Bilateral semidirect product decompositions of transformation monoids I

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July 29, 2009

Abstract

This is the first of a series of three papers involving bilateral semidirect product decompositions of monoids of transformations that preserve or reverse the order or the orientation on a finite set. In this paper we deal with the full transformation case. Namely, we consider the monoid $\mathcal{OR}_n$ of all full transformations on a chain with n elements that preserve or reverse the orientation, as well as its submonoids $\mathcal{OD}_n$ of all order-preserving or order-reversing elements, $\mathcal{OP}_n$ all orientation-preserving elements and $\mathcal{O}_n$ of all order-preserving elements.

2000 *Mathematics subject classification:* 20M05, 20M07, 20M20, 20M35.
Keywords: bilateral semidirect products, transformation semigroups, free monoids, presentations.

Introduction and preliminaries

In this paper we construct decompositions of certain monoids of transformations by means of bilateral semidirect products and quotients. The notion of a bilateral semidirect product of two semigroups was studied by Kunze in [11] and was strongly motivated by automata theoretic ideas (see [12, 13] for applications in Automata Theory). In [14] Kunze proved that the full transformation semigroup on a finite set X is a quotient of a bilateral semidirect product of the symmetric group on X and the semigroup of all order preserving full transformations on X, for some linear order on X. On the other hand, in the same paper, Kunze showed that the semigroup of all order preserving full transformations on a finite chain is a quotient of a bilateral semidirect product of two its subsemigroups. These results as well as applications to Formal Languages are also discussed by Kunze in [15].

Our strategy to construct bilateral semidirect product decompositions is quite different from Kunze techniques. In fact, we first develop a general method which consists in the construction of a bilateral semidirect product of two free monoids that, under certain conditions, induces a bilateral semidirect product of two monoids defined by presentations associated to these free monoids. Then, we apply this method to some monoids of transformations that preserve or reverse the order or the orientation on a finite chain. In particular, we give a simpler, shorter and transparent proof of Kunze's result [14] on the semigroup of all order preserving full transformations on a finite chain.

Let S and T be two semigroups. Let

$$\begin{array}{rccccc} \delta: & T & \longrightarrow & \mathcal{T}(S) & & \\ & u & \longmapsto & \delta_u: & S & \longrightarrow & S \\ & & & & s & \longmapsto & u \cdot s \end{array}$$

be an anti-homomorphism of semigroups (i.e. $(uv)\cdot s = u\cdot(v\cdot s)$, for $s\in S$ and $u,v\in T$) and let

$$\begin{array}{rccccc} \varphi: & S & \longrightarrow & \mathcal{T}(T) & & \\ & s & \longmapsto & \varphi_s: & T & \longrightarrow & T \\ & & & & u & \longmapsto & u^s \end{array}$$

[1]The author gratefully acknowledges support of FCT and PIDDAC, within the project PTDC/MAT/69514/2006 of CAUL.
[2]The author gratefully acknowledges support of FCT and PIDDAC, within the project PTDC/MAT/69514/2006 of CAUL, and of ISEL.

be a homomorphism of semigroups (i.e. $u^{sr} = (u^s)^r$, for $s, r \in S$ and $u \in T$) such that:

(SPR) $(uv)^s = u^{v \cdot s} v^s$, for $s \in S$ and $u, v \in T$ (*Sequential Processing Rule*); and

(SCR) $u \cdot (sr) = (u \cdot s)(u^s \cdot r)$, for $s, r \in S$ and $u \in T$ (*Serial Composition Rule*).

Within these conditions, we say that δ is a *left action* of T on S and that φ is a *right action* of S on T.

In [11], Kunze proved that the set $S \times T$ is a semigroup with respect to the following multiplication:

$$(s, u)(r, v) = (s(u \cdot r), u^r v),$$

for $s, r \in S$ and $u, v \in T$. We denote this semigroup by $S_\delta \bowtie_\varphi T$ (or, if it is not ambiguous, simply by $S \bowtie T$) and call it the *bilateral semidirect product* of S and T associated with δ and φ.

If S and T are monoids and the actions δ and φ preserve the identity (i.e. $1 \cdot s = s$, for $s \in S$, and $u^1 = u$, for $u \in T$) and are *monoidal* (i.e. $u \cdot 1 = 1$, for $u \in T$, and $1^s = 1$, for $s \in S$), then $S \bowtie T$ is a monoid with identity $(1, 1)$.

From now on, we will just consider bilateral semidirect products of monoids associated to monoidal actions.

Notice that, if φ is a trivial action (i.e. $(S)\varphi = \{\mathrm{id}_T\}$) then $S \bowtie T = S * T$ is an usual semidirect product, if δ is a trivial action (i.e. $(T)\delta = \{\mathrm{id}_S\}$) then $S \bowtie T$ coincides with a reverse semidirect product $T *_r S$ (by interchanging the coordinates) and if both actions are trivial then $S \bowtie T$ is the usual direct product $S \times T$. Observe also that the bilateral semidirect product is quite different from the Rhodes and Tilson [19] *double* semidirect product, where the second components multiply always as in the direct product.

Now, recall that a pseudovariety of monoids is a class of finite monoids closed under formation of finite direct products, submonoids and homomorphic images. The *semidirect product* $\mathsf{V} * \mathsf{W}$ of the pseudovarieties of monoids V and W is the pseudovariety generated by all monoidal semidirect products $M * N$, where $M \in \mathsf{V}$ and $N \in \mathsf{W}$. Similarly, we define the *reverse semidirect product* $\mathsf{V} *_r \mathsf{W}$ and the *bilateral semidirect product* $\mathsf{V} \bowtie \mathsf{W}$ of the pseudovarieties of monoids V and W. Clearly, $\mathsf{V} * \mathsf{W} \subseteq \mathsf{V} \bowtie \mathsf{W}$ and $\mathsf{W} *_r \mathsf{V} \subseteq \mathsf{V} \bowtie \mathsf{W}$.

The problem of the decidability of the semidirect product of pseudovarieties in general and the decidability of iterated semidirect products, possibly for particular pseudovarieties, has interested many semigroup theorists for the past few decades. Notice that a positive answer for the question of the decidability of iterated semidirect products whose factors are the classes of all finite groups and of all finite aperiodic semigroups would solve the problem of the *decidability of the complexity*, which is undoubtedly the most famous problem in finite semigroup theory.

Let X be an alphabet and denote by X^+ the free semigroup generated by X and by X^* the free monoid generated by X. A *monoid presentation* is an ordered pair $\langle X \mid R \rangle$, where X is an alphabet and R is a subset of $X^* \times X^*$. An element (u, v) of $X^* \times X^*$ is called a *relation* and it is usually represented by $u = v$. To avoid confusion, given $u, v \in X^*$, we will write $u \equiv v$, instead of $u = v$, whenever we want to state precisely that u and v are identical words of X^*. A monoid S is said to be *defined by a presentation* $\langle X \mid R \rangle$ if S is isomorphic to X^*/ρ_R, where ρ_R denotes the smallest congruence on X^* containing R. Often, we identify the words of X^* with the elements of S they represent. In this context, for $w, w' \in X^*$, saying that $w = w'$ in S means that $(w, w') \in \rho_R$. Notice that, if $w = w'$ in S then there exists a sequence $w \equiv w_0 \to w_1 \to \cdots \to w_n \equiv w'$ of elementary transitions of R, i.e., for each $i \in \{0, 1, \ldots, n-1\}$, there exist $x, y \in X^*$ and $(u = v) \in R$ such that either $w_i \equiv xuy$ and $w_{i+1} \equiv xvy$ or $w_i \equiv xvy$ and $w_{i+1} \equiv xuy$ (for more details, see [16] or [20]). We say that $\langle X \mid R \rangle$ is *letter-irredundant* if $x \neq 1$ in S and $x = y$ in S if and only if $x \equiv y$, for $x, y \in X$.

Now, let S be a monoid and let X be a set of generators of S. Let s be an element of S. The *length of s with respect to X* is the minimum of the set of positive integers $\{n \mid s = x_1 \cdots x_n, \text{ for some } x_1 \ldots, x_n \in X\}$, if s is not the identity, or zero, otherwise. We denote this non-negative integer by $|s|_X$ or, if it is not ambiguous, simply by $|s|$. Naturally, this number coincide with the usual notion of length of a word in a free monoid.

Denote by $\mathcal{T}_n$ the monoid of all full transformations of a set with n elements, say $X_n = \{1, 2, \ldots, n\}$. Consider X_n as a chain with the usual order: $X_n = \{1 < 2 < \cdots < n\}$. We say that a transformation s in $\mathcal{T}_n$ is *order-preserving* [*order-reversing*] if, for all $x, y \in \mathrm{Dom}(s)$, $x \leq y$ implies $xs \leq ys$ [$xs \geq ys$]. Clearly, the product of two order-preserving transformations or of two order-reversing transformations is order-preserving and the product of an order-preserving transformation by an order-reversing transformation is order-reversing. Denote by $\mathcal{O}_n$ the submonoid of $\mathcal{T}_n$ whose elements are order-preserving and by $\mathcal{OD}_n$ the submonoid of $\mathcal{T}_n$ whose elements are either order-preserving or order-reversing.

Next, let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t ($t \geq 0$) elements from the chain X_n. We say that a is *cyclic* [*anti-cyclic*] if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ [$a_i < a_{i+1}$], where a_{t+1} denotes a_1. Let $s \in \mathcal{T}_n$ and suppose that $\mathrm{Dom}(s) = \{a_1, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < \cdots < a_t$. We say that s is an *orientation-preserving* [*orientation-reversing*] transformation if the sequence of its images $(a_1 s, \ldots, a_t s)$ is cyclic [anti-cyclic]. It is also clear that the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving and the product of an orientation-preserving transformation by an orientation-reversing transformation is orientation-reversing. Denote by $\mathcal{OP}_n$ the submonoid of $\mathcal{T}_n$ whose elements are orientation-preserving and by $\mathcal{OR}_n$ the submonoid of $\mathcal{T}_n$ whose elements are either orientation-preserving or orientation-reversing.

Semigroups of order-preserving transformations have long been considered in the literature. In 1962, Aĭzenštat [1] exhibited a presentation for $\mathcal{O}_n$. Some years later, in 1971, Howie [10] studied some combinatorial and algebraic properties of $\mathcal{O}_n$ and, in 1992, Gomes and Howie [8] revisited this monoid. On the other hand, the notion of an orientation-preserving transformation was introduced by McAlister in [17] and, independently, by Catarino and Higgins in [5]. The monoid $\mathcal{OP}_n$ was also considered by Catarino in [4] and by Arthur and Ruškuc in [3].

1 The general method

In this section we present a general technique to obtain a bilateral semidirect decomposition of a monoid in terms of two of its submonoids.

Constructing bilateral semidirect products using presentations

Let A and B be two alphabets. Suppose we have defined actions of and on the letters satisfying

$$b \cdot a \in A \cup \{1\}, \quad 1 \cdot a = a, \quad b \cdot 1 = 1, \quad 1 \cdot 1 = 1 \tag{1}$$

and

$$b^a \in B^*, \quad b^1 = b, \quad 1^a = 1, \quad 1^1 = 1 \ , \tag{2}$$

for $a \in A$ and $b \in B$. Then first, inductively on the length of $u \in B^+$, define

$$(ub) \cdot a = u \cdot (b \cdot a) \tag{3}$$

and

$$(ub)^a = u^{b \cdot a} b^a, \tag{4}$$

for $a \in A \cup \{1\}$ and $b \in B$. Secondly, inductively on the length of $s \in A^+$, define

$$u \cdot (as) = (u \cdot a)(u^a \cdot s) \tag{5}$$

and

$$u^{as} = (u^a)^s, \tag{6}$$

for $u \in B^*$ and $a \in A$. Thus, we have well defined mappings

$$\begin{array}{llllll} \delta: & B^* & \longrightarrow & \mathcal{T}(A^*) & & \\ & u & \longmapsto & \delta_u: & A^* & \longrightarrow & A^* \\ & & & & s & \longmapsto & u \cdot s \end{array}$$

and

$$
\begin{array}{rcccccc}
\varphi: & A^* & \longrightarrow & \mathcal{T}(B^*) & & & \\
 & s & \longmapsto & \varphi_s: & B^* & \longrightarrow & B^* \\
 & & & & u & \longmapsto & u^s\ .
\end{array}
$$

Lemma 1.1 *Let $s,t \in A^*$ and $u,v \in B^*$. Then:*

(a) $1 \cdot s = s$ *and* $1^s = 1$*;*

(b) $u \cdot 1 = 1$ *and* $u^1 = u$*.*

Proof. (a) For $|s| \leq 1$ both equalities follow directly from (1) and (2). Now, we proceed by induction on the length of s. Suppose that $|s| > 1$ and let $a \in A$ and $s' \in A^+$ be such that $s = as'$. As $1 \leq |s'| < |s|$, by the induction hypothesis, we have $1 \cdot s' = s'$ and $1^{s'} = 1$, whence

$$1 \cdot s = 1 \cdot (as') = (1 \cdot a)(1^a \cdot s') = a(1 \cdot s') = as' = s,$$

by applying (5), and

$$1^s = 1^{as'} = (1^a)^{s'} = 1^{s'} = 1,$$

by applying (6).

