) \qquad (50)$$

where a_j and b_{jk} are given by (16) with $b^*_{jk} = \lfloor b_{jk} \rfloor$, the r_j are given by (32) and (33) or (34) and

$$u_j = \begin{cases} r^*_j & j \neq \alpha q \\ r^*_j + r_{\alpha q} & j = \alpha q\,, \quad \alpha = 1, \ldots, \left\lfloor \frac{p}{q} \right\rfloor \end{cases}$$

where r^*_j are given by (19) and the $r_{\alpha q}$ are given by (32) and (33) or (34).

For odd q and any p (that is, either even or odd p) we may write the exact c.f. of W_5 as

$$\Phi_{W_5}(t) = \left.\begin{array}{l} \displaystyle\prod_{\substack{j=2\\ j\neq\alpha q}}^{p+q-1} \left(\frac{n-j/q}{n}\right)^{r_j} \left(\frac{n-j/q}{n} - \mathrm{i}t\right)^{-r_j} \\ \displaystyle\times \prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{u_j} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-u_j} \end{array}\right\} \Phi_{1,W_5}(t)$$
$$\left.\begin{array}{l} \displaystyle\times \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{k=1}^{q} \frac{\Gamma\left(a_j + b_{jk}\right) \Gamma\left(a_j + b^*_{jk} - n\mathrm{i}t\right)}{\Gamma\left(a_j + b^*_{jk}\right) \Gamma\left(a_j + b_{jk} - n\mathrm{i}t\right)} \\ \displaystyle\times \left(\prod_{k=1}^{q} \frac{\Gamma(a_p + b_{pk})}{\Gamma(a_p + b^*_{pk})} \frac{\Gamma\left(a_p + b^*_{pk} - \frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(a_p + b_{pk} - \frac{n}{2}\mathrm{i}t\right)} \right)^{p \perp\!\!\!\perp 2} \end{array}\right\} \Phi_{2,W_5}(t) \qquad (51)$$

and for even q and odd p as

$$\Phi_{W_5}(t) = \left.\begin{array}{l} \displaystyle\prod_{\substack{j=2\\ j\neq\alpha q}}^{p+q-1} \left(\frac{n-j/q}{n}\right)^{r_j} \left(\frac{n-j/q}{n}-\mathrm{i}t\right)^{-r_j} \\ \displaystyle\times \prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{u_j} \left(\frac{n-j}{n}-\mathrm{i}t\right)^{-u_j} \end{array}\right\} \Phi_{1,W_5}(t)$$

$$\left.\begin{array}{l} \displaystyle\times \frac{\Gamma\left(\frac{nq-1}{2}\right)\Gamma\left(\frac{nq-2}{2}-\frac{nq}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{nq-1}{2}-\frac{nq}{2}\mathrm{i}t\right)\Gamma\left(\frac{nq-2}{2}\right)} \\ \displaystyle\times \prod_{j=1}^{(p-1)/2}\prod_{k=1}^{q} \frac{\Gamma\left(a_j+b_{jk}\right)\Gamma\left(a_j+b^*_{jk}-n\mathrm{i}t\right)}{\Gamma\left(a_j+b^*_{jk}\right)\Gamma\left(a_j+b_{jk}-n\mathrm{i}t\right)} \\ \displaystyle\times \prod_{k=1}^{q} \frac{\Gamma(a_p+b_{pk})}{\Gamma(a_p+b^*_{pk})} \frac{\Gamma\left(a_p+b^*_{pk}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(a_p+b_{pk}-\frac{n}{2}\mathrm{i}t\right)} \end{array}\right\} \Phi_{2,W_5}(t)$$

where a_p, b_{pk} and b^*_{pk} are given by (16) and (17).

