On the monoids of transformations that preserve the order and a uniform partition

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Abstract

In this paper we consider the monoid $\mathcal{O}_{m\times n}$ of all order-preserving full transformations on a chain with mn elements that preserve a uniform m-partition and its submonoids $\mathcal{O}^+_{m\times n}$ and $\mathcal{O}^-_{m\times n}$ of all extensive transformations and of all co-extensive transformations, respectively. We give formulas for the number of elements of these monoids and determine their ranks. Moreover, we construct a bilateral semidirect product decomposition of $\mathcal{O}_{m\times n}$ in terms of $\mathcal{O}^-_{m\times n}$ and $\mathcal{O}^+_{m\times n}$.

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Introduction and preliminaries

Let X be a set and denote by $\mathcal{T}(X)$ the monoid (under composition) of all full transformations on X. Let ρ be an equivalence relation on X. We denote by $\mathcal{T}_\rho(X)$ the submonoid of $\mathcal{T}(X)$ of all transformations that preserve the equivalence relation ρ, i.e.

$$\mathcal{T}_\rho(X) = \{\alpha \in \mathcal{T}(X) \mid (a\alpha, b\alpha) \in \rho, \text{for all } (a,b) \in \rho\}.$$

This monoid was studied by Huisheng in [14] who determined its regular elements and described its Green relations.

For $n \in \mathbb{N}$, let X_n be a chain with n elements, say $X_n = \{1 < 2 < \cdots < n\}$, and denote the monoid $\mathcal{T}(X_n)$ simply by $\mathcal{T}_n$. Let

$$\mathcal{T}^+_n = \{\alpha \in \mathcal{T}_n \mid x \le x\alpha, \text{for all } x \in X_n\} \quad \text{and} \quad \mathcal{T}^-_n = \{\alpha \in \mathcal{T}_n \mid x\alpha \le x, \text{for all } x \in X_n\},$$

i.e. the submonoids of $\mathcal{T}_n$ of all extensive transformations and of all co-extensive transformations, respectively. Let

$$\mathcal{O}_n = \{\alpha \in \mathcal{T}_n \mid x \le y \text{ implies } x\alpha \le y\alpha, \text{ for all } x, y \in X_n\}$$

be the submonoid of $\mathcal{T}_n$ whose elements are the order-preserving transformations and let

$$\mathcal{O}^+_n = \mathcal{T}^+_n \cap \mathcal{O}_n \quad \text{and} \quad \mathcal{O}^-_n = \mathcal{T}^-_n \cap \mathcal{O}_n$$

be the submonoids of $\mathcal{O}_n$ of all extensive transformations and of all co-extensive transformations, respectively.

The monoid $\mathcal{O}_n$ has been extensively studied since the sixties. In fact, in 1962, Aĭzenštat [1, 2] showed that the congruences of $\mathcal{O}_n$ are exactly the Rees congruences and gave a monoid presentation for $\mathcal{O}_n$, in terms

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of $2n-2$ idempotent generators, from which it can be deduced that the only non-trivial automorphism of $\mathcal{O}_n$ where $n>1$ is that given by conjugation by the permutation $(1\ n)(2\ n-1)\cdots(\lfloor n/2\rfloor\ \lceil n/2\rceil+1)$. In 1971, Howie [12] calculated the cardinal and the number of idempotents of $\mathcal{O}_n$ and later (1992), jointly with Gomes [9], determined its rank and idempotent rank. Recall that the [idempotent] rank of a finite [idempotent generated] monoid is the cardinality of a least-size [idempotent] generating set. More recently, Fernandes et al. [8] described the endomorphisms of the semigroup $\mathcal{O}_n$ by showing that there are three types of endomorphism: automorphisms, constants, and a certain type of endomorphism with two idempotents in the image. The monoid $\mathcal{O}_n$ also played a main role in several other papers [11, 22, 3, 5, 20, 6] where the central topic concerns the problem of the decidability of the pseudovariety generated by the family $\{\mathcal{O}_n \mid n\in\mathbb{N}\}$. This question was posed by J.-E. Pin in 1987 in the "Szeged International Semigroup Colloquium" and is still unanswered.

Now, let $m,n\in\mathbb{N}$ and let ρ be the equivalence relation on X_{mn} defined by

$$\rho=(A_1\times A_1)\cup(A_2\times A_2)\cup\cdots\cup(A_m\times A_m),$$

where $A_i=\{(i-1)n+1,(i-1)n+2,\ldots,in\}$, for $i\in\{1,\ldots,m\}$. Notice that the ρ-classes A_i, with $1\le i\le m$, form a uniform m-partition of X_{mn}. Denote by $\mathcal{T}_{m\times n}$ the submonoid $\mathcal{T}_\rho(X_{mn})$ of $\mathcal{T}_{mn}$ and let

$$\mathcal{T}^+_{m\times n}=\mathcal{T}_{m\times n}\cap\mathcal{T}^+_{mn}\quad\text{and}\quad\mathcal{T}^-_{m\times n}=\mathcal{T}_{m\times n}\cap\mathcal{T}^-_{mn}$$

be the submonoids of $\mathcal{T}_{m\times n}$ of all extensive transformations and of all co-extensive transformations, respectively.

Regarding the rank of $\mathcal{T}_{m\times n}$, first, Huisheng [13] proved that it is at most 6 and, later, Araújo and Schneider [4] improved this result by showing that, for $|X_{mn}|\ge 3$, the rank of $\mathcal{T}_{m\times n}$ is precisely 4.

Denote by $\mathcal{O}_{m\times n}$ the submonoid of $\mathcal{T}_{m\times n}$ of all order-preserving transformations that preserve the equivalence ρ, i.e.

$$\mathcal{O}_{m\times n}=\mathcal{T}_{m\times n}\cap\mathcal{O}_{mn},$$

and consider its submonoids

$$\mathcal{O}^+_{m\times n}=\mathcal{T}^+_{m\times n}\cap\mathcal{O}_{mn}\quad\text{and}\quad\mathcal{O}^-_{m\times n}=\mathcal{T}^-_{m\times n}\cap\mathcal{O}_{mn}$$

of all extensive transformations and of all co-extensive transformations, respectively.

Example 0.1 Let

$$\alpha_1=\left(\begin{array}{cccc|cccc|cccc}1&2&3&4&5&6&7&8&9&10&11&12\\1&3&3&2&9&12&10&10&5&6&6&8\end{array}\right),\ \alpha_2=\left(\begin{array}{cccc|cccc|cccc}1&2&3&4&5&6&7&8&9&10&11&12\\5&5&5&6&6&6&6&7&10&11&11&11\end{array}\right),$$

$$\alpha_3=\left(\begin{array}{cccc|cccc|cccc}1&2&3&4&5&6&7&8&9&10&11&12\\1&3&3&4&9&9&10&10&10&11&11&12\end{array}\right)\text{ and }\alpha_4=\left(\begin{array}{cccc|cccc|cccc}1&2&3&4&5&6&7&8&9&10&11&12\\1&1&2&2&3&3&4&4&5&6&8&8\end{array}\right).$$

Then, we have: $\alpha_1\in\mathcal{T}_{3\times4}$ but $\alpha_1\notin\mathcal{O}_{3\times4}$; $\alpha_2\in\mathcal{O}_{3\times4}$ but $\alpha_2\notin\mathcal{O}^+_{3\times4}$ and $\alpha_2\notin\mathcal{O}^-_{3\times4}$; and $\alpha_3\in\mathcal{O}^+_{3\times4}$ and $\alpha_4\in\mathcal{O}^-_{3\times4}$.

Notice that, as $\mathcal{O}^-_n$ and $\mathcal{O}^+_n$, the monoids $\mathcal{O}^-_{m\times n}$ and $\mathcal{O}^+_{m\times n}$ are isomorphic. In fact, the function which maps each transformation $\alpha\in\mathcal{O}^-_{m\times n}$ into the transformation $\alpha'\in\mathcal{O}^+_{m\times n}$ defined by $x\alpha'=mn+1-(mn+1-x)\alpha$, for all $x\in X_{mn}$, is an isomorphism of monoids. Moreover, for $\alpha\in\mathcal{O}_{m\times n}$, we have $\alpha=\alpha_1\alpha_2$, for some $\alpha_1\in\mathcal{O}^-_{m\times n}$ and $\alpha_2=\mathcal{O}^+_{m\times n}$. For instance, we may take the transformations α_1 and α_2 defined by

$$x\alpha_1=\begin{cases}x\alpha & \text{if } x\alpha\le x\\ x & \text{if } x\alpha\ge x\end{cases}\quad\text{and}\quad x\alpha_2=\begin{cases}x\alpha & \text{if } x\le x\alpha\\ x & \text{if } x\ge x\alpha\end{cases},$$

for all $x\in X_{mn}$. Notice that, in this case, we also have $\alpha=\alpha_2\alpha_1$.

The monoid $\mathcal{O}_{m\times n}$ was considered by Huisheng and Dingyu in [15] who described its Green relations. In this paper we determine the cardinals and the ranks of the monoids $\mathcal{O}_{m\times n}$, $\mathcal{O}^+_{m\times n}$ and $\mathcal{O}^-_{m\times n}$.

Next, let S and T be two semigroups. Let $\delta : T \longrightarrow \mathcal{T}(S)$ be an anti-homomorphism of semigroups and let $\varphi : S \longrightarrow \mathcal{T}(T)$ be a homomorphism of semigroups. For $s \in S$ and $u \in T$, denote $(s)(u)\delta$ by $u \centerdot s$ and $(u)(s)\varphi$ by u^s. We say that δ is a *left action* of T on S and that φ is a *right action* of S on T if they verify the following rules:

(SPR) $(uv)^s = u^{v \centerdot s} v^s$, for $s \in S$ and $u, v \in T$ (*Sequential Processing Rule*); and

(SCR) $u \centerdot (sr) = (u \centerdot s)(u^s \centerdot r)$, for $s, r \in S$ and $u \in T$ (*Serial Composition Rule*).

In [16] Kunze proved that the set $S \times T$ is a semigroup with respect to the following multiplication:

$$(s, u)(r, v) = (s(u \centerdot r), u^r v),$$

for $s, r \in S$ and $u, v \in T$. We denote this semigroup by $S_\delta \bowtie_\varphi T$ (or simply by $S \bowtie T$, if it is not ambiguous) and call it the *bilateral semidirect product* of S and T associated with δ and φ.

We notice that this concept was strongly motivated by automata theoretic ideas.

If S and T are monoids and the actions δ and φ preserve the identity (i.e. $1 \centerdot s = s$, for $s \in S$, and $u^1 = u$, for $u \in T$) and are *monoidal* (i.e. $u \centerdot 1 = 1$, for $u \in T$, and $1^s = 1$, for $s \in S$), then $S \bowtie T$ is a monoid with identity $(1, 1)$.

Observe that, if φ is a trivial action (i.e. $(S)\varphi = \{\mathrm{id}_T\}$) then $S \bowtie T = S * T$ is an usual semidirect product, if δ is a trivial action (i.e. $(T)\delta = \{\mathrm{id}_S\}$) then $S \bowtie T$ coincides with a reverse semidirect product $T *_r S$ (by interchanging the coordinates) and if both actions are trivial then $S \bowtie T$ is the usual direct product $S \times T$. Observe also that the bilateral semidirect product is quite different from the Rhodes and Tilson [19] *double* semidirect product, where the second components multiply always as a direct product.

In [17] Kunze proved that the monoid $\mathcal{O}_n$ is a quotient of a bilateral semidirect product of its subsemigroups $\mathcal{O}_n^-$ and $\mathcal{O}_n^+$. See also [18, 7]. We finish this paper by constructing a bilateral semidirect product decomposition of $\mathcal{O}_{m\times n}$ in terms of is submonoids $\mathcal{O}_{m\times n}^-$ and $\mathcal{O}_{m\times n}^+$, thus generalizing Kunze's result.

1 Wreath Products of Transformation Semigroups

In [4] Araújo and Schneider proved that the rank of $\mathcal{T}_{m\times n}$ is 4, by using the concept of wreath product of transformation semigroups. This approach will be also very useful in this paper.