(b) The proof of these properties is similar to (a) (by induction on the length of u, using this time (3) and (4)). ■

Next, we prove that δ and φ verify both the Sequential Processing Rule and the Serial Composition Rule.

Lemma 1.2 *Let $s,r \in A^*$ and $u,v \in B^*$. Then:*

(SCR) $u \cdot (sr) = (u \cdot s)(u^s \cdot r)$*;*

(SPR) $(uv)^s = u^{v \cdot s} v^s$*.*

Proof. (SCR) If $s = 1$ or $r = 1$, the equality follows from Lemma 1.1 (b). Hence, we admit that $s, r \in A^+$ and proceed by induction on the length of s. If $|s| = 1$ the equality follows from (5). Then, let $s = as'$, with $a \in A$ and $s' \in A^+$. Since $1 \leq |s'| < |s|$, we have

$$
\begin{array}{rcll}
u \cdot (sr) & = & u \cdot (as'r) & \\
 & = & (u \cdot a)(u^a \cdot (s'r)) & \text{(by (5))} \\
 & = & (u \cdot a)(u^a \cdot s')((u^a)^{s'} \cdot r) & \text{(by the induction hypothesis)} \\
 & = & (u \cdot (as'))(u^{as'} \cdot r) & \text{(by (5) and (6))} \\
 & = & (u \cdot s)(u^s \cdot r)\ . &
\end{array}
$$

(SPR) First, we show that $(uv)^a = u^{v \cdot a} v^a$, for $a \in A \cup \{1\}$. If $u = 1$ this equality follows from (2) (notice that $v \cdot a \in A \cup \{1\}$). So, admit that $|u| \geq 1$. We proceed by induction on the length of v. If $v = 1$ this equality follows from (1) and (2) and if $|v| = 1$ it follows from (4). Hence, let $v = v'b$, with $v' \in B^+$ and $b \in B$. Then, as $1 \leq |v'| < |v|$ and $b \cdot a \in A \cup \{1\}$, we have

$$
\begin{array}{rcll}
(uv)^a & = & (uv'b)^a & \\
 & = & (uv')^{b \cdot a} b^a & \text{(by (4))} \\
 & = & u^{v' \cdot (b \cdot a)} v'^{b \cdot a} b^a & \text{(by the induction hypothesis)} \\
 & = & u^{(v'b) \cdot a} (v'b)^a & \text{(by (3) and (4))} \\
 & = & u^{v \cdot a} v^a\ . &
\end{array}
$$

Now, we prove the equality for any $s \in A^*$ by induction on the length of s. Since we have just proved it for $|s| \leq 1$, take $s = as'$ with $a \in A$ and $s' \in A^+$. Then, as $1 \leq |s'| < |s|$ and $v \centerdot a \in A \cup \{1\}$, we have

$$\begin{array}{rcll} (uv)^s & = & (uv)^{as'} & \\ & = & ((uv)^a)^{s'} & \text{(by (6))} \\ & = & (u^{v \centerdot a} v^a)^{s'} & \text{(by the case } |s| = 1) \\ & = & (u^{v \centerdot a})^{v^a \centerdot s'} (v^a)^{s'} & \text{(by the induction hypothesis)} \\ & = & u^{(v \centerdot a)(v^a \centerdot s')} v^{as'} & \text{(by (6) and Lemma 1.1 (b))} \\ & = & u^{v \centerdot (as')} v^{as'} & \text{(by (5))} \\ & = & u^{v \centerdot s} v^s \ , & \end{array}$$

as required. ∎

Lemma 1.3 *Let $s, r \in A^*$ and $u, v \in B^*$. Then:*

(a) $(uv) \centerdot s = u \centerdot (v \centerdot s)$*;*

(b) $u^{sr} = (u^s)^r$.

Proof. (a) First we prove that $(uv) \centerdot a = u \centerdot (v \centerdot a)$, for $a \in A \cup \{1\}$. We proceed by induction on the length of v. For $|v| \leq 1$ the equality follows directly from (1) and (3). Then, suppose that $|v| > 1$ and let $b \in B$ and $v' \in B^+$ be such that $v = v'b$. As $1 \leq |v'| < |v|$ and $b \centerdot a \in A \cup \{1\}$, by (3) and the induction hypothesis, we have

$$(uv) \centerdot a = (uv'b) \centerdot a = (uv') \centerdot (b \centerdot a) = u \centerdot (v' \centerdot (b \centerdot a)) = u \centerdot ((v'b) \centerdot a) = u \centerdot (v \centerdot a).$$

Now, we proceed by induction on the length of s. So, suppose that $|s| > 1$ and let $a \in A$ and $s' \in A^+$ be such that $s = as'$. Then, as $1 \leq |s'| < |s|$, we have

$$\begin{array}{rcll} (uv) \centerdot s & = & (uv) \centerdot (as') & \\ & = & ((uv) \centerdot a)((uv)^a \centerdot s') & \text{(by (5))} \\ & = & (u \centerdot (v \centerdot a))((u^{v \centerdot a} v^a) \centerdot s') & \text{(by the case } |s| = 1 \text{ and (SPR))} \\ & = & (u \centerdot (v \centerdot a))(u^{v \centerdot a} \centerdot (v^a \centerdot s')) & \text{(by the induction hypothesis)} \\ & = & u \centerdot ((v \centerdot a)(v^a \centerdot s')) & \text{(by (SCR))} \\ & = & u \centerdot (v \centerdot (as')) & \text{(by (5))} \\ & = & u \centerdot (v \centerdot s) \ . & \end{array}$$

(b) If $s = 1$ or $r = 1$, the equality follows immediately from Lemma 1.1(b). Then, admit that $s, r \in A^+$. Now, we proceed by induction on the length of s. If $|s| = 1$ the equality follows from (6). So, let $s = as'$, with $a \in A$ and $s' \in A^+$. Since $1 \leq |s'| < |s|$, we have

$$u^{sr} = u^{as'r} = (u^a)^{s'r} = ((u^a)^{s'})^r = (u^{as'})^r = (u^s)^r,$$

by applying (6) in the second and fourth expressions and the induction hypothesis in the third expression, as required. ∎

Now, we have:

Proposition 1.4 *The mappings δ and φ are the unique left action of B^* on A^* and right action of A^* on B^*, respectively, extending the given actions of and on the letters.*

Proof. It follows immediately from Lemmas 1.1-1.3 that the operations defined by (1)-(6) are a left action of B^* on A^* and a right action of A^* on B^*. It remains to show the unicity.

Let δ' and φ' be a left action of B^* on A^* and a right action of A^* on B^*, respectively, such that $(a)\delta'_b = (a)\delta_b$ and $(b)\varphi'_a = (b)\varphi_a$, for $a \in A$ and $b \in B$. Let $s \in A^*$ and $u \in B^*$. We aim to show that $(s)\delta'_u = (s)\delta_u$ and $(u)\varphi'_s = (u)\varphi_s$. If $s = 1$ or $u = 1$, then both these equalities are valid, by definition. Thus, admit that $s \in A^+$ and $u \in B^+$.

We proceed by induction on the length of s.

Suppose that $|s| = 1$. Then, by induction on the length of u, we show that $(a)\delta'_u = (a)\delta_u$ and $(u)\varphi'_a = (u)\varphi_a$, for $a \in A$ and $u \in B^+$. If $|u| = 1$ we have precisely our main hypothesis. So, take $u = vb$, with $b \in B$ and $v \in B^+$. Also, let $a' = (a)\delta'_b = (a)\delta_b$. Notice that $a' \in A \cup \{1\}$. Then

$$(a)\delta'_u = (a)\delta'_{vb} = ((a)\delta'_b)\delta'_v = (a')\delta'_v = (a')\delta_v = ((a)\delta_b)\delta_v = (a)\delta_{vb} = (a)\delta_u$$

and

$$\begin{aligned}(u)\varphi'_a &= (vb)\varphi'_a = (v)\varphi'_{(a)\delta'_b}(b)\varphi'_a = (v)\varphi'_{a'}(b)\varphi'_a \\ &= (v)\varphi_{a'}(b)\varphi_a = (v)\varphi_{(a)\delta_b}(b)\varphi_a = (vb)\varphi_a = (u)\varphi_a,\end{aligned}$$

by applying in both chain of equalities the induction hypothesis in the fourth expression.

Now, by induction hypothesis, we assume that $(r)\delta'_u = (r)\delta_u$ and $(u)\varphi'_r = (u)\varphi_r$, for $u \in B^+$ and $r \in A^+$ such that $1 \le |r| < |s|$. So, take $u \in B^+$ and $s = ar$ with $a \in A$ and $r \in A^+$. Then

$$(s)\delta'_u = (ar)\delta'_u = (a)\delta'_u(r)\delta'_{(u)\varphi'_a} = (a)\delta'_u(r)\delta'_{(u)\varphi_a} = (a)\delta_u(r)\delta_{(u)\varphi_a} = (ar)\delta_u = (s)\delta_u$$

and

$$(u)\varphi'_s = (u)\varphi'_{ar} = ((u)\varphi'_a)\varphi'_r = ((u)\varphi_a)\varphi'_r = ((u)\varphi_a)\varphi_r = (u)\varphi_{ar} = (u)\varphi_s,$$

as required. ∎

Dually, suppose we have defined actions of and on the letters satisfying

$$b \centerdot a \in A^*, \quad 1 \centerdot a = a, \quad b \centerdot 1 = 1, \quad 1 \centerdot 1 = 1 \tag{7}$$

and

$$b^a \in B \cup \{1\}, \quad b^1 = b, \quad 1^a = 1, \quad 1^1 = 1\ , \tag{8}$$

for $a \in A$ and $b \in B$. Then first, inductively on the length of $s \in A^+$, define

$$b^{as} = (b^a)^s \tag{9}$$

and

$$b \centerdot (as) = (b \centerdot a)(b^a \centerdot s) \tag{10}$$

for $a \in A$ and $b \in B \cup \{1\}$, and secondly, inductively on the length of $u \in B^+$, define

$$(ub)^s = u^{b \centerdot s} b^s \tag{11}$$

and

$$(ub) \centerdot s = u \centerdot (b \centerdot s), \tag{12}$$

for $s \in A^*$ and $b \in B$.

Similarly, we have:

Proposition 1.5 *The mappings defined by (7)-(12) are the unique left action of B^* on A^* and right action of A^* on B^* extending the given actions of and on the letters.* ∎

Naturally, if we have both (1) and (8), then the actions defined by (3)-(6) and by (9)-(12) coincide.

Observe also that, as particular cases of both the propositions 1.4 and 1.5, we obtain constructions of semidirect products $A^* * B^*$ and of reverse semidirect products $B^* *_r A^*$ by just defining the actions on the letters (without any restriction for reverse semidirect products by Proposition 1.4 and for semidirect products by Proposition 1.5; and with the restriction (1) for semidirect products by Proposition 1.4 and the restriction (8) for reverse semidirect products by Proposition 1.5)..

Let δ be a left action of B^* on A^* and let φ be a right action of A^* on B^*.

We say that δ (resp., φ) *preserves letters* if it satisfies (1) (resp., (8)), i.e. the action of a letter on a letter is a letter or the empty word.

Let R be a set of relations on A^* and let U be a set of relations on B^*. Let S and T be the monoids defined by the presentations $\langle A \mid R\rangle$ and $\langle B \mid U\rangle$, respectively. We assume that these presentations are letter-irredundant.

We say that the action δ (resp., φ) *preserves* the presentations $\langle A \mid R\rangle$ and $\langle B \mid U\rangle$ if

$$b \cdot s = b \cdot r \text{ in } S \text{ (resp., } b^s = b^r \text{ in } T),$$

for all $(s = r) \in R$ and $b \in B$, and

$$u \cdot a = v \cdot a \text{ in } S \text{ (resp. } u^a = v^a \text{ in } T),$$

for all $(u = v) \in U$ and $a \in A$.

Now, we fix a left action of B^* on A^* and a right action of A^* on B^* that preserve letters and preserve the letter-irredundant presentations $\langle A \mid R\rangle$ and $\langle B \mid U\rangle$. We aim to show that these actions on free monoids induce a bilateral semidirect product $S \bowtie T$. First, we prove the following lemma.

Lemma 1.6 *Within the above conditions, let $z \in A^*$ and $w_1, w_2 \in B^*$ be such that $w_1 = w_2$ in T. Then, we have $w_1 \cdot z = w_2 \cdot z$ in S and $w_1^z = w_2^z$ in T.*

Proof. Clearly, for $z = 1$ the lemma follows by definition. Thus, we assume that $z \in A^+$ and proceed by induction on the length of z.

First, notice that, as the left action preserves letters and the presentation $\langle A \mid R\rangle$ is letter-irredundant, we have $u \cdot a = v \cdot a$ in S if and only if $u \cdot a \equiv v \cdot a$, for $u, v \in B^*$ and $a \in A \cup \{1\}$..