Thus, following the exposition in the previous section, near-exact distributions for Λ_5 may be obtained with p.d.f.'s and c.d.f.'s related with mixtures of GNIG distributions of depth $2p+q-3-\left\lfloor\frac{p}{q}\right\rfloor$, respectively given by

$$f_{\Lambda_5}(\ell) = \sum_{\nu=1}^{h/2} p_\nu\, f^{GNIG}\Bigg(-\log\ell \,\Big|\, \underbrace{r_3,\dots,r_j,\dots,r_{p+q-1}}_{j\neq\alpha q,\,\alpha\in I\!N}, u_2,\dots,u_p,s_\nu; \underbrace{\frac{n-3/q}{n},\dots,\frac{n-j/q}{n},\dots,\frac{n-(p+q-1)/q}{n}}_{j\neq\alpha q,\,\alpha\in I\!N}, \frac{n-2}{n},\dots,\frac{n-p}{n},\lambda; 2p+q-3-\left\lfloor\frac{p}{q}\right\rfloor\Bigg)\frac{1}{\ell}$$

and

$$F_{\Lambda_5}(\ell) = 1-\sum_{\nu=1}^{h/2} p_\nu\, F^{GNIG}\Bigg(-\log\ell \,\Big|\, \underbrace{r_3,\dots,r_j,\dots,r_{p+q-1}}_{j\neq\alpha q,\,\alpha\in I\!N}, u_2,\dots,u_p,s_\nu; \underbrace{\frac{n-3/q}{n},\dots,\frac{n-j/q}{n},\dots,\frac{n-(p+q-1)/q}{n}}_{j\neq\alpha q,\,\alpha\in I\!N}, \frac{n-2}{n},\dots,\frac{n-p}{n},\lambda; 2p+q-3-\left\lfloor\frac{p}{q}\right\rfloor\Bigg).$$

5 Numerical Studies

The measure

$$\Delta = \int_{-\infty}^{+\infty} \left| \frac{\Phi_W(t) - \Phi_n(t)}{t} \right| dt$$

where $\Phi_W(t)$ and $\Phi_n(t)$ represent respectively the c.f.'s of the r.v.'s W and W_n, with corresponding c.d.f.'s $F_W(w)$ and $F_n(w)$ and where $\Phi_n(t)$ may be seen as a function of some parameter n, is related to the Berry-Esseen upper bound (Berry, 1941; Esseen, 1945; Loève, 1977, Chap. VI, Sec. 21; Hwang, 1998), with

$$\Delta \geq \max_{w \in S} |F_W(w) - F_n(w)|$$

and

$$\Delta \underset{n\to\infty}{\longrightarrow} 0 \quad \Longleftrightarrow \quad W_n \overset{d}{\underset{n\to\infty}{\longrightarrow}} W\,,$$

where S represents the common support of W and W_n.

We will use the measure Δ to assess the closeness of the near-exact distributions (with c.f.'s $\Phi_n(t)$) to the exact distribution (with c.f. $\Phi_W(t)$) and we will show that when combining tests we either do not loose much precision or we may even in some cases gain some precision (that is, the near-exact distributions for the overall test statistic will have only slightly less good Δ values than the corresponding near-exact distributions for the elementary test statistics the overall test statistic is factorized into, or even in some cases, they may exhibit even lower values of Δ for the overall test statistic than at least for one or some of the elementary test statistics) or we will even gain precision whenever we have the exact distribution available for one of the elementary test statistics the overall test statistic is factorized into. Actually, the way the near-exact distributions for the l.r.t. statistic to test the equality of several p-multivariate Normal distributions were built enables us to take advantage from the fact that we have the exact distribution for the logarithm of the l.r.t. statistic to test the equality of the mean vectors for even p or odd q. We may note that if we have a l.r.t. statistic for the overall test whose logarithm $W = W_1 + W_2$ has a c.f.

$$\Phi_W(t) = \Phi_{W_1}(t)\, \Phi_{W_2}(t)$$

where we will only approximate the c.f. of W_2 by $\Phi_{W_2^*}(t)$, thus approximating the distribution of W by the distribution of $W^* = W_1 + W_2^*$, then we have that,

$$\begin{aligned}\Delta(W) &= \int_{-\infty}^{+\infty}\left|\frac{\Phi_W(t)-\Phi_{W^*}(t)}{t}\right|dt \;=\; \int_{-\infty}^{+\infty}\left|\frac{\Phi_{W_1}(t)\Phi_{W_2}(t)-\Phi_{W_1}(t)\Phi_{W_2^*}(t)}{t}\right|dt\\ &= \int_{-\infty}^{+\infty}\underbrace{|\Phi_{W_1}(t)|}_{\leq 1}\;\left|\frac{\Phi_{W_2}(t)-\Phi_{W_2^*}(t)}{t}\right|dt \qquad (52)\\ &\leq \int_{-\infty}^{+\infty}\left|\frac{\Phi_{W_2}(t)-\Phi_{W_2^*}(t)}{t}\right|dt \;=\; \Delta(W_2)\,.\end{aligned}$$