For simplicity, we define the wreath product $\mathcal{T}_n \wr \mathcal{T}_m$ of $\mathcal{T}_n$ and $\mathcal{T}_m$ as being the monoid with underlying set $\mathcal{T}_n^m \times \mathcal{T}_m$ and multiplication defined by

$$(\alpha_1, \ldots, \alpha_m; \beta)(\alpha'_1, \ldots, \alpha'_m; \beta') = (\alpha_1\alpha'_{1\beta}, \ldots, \alpha_m\alpha'_{m\beta}; \beta\beta'),$$

for all $(\alpha_1, \ldots, \alpha_m; \beta),\ (\alpha'_1, \ldots, \alpha'_m; \beta') \in \mathcal{T}_n^m \times \mathcal{T}_m$.

Let $\alpha \in \mathcal{T}_{m\times n}$ and let $\beta = \alpha/\rho \in \mathcal{T}_m$ be the *quotient* map of α by ρ, i.e. for all $j \in \{1, \ldots, m\}$, we have $A_j\alpha \subseteq A_{j\beta}$. For each $j \in \{1, \ldots, m\}$, define $\alpha_j \in \mathcal{T}_n$ by

$$k\alpha_j = ((j-1)n + k)\alpha - (j\beta - 1)n,$$

for all $k \in \{1, \ldots, n\}$. Let $\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m; \beta) \in \mathcal{T}_n^m \times \mathcal{T}_m$. With this notation, the function

$$\begin{array}{rccc} \psi : & \mathcal{T}_{m\times n} & \longrightarrow & \mathcal{T}_n \wr \mathcal{T}_m \\ & \alpha & \longmapsto & \overline{\alpha} \end{array}$$

is an isomorphism (see [4, Lemma 2.1]). From this fact, one can immediately conclude that the cardinality of $\mathcal{T}_{m\times n}$ is $n^{nm}m^m$.

Example 1.1 Consider the transformation

$$\alpha = \left(\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 3 & 3 & 2 & 9 & 12 & 10 & 10 & 5 & 6 & 6 & 8 \end{array}\right) \in \mathcal{T}_{3\times 4}.$$

Then, we have $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3; \beta)$, with $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 2 \end{pmatrix}$ and $\alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 4 \end{pmatrix}$.

Notice that the restriction of ψ to $\mathcal{O}_{m\times n}$ is not, in general, an isomorphism from $\mathcal{O}_{m\times n}$ into the wreath product $\mathcal{O}_n \wr \mathcal{O}_m$ (that may be defined similarly to $\mathcal{T}_n \wr \mathcal{T}_m$). For instance, for $m = n = 2$, take $\alpha = (\alpha_1, \alpha_2; \beta)$, with $\alpha_1 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Then $\alpha \in \mathcal{O}_2 \wr \mathcal{O}_2$ and $\alpha\psi^{-1} = \left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 1 \end{array}\right) \notin \mathcal{O}_{2\times 2}$.

In fact, the monoid $\mathcal{O}_{m\times n}$ is not, in general, isomorphic to $\mathcal{O}_m \wr \mathcal{O}_n$. For example, we have $|\mathcal{O}_{2\times 2}| = 19 < 27 = |\mathcal{O}_2 \wr \mathcal{O}_2|$.

Consider

$$\overline{\mathcal{O}}_{m\times n} = \{(\alpha_1, \ldots, \alpha_m; \beta) \in \mathcal{O}_n^m \times \mathcal{O}_m \mid j\beta = (j+1)\beta \text{ implies } n\alpha_j \le 1\alpha_{j+1}, \text{ for all } j \in \{1, \ldots, m-1\}\}.$$

Notice that, if $(\alpha_1, \ldots, \alpha_m; \beta) \in \overline{\mathcal{O}}_{m\times n}$ and $1 \le i < j \le m$ are such that $i\beta = j\beta$, then $n\alpha_i \le 1\alpha_j$.

Lemma 1.2 $\overline{\mathcal{O}}_{m\times n} = \mathcal{O}_{m\times n}\psi$.

Proof. First, let $(\alpha_1, \ldots, \alpha_m; \beta) \in \overline{\mathcal{O}}_{m\times n}$ and take $\alpha = (\alpha_1, \ldots, \alpha_m; \beta)\psi^{-1} \in \mathcal{T}_{m\times n}$. Let $x, y \in \{1, \ldots, mn\}$ be such that $x \le y$. Then $x \in A_i$ and $y \in A_j$, for some $1 \le i \le j \le m$. Hence, $x\alpha = (x - (i-1)n)\alpha_i + (i\beta - 1)n$ and $y\alpha = (y - (j-1)n)\alpha_j + (j\beta - 1)n$. If $i = j$ then

$$\begin{aligned} x \le y \quad &\Rightarrow \quad x - (j-1)n \le y - (j-1)n \\ &\Rightarrow \quad (x - (j-1)n)\alpha_j \le (y - (j-1)n)\alpha_j \\ &\Rightarrow \quad x\alpha = (x - (j-1)n)\alpha_j + (j\beta - 1)n \le (y - (j-1)n)\alpha_j + (j\beta - 1)n = y\alpha\ . \end{aligned}$$

If $i < j$ and $i\beta < j\beta$ then $x\alpha \le (i\beta)n \le (j\beta - 1)n < (j\beta - 1)n + 1 \le y\alpha$. Finally, if $i < j$ and $i\beta = j\beta$, then $(x - (i-1)n)\alpha_i \le n\alpha_i \le 1\alpha_j \le (x - (j-1)n)\alpha_j$, whence

$$x\alpha = (x - (i-1)n)\alpha_i + (i\beta - 1)n \le (y - (j-1)n)\alpha_j + (i\beta - 1)n = (y - (j-1)n)\alpha_j + (j\beta - 1)n = y\alpha.$$

Hence, α is an order-preserving transformation and so $\overline{\mathcal{O}}_{m\times n} \subseteq \mathcal{O}_{m\times n}\psi$.

Conversely, let $\alpha \in \mathcal{O}_{m\times n}$ and $(\alpha_1, \ldots, \alpha_m; \beta) = \alpha\psi$.

We start by showing that $\beta \in \mathcal{O}_m$. Let $i, j \in \{1, \ldots, m\}$ be such that $i \le j$. As $in \in A_i$ and $A_i\alpha \subseteq A_{i\beta}$, we have $(in)\alpha \in A_{i\beta}$. Similarly, $(jn)\alpha \in A_{j\beta}$. On the other hand, $i \le j$ implies $in \le jn$ and so $(in)\alpha \le (jn)\alpha$. It follows that $i\beta \le j\beta$.

Next, we prove that $\alpha_j \in \mathcal{O}_n$, for all $1 \le j \le m$. Let $j \in \{1, \ldots, m\}$ and let $x, y \in \{1, \ldots, n\}$ be such that $x \le y$. Then $(j-1)n + x \le (j-1)n + y$, whence $((j-1)n + x)\alpha \le ((j-1)n + y)\alpha$ and so $x\alpha_j = ((j-1)n + x)\alpha - (j\beta - 1)n \le ((j-1)n + y)\alpha - (j\beta - 1)n = y\alpha_j$.

Finally, let $j \in \{1, \ldots, m-1\}$ be such that $j\beta = (j+1)\beta$. Then, as $\alpha \in \mathcal{O}_{mn}$, we have

$$n\alpha_j = ((j-1)n+n)\alpha - (j\beta-1)n = (jn)\alpha - (j\beta-1)n \le (jn+1)\alpha - (j\beta-1)n = (jn+1)\alpha - ((j+1)\beta-1)n = 1\alpha_{j+1}.$$

Thus, $\mathcal{O}_{m\times n}\psi \subseteq \overline{\mathcal{O}}_{m\times n}$ and so $\overline{\mathcal{O}}_{m\times n} = \mathcal{O}_{m\times n}\psi$, as required. ∎

It follows immediately that:

Proposition 1.3 *The set* $\overline{\mathcal{O}}_{m\times n}$ *is a submonoid of* $\mathcal{T}_n \wr \mathcal{T}_m$ *(and of* $\mathcal{O}_n \wr \mathcal{O}_m$*) isomorphic to* $\mathcal{O}_{m\times n}$. ∎

Next, consider

$$\overline{\mathcal{T}}^+_{m\times n} = \{(\alpha_1,\ldots,\alpha_m;\beta) \in \mathcal{T}_n^m \times \mathcal{T}_m^+ \mid j\beta = j \text{ implies } \alpha_j \in \mathcal{T}_n^+, \text{ for all } j \in \{1,\ldots,m\}\}.$$

Notice that, as $\beta \in \mathcal{T}_m^+$ implies $m\beta = m$, then $\overline{\mathcal{T}}^+_{m\times n} \subseteq \mathcal{T}_n^{m-1} \times \mathcal{T}_n^+ \times \mathcal{T}_m^+$.

Lemma 1.4 $\overline{\mathcal{T}}^+_{m\times n} = \mathcal{T}^+_{m\times n}\psi$.

Proof. In order to show that $\overline{\mathcal{T}}^+_{m\times n} \subseteq \mathcal{T}^+_{m\times n}\psi$, let $(\alpha_1,\ldots,\alpha_m;\beta) \in \overline{\mathcal{T}}^+_{m\times n}$ and take $\alpha = (\alpha_1,\ldots,\alpha_m;\beta)\psi^{-1}$. We aim to show that $\alpha \in \mathcal{T}^+_{mn}$. Let $x \in \{1,\ldots,mn\}$ and take $j \in \{1,\ldots,m\}$ such that $x \in A_j$. Then $x\alpha \in A_{j\beta}$ and, as $\beta \in \mathcal{T}_m^+$, we have $j \leq j\beta$. If $j < j\beta$ then $j \leq j\beta - 1$ and so $x \leq jn \leq (j\beta-1)n < (j\beta-1)n+1 \leq x\alpha$. If $j\beta = j$ then $\alpha_j \in \mathcal{T}_m^+$ and so $x = (x-(j-1)n)+(j-1)n \leq (x-(j-1)n)\alpha_j + (j-1)n = x\alpha$. Hence $\alpha \in \mathcal{T}^+_{mn}$.

Conversely, let $\alpha \in \mathcal{T}^+_{m\times n}$ and $\alpha\psi = (\alpha_1,\ldots,\alpha_m;\beta)$.

First, observe that, for all $j \in \{1,\ldots,m\}$, as $A_j\alpha \subseteq A_{j\beta}$ and $\alpha \in \mathcal{T}^+_{m\times n}$, we have $jn \leq (jn)\alpha \leq (j\beta)n$ and so $j \leq j\beta$. Hence $\beta \in \mathcal{T}_m^+$.

Next, let $j \in \{1,\ldots,m\}$ be such that $j\beta = j$ and take $k \in \{1,\ldots,n\}$. Then

$$k\alpha_j = ((j-1)n+k)\alpha - (j\beta-1)n \geq (j-1)n+k-(j\beta-1)n = (j-1)n+k-(j-1)n = k.$$

Hence, $\alpha_j \in \mathcal{T}_n^+$ and so $\mathcal{T}^+_{m\times n}\psi \subseteq \overline{\mathcal{T}}^+_{m\times n}$, as required. ■

Thus, we have:

Proposition 1.5 *The set $\overline{\mathcal{T}}^+_{m\times n}$ is a submonoid of $\mathcal{T}_n \wr \mathcal{T}_m$ isomorphic to $\mathcal{T}^+_{m\times n}$.* ■

Now, let

$$\begin{aligned}\overline{\mathcal{O}}^+_{m\times n} &= \overline{\mathcal{O}}_{m\times n} \cap \overline{\mathcal{T}}^+_{m\times n}\\ &= \{(\alpha_1,\ldots,\alpha_m;\beta) \in \mathcal{O}_n^{m-1} \times \mathcal{O}_n^+ \times \mathcal{O}_m^+ \mid j\beta = (j+1)\beta \text{ implies } n\alpha_j \leq 1\alpha_{j+1} \text{ and}\\ &\qquad j\beta = j \text{ implies } \alpha_j \in \mathcal{O}_n^+, \text{ for all } j \in \{1,\ldots,m-1\}\}\ .\end{aligned}$$

As ψ is injective, by propositions 1.3 and 1.5, we have

$$\overline{\mathcal{O}}^+_{m\times n} = \mathcal{O}_{m\times n}\psi \cap \mathcal{T}^+_{m\times n}\psi = (\mathcal{O}_{m\times n} \cap \mathcal{T}^+_{m\times n})\psi = \mathcal{O}^+_{m\times n}\psi$$

and so:

Corollary 1.6 *The set $\overline{\mathcal{O}}^+_{m\times n}$ is a submonoid of $\mathcal{T}_n \wr \mathcal{T}_m$ (and of $\mathcal{O}_n \wr \mathcal{O}_m$) isomorphic to $\mathcal{O}^+_{m\times n}$.* ■

Similarly, being

$$\begin{aligned}\overline{\mathcal{O}}^-_{m\times n} &= \overline{\mathcal{O}}_{m\times n} \cap \overline{\mathcal{T}}^-_{m\times n}\\ &= \{(\alpha_1,\ldots,\alpha_m;\beta) \in \mathcal{O}_n^- \times \mathcal{O}_n^{m-1} \times \mathcal{O}_m^- \mid (j-1)\beta = j\beta \text{ implies } n\alpha_{j-1} \leq 1\alpha_j \text{ and}\\ &\qquad j\beta = j \text{ implies } \alpha_j \in \mathcal{O}_n^-, \text{ for all } j \in \{2,\ldots,m\}\},\end{aligned}$$

we have:

Proposition 1.7 *The set $\overline{\mathcal{O}}^-_{m\times n}$ is a submonoid of $\mathcal{T}_n \wr \mathcal{T}_m$ (and of $\mathcal{O}_n \wr \mathcal{O}_m$) isomorphic to $\mathcal{O}^-_{m\times n}$.* ■

2 Cardinals

In this section we use the previous bijections to obtain formulas for the number of elements of the monoids $\mathcal{O}_{m\times n}$, $\mathcal{O}^+_{m\times n}$ and $\mathcal{O}^-_{m\times n}$.

