First variation formula and conservation laws in several independent discrete variables

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Abstract

First variation formula for Lagrangian densities is a central element of the calculus of variations, that relates the differential of the action functional with both the Euler and Cartan forms, that are geometric objects, that is, as tensors, they do not depend on the choice of a coordinate system for the dependent or independent variables, or seen from a different viewpoint, they are invariant with respect to local automorphisms of these variables. First variation formula leads straightforward to Euler equations that characterize critical points of the action functional, as well as to Noether currents and conservation laws associated to any infinitesimal symmetry of the Lagrangian.

This paper sets the scene for discrete variational problems on an abstract cellular complex that serves as discrete model of $\mathbb{R}^p$ and for the discrete theory of partial differential operators that are commom in the Calculus of Variations. A central result is the construction of a unique decomposition of certain partial difference operators into two components, one that is a vector bundle morphism and other one that leads to boundary terms. Application of this result to the differential of the discrete Lagrangian leads to unique discrete Euler and momentum forms not depending either on the choice of reference on the base lattice or on the choice of coordinates on the configuration manifold, and satisfying the corresponding discrete first variation formula. This formula leads to discrete Euler equations for critical points and to exact discrete conservation laws for infinitesimal symmetries of the Lagrangian density, with a clear physical interpretation.

Key words: Discrete Calculus of Variations, Partial difference operators, Conservation laws, Abstract cellular complex, First variation formula, Adjoint difference operator, Discrete lagrangian density.
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1. Introduction

Discrete models in geometry were always present in mathematics both for its intrinsic interest and for the insight they let for more complicated related problems. This area is receiving in modern mathematics ([1, 5, 8, 14, 28, 32] and references therein) an increasing interest due to the possibilities it offers to, when combined with modern computational tools, analyze and solve problems with physical or geometrical origin that involve partial or ordinary differential equations, both if we look for numerical solutions, or if we try to determine its properties and how they reflect on the smooth case. An important part of these equations have its origin in a variational principle, with or without constraints [4, 9, 10, 11, 12, 22, 26, 27, 33].

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Historically, the first variational problem introduced by Johann Bernoulli (the determination of the braquistochrone curve) was solved by him precisely through an approximation with an associated discrete problem, taking the curve as composed of several rectilinear elements [17]. After a more rigorous formalization of the subject by Euler and later by Lagrange, within the language of differential calculus, the doctrine evolved to its modern state: on the one hand, and thanks to Weierstrass' foundational work, it represents a fruitful branch of functional analysis, and on the other hand, following the path marked among others by Lie, Noether and Cartan, there is the possibility to develop a geometrical study of the equations from a Lagrangian or Hamiltonian point of view, with a language that is independent of coordinates. The formalizing work by Lagrange and later Weierstrass gave it a sound theoretical foundation, while Lie's theory of continuous groups led to different key results of ODEs and PDEs theory, for example Noether's conservation laws. Application of Cartan's calculus brought geometrical character to the objects in the theory.

In the modern theory of numerical algorithms that model physical problems it is becoming clear [4, 13, 24, 27, 28, 29, 32, 33] the need of a formalization of the variational theory for a discrete Lagrangian and the consequent possibility to recover within this theory the discrete objects and its properties that may allow for a comparison with the corresponding objects that are present in the classical smooth theory. Such a study is not new [19, 24] but in the past decades has given rise to new numerical integration procedures of mechanical systems with interesting geometrical and long-term properties [4, 13, 20, 21, 27, 28, 29, 31, 34]. However, for several independent variables, that is, for PDEs of field theories, discrete variational integrators have been studied only in recent years [2, 13, 22, 23, 26].