In the tables ahead we will use W_4 to denote the logarithm of the l.r.t. statistic to test the equality of several (more precisely q) p-multivariate Normal distributions, that is, the null hypothesis H_{05} in (48), and W_1 and W_2 to denote the logarithm of the l.r.t. statistics to test the null hypotheses $H_{01|02}$ and H_{02} in (49), respectively.

For the cases where we have the exact distribution of W_1 under the form of a GIG distribution, that is, for even p or odd q, we have, of course, computed the values of Δ only for W_2 and W_4.

We will denote hereon respectively by GNIG, M2GNIG and M3GNIG the near-exact distributions corresponding to a GNIG or a mixture of two or three GNIG distributions.

Table 1 – values of Δ for $p=5$ and $q=4,12$

	$p=5$					
	$q=4$			$q=12$		
	W_1	W_2	W_4	W_1	W_2	W_4
$n=p+2$						
GNIG	9.78×10^{-7}	4.15×10^{-5}	4.69×10^{-5}	1.36×10^{-8}	1.61×10^{-5}	1.47×10^{-5}
M2GNIG	3.36×10^{-9}	1.84×10^{-7}	2.49×10^{-7}	4.51×10^{-12}	3.65×10^{-8}	3.21×10^{-8}
M3GNIG	6.37×10^{-12}	1.06×10^{-9}	1.82×10^{-9}	1.63×10^{-15}	1.36×10^{-10}	1.12×10^{-10}
$n=p+20$						
GNIG	8.52×10^{-8}	4.02×10^{-6}	4.41×10^{-6}	1.24×10^{-9}	1.46×10^{-6}	1.18×10^{-6}
M2GNIG	9.62×10^{-11}	1.47×10^{-9}	1.70×10^{-9}	1.31×10^{-13}	6.21×10^{-10}	3.94×10^{-10}
M3GNIG	1.17×10^{-13}	8.33×10^{-13}	2.81×10^{-12}	1.72×10^{-17}	5.63×10^{-13}	2.50×10^{-13}

Table 2 – values of Δ for $p = 11$ and $q = 4, 12$

	$p=11$					
	$q=4$			$q=12$		
	W_1	W_2	W_4	W_1	W_2	W_4
$n=p+2$						
GNIG	7.25×10^{-8}	2.52×10^{-6}	2.72×10^{-6}	1.24×10^{-9}	2.57×10^{-7}	2.65×10^{-7}
M2GNIG	6.01×10^{-11}	1.93×10^{-9}	2.21×10^{-9}	1.05×10^{-13}	3.02×10^{-11}	3.23×10^{-11}
M3GNIG	4.71×10^{-14}	2.31×10^{-12}	2.71×10^{-12}	1.83×10^{-18}	4.47×10^{-15}	4.71×10^{-15}
$n=p+20$						
GNIG	1.51×10^{-8}	1.12×10^{-6}	1.16×10^{-6}	2.41×10^{-10}	1.59×10^{-7}	1.51×10^{-7}
M2GNIG	6.11×10^{-12}	3.37×10^{-10}	3.38×10^{-10}	9.26×10^{-15}	9.58×10^{-12}	8.43×10^{-12}
M3GNIG	2.85×10^{-15}	1.77×10^{-13}	1.24×10^{-13}	4.53×10^{-19}	9.79×10^{-16}	5.76×10^{-16}

Tables 1 and 2 refer to cases where we do not have the exact distribution available under a manageable form for the statistic W_1. From the results in these tables we may see how the values of Δ for W_4 are in most cases only slightly higher than the values of Δ for W_2, hapenning that in the other cases, that is, for $p = 5$ and $q = 12$ the values of Δ are even slightly smaller for W_4 than for W_2, what also happens for $p = 11$, for the larger sample size for $q = 12$ and also for $q = 4$, in this latter case only for the M3GNIG distribution.