In order to count the elements of $\mathcal{O}_{m\times n}$, on one hand, for each transformation $\beta\in\mathcal{O}_m$, we determine the number of sequences $(\alpha_1,\ldots,\alpha_m)\in\mathcal{O}_n^m$ such that $(\alpha_1,\ldots,\alpha_m;\beta)\in\overline{\mathcal{O}}_{m\times n}$ and, on the other hand, we notice that this last number just depends of the kernel of β (and not of β itself).

With this purpose, let $\beta\in\mathcal{O}_m$. Suppose that $\operatorname{Im}\beta=\{b_1<b_2<\cdots<b_t\}$, for some $1\le t\le m$, and define $k_i=|b_i\beta^{-1}|$, for $i=1,\ldots,t$. Being β an order-preserving transformation, the sequence $(k_1,\ldots,k_t)$ determines the kernel of β: we have $\{k_1+\cdots+k_{i-1}+1,\ldots,k_1+\cdots+k_i\}\beta=\{b_i\}$, for $i=1,\ldots,t$ (considering $k_1+\cdots+k_{i-1}+1=1$, with $i=1$). We define the *kernel type* of β as being the sequence $(k_1,\ldots,k_t)$. Notice that $1\le k_i\le m$, for $i=1,\ldots,t$, and $k_1+k_2+\cdots+k_t=m$.

Now, recall that the number of non-decreasing sequences of length k from a chain with n elements (which is the same as the number of k-combinations with repetition from a set with n elements) is $\binom{n+k-1}{k}=\binom{n+k-1}{n-1}$ (see [10], for example). Next, notice that, as a sequence $(\alpha_1,\ldots,\alpha_k)\in\mathcal{O}_n^k$ satisfies the condition $n\alpha_j\le 1\alpha_{j+1}$, for all $1\le j\le k-1$, if and only if the concatenation sequence of the images of the transformations $\alpha_1,\ldots,\alpha_k$ (by this order) is still a non-decreasing sequence, then we have $\binom{n+kn-1}{n-1}$ such sequences.

Since $(\alpha_1,\ldots,\alpha_m;\beta)\in\overline{\mathcal{O}}_{m\times n}$ if and only if, for all $1\le i\le t$, $\alpha_{k_1+\cdots+k_{i-1}+1},\ldots,\alpha_{k_1+\cdots+k_i}$ are k_i order-preserving transformations such that the concatenation sequence of their images (by this order) is still a non-decreasing sequence, then we have $\prod_{i=1}^t\binom{k_in+n-1}{n-1}$ elements in $\overline{\mathcal{O}}_{m\times n}$ whose $(m+1)$-component is β.

Finally, now it is also clear that if β and β' are two elements of $\mathcal{O}_m$ with the same kernel type then $(\alpha_1,\ldots,\alpha_m;\beta)\in\overline{\mathcal{O}}_{m\times n}$ if and only if $(\alpha_1,\ldots,\alpha_m;\beta')\in\overline{\mathcal{O}}_{m\times n}$. Thus, as the number of transformations $\beta\in\mathcal{O}_m$ with kernel type of length t ($1\le t\le m$) coincides with the number of t-combinations (without repetition) from a set with m elements, it follows:

Theorem 2.1 $$|\mathcal{O}_{m\times n}|=\sum_{\substack{1\le k_1,\ldots,k_t\le m\\ k_1+\cdots+k_t=m\\ 1\le t\le m}}\binom{m}{t}\prod_{i=1}^t\binom{k_in+n-1}{n-1}.$$ ∎

The table below gives us an idea of the size of the monoid $\mathcal{O}_{m\times n}$.

$m\setminus n$	1	2	3	4	5	6
1	1	3	10	35	126	462
2	3	19	156	1555	17878	225820
3	10	138	2845	78890	2768760	115865211
4	35	1059	55268	4284451	454664910	61824611940
5	126	8378	1109880	241505530	77543615751	34003513468232
6	462	67582	22752795	13924561150	13556873588212	19134117191404027

Next, we describe a process to count the number of elements of $\mathcal{O}^+_{m\times n}$.
First, recall that the cardinal of $\mathcal{O}^+_n$ is the n^{th}-Catalan number, i.e. $|\mathcal{O}^+_n|=\frac{1}{n+1}\binom{2n}{n}$. See [21].
It is also useful to consider the following numbers:

$$\theta(n,i)=|\{\alpha\in\mathcal{O}^+_n \mid 1\alpha=i\}|,$$

for $1\le i\le n$. Clearly, we have $|\mathcal{O}^+_n|=\sum_{i=1}^n\theta(n,i)$. Moreover, for $2\le i\le n-1$, we have

$$\theta(n,i)=\theta(n,i+1)+\theta(n-1,i-1).$$

In fact, $\{\alpha \in \mathcal{O}_n^+ \mid\ 1\alpha = i\} = \{\alpha \in \mathcal{O}_n^+ \mid\ 1\alpha = i < 2\alpha\} \,\dot{\cup}\, \{\alpha \in \mathcal{O}_n^+ \mid\ 1\alpha = 2\alpha = i\}$ and it is easy to show that the function which maps each transformation $\beta \in \{\alpha \in \mathcal{O}_n^+ \mid\ 1\alpha = i < 2\alpha\}$ into the transformation

$$\begin{pmatrix} 1 & 2 & \dots & n \\ i+1 & 2\beta & \dots & n\beta \end{pmatrix} \in \{\alpha \in \mathcal{O}_n^+ \mid\ 1\alpha = i+1\}$$

and the function which maps each transformation $\beta \in \{\alpha \in \mathcal{O}_{n-1}^+ \mid\ 1\alpha = i-1\}$ into the transformation

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ i & i & 2\beta+1 & \dots & (n-2)\beta+1 & (n-1)\beta+1 \end{pmatrix} \in \{\alpha \in \mathcal{O}_n^+ \mid\ 1\alpha = 2\alpha = i\}$$

are bijections. Thus

$$\begin{aligned} \theta(n,i) &= |\{\alpha \in \mathcal{O}_n^+ \mid\ 1\alpha = i < 2\alpha\}| + |\{\alpha \in \mathcal{O}_n^+ \mid\ 1\alpha = 2\alpha = i\}| \\ &= |\{\alpha \in \mathcal{O}_n^+ \mid\ 1\alpha = i+1\}| + |\{\alpha \in \mathcal{O}_{n-1}^+ \mid\ 1\alpha = i-1\}| \\ &= \theta(n,i+1) + \theta(n-1,i-1). \end{aligned}$$

Also, it is not hard to prove that $\theta(n,2) = \theta(n,1) = \sum_{i=1}^{n-1} \theta(n-1,i) = |\mathcal{O}_{n-1}^+|$.

Now, we can prove:

Lemma 2.2 *For all* $1 \le i \le n$, $\theta(n,i) = \frac{i}{n}\binom{2n-i-1}{n-i} = \frac{i}{n}\binom{2n-i-1}{n-1}$.

Proof. We prove the lemma by induction on n.

For $n = 1$, it is clear that $\theta(1,1) = 1 = \frac{1}{1}\binom{2-1-1}{1-1}$.

Let $n \ge 2$ and suppose that the formula is valid for $n-1$.

Next, we prove the formula for n by induction on i.

For $i = 1$, as observed above, we have $\theta(n,1) = |\mathcal{O}_{n-1}^+| = \frac{1}{n}\binom{2n-2}{n-1}$.

For $i = 2$, we have $\theta(n,2) = \theta(n,1) = \frac{1}{n}\binom{2n-2}{n-1} = \frac{2}{n}\frac{(2n-2)!}{(n-1)!(n-1)!}\frac{n-1}{2n-2} = \frac{2}{n}\frac{(2n-3)!}{(n-1)!(n-2)!} = \frac{2}{n}\binom{2n-3}{n-1}$.

Now, suppose that the formula is valid for $i-1$, with $3 \le i \le n$. Then, using both induction hypothesis on i and on n in the second equality, we have $\theta(n,i) = \theta(n,i-1) - \theta(n-1,i-2) = \frac{i-1}{n}\binom{2n-i}{n-1} - \frac{i-2}{n-1}\binom{2n-i-1}{n-2} = \frac{i-1}{n}\frac{(2n-i)!}{(n-1)!(n-i+1)!} - \frac{i-2}{n-1}\frac{(2n-i-1)!}{(n-2)!(n-i+1)!} = \frac{i(n-i+1)}{n(2n-i)}\frac{(2n-i)!}{(n-1)!(n-i+1)!} = \frac{i}{n}\binom{2n-i-1}{n-1}$, as required. ∎

Recall that $(\alpha_1,\dots,\alpha_m;\beta) \in \overline{\mathcal{O}}_{m\times n}^+$ if and only if $\beta \in \mathcal{O}_m^+$, $\alpha_m \in \mathcal{O}_n^+$, $\alpha_1,\dots,\alpha_{m-1} \in \mathcal{O}_n$ and, for all $j \in \{1,\dots,m-1\}$, $j\beta = (j+1)\beta$ implies $n\alpha_j \le 1\alpha_{j+1}$ and $j\beta = j$ implies $\alpha_j \in \mathcal{O}_n^+$.

Let $\beta \in \mathcal{O}_m^+$. As for the monoid $\mathcal{O}_{m\times n}$, we aim to count the number of sequences $(\alpha_1,\dots,\alpha_m) \in \mathcal{O}_n^m$ such that $(\alpha_1,\dots,\alpha_m;\beta) \in \overline{\mathcal{O}}_{m\times n}^+$.

Let $(k_1,\dots,k_t)$ be the kernel type of β. Let $K_i = \{k_1+\cdots+k_{i-1}+1,\dots,k_1+\cdots+k_i\}$, for $i = 1,\dots,t$. Then, β fixes a point in K_i if and only if it fixes $k_1+\cdots+k_i$, for $i = 1,\dots,t$. It follows that $(\alpha_1,\dots,\alpha_m;\beta) \in \overline{\mathcal{O}}_{m\times n}^+$ if and only if, for all $1 \le i \le t$:

1. If β does not fix a point in K_i, then $\alpha_{k_1+\cdots+k_{i-1}+1},\dots,\alpha_{k_1+\cdots+k_i}$ are k_i order-preserving transformations such that the concatenation sequence of their images (by this order) is still a non-decreasing sequence (in this case, we have $\binom{k_in+n-1}{n-1}$ subsequences $(\alpha_{k_1+\cdots+k_{i-1}+1},\dots,\alpha_{k_1+\cdots+k_i})$ allowed);

2. If β fixes a point in K_i, then $\alpha_{k_1+\cdots+k_{i-1}+1},\dots,\alpha_{k_1+\cdots+k_i-1}$ are k_i-1 order-preserving transformations such that the concatenation sequence of their images (by this order) is still a non-decreasing sequence, $n\alpha_{k_1+\cdots+k_i-1} \le 1\alpha_{k_1+\cdots+k_i}$ and $\alpha_{k_1+\cdots+k_i} \in \mathcal{O}_n^+$ (in this case, we have $\sum_{j=1}^n \binom{(k_i-1)n+j-1}{j-1}\theta(n,j)$ subsequences $(\alpha_{k_1+\cdots+k_{i-1}+1},\dots,\alpha_{k_1+\cdots+k_i})$ allowed).