We briefly recall the basic initial steps of the Variational Theory in several independent variables. The interested reader can find details and technicalities for this presentation in [11], other references for this approach are [9, 10, 12]. Given a fibred manifold $\pi\colon Y \to X$ over a p-dimensional manifold with volume element $\mathrm{vol}_X \in \Omega^p(X)$, any function $L\colon J^1Y \to X$ on the associated first jet bundle and any compact domain $A \subset X$ of integration leads to a functional $\mathbb{L}_A\colon y(x) \in \Gamma(A, Y) \mapsto \mathbb{R}$ given by $\mathbb{L}_A(y(x)) = \int_A L(j^1_x y)\mathrm{vol}_X \in \mathbb{R}$. If we consider a 1-parameter deformation $\{y_t(x)\}_{t\in(-\epsilon,\epsilon)}$ of a given section $y = y_0$, it induces an infinitesimal variation $\delta y = \frac{\mathrm{d}}{\mathrm{d}t} y_t(x)\big|_{t=0} \in \Gamma(A, y^*VY)$, that is, a section of the vertical bundle of Y restricted to $y(A)$. The value $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}_A(y_t(x))\big|_{t=0}$ only depends on δy, through a first order differential operator:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}_A(y_t(x))\bigg|_{t=0} = \int_A (\mathrm{d}L \circ j^1 y(x))(j^1(\delta y))\mathrm{vol}_X = \int_A (\mathrm{d}_{y(x)}\mathbb{L})(\delta y)$$

where $\mathrm{d}_{y(x)}\mathbb{L}\colon \Gamma(X, y^*VY) \to \Omega^p(X)$ is a first order differential operator taking $\delta y \in \Gamma(X, y^*VY)$ to the p-form $(\mathrm{d}L \circ j^1 y)(j^1(\delta y))\mathrm{vol}_X \in \Omega^p(X)$ on X (where j^1 represents the natural extension of sections and of vector fields to the first jet bundle). The key step to determine if $y(x)$ is stationary for $\mathbb{L}_A$ for any possible variation with compact support is to decompose this first order differential operator with values in densities into two components, one of them linear, and the other one an exact differential. This can be done with the introduction of the adjoint operator of $\mathrm{d}_{y(x)}\mathbb{L}$, a linear operator $\mathcal{E}(y(x)) \in \Gamma(X, y^*V^*Y)$ satisfying:

$$(\mathrm{d}_{y(x)}\mathbb{L})(w) = \langle w, \mathcal{E}(y(x)) \cdot \mathrm{vol}_X\rangle + \mathrm{d}\left(\langle \omega_{y(x)}, w\rangle\right)$$

where $\mathcal{E}(y(x)) \cdot \mathrm{vol}_X$ is a n-form on X with values in y^*V^*Y, called Euler form associated to the Lagrangian density $L \cdot \mathrm{vol}_X$ and to the section $y(x) \in \Gamma(X, Y)$ and where $\omega_{y(x)}$ is a $(n-1)$-form on X with values on y^*V^*Y, the dual vertical bundle along $y(x)$. This formula represents essentially an integration by parts procedure taking some terms to the boundary while leaving in $\mathcal{E}(y(x))$ the essential part that acts on δy as a 0-order differential operator, allowing then to apply the main lemma of the calculus of variations: $\langle w, \mathcal{E}(y(x)) \cdot \mathrm{vol}_x\rangle = 0\, (\forall w \in \Gamma(X, y^*VY)) \Leftrightarrow \mathcal{E}(y(x)) = 0$ which produces Euler-Lagrange equations that characterize critical sections. The boundary term (integration of $\mathrm{d}\left(\langle \omega_{y(x)}, w\rangle\right)$ goes to the boundary due to Stokes' Theorem) allows then to recover conserved quantities if we have symmetries of the Lagrangian density [9, 10, 11, 12]. This boundary term defines the momentum form $\Omega(j^1_x y) = \omega_{y(x)}(x)$, which is an horizontal V^*Y-valued $(p-1)$-form on J^1Y that leads to the associated Cartan form of the problem, an ordinary p-form defined as $\Theta = \theta \wedge \Omega + L\mathrm{vol}_X$, where θ is the VY-valued structure 1-form on J^1Y. In this way one may recover the whole Hamilton-Cartan theory, with the characterization of critical sections by means of the de Donder-Weyl equations, Poisson bracket for field theories, etc. (see [11, 12]). It is known that Θ and $\mathcal{E}$ can be constructed univocally and covariate with the Lagrangian when we consider the natural action of any bundle automorphism on these objects.