Table 3 – values of Δ for $p = 4$ and $q = 4, 12$

	$p=4$			
	$q=4$		$q=12$	
	W_2	W_4	W_2	W_4
$n=p+2$				
GNIG	1.15×10^{-5}	8.90×10^{-6}	8.69×10^{-6}	7.11×10^{-6}
M2GNIG	1.70×10^{-8}	1.07×10^{-8}	1.13×10^{-8}	8.02×10^{-9}
M3GNIG	3.84×10^{-10}	1.94×10^{-10}	2.45×10^{-11}	1.50×10^{-11}
$n=p+20$				
GNIG	1.25×10^{-6}	7.87×10^{-7}	7.45×10^{-7}	4.79×10^{-7}
M2GNIG	8.18×10^{-10}	3.81×10^{-10}	1.80×10^{-10}	8.62×10^{-11}
M3GNIG	1.07×10^{-12}	3.74×10^{-13}	1.33×10^{-13}	4.77×10^{-14}

Table 4 – values of Δ for $p = 10$ and $q = 4, 12$

	$p=10$			
	$q=4$		$q=12$	
	W_2	W_4	W_2	W_4
$n=p+2$				
GNIG	$1.58{\times}10^{-6}$	$1.47{\times}10^{-6}$	$1.34{\times}10^{-7}$	$1.25{\times}10^{-7}$
M2GNIG	$7.83{\times}10^{-10}$	$6.85{\times}10^{-10}$	$6.64{\times}10^{-12}$	$5.93{\times}10^{-12}$
M3GNIG	$5.18{\times}10^{-13}$	$4.27{\times}10^{-13}$	$6.85{\times}10^{-17}$	$5.84{\times}10^{-17}$
$n=p+20$				
GNIG	$7.24{\times}10^{-7}$	$5.95{\times}10^{-7}$	$8.34{\times}10^{-8}$	$6.90{\times}10^{-8}$
M2GNIG	$1.12{\times}10^{-10}$	$8.08{\times}10^{-11}$	$1.18{\times}10^{-12}$	$8.59{\times}10^{-13}$
M3GNIG	$1.99{\times}10^{-14}$	$1.26{\times}10^{-14}$	$3.35{\times}10^{-16}$	$2.15{\times}10^{-16}$

Tables 3 through 6 refer to cases where we have the exact distribution of W_1 available under the form of a GIG distribution, either because p is even (tables 3 and 4) or because q is odd (tables 5 and 6). In all these cases, the way the near-exact distributions for W_4 were built, being able to profit from this fact, leads these distributions to exhibit smaller values of Δ than the corresponding near-exact distributions for W_2, as it was actually expected, from the result in (52).

Table 5 – values of Δ for $p = 5$ and $q = 5, 11$

	$p=5$			
	$q=5$		$q=11$	
	W_2	W_4	W_2	W_4
$n=p+2$				
GNIG	$3.68{\times}10^{-5}$	$3.08{\times}10^{-5}$	$1.79{\times}10^{-5}$	$1.53{\times}10^{-5}$
M2GNIG	$1.53{\times}10^{-7}$	$1.12{\times}10^{-7}$	$4.36{\times}10^{-8}$	$3.34{\times}10^{-8}$
M3GNIG	$9.43{\times}10^{-10}$	$6.02{\times}10^{-10}$	$1.74{\times}10^{-10}$	$1.19{\times}10^{-10}$
$n=p+20$				
GNIG	$3.50{\times}10^{-6}$	$2.40{\times}10^{-6}$	$1.63{\times}10^{-6}$	$1.12{\times}10^{-6}$
M2GNIG	$1.72{\times}10^{-9}$	$9.19{\times}10^{-10}$	$7.29{\times}10^{-10}$	$3.95{\times}10^{-10}$
M3GNIG	$2.25{\times}10^{-12}$	$9.45{\times}10^{-13}$	$7.03{\times}10^{-13}$	$2.99{\times}10^{-13}$