Define

$$\mathfrak{d}(\beta,i) = \begin{cases} \binom{k_in+n-1}{n-1}, & \text{if } (k_1+\cdots+k_i)\beta \ne k_1+\cdots+k_i \\ \sum_{j=1}^n \frac{j}{n}\binom{2n-j-1}{n-1}\binom{(k_i-1)n+j-1}{j-1}, & \text{if } (k_1+\cdots+k_i)\beta = k_1+\cdots+k_i, \end{cases}$$

for all $1 \le i \le t$.

Thus, we have:

Proposition 2.3 $|\mathcal{O}^+_{m\times n}| = \sum_{\beta\in\mathcal{O}^+_m}\prod_{i=1}^t \mathfrak{d}(\beta,i)$. ∎

Next, we obtain a formula for $|\mathcal{O}^+_{m\times n}|$ which does not depend of $\beta\in\mathcal{O}^+_m$.

Let β be an element of $\mathcal{O}^+_m$ with kernel type $(k_1,\ldots,k_t)$. Define $s_\beta=(s_1,\ldots,s_t)\in\{0,1\}^{t-1}\times\{1\}$ by $s_i=1$ if and only if $(k_1+\cdots+k_i)\beta = k_1+\cdots+k_i$, for all $1\le i\le t-1$.

Let $1\le t,k_1,\ldots,k_t\le m$ be such that $k_1+\cdots+k_t=m$ and let $(s_1,\ldots,s_t)\in\{0,1\}^{t-1}\times\{1\}$. Let $k=(k_1,\ldots,k_t)$ and $s=(s_1,\ldots,s_t)$. Define

$$\Delta(k,s) = |\{\beta\in\mathcal{O}^+_m \mid \beta \text{ has kernel type } k \text{ and } s_\beta = s\}|.$$

In order to get a formula for $\Delta(k,s)$, we count the number of distinct restrictions to unions of partition classes of the kernel of transformations β of $\mathcal{O}^+_m$ with kernel type k and $s_\beta=s$ corresponding to maximal subsequences of consecutive zeros of s.

Let β be an element of $\mathcal{O}^+_m$ with kernel type k and $s_\beta=s$.

First, notice that, given $i\in\{1,\ldots,t\}$, if $s_i=1$ then $K_i\beta=\{k_1+\cdots+k_i\}$ and if $s_i=0$ then the (unique) element of $K_i\beta$ belongs to K_j, for some $i<j\le t$.

Next, let $i\in\{1,\ldots,t\}$ and $r\in\{1,\ldots,t-i\}$ be such that $s_j=0$, for all $j\in\{i,\ldots,i+r-1\}$, $s_{i+r}=1$ and, if $i>1$, $s_{i-1}=1$ (i.e. $(s_i,\ldots,s_{i+r-1})$ is a maximal subsequence of consecutive zeros of s). Then

$$(K_i\cup\cdots\cup K_{i+r-2}\cup K_{i+r-1})\beta \subseteq K_{i+1}\cup\cdots\cup K_{i+r-1}\cup(K_{i+r}\setminus\{k_1+\cdots+k_{i+r}\}).$$

Let $\ell_j=|K_{i+j}\cap(K_i\cup\cdots\cup K_{i+r-1})\beta|$, for $1\le j\le r$. Hence, we have $\ell_1,\ldots,\ell_{r-1}\ge 0$, $\ell_r\ge 1$, $\ell_1+\cdots+\ell_r=r$ and $0\le \ell_1+\cdots+\ell_j\le j$, for all $1\le j\le r-1$.

On the other hand, given $\ell_1,\ldots,\ell_r$ such that $\ell_1,\ldots,\ell_{r-1}\ge 0$, $\ell_r\ge 1$, $\ell_1+\cdots+\ell_r=r$ and $0\le\ell_1+\cdots+\ell_j\le j$, for all $1\le j\le r-1$, we have precisely

$$\binom{k_{i+1}}{\ell_1}\binom{k_{i+2}}{\ell_2}\cdots\binom{k_{i+r-1}}{\ell_{r-1}}\binom{k_{i+r}-1}{\ell_r} = \binom{k_{i+r}-1}{\ell_r}\prod_{j=1}^{r-1}\binom{k_{i+j}}{\ell_j}$$

distinct restrictions to $K_i\cup\cdots\cup K_{i+r-1}$ of transformations β of $\mathcal{O}^+_m$, with kernel type k and $s_\beta=s$, such that $\ell_j=|K_{i+j}\cap(K_i\cup\cdots\cup K_{i+r-1})\beta|$, for $1\le j\le r$. It follow that the number of distinct restrictions to $K_i\cup\cdots\cup K_{i+r-1}$ of transformations β of $\mathcal{O}^+_m$ with kernel type k and $s_\beta=s$ is

$$\sum_{\substack{\ell_1+\cdots+\ell_r=r\\ 0\le\ell_1+\cdots+\ell_j\le j,\ 1\le j\le r-1\\ \ell_1,\ldots,\ell_{r-1}\ge 0,\ \ell_r\ge 1}} \binom{k_{i+r}-1}{\ell_r}\prod_{j=1}^{r-1}\binom{k_{i+j}}{\ell_j}.$$

Now, let p be the number of distinct maximal subsequences of consecutive zeros of s. Clearly, if $p=0$ then $\Delta(k,s)=1$. Hence, suppose that $p\ge 1$ and let $1\le u_1<v_1<u_2<v_2<\cdots<u_p<v_p\le t$ be such that

$$\{j\in\{1,\ldots,t\}\mid\ s_j=0\} = \bigcup_{i=1}^p\{u_i,\ldots,v_i-1\}$$

(i.e. $(s_{u_i},\ldots,s_{v_i-1})$, with $1\le i\le p$, are the p distinct maximal subsequences of consecutive zeros of s). Then, being $r_i=v_i-u_i$, for $1\le i\le p$, we have

$$\Delta(k,s) = \prod_{i=1}^p \sum_{\substack{\ell_1+\cdots+\ell_{r_i}=r_i\\ 0\le\ell_1+\cdots+\ell_j\le j\ \ 1\le j\le r_i-1\\ \ell_1,\ldots,\ell_{r_i-1}\ge 0,\ \ell_{r_i}\ge 1}} \binom{k_{u_i+r_i}-1}{\ell_{r_i}}\prod_{j=1}^{r_i-1}\binom{k_{u_i+j}}{\ell_j}.$$

Finally, notice that, if β and β' two elements of $\mathcal{O}_m^+$ with kernel type $k=(k_1,\ldots,k_t)$ such that $s_{\beta'}=s_\beta$, then $\mathfrak{d}(\beta,i)=\mathfrak{d}(\beta',i)$, for all $1\le i\le t$. Thus, defining

$$\Lambda(k,s)=\prod_{i=1}^{t}\mathfrak{d}(\beta,i),$$

where β is any transformation of $\mathcal{O}_m^+$ with kernel type k and $s_\beta=s$, we have:

Theorem 2.4 $|\mathcal{O}_{m\times n}^+| = \displaystyle\sum_{\substack{k=(k_1,\ldots,k_t)\\ 1\le k_1,\ldots,k_t\le m\\ k_1+\cdots+k_t=m\\ 1\le t\le m}}\ \sum_{s\in\{0,1\}^{t-1}\times\{1\}}\Delta(k,s)\Lambda(k,s).$ ■

We finish this section with a table that gives us an idea of the size of the monoid $\mathcal{O}_{m\times n}^+$.

$m\setminus n$	1	2	3	4	5	6
1	1	2	5	14	42	132
2	2	8	35	306	2401	21232
3	5	42	569	10024	210765	5089370
4	14	252	8482	410994	25366480	1847511492
5	42	1636	138348	18795636	3547275837	839181666224
6	132	11188	2388624	913768388	531098927994	415847258403464

Despite the unpleasant appearance, the previous formula allows us to calculate the cardinal of $\mathcal{O}_{m\times n}^+$, even for larger m and n. For instance, we have $|\mathcal{O}_{10\times 10}^+| = 47016758951069862896388976221392645550606752244$ and $|\mathcal{O}_{10\times 10}| = 50120434239662576358898758426196210942315027691269$.

3 Ranks

Our aim in this section is to determine the ranks of the monoids $\mathcal{O}_{m\times n}$, $\mathcal{O}_{m\times n}^+$ and $\mathcal{O}_{m\times n}^-$.

First, we recall some well known facts on the monoids $\mathcal{O}_n$, $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$ (see [1, 9, 21]).

Let

$$a_j=\begin{pmatrix}1&\cdots&j&j+1&j+2&\cdots&n\\1&\cdots&j&j&j+2&\cdots&n\end{pmatrix}\quad\text{and}\quad b_j=\begin{pmatrix}1&\cdots&j-1&j&j+1&\cdots&n\\1&\cdots&j-1&j+1&j+1&\cdots&n\end{pmatrix},$$

for $1\le j\le n-1$. Then $\{a_j\mid 1\le j\le n-1\}$, $\{b_j\mid 1\le j\le n-1\}$ and $\{a_j,b_j\mid 1\le j\le n-1\}$ are idempotent generating sets of $\mathcal{O}_n^-$, $\mathcal{O}_n^+$ and $\mathcal{O}_n$, respectively. Moreover, it was proved by Gomes and Howie [9] that $\{a_j,b_j\mid 1\le j\le n-1\}$ is a least-size idempotent generating set of $\mathcal{O}_n$, from which it follows that the idempotent rank of $\mathcal{O}_n$ is $2n-2$. On the other hand, it is easy to show that the transformations a_j, $1\le j\le n-1$, and b_j, $1\le j\le n-1$, are indecomposable elements (i.e. which are not product of elements distinct of themselves) of $\mathcal{O}_n^-$ and $\mathcal{O}_n^+$, respectively. It follows immediately that the rank and the idempotent rank of $\mathcal{O}_n^-$ and of $\mathcal{O}_n^+$ are equal to $n-1$. Next, consider the transformation

$$c=\begin{pmatrix}1&2&3&\cdots&n\\1&1&2&\cdots&n-1\end{pmatrix}\in\mathcal{O}_n^-.$$

Also in [9], Gomes and Howie proved that $\{b_1,\ldots,b_{n-1},c\}$ is a least-size generating set of $\mathcal{O}_n$, from which it follows that the rank of $\mathcal{O}_n$ is n.

Now, for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n-1\}$, let

$$b_{i,j} = \left(\begin{array}{c|ccccccc|c} \cdots & (i-1)n+1 & \cdots & (i-1)n+j-1 & (i-1)n+j & (i-1)n+j+1 & \cdots & in & \cdots \\ \cdots & (i-1)n+1 & \cdots & (i-1)n+j-1 & (i-1)n+j+1 & (i-1)n+j+1 & \cdots & in & \cdots \end{array} \right) \in \mathcal{O}^+_{m\times n}.$$

We are considering the non-represented elements of X_{mn} fixed by the transformation, i.e. $(x)b_{i,j} = x$, for all $x \in A_\ell$, with $1 \leq \ell \leq m$, $\ell \neq i$, $1 \leq i \leq m$ and $1 \leq j \leq n-1$. We use this convention in other definitions below.

Notice that, for $1 \leq i \leq m$ and $1 \leq j \leq n-1$,

$$\bar{b}_{i,j} = b_{i,j}\psi = (1, \dots, 1, b_j, 1, \dots, 1; 1) \in \overline{\mathcal{O}}^+_{m\times n},$$

with $b_j \in \mathcal{O}^+_n$ in the i^{th} component and 1 representing the identity map (of $\mathcal{T}_n$ or of $\mathcal{T}_m$).