Let $a \in A$. We aim to prove that $w_1 \cdot a = w_2 \cdot a$ in S (i.e. $w_1 \cdot a \equiv w_2 \cdot a$) and ${w_1}^a = {w_2}^a$ in T. It is a routine matter to show that it suffices to just consider elementary transitions. Therefore, without loss of generality, let $w_1 \equiv guh$ and $w_2 \equiv gvh$, with $g, h \in B^*$ and $(u = v) \in U$. Let $a' = h \cdot a \in A \cup \{1\}$. Then $u \cdot a' = v \cdot a'$ in S and so $u \cdot a' \equiv v \cdot a'$, whence $g \cdot (u \cdot a') \equiv g \cdot (v \cdot a')$, i.e. $w_1 \cdot a \equiv w_2 \cdot a$. On the other hand, $g^{u \cdot a'} \equiv g^{v \cdot a'}$ and $u^{a'} = v^{a'}$ in T, whence

$$w_1^a \equiv g^{u\cdot(h\cdot a)} u^{h\cdot a} h^a \equiv g^{u\cdot a'} u^{a'} h^a = g^{v\cdot a'} v^{a'} h^a \equiv g^{v\cdot(h\cdot a)} v^{h\cdot a} h^a \equiv w_2^a.$$

Now, let $z = az'$, with $a \in A$ and $z' \in A^+$. As $w_1^a = w_2^a$ in T and $1 \leq |z'| < |z|$, by the induction hypothesis, we have $w_1^a \cdot z' = w_2^a \cdot z'$ in S and $(w_1^a)^{z'} = (w_2^a)^{z'}$ in T. Hence

$$w_1^z \equiv w_1^{az'} \equiv (w_1^a)^{z'} = (w_2^a)^{z'} \equiv w_2^{az'} \equiv w_2^z$$

and, as also $w_1 \cdot a = w_2 \cdot a$ in S,

$$w_1 \cdot z \equiv w_1 \cdot (az') \equiv (w_1 \cdot a)(w_1^a \cdot z') = (w_2 \cdot a)(w_2^a \cdot z') \equiv w_2 \cdot (az') \equiv w_2 \cdot z,$$

as required. ∎

Similarly, by duality, we have:

Lemma 1.7 *Within the above conditions, let $z_1, z_2 \in A^*$ and $w \in B^*$ be such that $z_1 = z_2$ in S. Then, we have $w \cdot z_1 = w \cdot z_2$ in S and $w^{z_1} = w^{z_2}$ in T.* ∎

Clearly, by combining the previous two lemmas, we have $w_1 \cdot z_1 = w_2 \cdot z_2$ in S and ${w_1}^{z_1} = {w_2}^{z_2}$ in T, for all words $z_1, z_2 \in A^*$ and $w_1, w_2 \in B^*$ such that $z_1 = z_2$ in S and $w_1 = w_2$ in T, which proves the next result, announced above:

Theorem 1.8 *If a left action of B^* on A^* and a right action of A^* on B^* preserve letters and preserve the letter-irredundant presentations $\langle A \mid R \rangle$ and $\langle B \mid U \rangle$ then they induce a left action of T on S and a right action of S on T.* ∎

The decomposition

Now, let M be a monoid and let S and T be two submonoids of M. Let A and B be sets of generators of S and T, respectively. Consider a left action of T on S and a right action of S on T.

We say that the left (resp., right) action of T on S (resp., S on T) *preserves* A (resp., B) if $b \cdot a \in A \cup \{1\}$ (resp., $b^a \in B \cup \{1\}$), for $a \in A$ and $b \in B$. Notice that, if the left action preserves A then, clearly, $u \cdot a \in A \cup \{1\}$, for $a \in A$ and $u \in T$. Naturally, a similar property holds if the right action preserves B.

Lemma 1.9 *Within the above conditions, suppose that $ba = (b \cdot a)b^a$ in M, for $a \in A$ and $b \in B$. If either the left action preserves A or the right action preserves B, then $us = (u \cdot s)u^s$ in M, for $s \in S$ and $u \in T$.*

Proof. We prove the lemma by admitting that the left action preserves A. The other case is similar.

Let $s \in S$ and $u \in T$. First, we proceed by induction on the length of s (with respect to A). If $|s| = 0$ then the equality follows immediately by definition. We need to prove also the case $|s| = 1$, i.e. $ua = (u \cdot a)u^a$, for $a \in A$ and $u \in T$.

If $|u| = 0$ or $|u| = 1$ this equality follows by definition or by the main hypothesis, respectively. So, proceeding by induction on the length of u (with respect to B), we admit the equality for $1 \le |u| < k$. Let $u \in T$ be such that $|u| = k$. Then $u = bv$, for some $b \in B$ and some $v \in T$ with length $k-1$. Let $a' = v \cdot a \in A \cup \{1\}$. Hence

$$\begin{aligned} ua &= b(va) = b((v \cdot a)v^a) = (ba')v^a = (b \cdot a')b^{a'}v^a = (b \cdot (v \cdot a))b^{v \cdot a}v^a \\ &= ((bv) \cdot a)(bv)^a = (u \cdot a)u^a, \end{aligned}$$

by applying the induction hypothesis in the second expression and (SPR) in the sixth expression.

Now, by induction hypothesis, we assume that $us = (u \cdot s)u^s$, for $u \in T$ and $s \in S$ such that $1 \le |s| < n$. Let s be an element of S with length n and let $u \in T$. Then $s = ra$, for some $a \in A$ and some $r \in S$ with length $n-1$. Let $v = u^r \in T$. Thus, we have

$$\begin{aligned} us &= (ur)a = ((u \cdot r)u^r)a = (u \cdot r)(va) = (u \cdot r)(v \cdot a)v^a = (u \cdot r)(u^r \cdot a)(u^r)^a \\ &= (u \cdot (ra))u^{ra} = (u \cdot s)u^s, \end{aligned}$$

by applying the induction hypothesis in the second expression, the case $|s| = 1$ in the fourth expression and (SCR) in the sixth expression, as required. ∎

Theorem 1.10 *Let M be a monoid and let S and T be two submonoids of M generated by A and B, respectively. Let $S \bowtie T$ be a bilateral semidirect product of S and T such that either the left action preserves A or the right action preserves B. If $A \cup B$ generates M and $ba = (b \cdot a)b^a$ in M, for $a \in A$ and $b \in B$, then M is a homomorphic image of $S \bowtie T$.*

Proof. We prove that the mapping

$$\begin{array}{rccc} \mu: & S \bowtie T & \longrightarrow & M \\ & (s,u) & \longmapsto & su \end{array}$$

is a surjective homomorphism.

First, we show that μ is a homomorphism. Let $(s,u),(r,v) \in S \bowtie T$. Then

$$(s,u)\mu(r,v)\mu = surv = s(u \centerdot r)u^r v = (s(u \centerdot r), u^r v)\mu = ((s,u)(r,v))\mu \ ,$$

by applying the Lemma 1.9 in the second expression.

Now, we show that μ is onto. Let $x \in M$. As $A \cup B$ generates M, we may write $x = s_1 u_1 \cdots s_k u_k$, for some $s_1, \ldots, s_k \in S$ and $u_1, \ldots, u_k \in T$. Also, we may assume that k is the least positive integer for which such a decomposition exists. Suppose that $k \geq 2$. Then, by applying the Lemma 1.9, we have

$$x = s_1 u_1 \cdots s_{k-1}(u_{k-1} s_k) u_k = s_1 u_1 \cdots s_{k-1}(u_{k-1} \centerdot s_k) u_{k-1}^{s_k} u_k \ ,$$

which contradicts the minimality of k, as $s_{k-1}(u_{k-1} \centerdot s_k) \in S$ and $u_{k-1}^{s_k} u_k \in T$. Therefore $k = 1$, as required. ■

As an immediate consequence, for semidirect products, we have:

Corollary 1.11 *Let M be a monoid and let S and T be two submonoids of M generated by A and B, respectively. Let $S * T$ (resp., $S *_r T$) be a (resp., reverse) semidirect product of S and T. If $A \cup B$ generates M and $ba = (b \centerdot a)b$ (resp., $ab = ba^b$) in M, for $a \in A$ and $b \in B$, then M is a homomorphic image of $S * T$ (resp., $S *_r T$).* ■

2 Applications

Let $n \in \mathbb{N}$. In this section, we construct bilateral semidirect decompositions of the monoids $\mathcal{O}_n$, $\mathcal{OD}_n$, $\mathcal{OP}_n$ and $\mathcal{OR}_n$, by using the technique presented in the last section.

On the monoid $\mathcal{O}_n$

Our first application is a new proof of the Kunze [14] bilateral semidirect decomposition of the monoid $\mathcal{O}_n$ in terms of its submonoids $\mathcal{O}_n^- = \{s \in \mathcal{O}_n \mid (x)s \leq x, \text{ for } x \in X_n\}$ and $\mathcal{O}_n^+ = \{s \in \mathcal{O}_n \mid x \leq (x)s, \text{ for } x \in X_n\}$.

First, notice that $\mathcal{O}_n^-$ and $\mathcal{O}_n^+$ are isomorphic monoids: the mapping from $\mathcal{O}_n^-$ onto $\mathcal{O}_n^+$, which maps each transformation $s \in \mathcal{O}_n^-$ in the transformation $\bar{s} \in \mathcal{O}_n^+$ defined by $(x)\bar{s} = n+1-(n+1-x)s$, for $x \in X_n$, is an isomorphism of monoids.

For $i \in \{1, \ldots, n-1\}$, let

$$a_i = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & n \\ 1 & 2 & \cdots & i & i & i+2 & \cdots & n \end{pmatrix}$$

and

$$b_i = \bar{a}_{n-i} = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i+1 & i+1 & \cdots & n \end{pmatrix} .$$

Let $A = \{a_1, \ldots, a_{n-1}\}$ and $B = \{b_1, \ldots, b_{n-1}\}$. Then A and B are generating sets of $\mathcal{O}_n^-$ and $\mathcal{O}_n^+$, respectively. Furthermore, being R^- the set of the relations

- $a_i^2 = a_i$, for $1 \leq i \leq n-1$,
- $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} = a_{i+1} a_i$, for $1 \leq i \leq n-2$, and

- $a_ia_j = a_ja_i$, for $1 \leq i, j \leq n-1$ and $|i-j| \geq 2$,

and R^+ the set of the relations

- $b_i^2 = b_i$, for $1 \leq i \leq n-1$,
- $b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1} = b_ib_{i+1}$, for $1 \leq i \leq n-2$, and
- $b_ib_j = b_jb_i$, for $1 \leq i, j \leq n-1$ and $|i-j| \geq 2$,

the monoids $\mathcal{O}_n^-$ and $\mathcal{O}_n^+$ are defined by the presentations $\langle A \mid R^- \rangle$ and $\langle B \mid R^+ \rangle$, respectively. On the other hand, the monoid $\mathcal{O}_n$ is generated by $A \cup B$ and, being R the set of the relations

- $a_ib_i = b_ia_{i-1}$, for $2 \leq i \leq n-1$,
- $b_ia_i = a_ib_{i+1}$, for $1 \leq i \leq n-2$,
- $a_ib_i = b_i$, for $1 \leq i \leq n-1$,
- $b_ia_i = a_i$, for $1 \leq i \leq n-1$,
- $b_ja_i = a_ib_j$, for $1 \leq i, j \leq n-1$ and $j \notin \{i, i+1\}$,
- $a_{n-1}a_{n-2}a_{n-1} = a_{n-1}a_{n-2}$, and
- $b_1b_2b_1 = b_1b_2$,

is defined by the presentation $\langle A \cup B \mid R \rangle$. This presentation was given by Aĭzenštat in 1962 [1]. See also [21, 6].

Now, by applying the Proposition 1.4 (or 1.5), consider the left action δ of B^* on A^* and the right action φ of A^* on B^* that extend the following actions of and on the letters:

$$b_j \centerdot a_i = \begin{cases} 1 & \text{if } j = i+1 \\ a_i & \text{otherwise} \end{cases}$$

and

$$b_j^{a_i} = \begin{cases} 1 & \text{if } j = i \\ b_j & \text{otherwise} \end{cases},$$

for $1 \leq i, j \leq n-1$.