Table 6 – values of Δ for $p = 11$ and $q = 5, 11$

	$p = 11$			
	$q = 5$		$q = 11$	
	W_2	W_4	W_2	W_4
$n = p + 2$				
GNIG	5.60×10^{-7}	5.25×10^{-7}	2.71×10^{-7}	2.56×10^{-7}
M2GNIG	8.73×10^{-11}	7.81×10^{-11}	3.29×10^{-11}	2.97×10^{-11}
M3GNIG	3.60×10^{-15}	3.07×10^{-15}	4.95×10^{-15}	4.29×10^{-15}
$n = p + 20$				
GNIG	3.57×10^{-7}	2.99×10^{-7}	1.68×10^{-7}	1.42×10^{-7}
M2GNIG	3.29×10^{-11}	2.97×10^{-11}	1.02×10^{-11}	7.68×10^{-12}
M3GNIG	1.64×10^{-14}	1.08×10^{-14}	1.05×10^{-15}	7.06×10^{-16}

Analysing the overall set of tables we may see how the near-exact distributions proposed show an asymptotic behavior, for all of the statistics involved, not only for increasing sample sizes but also for increasing values of p (the number of variables involved) and q (the number of Normal distributions involved for W_4 or covariance matrices for W_2 or mean vectors for W_1). Only for the cases of $p = 10$ and $q = 12$ or $p = 11$ and $q = 5$ it seems that the distribution M3GNIG somehow exhibits too small values of Δ for the smaller sample sizes.

6 Conclusions

Although other authors had already shown or referred that all the l.r.t. statistics considered in this paper have the same distribution as the product of a given number of independent r.v.'s, which are either Beta r.v.'s or functions of Beta r.v.'s (see for example, Anderson (2003), Hsieh (1979)), we have shown that indeed all those statistics have the same distribution as the product of independent Beta r.v.'s with different parameters, according to the statistic considered.

On the other hand, we have also shown that not only can one obtain near-exact distributions for all the 'basic' l.r.t. statistics used in Multivariate Analysis in a uniform manner, but also the same general strategy permits the development of near-exact distributions for more elaborate l.r.t. statistics, retaining a similar structure and formulation.

The development of near-exact distributions as very close, although manageable, approximations to the exact distribution of these l.r.t. statistics seems most useful since their exact distributions do not have manageable closed forms (see for example, Aslam & Rocke (2005)).

Two good features of the near-exact distributions are the facts that, opposite to the usual asymptotic distributions, they show a very good fit even for small samples and their closeness to the exact distribution even improves when the dimension increases.

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Appendix A

The Gamma, GIG (Generalized Integer Gamma) and GNIG (Genealized Near-Integer Gamma) distributions

We will use this Appendix to establish some notation concerning distributions used in the paper, as well as to give the expressions for the p.d.f.'s (probability density functions) and c.d.f.'s (cumulative distribution functions) of the GIG (Generalized Integer Gamma) and GNIG (Generalized Near-Integer Gamma) distributions.

We will say that the r.v. X has a Gamma distribution with rate parameter $\lambda > 0$ and shape parameter $r > 0$, if its p.d.f. may be written as

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)}\, e^{-\lambda x}\, x^{r-1}\,, \qquad (x > 0)$$

and we will denote this fact by

$$X \sim \Gamma(r, \lambda)\,.$$

Let

$$X_j \sim \Gamma(r_j, \lambda_j) \qquad j = 1, \ldots, p$$

be p independent r.v.'s with Gamma distributions with shape parameters $r_j \in \mathbb{N}$ and rate parameters $\lambda_j > 0$, with $\lambda_j \neq \lambda_{j'}$, for all $j, j' \in \{1, \ldots, p\}$. We will say that then the r.v.