Next, for $i \in \{1, \dots, m-1\}$ and $j \in \{1, \dots, n\}$, let

$$t_{i,j} = \left(\begin{array}{c|cccccc|} \cdots & (i-1)n+1 & \cdots & in-j+1 & in-j+2 & \cdots & in \\ \cdots & in+1 & \cdots & in+1 & in+2 & \cdots & in+j \end{array} \right.$$

$$\left. \begin{array}{|cccccc|c} in+1 & \cdots & in+j & in+j+1 & \cdots & (i+1)n & \cdots \\ in+j & \cdots & in+j & in+j+1 & \cdots & (i+1)n & \cdots \end{array} \right) \in \mathcal{O}^+_{m\times n}.$$

For $1 \leq j \leq n$, being

$$s_j = \begin{pmatrix} 1 & \cdots & n-j+1 & n-j+2 & \cdots & n \\ 1 & \cdots & 1 & 2 & \cdots & j \end{pmatrix} \in \mathcal{O}^-_n \quad \text{and} \quad t_j = \begin{pmatrix} 1 & \cdots & j & j+1 & \cdots & n \\ j & \cdots & j & j+1 & \cdots & n \end{pmatrix} \in \mathcal{O}^+_n,$$

(notice that $s_n = 1$ and t_n is the constant map with value n), we have

$$\bar{t}_{i,j} = t_{i,j}\psi = (1, \dots, 1, s_j, t_j, 1, \cdots, 1; b_i) \in \overline{\mathcal{O}}^+_{m\times n},$$

with $b_i \in \mathcal{O}^+_m$ (notice that we may unambiguously use the same notation for the generators of $\mathcal{O}^+_m$ and $\mathcal{O}^+_n$) and s_j in the i^{th} component.

Example 3.1 Regarding the monoid $\mathcal{O}^+_{3\times 4}$, we have:

$$b_{1,1} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 2&2&3&4&5&6&7&8&9&10&11&12 \end{array} \right) \qquad t_{1,1} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 5&5&5&5&5&6&7&8&9&10&11&12 \end{array} \right)$$

$$b_{1,2} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&3&3&4&5&6&7&8&9&10&11&12 \end{array} \right) \qquad t_{1,2} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 5&5&5&6&6&6&7&8&9&10&11&12 \end{array} \right)$$

$$b_{1,3} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&4&4&5&6&7&8&9&10&11&12 \end{array} \right) \qquad t_{1,3} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 5&5&6&7&7&7&7&8&9&10&11&12 \end{array} \right)$$

$$b_{2,1} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&3&4&6&6&7&8&9&10&11&12 \end{array} \right) \qquad t_{1,4} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 5&6&7&8&8&8&8&8&9&10&11&12 \end{array} \right)$$

$$b_{2,2} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&3&4&5&7&7&8&9&10&11&12 \end{array} \right) \qquad t_{2,1} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&3&4&9&9&9&9&9&10&11&12 \end{array} \right)$$

$$b_{2,3} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&3&4&5&6&8&8&9&10&11&12 \end{array} \right) \qquad t_{2,2} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&3&4&9&9&9&10&10&10&11&12 \end{array} \right)$$

$$b_{3,1} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&3&4&5&6&7&8&10&10&11&12 \end{array} \right) \qquad t_{2,3} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&3&4&9&9&10&11&11&11&11&12 \end{array} \right)$$

$$b_{3,2} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&3&4&5&6&7&8&9&11&11&12 \end{array} \right) \qquad t_{2,4} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&3&4&9&10&11&12&12&12&12&12 \end{array} \right)$$

$$b_{3,3} = \left(\begin{array}{cccc|cccc|cccc} 1&2&3&4&5&6&7&8&9&10&11&12 \\ 1&2&3&4&5&6&7&8&9&10&12&12 \end{array} \right)$$

Let $M = \{\alpha \in \mathcal{O}^+_{m\times n} \mid A_i\alpha \subseteq A_i, \text{for all } 1 \le i \le m\}$. Then $M\psi = \{(\alpha_1, \dots, \alpha_m; 1) \mid \alpha_1, \dots, \alpha_m \in \mathcal{O}^+_n\}$, which is clearly a monoid isomorphic to $(\mathcal{O}^+_n)^m$. As the set $\{b_j \mid 1 \le j \le n-1\}$ generates $\mathcal{O}^+_n$, then the set $\{\bar{b}_{i,j} \mid 1 \le i \le m, 1 \le j \le n-1\}$ generates $M\psi$ and so $\{b_{i,j} \mid 1 \le i \le m, 1 \le j \le n-1\}$ is a generating set of the submonoid M of $\mathcal{O}^+_{m\times n}$.

Lemma 3.2 *The monoid $\mathcal{O}^+_{2\times n}$ is generated by $\{b_{1,j}, b_{2,j}, t_{1,\ell} \mid 1 \le j \le n-1, 1 \le \ell \le n\}$.*

Proof. Let N be the submonoid of $\overline{\mathcal{O}}^+_{2\times n}$ generated by $\{\bar{b}_{1,j}, \bar{b}_{2,j}, \bar{t}_{1,\ell} \mid 1 \le j \le n-1, 1 \le \ell \le n\}$. In order to prove the lemma, we show that $N = \overline{\mathcal{O}}^+_{2\times n}$.

Notice that, an element of $\overline{\mathcal{O}}^+_{2\times n}$ has the form $(\alpha_1, \alpha_2; 1)$, with $\alpha_1, \alpha_2 \in \mathcal{O}^+_n$, or the form $(\alpha_1, \alpha_2; \beta)$, with $\beta = \binom{1\ 2}{2\ 2}$, $n\alpha_1 \le 1\alpha_2$, $\alpha_1 \in \mathcal{O}_n$ and $\alpha_2 \in \mathcal{O}^+_n$. By the above observation, the elements of the first form belong to N, whence it remains to show that the elements of the second form also belong to N. We perform this task by considering first two particular cases. Observe that $\bar{t}_{1,\ell} = (s_\ell, t_\ell; \beta)$, for $1 \le \ell \le n$.

CASE 1. Let $\alpha = (\alpha_1, t_j; \beta)$, with $1 \le j \le n$ and $\alpha_1 \in \mathcal{O}_n$ such that $\operatorname{Im}\alpha_1 = \{1, \dots, j\}$.

Then, it is easy to show that $n\alpha_1 = j$ and, for $1 \le i \le n-1$, $i\alpha_1 \le (i+1)\alpha_1 \le i\alpha_1 + 1$.

Take $s'_j = \begin{pmatrix} 1 & 2 & \cdots & j & j+1 & \cdots & n \\ n-j+1 & n-j+2 & \cdots & n & n & \cdots & n \end{pmatrix} \in \mathcal{O}^+_n$ and let $\theta = \alpha_1 s'_j$. Clearly, $\theta \in \mathcal{O}_n$. Moreover, $\theta \in \mathcal{O}^+_n$. In fact, for $1 \le i \le n$, as $i\alpha_1 \le j$, then $i\theta = i\alpha_1 s'_j = n - j + i\alpha_1$. As $n\theta = n$, if $\theta \notin \mathcal{O}^+_n$, then we may find $i \in \{1, \dots, n-1\}$ such that $i\theta < i < (i+1)\theta$, whence $n - j + i\alpha_1 < i < n - j + (i+1)\alpha_1$ and so $i\alpha_1 + 1 < (i+1)\alpha_1$, a contradiction. Hence $\theta \in \mathcal{O}^+_n$. Then, we have $(\theta, 1; 1) \in N$ and, as $\alpha_1 s'_j s_j = \alpha_1$, it follows that

$$\alpha = (\alpha_1, t_j; \beta) = (\theta s_j, t_j; \beta) = (\theta, 1; 1)(s_j, t_j; \beta) = (\theta, 1; 1)\bar{t}_{1,j} \in N.$$

CASE 2. Let $\alpha = (\alpha_1, t_{n\alpha_1}; \beta)$, with $\alpha_1 \in \mathcal{O}_n$.

Suppose that $\operatorname{Im}\alpha_1 = \{i_1 < i_2 < \cdots < i_j = n\alpha_1\}$, with $1 \le j \le n$. Take θ as being the unique element of $\mathcal{O}_n$ such that $\operatorname{Im}\theta = \{1, \dots, j\}$ and $\operatorname{Ker}\theta = \operatorname{Ker}\alpha_1$ (i.e. $(i_k\alpha_1^{-1})\theta = \{k\}$, for $1 \le k \le j$). As $k \le i_k$, for $1 \le k \le j$, the transformation

$$\theta' = \begin{pmatrix} 1 & 2 & \cdots & j & \cdots & i_j & i_j+1 & \cdots & n \\ i_1 & i_2 & \cdots & i_j & \cdots & i_j & i_j+1 & \cdots & n \end{pmatrix}$$

belongs to $\mathcal{O}^+_n$. Now, let $x \in \{1, \dots, n\}$ and $k \in \{1, \dots, j\}$. As $x \in i_k\alpha_1^{-1}$ if and only if $x\theta = k$, we deduce that $\theta\theta' = \alpha_1$. Moreover, clearly $t_j\theta' = t_{n\alpha_1}$. Hence, as $(\theta', \theta'; 1) \in N$ and, by the CASE 1, $(\theta, t_j; \beta) \in N$, we have

$$\alpha = (\alpha_1, t_{n\alpha_1}; \beta) = (\theta\theta', t_j\theta'; \beta) = (\theta, t_j; \beta)(\theta', \theta'; 1) \in N.$$

GENERAL CASE. Let $\alpha = (\alpha_1, \alpha_2; \beta)$, with $n\alpha_1 \le 1\alpha_2$, $\alpha_1 \in \mathcal{O}_n$ and $\alpha_2 \in \mathcal{O}^+_n$.

Consider the canonical decomposition (mentioned in the introductory section) $\alpha_1 = \theta_1\varepsilon_1$, with $\theta_1 \in \mathcal{O}^+_n$ and $\varepsilon_1 \in \mathcal{O}^-_n$ being the transformations defined by

$$i\theta_1 = \begin{cases} i & \text{if } i\alpha_1 \le i \\ i\alpha_1 & \text{if } i\alpha_1 \ge i \end{cases} \quad \text{and} \quad i\varepsilon_1 = \begin{cases} i\alpha_1 & \text{if } i\alpha_1 \le i \\ i & \text{if } i\alpha_1 \ge i \end{cases},$$

for $1 \le i \le n$. As $n\varepsilon_1 = n\alpha_1 \le 1\alpha_2$, then we have $\alpha_2 t_{n\varepsilon_1} = \alpha_2$. Hence, since $(\theta_1, \alpha_2; 1) \in N$ and, by the CASE 2, $(\varepsilon_1, t_{n\varepsilon_1}; \beta) \in N$, it follows

$$\alpha = (\alpha_1, \alpha_2; \beta) = (\theta_1\varepsilon_1, \alpha_2 t_{n\varepsilon_1}; \beta) = (\theta_1, \alpha_2; 1)(\varepsilon_1, t_{n\varepsilon_1}; \beta) \in N,$$

as required. ■

Next, let $k \in \{1, \dots, m-1\}$ and consider the submonoid

$$S_k = \{\alpha \in \mathcal{O}^+_{m\times n} \mid (A_k \cup A_{k+1})\alpha \subseteq A_k \cup A_{k+1} \text{ and } x\alpha = x, \text{ for all } x \in X_{mn} \setminus (A_k \cup A_{k+1})\}$$

of $\mathcal{O}^+_{m\times n}$. Clearly, S_k is isomorphic to $\mathcal{O}^+_{2\times n}$ and so, in view of Lemma 3.2, it is generated by

$$\{b_{k,j}, b_{k+1,j}, t_{k,\ell} \mid 1 \le j \le n-1, 1 \le \ell \le n\}.$$

Now, we can prove:

Proposition 3.3 *The set* $B = \{b_{i,j},\ t_{k,\ell} \mid\ 1 \le i \le m,\ 1 \le j \le n-1,\ 1 \le k \le m-1,\ 1 \le \ell \le n\}$ *is a generating set, with* $2mn - m - n$ *elements, of the monoid* $\mathcal{O}^+_{m\times n}$.

Proof. Denote by N the submonoid of $\mathcal{O}^+_{m\times n}$ generated by B. Then, we already proved that the submonoids $S_1, \dots, S_{m-1}, M$ of $\mathcal{O}^+_{m\times n}$ are contained in N. For each $\alpha \in \mathcal{O}^+_{m\times n}$, let $\mathtt{d}(\alpha) = |\{i \in \{1, \dots, m\} \mid A_i\alpha \not\subseteq A_i\}|$. In order to prove the result, we show that $\alpha \in N$, for all $\alpha \in \mathcal{O}^+_{m\times n}$, by induction on $\mathtt{d}(\alpha)$.

Let $\alpha \in \mathcal{O}^+_{m\times n}$ be such that $\mathtt{d}(\alpha) = 0$. Then $\alpha \in M$ and so $\alpha \in N$.