Notice that both δ and φ preserve letters. On the other hand, clearly, both the presentations $\langle A \mid R^- \rangle$ and $\langle B \mid R^+ \rangle$ are letter-irredundant. Moreover, we have:

Lemma 2.1 *The actions δ and φ preserve the presentations $\langle A \mid R^- \rangle$ and $\langle B \mid R^+ \rangle$.*

Proof. We have to prove the following relations:

(i) For $1 \leq j \leq n-1$,

$$\begin{cases} b_j \centerdot a_i^2 = b_j \centerdot a_i, & \text{for } 1 \leq i \leq n-1, \\ b_j \centerdot (a_ia_{i+1}a_i) = b_j \centerdot (a_{i+1}a_ia_{i+1}) = b_j \centerdot (a_{i+1}a_i), & \text{for } 1 \leq i \leq n-2, \\ b_j \centerdot (a_ia_k) = b_j \centerdot (a_ka_i), & \text{for } 1 \leq i, k \leq n-1 \text{ and } |i-k| \geq 2, \\ & \\ b_j^{a_i^2} = b_j^{a_i}, & \text{for } 1 \leq i \leq n-1, \\ b_j^{a_ia_{i+1}a_i} = b_j^{a_{i+1}a_ia_{i+1}} = b_j^{a_{i+1}a_i}, & \text{for } 1 \leq i \leq n-2, \\ b_j^{a_ia_k} = b_j^{a_ka_i}, & \text{for } 1 \leq i, k \leq n-1 \text{ and } |i-k| \geq 2; \end{cases}$$

(ii) And, for $1 \le i \le n-1$,

$$\begin{cases} b_j^2 \cdot a_i = b_j \cdot a_i, & \text{for } 1 \le j \le n-1, \\ (b_j b_{j+1} b_j) \cdot a_i = (b_{j+1} b_j b_{j+1}) \cdot a_i = (b_{j+1} b_j) \cdot a_i, & \text{for } 1 \le j \le n-2, \\ (b_j b_k) \cdot a_i = (b_k b_j) \cdot a_i, & \text{for } 1 \le j, k \le n-1 \text{ and } |j-k| \ge 2, \\ & \\ (b_j^2)^{a_i} = b_j^{a_i}, & \text{for } 1 \le j \le n-1, \\ (b_j b_{j+1} b_j)^{a_i} = (b_{j+1} b_j b_{j+1})^{a_i} = (b_{j+1} b_j)^{a_i}, & \text{for } 1 \le j \le n-2, \\ (b_j b_k)^{a_i} = (b_k b_j)^{a_i}, & \text{for } 1 \le j, k \le n-1 \text{ and } |j-k| \ge 2. \end{cases}$$

We just present the proof of (i). The proof of (ii) is similar.
Let $1 \le j \le n-1$. Then: for $1 \le i \le n-1$,

$$\begin{aligned} b_j \cdot a_i^2 &= (b_j \cdot a_i)(b_j^{a_i} \cdot a_i) = \begin{cases} 1(b_j \cdot a_i) & \text{if } j = i+1 \\ a_i(1 \cdot a_i) & \text{if } j = i \\ a_i(b_j \cdot a_i) & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } j = i+1 \\ a_i^2 & \text{if } j = i \\ a_i^2 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } j = i+1 \\ a_i & \text{otherwise} \end{cases} = b_j \cdot a_i \ ; \end{aligned}$$

for $1 \le i \le n-2$,

$$\begin{aligned} & b_j \cdot (a_i a_{i+1} a_i) = (b_j \cdot a_i)(b_j^{a_i} \cdot (a_{i+1} a_i)) = \begin{cases} 1(b_j \cdot (a_{i+1} a_i)) & \text{if } j = i+1 \\ a_i(a_{i+1} a_i) & \text{if } j = i \\ a_i(b_j \cdot (a_{i+1} a_i)) & \text{otherwise} \end{cases} \\ &= \begin{cases} (b_j \cdot a_{i+1})(b_j^{a_{i+1}} \cdot a_i) & \text{if } j = i+1 \\ a_i a_{i+1} a_i & \text{if } j = i \\ a_i(b_j \cdot a_{i+1})(b_j^{a_{i+1}} \cdot a_i) & \text{otherwise} \end{cases} = \begin{cases} a_{i+1}(1 \cdot a_i) & \text{if } j = i+1 \\ a_i a_{i+1} a_i & \text{if } j = i \\ a_i(b_j \cdot a_{i+1})(b_j \cdot a_i) & \text{otherwise} \end{cases} \\ &= \begin{cases} a_{i+1} a_i & \text{if } j = i+1 \\ a_i a_{i+1} a_i & \text{if } j = i \\ a_i 1 a_i & \text{if } j = i+2 \\ a_i a_{i+1} a_i & \text{otherwise} \end{cases} = \begin{cases} a_i & \text{if } j = i+2 \\ a_{i+1} a_i & \text{otherwise} \end{cases} \\ &= (b_j \cdot a_{i+1})(b_j^{a_{i+1}} \cdot a_i) = b_j \cdot (a_{i+1} a_i) \end{aligned}$$

and, similarly, $b_j \cdot (a_{i+1} a_i a_{i+1}) = b_j \cdot (a_{i+1} a_i)$; for $1 \le i, k \le n-1$ and $|i-k| \ge 2$,

$$\begin{aligned} b_j \cdot (a_i a_k) &= (b_j \cdot a_i)(b_j^{a_i} \cdot a_k) = \begin{cases} 1(b_j \cdot a_k) & \text{if } j = i+1 \\ a_i a_k & \text{if } j = i \\ a_i(b_j \cdot a_k) & \text{otherwise} \end{cases} = \begin{cases} a_k & \text{if } j = i+1 \\ a_i & \text{if } j = k+1 \\ a_i a_k & \text{otherwise} \end{cases} \\ &= \begin{cases} a_k & \text{if } j = i+1 \\ a_i & \text{if } j = k+1 \\ a_k a_i & \text{otherwise} \end{cases} = b_j \cdot (a_k a_i) \ ; \end{aligned}$$

for $1 \le i \le n-1$,

$$b_j^{a_i^2} = (b_j^{a_i})^{a_i} = \begin{cases} 1 & \text{if } j = i \\ b_j & \text{otherwise} \end{cases} = b_j^{a_i} \ ;$$

for $1 \le i \le n-2$,

$$b_j^{a_i a_{i+1} a_i} = ((b_j^{a_i})^{a_{i+1}})^{a_i} = \begin{cases} 1 & \text{if } j = i \text{ or } j = i+1 \\ b_j & \text{otherwise} \end{cases} = (b_j^{a_{i+1}})^{a_i} = b_j^{a_{i+1} a_i}$$

and, similarly, $b_j^{a_{i+1} a_i a_{i+1}} = b_j^{a_{i+1} a_i}$; finally, for $1 \le i, k \le n-1$ and $|i-k| \ge 2$,

$$b_j^{a_i a_k} = (b_j^{a_i})^{a_k} = \begin{cases} 1 & \text{if } j = i \text{ or } j = k \\ b_j & \text{otherwise} \end{cases} = (b_j^{a_k})^{a_i} = b_j^{a_k a_i},$$

as required. ∎

Now, accordingly with the Theorem 1.8, we have a well defined bilateral semidirect product $\mathcal{O}_n^- \bowtie \mathcal{O}_n^+$ induced by the actions δ and φ. Furthermore:

Lemma 2.2 *One has* $b_j a_i = (b_j \centerdot a_i) b_j^{a_i}$ *in* $\mathcal{O}_n$, *for* $1 \le i, j \le n-1$.

Proof. Let $1 \le i, j \le n-1$. If $j = i$ then $b_i a_i = a_i = a_i 1 = (b_i \centerdot a_i) b_i^{a_i}$ and if $j = i+1$ then $b_{i+1} a_i = a_{i+1} b_{i+1} = b_{i+1} = 1 b_{i+1} = (b_{i+1} \centerdot a_i) b_{i+1}^{a_i}$. Otherwise, $b_j a_i = a_i b_j = (b_j \centerdot a_i) b_j^{a_i}$, as required. ∎

As the left action in $\mathcal{O}_n^- \bowtie \mathcal{O}_n^+$ preserves A (in fact, the right action also preserves B) and $A \cup B$ generates $\mathcal{O}_n$, then all the hypothesis of the Theorem 1.10 are satisfied and so we have:

Theorem 2.3 *The monoid* $\mathcal{O}_n$ *is a homomorphic image of* $\mathcal{O}_n^- \bowtie \mathcal{O}_n^+$. ∎

Let O be the pseudovariety of monoids generated by $\{\mathcal{O}_n \mid n \in \mathbb{N}\}$ and let J be the pseudovariety of monoids generated by $\{\mathcal{O}_n^+ \mid n \in \mathbb{N}\}$ (or by $\{\mathcal{O}_n^- \mid n \in \mathbb{N}\}$). Notice that it is well-known that J is the pseudovariety of $\mathcal{J}$-trivial monoids, which are the syntactic monoids of the piecewise testable languages (see e.g. [18]). As an immediate consequence of the last result, we obtain:

Corollary 2.4 $\mathsf{O} \subset \mathsf{J} \bowtie \mathsf{J}$. ∎

Unfortunately, this inclusion is strict. In fact, being R the pseudovariety of all $\mathcal{R}$-trivial monoids, Higgins [9] showed that $\mathsf{R} \not\subseteq \mathsf{O}$ and, on the other hand, the equality $\mathsf{J} * \mathsf{J} = \mathsf{J} * \mathsf{R}$ is a particular instance of a result of Almeida and Weil [2, Corollary 8.6]. Hence, as $\mathsf{R} \subseteq \mathsf{J} * \mathsf{R} = \mathsf{J} * \mathsf{J} \subseteq \mathsf{J} \bowtie \mathsf{J}$, we have $\mathsf{J} \bowtie \mathsf{J} \not\subseteq \mathsf{O}$.

On the monoid $\mathcal{OD}_n$

The monoid $\mathcal{OD}_n$ was considered by the first author together with Gomes and Jesus in [7]. They have showed that $\mathcal{OD}_n$ is generated by its submonoid $\mathcal{O}_n$ together with the reflexion permutation

$$h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}$$

and, moreover, being A, B and R as above, by adding to R the relations

- $h^2 = 1$,
- $h a_i = b_{n-i} h$, for $1 \le i \le n-1$, and
- $a_{n-1} a_{n-2} \cdots a_1 h = a_{n-1} a_{n-2} \cdots a_1 b_1 b_2 \cdots b_{n-1}$,

we obtain a presentation of $\mathcal{OD}_n$ in terms of the generating set $A \cup B \cup \{h\}$. Notice that, as $b_1 b_2 \cdots b_{n-1} = a_{n-1} a_{n-2} \cdots a_1 b_1 b_2 \cdots b_{n-1}$ in $\mathcal{O}_n$, we may replace the relation $a_{n-1} a_{n-2} \cdots a_1 h = a_{n-1} a_{n-2} \cdots a_1 b_1 b_2 \cdots b_{n-1}$ by the simpler relation $a_{n-1} a_{n-2} \cdots a_1 h = b_1 b_2 \cdots b_{n-1}$.

Let $\mathcal{C}_2$ be a cyclic group of order two. Then, $\mathcal{C}_2$ is defined by the presentation $\langle h \mid h^2 = 1 \rangle$ and we may take $\mathcal{C}_2$ as being the submonoid of $\mathcal{OD}_n$ generated by the transformation h.

Now, we aim to construct a semidirect decomposition and a reverse semidirect decomposition of $\mathcal{OD}_n$ in terms of its submonoids $\mathcal{C}_2$ and $\mathcal{O}_n$.

First, by applying the Proposition 1.4 (or 1.5), and considering a trivial right action, let δ_1 be the left action of $\{h\}^*$ on $(A \cup B)^*$ that extends the following action of and on the letters:

$$h \centerdot a_i = b_{n-i} \text{ and } h \centerdot b_i = a_{n-i},$$

for $1 \le i \le n-1$.

Notice that δ_1 preserves letters (as well as the trivial right action) and the presentations $\langle A \cup B \mid R \rangle$ (of $\mathcal{O}_n$) and $\langle h \mid h^2 = 1 \rangle$ (of $\mathcal{C}_2$) are letter-irredundant. Moreover, it is routine matter to show that:

Lemma 2.5 *The action* δ_1 *preserves the presentations* $\langle A \cup B \mid R \rangle$ *and* $\langle h \mid h^2 = 1 \rangle$. ∎

Hence, by the Theorem 1.8 (considering a trivial right action), we have a well defined semidirect product $\mathcal{O}_n * \mathcal{C}_2$ induced by the action δ_1. On the other hand, we have $ha_i = b_{n-i}h$ in $\mathcal{OD}_n$, for $1 \le i \le n-1$, and from these relations and $h^2 = 1$, it follows also $hb_i = a_{n-i}h$ in $\mathcal{OD}_n$, for $1 \le i \le n-1$. Thus, we have:

Lemma 2.6 *For $1 \le i \le n-1$, $ha_i = (h \centerdot a_i)h$ and $hb_i = (h \centerdot b_i)h$ in $\mathcal{OD}_n$.* ∎

And, by the Corollary 1.11, it follows:

Theorem 2.7 *The monoid $\mathcal{OD}_n$ is a homomorphic image of $\mathcal{O}_n * \mathcal{C}_2$.* ∎

Let OD be the pseudovariety of monoids generated by $\{\mathcal{OD}_n \mid n \in \mathbb{N}\}$. Let Ab_2 be the pseudovariety of monoids generated by $\mathcal{C}_2$ (a pseudovariety of abelian groups). Then:

Corollary 2.8 $\mathsf{OD} \subseteq \mathsf{O} * \mathsf{Ab}_2$. ∎

Now, as $\mathcal{C}_2$ is a commutative monoid, the left action of $\mathcal{C}_2$ on $\mathcal{O}_n$ may also be considered as a right action (which coincides with the one induced by the right action of $\{h\}^*$ on $(A \cup B)^*$ that extends the following action of and on the letters: $a_i^h = b_{n-i}$ and $b_i^h = a_{n-i}$, for $1 \le i \le n-1$) and so we also have a well defined reverse semidirect product $\mathcal{O}_n *_r \mathcal{C}_2$. As, clearly, $a_ih = hb_{n-i}$ and $b_ih = ha_{n-i}$ in $\mathcal{OD}_n$, i.e. $a_ih = ha_i^h$ and $b_ih = hb_i^h$ in $\mathcal{OD}_n$, for $1 \le i \le n-1$, again by Corollary 1.11, we have:

Theorem 2.9 *The monoid $\mathcal{OD}_n$ is a homomorphic image of $\mathcal{O}_n *_r \mathcal{C}_2$.* ∎

It follows immediately:

Corollary 2.10 $\mathsf{OD} \subseteq \mathsf{O} *_r \mathsf{Ab}_2$. ∎

On the monoid $\mathcal{OP}_n$

A presentation for the monoid $\mathcal{OP}_n$ was given by Catarino in [4]: being $A \cup B$ the set of generators of $\mathcal{O}_n$ considered above and g the n-cycle permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix},$$

then $A \cup B \cup \{g\}$ generates $\mathcal{OP}_n$ and, by adding the relations

- $g^n = 1$,
- $a_ig = ga_{i+1}$, for $1 \le i \le n-2$,
- $b_ig = gb_{i+1}$, for $1 \le i \le n-2$,
- $a_{n-1}g = b_{n-1}b_{n-2}\cdots b_1$,
- $b_{n-1}g = g^2a_1a_2\cdots a_{n-1}$, and
- $ga_{n-1}a_{n-2}\cdots a_1 = a_{n-1}a_{n-2}\cdots a_1$,

to any set of defining relations of $\mathcal{O}_n$ in terms of the generating set $A \cup B$, we obtain a presentation for $\mathcal{OP}_n$ with $A \cup B \cup \{g\}$ as set of generators. See also [17, 5, 3].