It is worth mentioning that already 30 years ago [24], an attemp was made to derive the discrete Euler equation in one independent variable via a variational approach and thereby to generate the proper boundary terms which are necessary for obtaining conservation theorems. As it was indicated in the work "The catalytic factor in formulating this approach is a discrete version of Lagrange's identity for difference operators and their adjoints".

Our work, within the spirit of the geometric calculus of variations [9, 10, 11], pretends to formulate, in the simplest possible context, a variational theory in several discrete independent variables which allows to recover the basic element of the calculus of variations: the integration by parts mechanism for densities (depending on a vector field), which allows to decompose the density into a linear term (that does not depend on derivatives of the vector field) and another term that is an exterior differential, which, using Stokes theorem, after integration, should depend only on boundary terms. In order to obtain the unique decomposition the emphasis is set on the geometrical nature of the object, its invariance with respect to possible automorphisms (change of coordinates) both for the discrete independent variables and for the smooth dependent ones.

Section 2 introduces the base discrete space where the discrete independent coordinates live, and the important example of the discrete Euclidean space together with its group of symmetries. In this work we shall visualize a discrete space as a collection of vertices, oriented line segments, oriented surface elements, and so on, by which we don't mean any realization in $\mathbb{R}^p$ but rather an abstract cellular complex in the sense of [3, 18], giving a set of cells, a function that allows to determine the dimension of each cell, and an incidence mapping determining if one cell is on the boundary of another and if the orientations are compatible, which allows to compute the oriented boundary. This is the basis for topological computations, like the determination of adherence, interior and boundary, for the introduction of oriented domains of integration (or chains) and its oriented boundaries, and for the consideration of discrete forms (or cochains) together with the duality product (integration) of forms on a domain, satisfying Stokes' formula. This is the minimal machinery required to introduce the notion of Lagrangian density and the variational problem associated to it. It must be indicated that variational problems on cell complexes where also considered in [14, 20], with preference given to simplicial complexes, and where a central role is given to the discrete Hodge operator on a simplicial complex and to the dual cell complex, which play no role in our theory. Though introduction of dual meshes and Hodge operator is seen as essential in these works ("dual meshes are essential as a means of encoding physically relevant phenomena such as fluxes across boundaries" [20]), we believe the choice to operate with this machinery to be more a matter of taste.

We concentrate ourselves then on the group of automorphisms: Abstract cellular complexes usually employed to discretize bodies (which are usually seen through a spatial model rather than in an abstract sense), as for example in the well established theory of finite elements, tend to be too rigid, in the sense that its group of automorphisms is not transitive, there is the possibility that different vertices don't have the same number of adherent cells, and thus different vertices or k-cells need different treatment when automorphisms of the grid are considered. In many cases the only symmetry for these cellular complexes is the identity. We shall avoid this by considering an example of p-dimensional homogeneous abstract cellular complex with well-known symmetry group: the discrete Euclidean space. The group of symmetries of this abstract cellular complex, namely the integer Euclidean group, is then studied, with special emphasis on the natural action of the group on each element of our discrete topological structure.

Section 3 is devoted to the formulation of variational problems on bundles over discrete cellular complexes. The introduction of a discrete Lagrangian density for these bundles (Definition 3.4) together with the choice of a domain where variations are taken lead to an action functional that is a function on a finite dimensional manifold. In the case of the Euclidean space a basic result (Lemma 3.5) shows how to interpolate any configuration of the discrete theory as a continuous configuration of the smooth theory, and therefore how to create a discrete Lagrangian density from a continuous one, which is the first step for the comparison of results of both formalisms. Critical configurations (Definition 3.8) are defined and variations of the action functional are expressed in terms of a difference operator (formula (3.3) and Lemma 3.9).