Hence, let $p \ge 0$ and suppose, by induction hypothesis, that $\alpha \in N$, for all $\alpha \in \mathcal{O}^+_{m\times n}$ with $\mathtt{d}(\alpha) = p$. Let $\alpha \in \mathcal{O}^+_{m\times n}$ be such that $\mathtt{d}(\alpha) = p + 1$. Let $i \in \{1, \dots, m-1\}$ be the least index such that $A_i\alpha \not\subseteq A_i$ and let $k \in \{i+1, \dots, m\}$ be such that $A_i\alpha \subseteq A_k$. Take

$$\alpha_1 = \left(\begin{array}{ccc|c|ccc|ccc|ccc|c|ccc} 1 & \cdots & n & \cdots & (i-2)n+1 & \cdots & (i-1)n & (i-1)n+1 & \cdots & in & in+1 & \cdots & (i+1)n & \cdots & (m-1)n+1 & \cdots & mn \\ 1\alpha & \cdots & n\alpha & \cdots & ((i-2)n+1)\alpha & \cdots & ((i-1)n)\alpha & (i-1)n+1 & \cdots & in & (in+1)\alpha & \cdots & ((i+1)n)\alpha & \cdots & ((m-1)n+1)\alpha & \cdots & (mn)\alpha \end{array}\right)$$

and

$$\alpha_2 = \left(\begin{array}{c|ccc|ccc|ccccccc|ccc|c} \cdots & (k-3)n+1 & \cdots & (k-2)n & (k-2)n+1 & \cdots & (k-1)n & (k-1)n+1 & \cdots & (in)\alpha & (in)\alpha+1 & \cdots & kn & kn+1 & \cdots & (k+1)n & \cdots \\ \cdots & (k-3)n+1 & \cdots & (k-2)n & ((i-1)n+1)\alpha & \cdots & (in)\alpha & (in)\alpha & \cdots & (in)\alpha & (in)\alpha+1 & \cdots & kn & kn+1 & \cdots & (k+1)n & \cdots \end{array}\right).$$

Then $\alpha_1 \in \mathcal{O}^+_{m\times n}$ and $\mathtt{d}(\alpha_1) = p$, whence $\alpha_1 \in N$, by induction hypothesis. Moreover, we also have $\alpha_2 \in N$, since $\alpha_2 \in S_{k-1}$. Finally, it is routine to show that $\alpha = \alpha_1 t_{i,n} \cdots t_{k-2,n}\alpha_2$ and so $\alpha \in N$, as required. ∎

Next, we prove that B is a least-size generating set of $\mathcal{O}^+_{m\times n}$.

Theorem 3.4 *The rank of* $\mathcal{O}^+_{m\times n}$ *is* $2mn - m - n$.

Proof. It suffices to show that all the elements of $B\psi$ are indecomposable in $\overline{\mathcal{O}}^+_{m\times n}$.

Let $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n-1\}$. Recall that $\bar{b}_{i,j} = (1, \dots, 1, b_j, 1, \dots, 1; 1)$, with $b_j \in \mathcal{O}^+_n$ in the i^{th} component. As the identity is indecomposable (in $\mathcal{O}^+_n$ and in $\mathcal{O}^+_m$) and b_j is indecomposable in $\mathcal{O}^+_n$, it follows immediately that $\bar{b}_{i,j}$ is indecomposable in $\overline{\mathcal{O}}^+_{m\times n}$.

Now, let $i \in \{1, \dots, m-1\}$ and $j \in \{1, \dots, n\}$. We prove that $\bar{t}_{i,j} = (1, \dots, 1, s_j, t_j, 1, \dots, 1; b_i)$ also is indecomposable in $\overline{\mathcal{O}}^+_{m\times n}$ (notice that s_j is the i^{th} component of $\bar{t}_{i,j}$). Let $\alpha = (\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_m; \beta), \alpha' = (\alpha'_1, \dots, \alpha'_i, \alpha'_{i+1}, \dots, \alpha'_m; \beta') \in \overline{\mathcal{O}}^+_{m\times n}$ be such that $\bar{t}_{i,j} = \alpha\alpha' = (\alpha_1\alpha'_{1\beta}, \dots, \alpha_i\alpha'_{i\beta}, \alpha_{i+1}\alpha'_{(i+1)\beta}, \dots, \alpha_m\alpha'_{m\beta}; \beta\beta')$. As $\beta, \beta' \in \mathcal{O}^+_m$ and $\beta\beta' = b_i$, we have $\beta, \beta' \in \{1, b_i\}$. Hence, $\bar{t}_{i,j} = (\alpha_1\alpha'_1, \dots, \alpha_i\alpha'_{i\beta}, \alpha_{i+1}\alpha'_{i+1}, \dots, \alpha_m\alpha'_m; b_i)$ and so $\alpha_k = \alpha'_k = 1$, for $k \in \{1, \dots, m\} \setminus \{i, i+1\}$, $\alpha_{i+1}\alpha'_{i+1} = t_j$ and $\alpha_{i+1}, \alpha'_{i+1} \in \mathcal{O}^+_n$. Notice that, from the equality $\alpha_{i+1}\alpha'_{i+1} = t_j$ we deduce that $\{j, \dots, n\} = \operatorname{Im} t_j \subseteq \operatorname{Im} \alpha'_{i+1}$.

Suppose that $\beta = b_i$. Then $i\beta = i+1$, whence $\alpha_i\alpha'_{i+1} = s_j$ and so $\{1, \dots, j\} = \operatorname{Im} s_j \subseteq \operatorname{Im} \alpha'_{i+1}$. Hence $\operatorname{Im} \alpha'_{i+1} = \{1, \dots, n\}$, which implies that $\alpha'_{i+1} = 1$. Thus, $\alpha_i = s_j$ and $\alpha_{i+1} = t_j$ and so $\alpha = \bar{t}_{i,j}$.

On the other hand, admit that $\beta = 1$. Then $\beta' = b_i$, $\alpha_i \in \mathcal{O}_n^+$ and $\alpha_i\alpha_i' = s_j$.

First, we prove that $\alpha_i' = s_j$. As $\alpha_i \in \mathcal{O}_n^+$, we have $1 = (n-j+1)s_j = (n-j+1)\alpha_i\alpha_i' \geq (n-j+1)\alpha_i'$, whence $(n-j+1)\alpha_i' = 1$. Moreover, from the equality $\alpha_i\alpha_i' = s_j$ we deduce that $\{1,\ldots,j\} = \operatorname{Im} s_j \subseteq \operatorname{Im}\alpha_i'$ and so we have $\alpha_i' = s_j$.

Finally, we prove that $\alpha_{i+1}' = t_j$. As $\alpha_i \in \mathcal{O}_n^+$, we have $n\alpha_i = n$ and so $j = ns_j = n\alpha_i\alpha_i' = n\alpha_i' \leq 1\alpha_{i+1}'$, from which we deduce that $\operatorname{Im}\alpha_{i+1}' \subseteq \{j,\ldots,n\}$. Thus $\operatorname{Im}\alpha_{i+1}' = \{j,\ldots,n\}$. Moreover, as $\alpha_{i+1}, \alpha_{i+1}' \in \mathcal{O}_n^+$, we have $j \leq j\alpha_{i+1} \leq j\alpha_{i+1}\alpha_{i+1}' = jt_j = j$, whence $j = j\alpha_{i+1}$ and so $j\alpha_{i+1}' = j\alpha_{i+1}\alpha_{i+1}' = jt_j = j$. Thus, we have $\alpha_{i+1}' = t_j$.

Hence, we also proved that, if $\beta = 1$ then $\alpha' = \bar{t}_{i,j}$. Thus $\bar{t}_{i,j}$ is indecomposable in $\overline{\mathcal{O}}_{m\times n}^+$, as required. ∎

Now, recall that the monoid $\mathcal{O}_{m\times n}^-$ is isomorphic to $\mathcal{O}_{m\times n}^+$. Therefore, $\mathcal{O}_{m\times n}^-$ as rank equal to $2mn-m-n$ and a least-size generating set of $\mathcal{O}_{m\times n}^-$ can be obtained from B by isomorphism. Next, we describe explicitly such generating set of $\mathcal{O}_{m\times n}^-$.

For $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n-1\}$, let

$$a_{i,j} = \left(\begin{array}{c|cccccccc|c} \cdots & (i-1)n+1 & \cdots & (i-1)n+j & (i-1)n+j+1 & (i-1)n+j+2 & \cdots & in & \cdots \\ \cdots & (i-1)n+1 & \cdots & (i-1)n+j & (i-1)n+j & (i-1)n+j+2 & \cdots & in & \cdots \end{array}\right).$$

For $i \in \{1,\ldots,m-1\}$ and $j \in \{1,\ldots,n\}$, let

$$s_{i,j} = \left(\begin{array}{c|cccccc|} \cdots & (i-1)n+1 & \cdots & in-j+1 & in-j+2 & \cdots & in \\ \cdots & (i-1)n+1 & \cdots & in-j+1 & in-j+1 & \cdots & in-j+1 \end{array}\right.$$
$$\left.\begin{array}{|cccccc|c} in+1 & in+2 & \cdots & in+j & \cdots & (i+1)n & \cdots \\ in-j+1 & in-j+2 & \cdots & in & \cdots & in & \cdots \end{array}\right).$$

Then, we have that $A = \{a_{i,j}, s_{k,\ell} \mid 1 \leq i \leq m, 1 \leq j \leq n-1, 1 \leq k \leq m-1, 1 \leq \ell \leq n\}$ is a least-size generating set of $\mathcal{O}_{m\times n}^-$.

Next, for $i \in \{1,\ldots,m\}$, consider

$$c_i = \left(\begin{array}{c|ccccc|c} \cdots & (i-1)n+1 & (i-1)n+2 & (i-1)n+3 & \cdots & in & \cdots \\ \cdots & (i-1)n+1 & (i-1)n+1 & (i-1)n+2 & \cdots & in-1 & \cdots \end{array}\right) \in \mathcal{O}_{m\times n}^-.$$

For instance, in $\mathcal{O}_{2\times 4}^-$, we have

$$c_1 = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 2 & 3 & 5 & 6 & 7 & 8 \end{array}\right) \quad \text{and} \quad c_2 = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 5 & 6 & 7 \end{array}\right).$$

We now focus our attention on the monoid $\mathcal{O}_{m\times n}$.

As observed in the introductory section, we have $\mathcal{O}_{m\times n} = \mathcal{O}_{m\times n}^-\mathcal{O}_{m\times n}^+$, whence $A \cup B$ is a generating set of $\mathcal{O}_{m\times n}$.

Let $i \in \{1,\ldots,m\}$. It is easy to show that $T_i = \{\alpha \in \mathcal{O}_{m\times n} \mid A_i\alpha \subseteq A_i \text{ and } x\alpha = x, \text{ for all } x \in X_{mn} \setminus A_i\}$ is a submonoid of $\mathcal{O}_{m\times n}$ isomorphic to $\mathcal{O}_n$. As $\{a_j, b_j \mid 1 \leq j \leq n-1\}$ and $\{c, b_1, \ldots, b_{n-1}\}$ are generating sets of $\mathcal{O}_n$ [9], then $\{a_{i,j}, b_{i,j} \mid \ 1 \leq j \leq n-1\}$ and $\{c_i, b_{i,j} \mid \ 1 \leq j \leq n-1\}$ are generating sets of T_i. Hence

$$\{c_i, s_{k,\ell} \mid 1 \leq i \leq m, 1 \leq k \leq m-1, 1 \leq \ell \leq n\} \cup B$$

generates $\mathcal{O}_{m\times n}$.

On the other hand, it is a routine matter to show that $t_{k,1} = s_{k,n}t_{k,n}$, $s_{k,1} = t_{k,n}s_{k,n}$ and

$$s_{k,\ell} = (b_{k,n-\ell+1}\cdots b_{k,2})(b_{k,n-\ell+2}\cdots b_{k,3})\cdots(b_{k,n-1}\cdots b_{k,\ell})(b_{k+1,\ell}\cdots b_{k+1,2})(b_{k+1,\ell+1}\cdots b_{k+1,3})\cdots$$
$$\cdots(b_{k+1,n-1}\cdots b_{k+1,n-\ell+1})t_{k,n-\ell+1}s_{k,n}\ ,$$

for $1 \leq k \leq m-1$ and $2 \leq \ell \leq n-1$.

Therefore, we have:

Proposition 3.5 *The set* $C = \{c_i, b_{i,j}, s_{k,n}, t_{k,\ell} \mid 1 \le i \le m, 1 \le j \le n-1, 1 \le k \le m-1, 2 \le \ell \le n\}$ *is a generating set, with* $2mn - n$ *elements, of the monoid* $\mathcal{O}_{m\times n}$. ■

We finish this section by proving that C is a least-size generating set of $\mathcal{O}_{m\times n}$.