Let $\mathcal{C}_n$ be a cyclic group of order n. Clearly, $\mathcal{C}_n$ is defined by the presentation $\langle g \mid g^n = 1\rangle$ and may be considered as the submonoid of $\mathcal{OP}_n$ generated by the n-cycle g.

Our objective is to construct a bilateral semidirect decomposition of $\mathcal{OP}_n$ in terms of its submonoids $\mathcal{C}_n$ and $\mathcal{O}_n$.

By convenience, we consider the (obviously letter-irredundant) presentation

$$\langle C \mid N\rangle = \langle g_1, \ldots, g_{n-1} \mid g_1^n = 1, g_1^k = g_k, 2 \le k \le n-1\rangle$$

of $\mathcal{C}_n$ (with $g_1 = g$, as elements of $\mathcal{C}_n$) and, being a_n and b_n two symbols not in $A \cup B$ and R as above, the presentation

$$\langle X \mid R'\rangle = \langle A \cup B \cup \{a_n, b_n\} \mid R, a_1 a_2 \cdots a_{n-1} = a_n, b_{n-1} b_{n-2} \cdots b_1 = b_n\rangle$$

of $\mathcal{O}_n$. Notice that, as elements of $\mathcal{O}_n$, we have

$$a_n = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 2 & \cdots & n-1 \end{pmatrix} \text{ and } b_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & n \end{pmatrix},$$

whence $\langle X \mid R'\rangle$ is also letter-irredundant.

Now, consider the left action δ_2 of X^* on C^* and the right action φ_2 of C^* on X^* that extend, by Proposition 1.4 (or 1.5), the following actions of and on the letters:

$$a_i \cdot g_k = \begin{cases} 1 & \text{if } k = 1 \text{ and } i \in \{n-1, n\} \\ g_{k-1} & \text{if } k \ge 2 \text{ and } i \in \{n-k, n\} \\ g_k & \text{otherwise ,} \end{cases} \qquad b_i \cdot g_k = \begin{cases} 1 & \text{if } k = n-1 \text{ and } i \in \{1, n\} \\ g_{k+1} & \text{if } k < n-1 \text{ and } i \in \{n-k, n\} \\ g_k & \text{otherwise ,} \end{cases}$$

$$a_i^{g_k} = \begin{cases} a_{i+k} & \text{if } i < n-k \\ b_n & \text{if } i = n-k \\ a_{i+k-n} & \text{if } n-k+1 \le i \le n-1 \\ b_k & \text{if } i = n \text{ ,} \end{cases} \qquad b_i^{g_k} = \begin{cases} b_{i+k} & \text{if } i < n-k \\ a_n & \text{if } i = n-k \\ b_{i+k-n} & \text{if } n-k+1 \le i \le n-1 \\ a_k & \text{if } i = n \text{ ,} \end{cases}$$

for $1 \le i \le n$ and $1 \le k \le n-1$.

Notice that both δ_2 and φ_2 preserve letters. Moreover, we have:

Lemma 2.11 *The actions δ_2 and φ_2 preserve the presentations $\langle C \mid N\rangle$ and $\langle X \mid R'\rangle$.*

Proof. We begin by showing that the actions preserve $\langle C \mid N\rangle$.

(1) First, we prove that $x \cdot g_1^k = x \cdot g_k$ in $\mathcal{C}_n$, for $x \in X$ and $2 \le k \le n-1$, by induction on k. Let $k \ge 2$. Let $i \in \{1, \ldots, n\}$. Then

$$a_i \cdot g_1^k = (a_i \cdot g_1)(a_i^{g_1} \cdot g_1^{k-1}) = \begin{cases} 1(b_1 \cdot g_1^{k-1}) & \text{if } i = n \\ 1(b_n \cdot g_1^{k-1}) & \text{if } i = n-1 \\ g_1(a_{i+1} \cdot g_1^{k-1}) & \text{otherwise} \end{cases} = \begin{cases} b_1 \cdot g_1^{k-1} & \text{if } i = n \\ b_n \cdot g_1^{k-1} & \text{if } i = n-1 \\ g_1(a_{i+1} \cdot g_1^{k-1}) & \text{otherwise .} \end{cases}$$

Thus, for $k = 2$, we have

$$a_i \cdot g_1^2 = \begin{cases} b_1 \cdot g_1 & \text{if } i = n \\ b_n \cdot g_1 & \text{if } i = n-1 \\ g_1(a_{i+1} \cdot g_1) & \text{otherwise} \end{cases} = \begin{cases} g_1 & \text{if } i = n \\ g_2 & \text{if } i = n-1 \\ g_1 1 & \text{if } i = n-2 \\ g_1^2 & \text{otherwise} \end{cases} = \begin{cases} g_1 & \text{if } i = n \text{ or } i = n-2 \\ g_2 & \text{otherwise} \end{cases} = a_i \cdot g_2 \text{ .}$$

Similarly, we may prove that $b_i \cdot g_1^2 = b_i \cdot g_2$.

Next, we admit by induction hypothesis that, for some $2 \leq k < n-1$, $x \cdot g_1^k = x \cdot g_k$ in $\mathcal{C}_n$, for $x \in X$. Then

$$\begin{aligned} a_i \cdot g_1^{k+1} &= \begin{cases} b_1 \cdot g_1^k & \text{if } i = n \\ b_n \cdot g_1^k & \text{if } i = n-1 \\ g_1(a_{i+1} \cdot g_1^k) & \text{otherwise} \end{cases} = \begin{cases} b_1 \cdot g_k & \text{if } i = n \\ b_n \cdot g_k & \text{if } i = n-1 \\ g_1(a_{i+1} \cdot g_k) & \text{otherwise} \end{cases} \\ &= \begin{cases} g_k & \text{if } i = n \\ g_{k+1} & \text{if } i = n-1 \\ g_1 g_{k-1} & \text{if } i = n-(k+1) \\ g_1 g_k & \text{otherwise} \end{cases} = \begin{cases} g_k & \text{if } i = n \text{ or } i = n-(k+1) \\ g_{k+1} & \text{otherwise} \end{cases} = a_i \cdot g_{k+1}\ . \end{aligned}$$

Similarly, we may prove that $b_i \cdot g_1^{k+1} = b_i \cdot g_{k+1}$.

(2) Next, we prove that $x \cdot g_1^n = x \cdot 1 (= 1)$ in $\mathcal{C}_n$, for $x \in X$. Let $i \in \{1, \ldots, n\}$. Then, by using the relations proved above, we have

$$\begin{aligned} a_i \cdot g_1^n &= \begin{cases} b_1 \cdot g_1^{n-1} & \text{if } i = n \\ b_n \cdot g_1^{n-1} & \text{if } i = n-1 \\ g_1(a_{i+1} \cdot g_1^{n-1}) & \text{otherwise.} \end{cases} = \begin{cases} b_1 \cdot g_{n-1} & \text{if } i = n \\ b_n \cdot g_{n-1} & \text{if } i = n-1 \\ g_1(a_{i+1} \cdot g_{n-1}) & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1 & \text{if } i = n \text{ or } i = n-1 \\ g_1 g_{n-1} & \text{otherwise.} \end{cases} = 1 = a_i \cdot 1 \end{aligned}$$

and, similarly, $b_i \cdot g_1^n = 1 = b_i \cdot 1$.

(3) Now, we prove that $x^{g_1^k} = x^{g_k}$ in $\mathcal{O}_n$, for $x \in X$ and $2 \leq k \leq n-1$, by induction on k. Let $k \geq 2$. Let $i \in \{1, \ldots, n\}$. Then

$$b_i^{g_1^k} = (b_i^{g_1})^{g_1^{k-1}} = \begin{cases} a_1^{g_1^{k-1}} & \text{if } i = n \\ a_n^{g_1^{k-1}} & \text{if } i = n-1 \\ b_{i+1}^{g_1^{k-1}} & \text{otherwise .} \end{cases}$$

So, for $k = 2$, we have

$$b_i^{g_1^2} = \begin{cases} a_1^{g_1} & \text{if } i = n \\ a_n^{g_1} & \text{if } i = n-1 \\ b_{i+1}^{g_1} & \text{otherwise} \end{cases} = \begin{cases} a_2 & \text{if } i = n \\ b_1 & \text{if } i = n-1 \\ a_n & \text{if } i = n-2 \\ b_{i+2} & \text{otherwise} \end{cases} = b_i^{g_2}$$

and, similarly, $a_i^{g_1^2} = a_i^{g_2}$.

Next, we admit by induction hypothesis that, for some $2 \leq k < n-1$, $x^{g_1^k} = x^{g_k}$ in $\mathcal{O}_n$, for $x \in X$. Then

$$\begin{aligned} b_i^{g_1^{k+1}} &= \begin{cases} a_1^{g_1^k} & \text{if } i = n \\ a_n^{g_1^k} & \text{if } i = n-1 \\ b_{i+1}^{g_1^k} & \text{otherwise} \end{cases} = \begin{cases} a_1^{g_k} & \text{if } i = n \\ a_n^{g_k} & \text{if } i = n-1 \\ b_{i+1}^{g_k} & \text{otherwise} \end{cases} = \begin{cases} a_{k+1} & \text{if } i = n \\ b_k & \text{if } i = n-1 \\ b_{i+k+1-n} & \text{if } n-k \leq i \leq n-2 \\ a_n & \text{if } i = n-(k+1) \\ b_{i+k+1} & \text{otherwise} \end{cases} \\ &= \begin{cases} a_{k+1} & \text{if } i = n \\ b_{i+k+1-n} & \text{if } n-k \leq i \leq n-1 \\ a_n & \text{if } i = n-(k+1) \\ b_{i+k+1} & \text{otherwise} \end{cases} = b_i^{g_{k+1}} \end{aligned}$$

and, similarly, $a_i^{g_1^{k+1}} = a_i^{g_{k+1}}$.

(4) Finally, we prove that $x^{g_1^n} = x^1 (= x)$ in $\mathcal{O}_n$, for $x \in X$. Let $i \in \{1, \ldots, n\}$. Then, by using the relations proved above, we have

$$b_i^{g_1^n} = \begin{cases} a_1^{g_1^{n-1}} & \text{if } i = n \\ a_n^{g_1^{n-1}} & \text{if } i = n-1 \\ b_{i+1}^{g_1^{n-1}} & \text{otherwise .} \end{cases} = \begin{cases} a_1^{g_{n-1}} & \text{if } i = n \\ a_n^{g_{n-1}} & \text{if } i = n-1 \\ b_{i+1}^{g_{n-1}} & \text{otherwise .} \end{cases} = b_i = b_i^1$$

and, similarly, $a_i^{g_1^n} = a_i = a_i^1$.

Now, it remains to prove that the actions preserve the presentation $\langle X \mid R' \rangle$. We just present the proof for the actions of and on the letter $g \equiv g_1$ of C. For the other letters the proof is analogous (though a little bit more involved).