$$Y = \sum_{j=1}^{p} X_j$$

has a GIG (Generalized Integer Gamma) distribution of depth p, with shape parameters r_j and rate parameters λ_j, $(j = 1, \ldots, p)$, and we will denote this fact by

$$Y \sim GIG(r_j, \lambda_j;\, p)\,.$$

The p.d.f. and c.d.f. (cumulative distribution function) of Y are respectively given by (Coelho, 1998)

$$f^{GIG}(y|r_1,\ldots,r_p;\lambda_1,\ldots,\lambda_p;p) \;=\; K\sum_{j=1}^{p} P_j(y)\;\mathrm{e}^{-\lambda_j\, y}\,, \qquad (y>0) \tag{53}$$

and

$$F^{GIG}(y|r_1,\ldots,r_j;\lambda_1,\ldots,\lambda_p;p) \;=\; 1-K\sum_{j=1}^{p} P_j^*(y)\;\mathrm{e}^{-\lambda_j\, y}\,, \qquad (y>0) \tag{54}$$

where

$$K=\prod_{j=1}^{p}\lambda_j^{r_j}\;, \qquad P_j(y)=\sum_{k=1}^{r_j} c_{j,k}\, y^{k-1} \tag{55}$$

and

$$P_j^*(y)=\sum_{k=1}^{r_j} c_{j,k}\,(k-1)!\sum_{i=0}^{k-1}\frac{y^i}{i!\,\lambda_j^{k-i}}$$

with

$$c_{j,r_j}=\frac{1}{(r_j-1)!}\prod_{\substack{i=1\\ i\neq j}}^{p}(\lambda_i-\lambda_j)^{-r_i}\;, \qquad j=1,\ldots,p\,, \tag{56}$$

and

$$c_{j,r_j-k}=\frac{1}{k}\sum_{i=1}^{k}\frac{(r_j-k+i-1)!}{(r_j-k-1)!}\,R(i,j,p)\,c_{j,r_j-(k-i)}\,, \qquad (k=1,\ldots,r_j-1;\; j=1,\ldots,p) \tag{57}$$

where

$$R(i,j,p)=\sum_{\substack{k=1\\ k\neq j}}^{p} r_k\,(\lambda_j-\lambda_k)^{-i} \qquad (i=1,\ldots,r_j-1)\,. \tag{58}$$

The GNIG (Generalized Near-Integer Gamma) distribution of depth $p+1$ (Coelho, 2004) is the distribution of the r.v.

$$Z = Y_1 + Y_2$$

where Y_1 and Y_2 are independent, Y_1 having a GIG distribution of depth p and Y_2 with a Gamma distribution with a non-integer shape parameter r and a rate parameter $\lambda \neq \lambda_j$ $(j = 1, \ldots, p)$. The p.d.f. (probability density function) of Z is given by

$$f^{GNIG}(z|r_1, \ldots, r_p, r;\, \lambda_1, \ldots, \lambda_p, \lambda; p+1) = \\ K\lambda^r \sum_{j=1}^{p} \mathrm{e}^{-\lambda_j z} \sum_{k=1}^{r_j} \left\{ c_{j,k} \frac{\Gamma(k)}{\Gamma(k+r)} z^{k+r-1} {}_1F_1(r, k+r, -(\lambda-\lambda_j)z) \right\}, \quad (z > 0) \tag{59}$$

and the c.d.f. (cumulative distribution function) given by

$$F^{GNIG}(z|r_1, \ldots, r_p, r;\, \lambda_1, \ldots, \lambda_p, \lambda; p+1) = \frac{\lambda^r\, z^r}{\Gamma(r+1)} {}_1F_1(r, r+1, -\lambda z) \\ -K\lambda^r \sum_{j=1}^{p} \mathrm{e}^{-\lambda_j z} \sum_{k=1}^{r_j} c^*_{j,k} \sum_{i=0}^{k-1} \frac{z^{r+i}\lambda_j^i}{\Gamma(r+1+i)} {}_1F_1(r, r+1+i, -(\lambda-\lambda_j)z) \quad (z > 0) \tag{60}$$

where

$$c^*_{j,k} = \frac{c_{j,k}}{\lambda_j^k} \Gamma(k)$$

with $c_{j,k}$ given by (56) through (58) above. In the above expressions ${}_1F_1(a, b; z)$ is the Kummer confluent hypergeometric function. This function typically has very good convergence properties and is nowadays easily handled by a number of software packages.

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