Theorem 3.6 *The rank of* $\mathcal{O}_{m\times n}$ *is* $2mn - n$.

Proof. For $i \in \{1,\dots,m-1\}$ and $j \in \{1,\dots,n\}$, let

$$\alpha = \alpha_{i,j} = \left(\begin{array}{c|cccccc|} \cdots & (i-1)n+1 & \cdots & (i-1)n+j-1 & (i-1)n+j & \cdots & in \\ \cdots & (i-1)n+1 & \cdots & (i-1)n+j-1 & (i-1)n+j & \cdots & (i-1)n+j \end{array} \right.$$

$$\left. \begin{array}{|cccccc|c} in+1 & \cdots & in+j & in+j+1 & \cdots & (i+1)n & \cdots \\ (i-1)n+j & \cdots & (i-1)n+j & (i-1)n+j+1 & \cdots & in & \cdots \end{array} \right).$$

Notice that α fixes all elements of A_k, for all $k \in \{1,\dots,m\} \setminus \{i, i+1\}$, and $\operatorname{Im}\alpha = X_{mn} \setminus A_{i+1}$.

Take $\alpha_1, \alpha_2 \in \mathcal{O}_{m\times n}$ such that $\alpha = \alpha_1\alpha_2$. As $|\operatorname{Im}\alpha| = (m-1)n$, then $|\operatorname{Im}\alpha_1| \ge (m-1)n$ and $\operatorname{Im}\alpha \subseteq \operatorname{Im}\alpha_2$.

CASE 1. Suppose that $\operatorname{Im}\alpha_2 \cap A_{i+1} \ne \emptyset$. Then $A_k\alpha_2 \subseteq A_{i+1}$, for some $k \in \{1,\dots,m\}$. As $X_{mn} \setminus A_{i+1} \subseteq \operatorname{Im}\alpha_2$, we must have $A_1 \cup \cdots \cup A_i \subseteq (A_1 \cup \cdots \cup A_{k-1})\alpha_2$ and $A_{i+2} \cup \cdots \cup A_m \subseteq (A_{k+1} \cup \cdots \cup A_m)\alpha_2$. Then $i \le k-1$ and $i+2 \ge k+1$, whence $k = i+1$. Moreover, α_2 maps $X_{mn} \setminus A_{i+1}$ onto $X_{mn} \setminus A_{i+1}$ and so it fixes all elements of $X_{mn} \setminus A_{i+1}$. Now, let $x \in X_{mn}$. If $x\alpha_1 \in A_{i+1}$ then $x\alpha = x\alpha_1\alpha_2 \in A_{i+1}$, a contradiction. Hence $x\alpha_1 \in X_{mn} \setminus A_{i+1}$ and so $x\alpha = x\alpha_1\alpha_2 = x\alpha_1$. Thus $\alpha = \alpha_1$.

CASE 2. On the other hand, suppose that $\operatorname{Im}\alpha_2 \cap A_{i+1} = \emptyset$. Then $\operatorname{Im}\alpha_2 \subseteq X_{mn} \setminus A_{i+1}$ and so $\operatorname{Im}\alpha_2 = X_{mn} \setminus A_{i+1}$.

Let $Y = A_1 \cup \cdots \cup A_{i-1} \cup \{(i-1)n+1, \dots, (i-1)n+j\} \cup \{in+j+1, \dots, (i+1)n\} \cup A_{i+2} \cup \cdots \cup A_m$. Notice that $|Y| = (m-1)n$. As α is injective in Y, then α_1 must also be injective in Y. It follows that $A_i\alpha_1 \subseteq A_k$ and $A_{i+1}\alpha_1 \subseteq A_\ell$, for some $i \le k \le \ell \le i+1$ (observe that $(i-1)n+1 \in A_i \cap Y$ and $(i+1)n \in A_{i+1} \cap Y$).

If $k = i$ and $\ell = i+1$ then $(in)\alpha_1 \le in$ and $(in+1)\alpha_1 \ge in+1$, whence

$$(i-1)n+j = (in)\alpha = (in)\alpha_1\alpha_2 \le (in)\alpha_2 \le (in+1)\alpha_2 \le (in+1)\alpha_1\alpha_2 = (in+1)\alpha = (i-1)n+j$$

and so $(in)\alpha_2 = (in+1)\alpha_2 = (i-1)n+j$.

On the other hand, if $k = \ell$ then $|\operatorname{Im}\alpha_1| = (m-1)m = |Y|$, which implies that

$$\begin{aligned} ((i-1)n+1)\alpha_1 < \cdots < ((i-1)n+j-1)\alpha_1 < ((i-1)n+j)\alpha_1 = \cdots = (in)\alpha_1 = \\ = (in+1)\alpha_1 = \cdots = (in+j)\alpha_1 < (in+j+1)\alpha_1 < \cdots < ((i+1)n)\alpha_1. \end{aligned}$$

Then $(in)\alpha_1 = (in+1)\alpha_1 = (i-1)n+j$, if $k = i = \ell$, and $(in)\alpha_1 = (in+1)\alpha_1 = in+j$, if $k = i+1 = \ell$.

Therefore, we proved that, in order to write $\alpha_{i,j}$ as a product of elements of $\mathcal{O}_{m\times n}$, we must have a factor $\alpha'_{i,j}$ with $|\operatorname{Im}\alpha'_{i,j}| = (m-1)n$ such that $(in)\alpha'_{i,j} = (in+1)\alpha'_{i,j} = (i-1)n+j$ or $(in)\alpha'_{i,j} = (in+1)\alpha'_{i,j} = in+j$.

Observe that, given $i, k \in \{1,\dots,m-1\}$ and $j, \ell \in \{1,\dots,n\}$ such that $(i,j) \ne (k,\ell)$, then $\alpha'_{i,j} \ne \alpha'_{k,\ell}$. In fact, it is clear that, if $i = k$ and $j \ne \ell$ then $\alpha'_{i,j} \ne \alpha'_{i,\ell}$. On the other hand, if $i \ne k$ then $\alpha'_{i,j} = \alpha'_{k,\ell}$ implies that $|\operatorname{Im}\alpha'_{i,j}| < (m-1)n$, a contradiction.

Thus, each generating set of $\mathcal{O}_{m\times n}$ must have $(m-1)n$ distinct elements with image size equal to $(m-1)n$.

Next, observe that, for $i \in \{1,\dots,m\}$, the elements of $T_i\psi$ are of the form $(1,\dots,1,\alpha_i,1,\dots,1;1)$, with $\alpha_i \in \mathcal{O}_n$ in the i^{th} component. Then, as the identity is indecomposable (in $\mathcal{O}_n$ and in $\mathcal{O}_m$), given $\alpha \in T_i$ and $\alpha', \alpha'' \in \mathcal{O}_{m\times n}$, it is clear that $\alpha = \alpha'\alpha''$ implies $\alpha', \alpha'' \in T_i$. On the other hand, since $\mathcal{O}_n$ has rank n and T_i is isomorphic to $\mathcal{O}_n$, in order to generate in $\mathcal{O}_{m\times n}$ all the elements of T_i, we need at least n distinct (non-identity) elements of T_i, for $i \in \{1,\dots,m\}$. Hence, each generating set of $\mathcal{O}_{m\times n}$ must have mn distinct elements with image size greater than or equal to $(m-1)n+1$.

Therefore, we proved that each generating set of $\mathcal{O}_{m\times n}$ must have $(m-1)n + mn$ distinct elements and so, in view of Proposition 3.5, we conclude that $\mathcal{O}_{m\times n}$ has rank $2mn - n$, as required. ■

4 A bilateral semidirect product decomposition of $\mathcal{O}_{m\times n}$

In this section, we present a bilateral semidirect product decomposition of $\mathcal{O}_{m\times n}$ in terms of is submonoids $\mathcal{O}^-_{m\times n}$ and $\mathcal{O}^+_{m\times n}$. This result generalizes the Kunze's bilateral semidirect product decomposition [17] of the monoid $\mathcal{O}_n$ in terms of $\mathcal{O}^-_n$ and $\mathcal{O}^+_n$. Our strategy is to use Kunze's actions on $\mathcal{O}^-_{mn}$ and $\mathcal{O}^+_{mn}$ to induce a left action of $\mathcal{O}^+_{m\times n}$ on $\mathcal{O}^-_{m\times n}$ and a right action of $\mathcal{O}^-_{m\times n}$ on $\mathcal{O}^+_{m\times n}$.

Let S be a monoid and let S^- and S^+ be two submonoids of S. Let us consider a left action δ of S^+ on S^- and a right action φ of S^- on S^+ such that the function

$$\begin{array}{ccc} S^-\bowtie S^+ & \longrightarrow & S \\ (s,u) & \mapsto & su \end{array}$$

is a homomorphism. For $s\in S^-$ and $u\in S^+$, denote $(s)(u)\delta$ by $u\cdot s$ and $(u)(s)\varphi$ by u^s.

Now, let T be a submonoid of S, T^- a submonoid of S^- and T^+ a submonoid of S^+. It is a routine matter to check that, if $u\cdot s\in T^-$ and $u^s\in T^+$, for all $s\in T^-$ and $u\in T^+$, then δ induces a left action of T^+ on T^- and φ induces a right action of T^- on T^+. If, in addition, $T=T^-T^+$ then

$$\begin{array}{ccc} T^-\bowtie T^+ & \longrightarrow & T \\ (s,u) & \mapsto & su \end{array}$$

is a surjective homomorphism.

Next, we recall, in slightly different way, some aspects of the original construction made by Kunze in [17], in order to prove that the monoid $\mathcal{O}_n$ is a quotient of a bilateral semidirect product of $\mathcal{O}^-_n$ and $\mathcal{O}^+_n$. The reader will also benefit from reading the authors's paper [7], where a more sophisticated and transparent construction is presented.

Let $i\in\{1,\ldots,n-1\}$ and $j\in\{2,\ldots,n\}$. We define the transformations $\sigma_{i,j}\in\mathcal{O}^-_n$ and $\varepsilon_{i,j}\in\mathcal{O}^+_n$ by

$$x\sigma_{i,j}=\begin{cases} i & \text{if } i\le x\le j\\ x & \text{otherwise}\end{cases}\qquad\text{and}\qquad x\varepsilon_{i,j}=\begin{cases} j & \text{if } i\le x\le j\\ x & \text{otherwise}\end{cases},$$

for all $x\in\{1,\ldots,n\}$.

Observe that, for $i\neq j$ and $k\neq\ell$, we have $\sigma_{i,j}=\sigma_{k,\ell}$ if and only if $i=k$ e $j=\ell$. The same holds for $\varepsilon_{i,j}$.

These transformations allow us to represent in a canonical form the elements of $\mathcal{O}^-_n$ and $\mathcal{O}^+_n$: given $\sigma\in\mathcal{O}^-_n$ and $\varepsilon\in\mathcal{O}^+_n$, we have

$$\sigma=\sigma_{1,a_1}\cdots\sigma_{n-1,a_{n-1}},$$

with $a_i=\max(\{1,\ldots,i\}\alpha^{-1})$, for $i\in\{1,\ldots,n-1\}$, and

$$\varepsilon=\varepsilon_{b_n,n}\cdots\varepsilon_{b_2,2},$$

with $b_j=\min(\{j,\ldots,n\}\alpha^{-1})$, for $j\in\{2,\ldots,n\}$.

For instance, given $\sigma=\begin{pmatrix}1&2&3&4&5&6&7\\1&1&2&2&3&5&7\end{pmatrix}\in\mathcal{O}^-_7$ and $\varepsilon=\begin{pmatrix}1&2&3&4&5&6&7\\3&3&5&6&6&7&7\end{pmatrix}\in\mathcal{O}^-_7$, we have $\sigma=\sigma_{1,2}\sigma_{2,4}\sigma_{3,5}\sigma_{4,5}\sigma_{5,6}\sigma_{6,6}$ and $\varepsilon=\varepsilon_{6,7}\varepsilon_{4,6}\varepsilon_{3,5}\varepsilon_{3,4}\varepsilon_{1,3}\varepsilon_{1,2}$.