(i) On the relations $a_i b_i = b_i a_{i-1}$, with $2 \leq i \leq n-1$:

$$(a_i b_i) \centerdot g = a_i \centerdot (b_i \centerdot g) = \begin{cases} a_i \centerdot g_2 & \text{if } i = n-1 \\ a_i \centerdot g & \text{if } i \leq n-2 \end{cases} = \begin{cases} g_2 & \text{if } i = n-1 \\ g & \text{if } i \leq n-2 \end{cases} = b_i \centerdot g = b_i \centerdot (a_{i-1} \centerdot g) = (b_i a_{i-1}) \centerdot g$$

and, as $a_1 a_n = a_n = a_n a_{n-1}$ in $\mathcal{O}_n$,

$$\begin{aligned} (a_i b_i)^g &= a_i^{b_i \centerdot g} b_i^g = \begin{cases} a_{n-1}^{g_2} a_n & \text{if } i = n-1 \\ a_i^g b_{i+1} & \text{if } i \leq n-2 \end{cases} = \begin{cases} a_1 a_n & \text{if } i = n-1 \\ a_{i+1} b_{i+1} & \text{if } i \leq n-2 \end{cases} \\ &= \begin{cases} a_n a_{n-1} & \text{if } i = n-1 \\ b_{i+1} a_i & \text{if } i \leq n-2 \end{cases} = b_i^g a_i = b_i^{a_{i-1} \centerdot g} a_{i-1}^g = (b_i a_{i-1})^g \ . \end{aligned}$$

(ii) On the relations $b_i a_i = a_i b_{i+1}$, with $1 \leq i \leq n-2$:

$$(b_i a_i) \centerdot g = b_i \centerdot (a_i \centerdot g) = b_i \centerdot g = g = \begin{cases} a_{n-2} \centerdot g_2 & \text{if } i = n-2 \\ a_i \centerdot g & \text{if } i \leq n-3 \end{cases} = a_i \centerdot (b_{i+1} \centerdot g) = (a_i b_{i+1}) \centerdot g$$

and, as $b_n a_n = a_{n-1} = b_{n-1} a_{n-1}$ in $\mathcal{O}_n$,

$$\begin{aligned} (b_i a_i)^g &= b_i^{a_i \centerdot g} a_i^g = b_i^g a_{i+1} = b_{i+1} a_{i+1} = \begin{cases} b_{n-1} a_{n-1} & \text{if } i = n-2 \\ a_{i+1} b_{i+2} & \text{if } i \leq n-3 \end{cases} = \begin{cases} b_n a_n & \text{if } i = n-2 \\ a_i^g b_{i+2} & \text{if } i \leq n-3 \end{cases} \\ &= \begin{cases} a_{n-2}^{g_2} a_n & \text{if } i = n-2 \\ a_i^g b_{i+2} & \text{if } i \leq n-3 \end{cases} = a_i^{b_{i+1} \centerdot g} b_{i+1}^g = (a_i b_{i+1})^g \ . \end{aligned}$$

(iii) On the relations $a_i b_i = b_i$, for $1 \leq i \leq n-1$:

$$(a_i b_i) \centerdot g = a_i \centerdot (b_i \centerdot g) = \begin{cases} a_i \centerdot g_2 & \text{if } i = n-1 \\ a_i \centerdot g & \text{if } i \leq n-2 \end{cases} = \begin{cases} g_2 & \text{if } i = n-1 \\ g & \text{if } i \leq n-2 \end{cases} = b_i \centerdot g$$

and, as $a_1 a_n = a_n$ in $\mathcal{O}_n$,

$$(a_i b_i)^g = a_i^{b_i \centerdot g} b_i^g = \begin{cases} a_{n-1}^{g_2} a_n & \text{if } i = n-1 \\ a_i^g b_{i+1} & \text{if } i \leq n-2 \end{cases} = \begin{cases} a_1 a_n & \text{if } i = n-1 \\ a_{i+1} b_{i+1} & \text{if } i \leq n-2 \end{cases} = \begin{cases} a_n & \text{if } i = n-1 \\ b_{i+1} & \text{if } i \leq n-2 \end{cases} = b_i^g \ .$$

(iv) On the relations $b_i a_i = a_i$, for $1 \leq i \leq n-1$:

$$(b_i a_i) \centerdot g = b_i \centerdot (a_i \centerdot g) = \begin{cases} b_i \centerdot 1 & \text{if } i = n-1 \\ b_i \centerdot g & \text{if } i \leq n-2 \end{cases} = \begin{cases} 1 & \text{if } i = n-1 \\ g & \text{if } i \leq n-2 \end{cases} = a_i \centerdot g$$

and, as $b_{n-1} b_n = b_n$ in $\mathcal{O}_n$,

$$(b_i a_i)^g = b_i^{a_i \centerdot g} a_i^g = \begin{cases} b_{n-1}^1 b_n & \text{if } i = n-1 \\ b_i^g a_{i+1} & \text{if } i \leq n-2 \end{cases} = \begin{cases} b_n & \text{if } i = n-1 \\ b_{i+1} a_{i+1} & \text{if } i \leq n-2 \end{cases} = \begin{cases} b_n & \text{if } i = n-1 \\ a_{i+1} & \text{if } i \leq n-2 \end{cases} = a_i^g \ .$$

(v) On the relations $b_j a_i = a_i b_j$, for $1 \leq i, j \leq n-1$ and $j \notin \{i, i+1\}$:

$$\begin{aligned} (b_j a_i) \centerdot g &= b_j \centerdot (a_i \centerdot g) = \begin{cases} b_j \centerdot 1 & \text{if } i = n-1 \\ b_j \centerdot g & \text{if } i \leq n-2 \end{cases} = \begin{cases} 1 & \text{if } i = n-1,\ j \leq n-2 \\ g_2 & \text{if } i < n-2,\ j = n-1 \\ g & \text{if } i, j \leq n-2,\ j \notin \{i, i+1\} \end{cases} \\ &= \begin{cases} a_i \centerdot g_2 & \text{if } j = n-1 \\ a_i \centerdot g & \text{if } j \leq n-2 \end{cases} = a_i \centerdot (b_j \centerdot g) = (a_i b_j) \centerdot g \end{aligned}$$

and, as $b_jb_n = b_nb_{j+1}$, $1 \le j \le n-2$, and $a_na_i = a_{i+1}a_n$, $1 \le i \le n-2$, in $\mathcal{O}_n$,

$$\begin{aligned}(b_ja_i)^g &= b_j^{a_i \cdot g}a_i^g = \begin{cases} b_j^1 b_n & \text{if } i = n-1 \\ b_j^g a_{i+1} & \text{if } i \le n-2 \end{cases} = \begin{cases} b_jb_n & \text{if } i = n-1,\ j \le n-2 \\ a_na_{i+1} & \text{if } i < n-2,\ j = n-1 \\ b_{j+1}a_{i+1} & \text{if } i,j \le n-2,\ j \notin \{i, i+1\} \end{cases} \\ &= \begin{cases} b_nb_{j+1} & \text{if } i = n-1,\ j \le n-2 \\ a_{i+2}a_n & \text{if } i < n-2,\ j = n-1 \\ a_{i+1}b_{j+1} & \text{if } i,j \le n-2,\ j \notin \{i, i+1\} \end{cases} = \begin{cases} a_i^{g_2}a_n & \text{if } j = n-1 \\ a_i^g b_{j+1} & \text{if } j \le n-2 \end{cases} = a_i^{b_j \cdot g}b_j^g = (a_ib_j)^g\ .\end{aligned}$$

(vi) On the relation $a_{n-1}a_{n-2}a_{n-1} = a_{n-1}a_{n-2}$:

$$(a_{n-1}a_{n-2}a_{n-1}) \cdot g = a_{n-1} \cdot (a_{n-2} \cdot (a_{n-1} \cdot g)) = a_{n-1} \cdot (a_{n-2} \cdot 1) = 1 = a_{n-1} \cdot g = a_{n-1} \cdot (a_{n-2} \cdot g) = (a_{n-1}a_{n-2}) \cdot g$$

and, as $a_{n-1}a_{n-2}b_n = b_na_{n-1}$ in $\mathcal{O}_n$,

$$\begin{aligned}(a_{n-1}a_{n-2}a_{n-1})^g &= (a_{n-1}a_{n-2})^{a_{n-1} \cdot g}a_{n-1}^g = (a_{n-1}a_{n-2})^1 b_n = a_{n-1}a_{n-2}b_n \\ &= b_na_{n-1} = a_{n-1}^g a_{n-1} = a_{n-1}^{a_{n-2} \cdot g}a_{n-2}^g = (a_{n-1}a_{n-2})^g\ .\end{aligned}$$

(vii) On the relation $b_1b_2b_1 = b_1b_2$:

$$(b_1b_2b_1) \cdot g = (b_1b_2) \cdot (b_1 \cdot g) = (b_1b_2) \cdot g$$

and, as $b_2b_3b_2 = b_2b_3$ in $\mathcal{O}_n$,

$$(b_1b_2b_1)^g = (b_1b_2)^{b_1 \cdot g}b_1^g = (b_1b_2)^g b_2 = b_1^{b_2 \cdot g}b_2^g b_2 = b_1^g b_3 b_2 = b_2b_3b_2 = b_2b_3 = b_1^g b_2^g = b_1^{b_2 \cdot g}b_2^g = (b_1b_2)^g\ .$$

(viii) On the relation $a_1a_2 \cdots a_{n-1} = a_n$:

$$(a_1a_2 \cdots a_{n-1}) \cdot g = (a_1a_2 \cdots a_{n-2}) \cdot (a_{n-1} \cdot g) = (a_1a_2 \cdots a_{n-2}) \cdot 1 = 1 = a_n \cdot g\ .$$

Similarly, also for $2 \le i \le n-1$, we have $(a_i \cdots a_{n-1}) \cdot g = 1$. On the other hand, as $a_1a_2 \cdots a_{n-2}b_n = b_1$ in $\mathcal{O}_n$,

$$(a_1a_2 \cdots a_{n-1})^g = a_1^{(a_2 \cdots a_{n-1}) \cdot g}a_2^{(a_3 \cdots a_{n-1}) \cdot g} \cdots a_{n-2}^{a_{n-1} \cdot g}a_{n-1}^g = a_1^1a_2^1 \cdots a_{n-2}^1 b_n = a_1a_2 \cdots a_{n-2}b_n = b_1 = a_n^g\ .$$

(ix) On the relation $b_{n-1}b_{n-2} \cdots b_1 = b_n$: as $b_i \cdot g = g$, for $1 \le i \le n-2$, then $(b_{n-2} \cdots b_1) \cdot g = g$ (indeed, we have $(b_i \cdots b_1) \cdot g = g$, for $1 \le i \le n-2$), whence

$$(b_{n-1}b_{n-2} \cdots b_1) \cdot g = b_{n-1} \cdot ((b_{n-2} \cdots b_1) \cdot g) = b_{n-1} \cdot g = g_2 = b_n \cdot g$$

and, as $a_nb_{n-1}b_{n-2} \cdots b_2 = a_1$ in $\mathcal{O}_n$,

$$(b_{n-1}b_{n-2} \cdots b_1)^g = b_{n-1}^{(b_{n-2} \cdots b_1) \cdot g}b_{n-2}^{(b_{n-3} \cdots b_1) \cdot g} \cdots b_2^{b_1 \cdot g}b_1^g = b_{n-1}^g b_{n-2}^g \cdots b_2^g b_1^g = a_nb_{n-1} \cdots b_3b_2 = a_1 = b_n^g\ ,$$

as required. ∎

Now, by applying the Theorem 1.8, we have a well defined bilateral semidirect product $\mathcal{C}_n \bowtie \mathcal{O}_n$ induced by the actions δ_2 and φ_2. Furthermore, as $a_ig = ga_{i+1}$ and $b_ig = gb_{i+1}$, for $1 \le i \le n-2$, $a_{n-1}g = b_n$, $b_{n-1}g = g^2a_n = g_2a_n$, $a_ng = a_1a_2 \cdots a_{n-2}a_{n-1}g = a_1a_2 \cdots a_{n-2}b_n = b_1$ and $b_ng = b_{n-1}b_{n-2} \cdots b_1g = b_{n-1}gb_{n-1} \cdots b_2 = g^2a_nb_{n-1} \cdots b_2 = g^2a_1 = g_2a_1$ in $\mathcal{OP}_n$, it follows immediately:

Lemma 2.12 *For $1 \le i \le n$, $a_ig = (a_i \cdot g)(a_i^g)$ and $b_ig = (b_i \cdot g)(b_i^g)$ in $\mathcal{OP}_n$.* ∎

As the right action in $\mathcal{C}_n \bowtie \mathcal{O}_n$ preserves X and $\{g\} \cup X$ generates $\mathcal{OP}_n$, by the Theorem 1.10, we have:

Theorem 2.13 *The monoid $\mathcal{OP}_n$ is a homomorphic image of $\mathcal{C}_n \bowtie \mathcal{O}_n$.* ∎

Let OP be the pseudovariety of monoids generated by $\{\mathcal{OP}_n \mid n \in \mathbb{N}\}$ and let Ab be the pseudovariety (of monoids) of all abelian groups. Then:

Corollary 2.14 $\mathsf{OP} \subseteq \mathsf{Ab} \bowtie \mathsf{O}$. ∎

On the monoid $\mathcal{OR}_n$

The monoid $\mathcal{OR}_n$ was studied by McAlister in [17] and by Catarino and Higgins in [5]. A presentation for $\mathcal{OR}_n$ was given by Arthur and Ruškuc [3] (see also [6]). We obtain a set of generators of $\mathcal{OR}_n$ by adding to a set of generators of $\mathcal{O}_n$ the n-cycle permutation g and the reflexion permutation h considered above. Therefore, $\mathcal{O}_n$, $\mathcal{OD}_n$, $\mathcal{OP}_n$, $\mathcal{C}_2$ and $\mathcal{C}_n$ are submonoids of $\mathcal{OR}_n$. Let $\mathcal{D}_{2n}$ be a dihedral group of order $2n$. Then, $\mathcal{D}_{2n}$ is defined by the presentation $\langle g, h \mid h^2 = 1, g^n = 1, hg = g^{n-1}h\rangle$ and, clearly, may be considered as the submonoid of $\mathcal{OR}_n$ generated by the permutations g and h.