Now, we may define a left action of $\mathcal{O}^+_n$ on $\mathcal{O}^-_n$ and a right action of $\mathcal{O}^-_n$ on $\mathcal{O}^+_n$ as follows: given $\sigma=\sigma_{1,a_1}\cdots\sigma_{n-1,a_{n-1}}\in\mathcal{O}^-_n$ and $\varepsilon=\varepsilon_{b_n,n}\cdots\varepsilon_{b_2,2}\in\mathcal{O}^-_n$ (canonically represented), we let

$$\varepsilon\cdot\sigma=\sigma_{1,a'_1}\cdots\sigma_{n-1,a'_{n-1}}\ ,$$

with $a'_i=\max\{i,\min\{a_i,b_{a_i+1}-1\}\}$ (where $b_{n+1}=n+1$ is assumed for the case $a_i=n$), for $1\le i\le n-1$, and

$$\varepsilon^\sigma=\varepsilon_{b'_n,n}\cdots\varepsilon_{b'_2,2}\ ,$$

with

$$b'_n = \begin{cases} b_n & \text{if } a_{n-1} = n-1 \\ n & \text{otherwise} \end{cases} \quad \text{and} \quad b'_j = \begin{cases} b_j & \text{if } a_{j-1} = j-1 \\ \min\{j, b_{a_{j-1}+1}\} & \text{if } j \le a_{j-1} < a_j \\ \min\{j, b'_{j+1}\} & \text{if } a_j = a_{j-1}\ , \end{cases}$$

(recursively defined) for $2 \le j \le n-1$. Notice that both expressions are canonical forms.

Example 4.1 Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 1 & 1 & 1 & 6 & 6 & 6 & 6 & 6 & 9 & 12 \end{pmatrix} = \sigma_{1,5}\sigma_{2,5}\sigma_{3,5}\sigma_{4,5}\sigma_{5,5}\sigma_{6,10}\sigma_{7,10}\sigma_{8,10}\sigma_{9,11}\sigma_{10,11}\sigma_{11,11} \in \mathcal{O}^-_{12}$$

(notice that $\sigma \notin \mathcal{O}^-_{3\times 4}$) and

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 5 & 5 & 8 & 8 & 8 & 8 & 8 & 8 & 12 & 12 & 12 & 12 \end{pmatrix} = \varepsilon_{9,12}\varepsilon_{9,11}\varepsilon_{9,10}\varepsilon_{9,9}\varepsilon_{3,8}\varepsilon_{3,7}\varepsilon_{3,6}\varepsilon_{1,5}\varepsilon_{1,4}\varepsilon_{1,3}\varepsilon_{1,2} \in \mathcal{O}^+_{12}$$

(notice that $\varepsilon \in \mathcal{O}^+_{3\times 4}$). Then

$$\varepsilon \cdot \sigma = \sigma_{1,2}\sigma_{2,2}\sigma_{3,3}\sigma_{4,4}\sigma_{5,5}\sigma_{6,8}\sigma_{7,8}\sigma_{8,8}\sigma_{9,9}\sigma_{10,10}\sigma_{11,11} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 3 & 4 & 5 & 6 & 6 & 6 & 9 & 10 & 11 & 12 \end{pmatrix} \in \mathcal{O}^-_{12}$$

(notice that $\varepsilon \cdot \sigma \in \mathcal{O}^-_{3\times 4}$) and

$$\varepsilon^\sigma = \varepsilon_{9,12}\varepsilon_{9,11}\varepsilon_{9,10}\varepsilon_{9,9}\varepsilon_{8,8}\varepsilon_{7,7}\varepsilon_{3,6}\varepsilon_{3,5}\varepsilon_{3,4}\varepsilon_{3,3}\varepsilon_{2,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 6 & 6 & 6 & 6 & 7 & 8 & 12 & 12 & 12 & 12 \end{pmatrix} \in \mathcal{O}^+_{12}$$

(notice that $\varepsilon^\sigma \notin \mathcal{O}^+_{3\times 4}$).

Regarding these actions, Kunze [17] proved that the function

$$\begin{array}{ccc} \mathcal{O}^-_n \bowtie \mathcal{O}^+_n & \longrightarrow & \mathcal{O}_n \\ (\sigma, \varepsilon) & \mapsto & \sigma\varepsilon \end{array}$$

is a surjective homomorphism. See [7] for a more clear and explicit presentation.

Next, we focus our attention on the monoids $\mathcal{O}_{m\times n}$, $\mathcal{O}^-_{m\times n}$ and $\mathcal{O}^+_{m\times n}$.

First, we characterize the canonical forms of the elements of $\mathcal{O}^-_{m\times n}$ and $\mathcal{O}^+_{m\times n}$.

Proposition 4.2 *Let $\sigma = \sigma_{1,a_1} \cdots \sigma_{mn-1,a_{mn-1}} \in \mathcal{O}^-_{mn}$ and $\varepsilon = \varepsilon_{b_{mn},mn} \cdots \varepsilon_{b_2,2} \in \mathcal{O}^+_{mn}$ canonically represented. Then:*

1. *$\sigma \in \mathcal{O}^-_{m\times n}$ if and only if $i \equiv 0 \pmod n$ implies $a_i \equiv 0 \pmod n$, for $i \in \{1, \dots, mn-1\}$;*
2. *$\varepsilon \in \mathcal{O}^+_{m\times n}$ if and only if $j \equiv 1 \pmod n$ implies $b_j \equiv 1 \pmod n$, for $j \in \{2, \dots, mn\}$.*

Proof. We only prove the first property, as the second one can be proved similarly.

Suppose that there exists $i \in \{1, \dots, mn-1\}$ such that $i \equiv 0 \pmod n$ and $a_i \not\equiv 0 \pmod n$. Regarding the canonical form of σ, we have $(a_i)\sigma \le i$ and $(a_i+1)\sigma > i$. As $i \equiv 0 \pmod n$, then $(a_i)\sigma, (a_i+1)\sigma \notin A_k$, for all $k \in \{1, \dots, m\}$. On the other hand, as $a_i \not\equiv 0 \pmod n$, then $a_i, a_i+1 \in A_k$, for some $k \in \{1, \dots, m\}$. Hence $\sigma \notin \mathcal{O}^-_{m\times n}$.

Conversely, suppose that $i \equiv 0 \pmod n$ implies $a_i \equiv 0 \pmod n$, for all $i \in \{1, \dots, mn-1\}$. Let $x, y \in X_{mn}$ be such that $x \le y$. Suppose that $x\sigma, y\sigma \notin A_k$, for all $k \in \{1, \dots, m\}$. Then $x\sigma < y\sigma$ and there exists $i \in \{x\sigma, \dots, y\sigma - 1\}$ such that $i \equiv 0 \pmod n$. It follows that $x \le a_{x\sigma} \le a_i < y$ and, by the hypothesis, $a_i \equiv 0 \pmod n$, whence $x, y \notin A_k$, for all $k \in \{1, \dots, m\}$. Thus $\sigma \in \mathcal{O}^-_{m\times n}$, as required. ∎

Lemma 4.3 *Let* $\sigma = \sigma_{1,a_1} \cdots \sigma_{mn-1,a_{mn-1}} \in \mathcal{O}^-_{m\times n}$ *and* $\varepsilon = \varepsilon_{b_{mn},mn} \cdots \varepsilon_{b_2,2} \in \mathcal{O}^+_{m\times n}$*. Then* $\varepsilon \centerdot \sigma \in \mathcal{O}^-_{m\times n}$ *and* $\varepsilon^\sigma \in \mathcal{O}^+_{m\times n}$*.*

Proof. We begin by proving that $\varepsilon \centerdot \sigma \in \mathcal{O}^-_{m\times n}$. Consider $\varepsilon \centerdot \sigma = \sigma_{1,a'_1} \cdots \sigma_{mn-1,a'_{mn-1}}$, as defined above. Let $i \in \{1,\dots,mn-1\}$ and suppose that $i \equiv 0 \pmod n$. Then, as $\sigma \in \mathcal{O}^-_{m\times n}$, we have $a_i \equiv 0 \pmod n$. If $a'_i = a_i$ or $a'_i = i$, then trivially $a'_i \equiv 0 \pmod n$. So, admit that $a'_i = b_{a_i+1} - 1$. As $a_i \equiv 0 \pmod n$, then $a_i + 1 \equiv 1 \pmod n$. Now, as $\varepsilon \in \mathcal{O}^+_{m\times n}$, it follows that $b_{a_i+1} \equiv 1 \pmod n$ and so $a'_i = b_{a_i+1} - 1 \equiv 0 \pmod n$. Hence $\varepsilon \centerdot \sigma \in \mathcal{O}^-_{m\times n}$.

Next, we prove that $\varepsilon^\sigma \in \mathcal{O}^+_{m\times n}$. Take $\varepsilon^\sigma = \varepsilon_{b'_{mn},mn} \cdots \varepsilon_{b'_2,2}$, as defined above. Let $j \in \{2,\dots,mn\}$ and suppose that $j \equiv 1 \pmod n$. Then, as $\varepsilon \in \mathcal{O}^+_{m\times n}$, we have $b_j \equiv 1 \pmod n$. Observe that $j < mn$.

If $a_{j-1} = j-1$ then $b'_j = b_j \equiv 1 \pmod n$.

If $j \le a_{j-1} < a_j$ then $b'_j = \min\{j, b_{a_{j-1}+1}\}$. If $b'_j = j$ then trivially $b'_j \equiv 1 \pmod n$. So, admit that $b'_j = b_{a_{j-1}+1}$. As $j-1 \equiv 0 \pmod n$ and $\sigma \in \mathcal{O}^-_{m\times n}$, then $a_{j-1} \equiv 0 \pmod n$, whence $a_{j-1}+1 \equiv 1 \pmod n$ and so $b'_j = b_{a_{j-1}+1} \equiv 1 \pmod n$.

It remains to consider $a_j = a_{j-1}$. In this case, $b'_j = \min\{j, b'_{j+1}\}$. If $j \le b'_{j+1}$ then $b'_j = j \equiv 1 \pmod n$. Therefore, admit that $j > b'_{j+1}$. Hence, $b'_j = b'_{j+1} < j$.

Let $k \in \{j,\dots,mn-1\}$ be the greater index such that $a_k = a_{k-1} = \cdots = a_j = a_{j-1}$.

First, we prove that $b'_{k+1} = b'_k = \cdots = b'_{j+1} = b'_j$. In order to obtain a contradiction, suppose there exists $t \in \{j+1,\dots,k+1\}$ such that $b'_t > b'_{t-1} = \cdots = b'_j$. Then, as $a_{t-1} = a_{t-2}$, we have $b'_t > b'_{t-1} = \min\{t-1, b'_t\}$ (notice that $t-1 \le k < mn$), whence $j \le t-1 = b'_{t-1} = b'_j < j$, a contradiction.

Next, recall that $a_{j-1} \equiv 0 \pmod n$. Hence, $a_k \equiv 0 \pmod n$. If $k = mn-1$ then, as $a_{mn-1} \ge mn-1$ and $a_{mn-1} \equiv 0 \pmod n$, we must have $a_{mn-1} = mn$ and so $j > b'_j = b'_{mn} = mn$, a contradiction. Hence $k < mn-1$. Moreover, we have $a_{k+1} > a_k = a_{k-1} = \cdots = a_j = a_{j-1}$.

Now, if $a_k = k$ then $b'_j = b'_{k+1} = b_{k+1} \equiv 1 \pmod n$, since $k+1 = a_k + 1 \equiv 1 \pmod n$ and $\varepsilon \in \mathcal{O}^+_{m\times n}$.

Finally, suppose that $a_{k+1} > a_k \ge k+1$. Then $b'_j = b'_{k+1} = \min\{k+1, b_{a_k+1}\}$. If $k+1 \le b_{a_k+1}$ then $j > b'_j = k+1 \ge j+1$, a contradiction. Thus, $k+1 > b_{a_k+1}$ and so $b'_j = b_{a_k+1}$. From $a_k + 1 \equiv 1 \pmod n$, it follows that $b'_j = b_{a_k+1} \equiv 1 \pmod n$, as required. ∎

The previous lemma allow us to consider the bilateral semidirect product $\mathcal{O}^-_{m\times n} \bowtie \mathcal{O}^+_{m\times n}$ induced by the bilateral semidirect product $\mathcal{O}^-_{mn} \bowtie \mathcal{O}^+_{mn}$. Furthermore, as $\mathcal{O}_{m\times n} = \mathcal{O}^-_{m\times n}\mathcal{O}^+_{m\times n}$, by the general observations made in the beginning of this section, we obtain:

Theorem 4.4 *The monoid* $\mathcal{O}_{m\times n}$ *is a homomorphic image of* $\mathcal{O}^-_{m\times n} \bowtie \mathcal{O}^+_{m\times n}$*.* ∎

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