In the remaining of this paper we construct some bilateral semidirect decompositions of $\mathcal{OR}_n$ in terms of its submonoids mentioned above.

First, we consider $\mathcal{D}_{2n}$ and $\mathcal{O}_n$. By convenience, we consider the letter-irredundant presentation

$$\langle D \mid N'\rangle = \langle g_1, \ldots, g_{n-1}, h \mid h^2 = 1, g_1^n = 1, hg_1 = g_{n-1}h, g_1^k = g_k, 2 \le k \le n-1\rangle$$

of $\mathcal{D}_{2n}$ and again the letter-irredundant presentation $\langle X \mid R'\rangle$ of $\mathcal{O}_n$.

Let δ_3 the left action of X^* on D^* and φ_3 the right action of D^* on X^* that extend, by Proposition 1.4 (or 1.5), the following actions of and on the letters:

- $x \cdot h$ and x^h as in $(A \cup B)^* *_r \{h\}^*$ (which induces the reverse semidirect product $\mathcal{O}_n *_r \mathcal{C}_2$ considered in the Theorem 2.9), for $x \in A \cup B$;
- $a_n \cdot h = b_n \cdot h = h$ (thus $x \cdot h = h$, for $x \in X$), $a_n^h = b_n$ and $b_n^h = a_n$;
- $x \cdot g_k$ and x^{g_k} as in $C^* \bowtie X^*$ (which induces $\mathcal{C}_n \bowtie \mathcal{O}_n$), for $x \in X$ and $1 \le k \le n-1$.

Observe that the actions δ_3 and φ_3 preserve letters. We also have:

Lemma 2.15 *The actions δ_3 and φ_3 preserve the presentations $\langle D \mid N'\rangle$ and $\langle X \mid R'\rangle$.*

Proof. By the Lemma 2.11 and the "dual" of Lemma 2.5 (see the paragraph between the Corollary 2.8 and the Theorem 2.9), it remains to show that:

(i) $(a_1 \cdots a_n) \cdot h = a_n \cdot h$, $(b_{n-1} \cdots b_1) \cdot h = b_n \cdot h$, $(a_1 \cdots a_n)^h = a_n^h$ and $(b_{n-1} \cdots b_1)^h = b_n^h$;

(ii) $a_n \cdot h^2 = a_n \cdot 1$, $b_n \cdot h^2 = b_n \cdot 1$, $a_n^{h^2} = a_n^1$ and $b_n^{h^2} = b_n^1$; and

(iii) $x \cdot (hg_1) = x \cdot (g_{n-1}h)$ and $x^{hg_1} = x^{g_{n-1}h}$, $x \in X$.

The first two relations from (i) are obvious (as $w \cdot h = h$, for $w \in X^*$). On the other hand,

$$(a_1a_2 \cdots a_{n-1})^h = a_1^{(a_2\cdots a_{n-1})\cdot h} a_2^{(a_3\cdots a_{n-1})\cdot h} \cdots a_{n-2}^{a_{n-1}\cdot h} a_{n-1}^h = a_1^h a_2^h \cdots a_{n-2}^h a_{n-1}^h = b_{n-1}b_{n-2} \cdots b_2b_1 = b_n = a_n^h$$

and, similarly, $(b_{n-1} \cdots b_1)^h = b_n^h$. The relations from (ii) are easy to prove. Next, we prove the relations from (iii) for $x \in \{a_1, \ldots a_n\}$. For $x \in \{b_1, \ldots, b_n\}$ the proof is similar. Then, for $i = 1$, we have

$$a_1 \cdot (hg_1) = (a_1 \cdot h)(a_1^h \cdot g_1) = h(b_{n-1} \cdot g_1) = hg_2 = g_{n-2}h = (a_1 \cdot g_{n-1})(a_1^{g_{n-1}} \cdot h) = a_1 \cdot (g_{n-1}h)$$

and

$$a_1^{hg_1} = (a_1^h)^{g_1} = b_{n-1}^{g_1} = a_n = b_n^h = (a_1^{g_{n-1}})^h = a_1^{g_{n-1}h}.$$

For $2 \le i \le n-1$, we have

$$a_i \cdot (hg_1) = (a_i \cdot h)(a_i^h \cdot g_1) = h(b_{n-i} \cdot g_1) = hg_1 = g_{n-1}h = (a_i \cdot g_{n-1})(a_i^{g_{n-1}} \cdot h) = a_i \cdot (g_{n-1}h)$$

and

$$a_i^{hg_1} = (a_i^h)^{g_1} = b_{n-i}^{g_1} = b_{n-i+1} = a_{i-1}^h = (a_i^{g_{n-1}})^h = a_i^{g_{n-1}h}.$$

Finally, for $i = n$, we have

$$a_n \cdot (hg_1) = (a_n \cdot h)(a_n^h \cdot g_1) = h(b_n \cdot g_1) = hg_2 = g_{n-2}h = (a_n \cdot g_{n-1})(a_n^{g_{n-1}} \cdot h) = a_n \cdot (g_{n-1}h)$$

and

$$a_n^{hg_1} = (a_n^h)^{g_1} = b_n^{g_1} = a_1 = b_{n-1}^h = (a_n^{g_{n-1}})^h,$$

as required. ■

Once again, by applying the Theorem 1.8, we have a well defined bilateral semidirect product $\mathcal{D}_{2n} \bowtie \mathcal{O}_n$ induced by the actions δ_3 and φ_3. Furthermore, as $a_n h = hb_n$ (and $b_n h = ha_n$), we have $a_n h = (a_n \cdot h)(a_n^h)$ and $b_n h = (b_n \cdot h)(b_n^h)$ in $\mathcal{OR}_n$. Hence, by taking in consideration the Lemma 2.12 and the observation before the Theorem 2.9, it follows immediately:

Lemma 2.16 *For $x \in X$, $xg = (x \cdot g)(x^g)$ and $xh = (x \cdot h)(x^h)$ in $\mathcal{OR}_n$.* ■

As the right action in $\mathcal{D}_{2n} \bowtie \mathcal{O}_n$ preserves X and $\{g, h\} \cup X$ generates $\mathcal{OR}_n$, by the Theorem 1.10, we have:

Theorem 2.17 *The monoid $\mathcal{OR}_n$ is a homomorphic image of $\mathcal{D}_{2n} \bowtie \mathcal{O}_n$.* ■

Let OR be the pseudovariety of monoids generated by $\{\mathcal{OR}_n \mid n \in \mathbb{N}\}$ and let Dih be the pseudovariety of monoids (groups) generated by $\{\mathcal{D}_{2n} \mid n \in \mathbb{N}\}$. Then:

Corollary 2.18 $\mathsf{OR} \subseteq \mathsf{Dih} \bowtie \mathsf{O}$. ■

Next, we consider the submonoids $\mathcal{C}_n$ and $\mathcal{OD}_n$ of $\mathcal{OR}_n$ together with the letter-irredundant presentations $\langle C \mid N \rangle$ of $\mathcal{C}_n$ and

$$\langle Y \mid R_1 \rangle = \langle X \cup \{h\} \mid R', h^2 = 1, a_{n-1}a_{n-2} \cdots a_1 h = b_1 b_2 \cdots b_{n-1}, ha_i = b_{n-i}h, 1 \le i \le n-1 \rangle$$

of $\mathcal{OD}_n$.

Let δ_4 the left action of Y^* on C^* and φ_4 the right action of C^* on Y^* that extend, by Proposition 1.4 (or 1.5), the following actions of and on the letters:

- $x \cdot g_k$ and x^{g_k} as in $C^* \bowtie X^*$ (which induces $\mathcal{C}_n \bowtie \mathcal{O}_n$), for $x \in X$ and $1 \le k \le n-1$;
- $h \cdot g_k = g_{n-k}$ and $h^{g_k} = h$, for $1 \le k \le n-1$.

Notice that, both the actions δ_4 and φ_4 preserve letters. Furthermore, we also have:

Lemma 2.19 *The actions δ_4 and φ_4 preserve the presentations $\langle C \mid N \rangle$ and $\langle Y \mid R_1 \rangle$.*

Proof. By the Lemma 2.11, it remains to show that:

(i) $h \cdot g_1^k = h \cdot g_k$ and $h^{g_1^k} = h^{g_k}$, for $2 \le k \le n-1$;

(ii) $h \cdot g_1^n = h \cdot 1$ and $h^{g_1^n} = h^1$;

(iii) $h^2 \cdot g_k = 1 \cdot g_k$, for $1 \le k \le n-1$;

(iv) $(ha_i) \cdot g_k = (b_{n-i}h) \cdot g_k$, for $1 \le i, k \le n-1$;

(v) $(a_{n-1}a_{n-2} \cdots a_1 h) \cdot g_k = (b_1 b_2 \cdots b_{n-1}) \cdot g_k$, for $1 \le k \le n-1$.

First, notice that, as $h^{g_k} = h$, for $1 \le k \le n-1$, the second equalities of (i) and (ii) are immediate.

We begin by proving that $h \cdot g_1^k = h \cdot g_k$, for $2 \le k \le n-1$, by induction on k. For $k = 2$, we have

$$h \cdot g_1^2 = (h \cdot g_1)(h^{g_1} \cdot g_1) = g_{n-1}(h \cdot g_1) = g_{n-1}g_{n-1} = g^{n-1}g^{n-1} = g^{n-2} = g_{n-2} = h \cdot g_2.$$

Then, assume by induction hypothesis that, $h \cdot g_1^{k-1} = h \cdot g_{k-1}$, for some $2 < k \le n-1$. Hence

$$h \cdot g_1^k = (h \cdot g_1)(h^{g_1} \cdot g_1^{k-1}) = g_{n-1}(h \cdot g_1^{k-1}) = g_{n-1}(h \cdot g_{k-1}) = g_{n-1}g_{n-k+1} = g^{n-1}g^{n-k+1} = g^{n-k} = g_{n-k} = h \cdot g_k.$$

Next, we finish the proof of (ii):

$$h \cdot g_1^n = (h \cdot g_1)(h^{g_1} \cdot g_1^{n-1}) = g_{n-1}(h \cdot g_1^{n-1}) = g_{n-1}(h \cdot g_{n-1}) = g_{n-1}g_1 = g^{n-1}g = 1 = h \cdot 1.$$

In order to prove (iii), let $1 \le k \le n-1$. Then $h^2 \cdot g_k = h \cdot (h \cdot g_k) = h \cdot g_{n-k} = g_k = 1 \cdot g_k$.

Now, we prove (iv): for $1 \le i, k \le n-1$, we have

$$\begin{array}{rcl}(ha_i) \cdot g_k & = & h \cdot (a_i \cdot g_k) = \left\{\begin{array}{ll} h \cdot 1 & \text{if } k=1,\ i=n-1 \\ h \cdot g_{k-1} & \text{if } k \ge 2,\ i=n-k \\ h \cdot g_k & \text{otherwise}\end{array}\right. = \left\{\begin{array}{ll} 1 & \text{if } k=1,\ i=n-1 \\ g_{n-k+1} & \text{if } k \ge 2,\ i=n-k \\ g_{n-k} & \text{otherwise}\end{array}\right. \\ & = & \left\{\begin{array}{ll} 1 & \text{if } n-k=n-1,\ n-i=1 \\ g_{n-k+1} & \text{if } n-k \le n-2,\ n-i=k \\ g_{n-k} & \text{otherwise}\end{array}\right. = b_{n-i} \cdot g_{n-k} = b_{n-i} \cdot (h \cdot g_k) = (b_{n-i}h) \cdot g_k.\end{array}$$

Finally, we prove the last equality. Let $1 \le k \le n-1$. Then

$$\begin{array}{rcl}(a_{n-1}a_{n-2}\cdots a_1 h) \cdot g_k & = & a_{n-1} \cdot (a_{n-2} \cdot (\cdots \cdot (a_1 \cdot (h \cdot g_k)))) = a_{n-1} \cdot (a_{n-2} \cdot (\cdots \cdot (a_1 \cdot g_{n-k}))) \\ & = & a_{n-1} \cdot (a_{n-2} \cdot (\cdots \cdot (a_k \cdot g_{n-k}))) = a_{n-1} \cdot (a_{n-2} \cdot (\cdots \cdot (a_{k+1} \cdot g_{n-(k+1)}))) \\ & = & a_{n-1} \cdot g_1 = 1 = b_1 \cdot g_{n-1} = b_1 \cdot (b_2 \cdot (\cdots \cdot (b_{n-(k+1)} \cdot g_{k+1}))) \\ & = & b_1 \cdot (b_2 \cdot (\cdots \cdot (b_{n-k} \cdot g_k))) = b_1 \cdot (b_2 \cdot (\cdots \cdot (b_{n-1} \cdot g_k))) = (b_1 b_2 \cdots b_{n-1}) \cdot g_k,\end{array}$$

as required. ■

Thus, by the Theorem 1.8, we have a well defined bilateral semidirect product $\mathcal{C}_n \bowtie \mathcal{OD}_n$ induced by the actions δ_4 and φ_4. On the other hand, as $hg = g^{n-1}h$, then $hg = (h \cdot g)h^g$ in $\mathcal{OR}_n$ and so, by taking also in consideration the Lemma 2.12, it follows immediately:

Lemma 2.20 *For $x \in Y$, $xg = (x \cdot g)(x^g)$.* ■

As the right action in $\mathcal{C}_n \bowtie \mathcal{OD}_n$ preserves Y and $\{g\} \cup Y$ generates $\mathcal{OR}_n$, by the Theorem 1.10, we have:

Theorem 2.21 *The monoid $\mathcal{OR}_n$ is a homomorphic image of $\mathcal{C}_n \bowtie \mathcal{OD}_n$.* ■

It follows that:

Corollary 2.22 $\mathsf{OR} \subseteq \mathsf{Ab} \bowtie \mathsf{OD}$. ■

Now, consider the submonoids $\mathcal{C}_2$ and $\mathcal{OP}_n$ of $\mathcal{OR}_n$ and the letter-irredundant presentations $\langle h \mid h^2 = 1\rangle$ of $\mathcal{C}_2$ and

$$\begin{array}{rl}\langle Z \mid R_2\rangle = \langle A \cup B \cup C \mid R, N, & a_{n-1}g_1 = b_{n-1}b_{n-2}\cdots b_1, g_1 a_{n-1}a_{n-2}\cdots a_1 = a_{n-1}a_{n-2}\cdots a_1, \\ & b_{n-1}g_1 = g_2 a_1 a_2 \cdots a_{n-1}, a_i g_1 = g_1 a_{i+1}, b_i g_1 = g_1 b_{i+1}, 1 \le i \le n-2\rangle\end{array}$$

of $\mathcal{OP}_n$.

By applying the Proposition 1.4, or 1.5, and considering a trivial right action, let δ_5 be the left action of $\{h\}^*$ on Z^* that extends the following action of and on the letters:

- $h \cdot a_i = b_{n-i}$ and $h \cdot b_i = a_{n-i}$, for $1 \leq i \leq n-1$ (as the left action δ_1 of $\{h\}^*$ on $(A \cup B)^*$);
- $h \cdot g_k = g_{n-k}$, for $1 \leq k \leq n-1$.

Notice that δ_5 preserves letters (as well as the trivial right action). Moreover, we have:

Lemma 2.23 *The action δ_5 preserves the presentations $\langle Z \mid R_2 \rangle$ and $\langle h \mid h^2 = 1 \rangle$.*

Proof. By the Lemma 2.5, it remains to show that:

(i) $h^2 \cdot g_k = 1 \cdot g_k$, for $1 \leq k \leq n-1$;

(ii) $h \cdot g_1^n = h \cdot 1$;

(iii) $h \cdot g_1^k = h \cdot g_k$, for $2 \leq k \leq n-1$;

(iv) $h \cdot (a_{n-1}g_1) = h \cdot (b_{n-1}b_{n-2} \cdots b_1)$;

(v) $h \cdot (g_1 a_{n-1}a_{n-2} \cdots a_1) = h \cdot (a_{n-1}a_{n-2} \cdots a_1)$;

(vi) $h \cdot (b_{n-1}g_1) = h \cdot (g_2 a_1 a_2 \cdots a_{n-1})$;

(vii) $h \cdot (a_i g_1) = h \cdot (g_1 a_{i+1})$ and $h \cdot (b_i g_1) = h \cdot (g_1 b_{i+1})$, for $1 \leq i \leq n-2$.

The proofs of (i) to (iii) are similar to the proofs of (i) to (iii) of Lemma 2.19.

First, we prove (iv): as $b_1 g_{n-1} = a_1 a_2 \cdots a_{n-1}$ in $\mathcal{OP}_n$, we have

$$\begin{aligned} h \cdot (a_{n-1}g_1) &= (h \cdot a_{n-1})(h^{a_{n-1}} \cdot g_1) = b_1(h \cdot g_1) = b_1 g_{n-1} = a_1 a_2 \cdots a_{n-1} = (h \cdot b_{n-1})(h \cdot b_{n-2}) \cdots (h \cdot b_1) \\ &= (h \cdot b_{n-1})(h^{b_{n-1}} \cdot b_{n-2})(h^{b_{n-1}b_{n-2}} \cdot b_{n-3}) \cdots (h^{b_{n-1}b_{n-2} \cdots b_2} \cdot b_1) = h \cdot (b_{n-1}b_{n-2} \cdots b_1). \end{aligned}$$

Next, as $b_1 b_2 \cdots b_{n-1} = g_{n-1} b_1 b_2 \cdots b_{n-1}$ (notice that $b_1 b_2 \cdots b_{n-1}$ is a right zero in $\mathcal{OP}_n$), then

$$\begin{aligned} h \cdot (g_1 a_{n-1}a_{n-2} \cdots a_1) &= (h \cdot g_1)(h^{g_1} \cdot a_{n-1})(h^{g_1 a_{n-1}} \cdot a_{n-2}) \cdots (h^{g_1 a_{n-1} \cdots a_2} \cdot a_1) \\ &= (h \cdot g_1)(h \cdot a_{n-1})(h \cdot a_{n-2}) \cdots (h \cdot a_1) = g_{n-1} b_1 b_2 \cdots b_{n-1} = b_1 b_2 \cdots b_{n-1} \\ &= (h \cdot a_{n-1})(h \cdot a_{n-2}) \cdots (h \cdot a_1) = (h \cdot a_{n-1})(h^{a_{n-1}} \cdot a_{n-2}) \cdots (h^{a_{n-1} \cdots a_2} \cdot a_1), \end{aligned}$$

which proves (v).

Now, we prove (vi), using the fact that $a_1 g_{n-1} = g_{n-2} b_{n-1} \cdots b_1$ in $\mathcal{OP}_n$. Hence

$$\begin{aligned} h \cdot (b_{n-1}g_1) &= (h \cdot b_{n-1})(h^{b_{n-1}} \cdot g_1) = a_1 g_{n-1} = g_{n-2} b_{n-1} \ldots b_1 = (h \cdot g_2)(h \cdot a_1)(h \cdot a_2) \cdots (h \cdot a_{n-1}) \\ &= (h \cdot g_2)(h^{g_2} \cdot a_1)(h^{g_2 a_1} \cdot a_2) \cdots (h^{g_2 a_1 \cdots a_{n-2}} \cdot a_{n-1}) = h \cdot (g_2 a_1 a_2 \cdots a_{n-1}). \end{aligned}$$

Finally, as $b_{n-i} g_{n-1} = g_{n-1} b_{n-i-1}$ and $a_{n-i} g_{n-1} = g_{n-1} a_{n-i-1}$, for $1 \leq i \leq n-2$, in $\mathcal{OP}_n$, it follows that $h \cdot (a_i g_1) = h \cdot (g_1 a_{i+1})$ and $h \cdot (b_i g_1) = h \cdot (g_1 b_{i+1})$, for $1 \leq i \leq n-2$, as required. ∎

Thus, by the Theorem 1.8 (considering a trivial right action), we have a well defined semidirect product $\mathcal{OP}_n * \mathcal{C}_2$ induced by the action δ_5. On the other hand, we have $hg^k = g^{n-k}h$, whence $hg_k = (h \cdot g_k)h$, for $1 \leq k \leq n-1$. So, by taking also in consideration the Lemma 2.6, it follows immediately:

Lemma 2.24 *For $x \in A \cup B \cup C$, $hx = (h \cdot x)h$ in $\mathcal{OR}_n$.* ∎

Now, by the Corollary 1.11, we have:

Theorem 2.25 *The monoid $\mathcal{OR}_n$ is a homomorphic image of $\mathcal{OP}_n * \mathcal{C}_2$.* ∎

It follows immediately:

Corollary 2.26 $\mathsf{OR} \subseteq \mathsf{OP} * \mathsf{Ab}_2$. ∎

Once again, as $\mathcal{C}_2$ is a commutative monoid, the left action of $\mathcal{C}_2$ on $\mathcal{OP}_n$ may also be considered as a right action (which coincides with the one induced by the right action of $\{h\}^*$ on $(A \cup B \cup C)^*$ that extends the following action of and on the letters: $a_i^h = b_{n-i}$ and $b_i^h = a_{n-i}$, $g_i^h = g_{n-i}$, for $1 \leq i \leq n-1$) and so we also have a well defined reverse semidirect product $\mathcal{OP}_n *_r \mathcal{C}_2$. As, clearly, $xh = hx^h$ in $\mathcal{OR}_n$, for $x \in A \cup B \cup C$, again by Corollary 1.11, we have:

Theorem 2.27 *The monoid $\mathcal{OR}_n$ is a homomorphic image of $\mathcal{OP}_n *_r \mathcal{C}_2$.* ∎

And, so:

Corollary 2.28 $\mathsf{OR} \subseteq \mathsf{OP} *_r \mathsf{Ab}_2$. ∎

Conjectures

We finish this paper by formulating some conjectures:

Conjecture 2.29 $\mathsf{OD} = \mathsf{O} * \mathsf{Ab}_2 = \mathsf{O} *_r \mathsf{Ab}_2$.

Conjecture 2.30 $\mathsf{OP} = \mathsf{Ab} \bowtie \mathsf{O}$.

Conjecture 2.31 $\mathsf{OR} = \mathsf{Dih} \bowtie \mathsf{O} = \mathsf{Ab} \bowtie \mathsf{OD} = \mathsf{OP} * \mathsf{Ab}_2 = \mathsf{OP} *_r \mathsf{Ab}_2$.

References

[1] A.Ya. Aĭzenštat, *The defining relations of the endomorphism semigroup of a finite linearly ordered set,* Sibirsk. Mat. **3** (1962), 161–169 (Russian).

[2] J. Almeida and P. Weil, *Profinite categories and semidirect products,* J. Pure Appl. Algebra **123** (1998), 1–50.

[3] R.E. Arthur and N. Ruškuc, *Presentations for two extensions of the monoid of order-preserving mappings on a finite chain,* Southeast Asian Bull. Math. **24** (2000), 1–7.

[4] P.M. Catarino, *Monoids of orientation-preserving transformations of a finite chain and their presentation,* Semigroups and Applications, eds. J.M. Howie & N. Ruškuc, World Scientific (1998), 39–46.

[5] P.M. Catarino and P.M. Higgins, *The monoid of orientation-preserving mappings on a chain,* Semigroup Forum **58** (1999), 190–206.

[6] V.H. Fernandes, *Presentations for some monoids of partial transformations on a finite chain: a survey,* Semigroups, Algorithms, Automata and Languages, eds. Gracinda M. S. Gomes & Jean-Éric Pin & Pedro V. Silva, World Scientific, (2002), 363–378.

[7] V.H. Fernandes, G.M.S. Gomes and M.M. Jesus, *Presentations for some monoids of partial transformations on a finite chain,* Comm. Algebra **33** (2005), 587–604.

[8] G.M.S. Gomes and J.M. Howie, *On the ranks of certain semigroups of order-preserving transformations,* Semigroup Forum **45** (1992), 272–282.

[9] P.M. Higgins, *Divisors of semigroups of order-preserving mappings on a finite chain,* Internat. J. Algebra Comput. **5** (1995), 725–742.

[10] J.M. Howie, *Product of idempotents in certain semigroups of transformations,* Proc. Edinburgh Math. Soc. **17** (1971), 223–236.

[11] M. Kunze, *Zappa products,* Acta Math. Hungar. **41** (1983), 225–239.

[12] M. Kunze, *Lineare Parallelrechner I* [Linear parallel processing machines I], Elektron. Informationsverarb. Kybernet. **20** (1984), 9–39 (German).

[13] M. Kunze, *Lineare Parallelrechner II* [Linear parallel processing machines II], Elektron. Informationsverarb. Kybernet. **20** (1984), 111–147 (German).

[14] M. Kunze, *Bilateral semidirect products of transformation semigroups,* Semigroup Forum **45** (1992), 166–182.

[15] M. Kunze, *Standard automata and semidirect products of transformation semigroups,* Theoret. Comput. Sci. **108** (1993), 151–171.

[16] G. Lallement, *Semigroups and Combinatorial Applications,* John Wiley & Sons, 1979.

[17] D.B. McAlister, *Semigroups generated by a group and an idempotent,* Comm. Algebra **26** (1998), 515–547.

[18] J.-E. Pin, *Varieties of Formal Languages,* Plenum, London, 1986.

[19] J. Rhodes and B. Tilson, *The kernel of monoid morphisms,* J. Pure Appl. Algebra **62** (1989), 277–268.

[20] N. Ruškuc, *Semigroup Presentations,* Ph. D. Thesis, University of St-Andrews, 1995.

[21] A. Solomon, *Catalan monoids, monoids of local endomorphisms, and their presentations,* Semigroup Forum **53** (1996), 351–368.

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