The objective of section 4 is a detailed study of difference operators (Definition 4.1) as indicated in section 3, and the main theorem (adjointness formula, Theorem 4.5) allows to decompose any such operator in two terms (formula (4.4)), one concentrated at the vertices and the other one being the differential of a $(p-1)$-cochain. The decomposition is unique if one requires the covariance of the objects. This result plays a central role in the whole discrete variational theory.

In section 5 this formula is successfully applied to obtain a first variation formula for the action functional of a discrete variational problem (Theorem 5.2) that relates the differential of the Lagrangian density with the discrete Euler and momentum forms (Definition 5.1) . As in the continuous case, this variation formula leads to a characterization of critical configurations (discrete Euler equations, Theorem 5.3) and the analysis of the boundary term in the case of symmetries of the problem leads to the corresponding [30] Noether current (Theorem 5.5), which is a $(p-1)$ form depending on the configuration, whose differential vanishes if the configuration is critical. We illustrate the meaning of these objects and first variation formula in the case of 1-dimension and 2-dimension discrete Euclidean spaces. Our result improves previous ones [6, 7, 15, 16, 22, 23, 25, 26] in being valid in any number of discrete independent or continuous dependent variables, having variational origin, being a formula on discrete forms (cochains), with local nature (and not an integral formula for the whole boundary of some domains), and is totally embedded in a geometric formalism, being independent of the choice of preferred directions and covariant for the whole group of symmetries.

Section 6 gives a physical interpretation of the discrete Euler equations as a condition of equilibrium of contact forces at each vertex and an interpretation of Noether theorem as vanishing of the total momentum of the forces pulling at the boundary from the inner side of certain domains, in the sense made clear in Theorem 6.4 and in the Remark thereafter. This result is related to previous ones: In [26] the authors, working in the case of 2 independent discrete variables, propose a way of deriving a first variation formula leading to various interesting results, among them a discrete theorem of Noether which, due to an interplay of a quadrilateral lattice and a Lagrangian depending on 3 vertices (which in fact means that a preferred direction is chosen on each quadrilateral) gives rise to a notion of the momentum that has three components at every face, loosing some of the symmetry of the base manifold. The resulting discrete Noether's formula [26, formula (5.7)] is given only in its integral version, which is a 2-dimensional analogue of (6.4) valid "on-shell" (i.e. for critical configurations), but not of its non-integral version given in Theorem (5.5), or of first variation formula (Theorem 5.2, which holds also "off-shell", valid for any configuration). We relate our Noether theorem with other conservation laws obtained in [15, 16].

Finally in section 7 we give a simple illustrative example from membrane theory and to some extent the main similarities and differences with the theory of Asynchronous Variational Integrators [23].

2. Discrete manifolds

Consider an oriented discrete p-dimensional manifold X (or oriented abstract cellular complex). By this we mean a set X (the cells) together with a mapping $\dim\colon X \to \{0, 1, \dots, p\}$ which allows to define $X_k = \{\beta \in X\colon \dim\beta = k\}$ so that $X = \bigsqcup_{k=0}^{p} X_k$ (X_0 is the set of 0-cells or vertices, X_1 is the set of oriented 1-cells or edges, and X_k is the set of oriented k-cells) and an incidence mapping $[\,:\,]\colon X \times X \to \{\pm 1, 0\}$ (see [3, 18]). We say a cell $\alpha \in X$ is incident to a cell $\beta \in X$ with compatible orientation if $[\beta\colon \alpha] = 1$, incident with opposite orientations if $[\beta\colon \alpha] = -1$ and non-incident if $[\beta\colon \alpha] = 0$. The incidence mapping is chosen so that each k-cell β admits only a finite number of incident cells, all with dimension $(k-1)$, that is, $[\beta\colon \alpha] \neq 0$ only for a finite number of cells α, all with $\dim\alpha = \dim\beta - 1$, and such that:

$$\sum_{\alpha \in X_k} [\beta_{k+1}\colon \alpha] \cdot [\alpha\colon \gamma_{k-1}] = 0, \qquad \forall k \in \{1, 2, \dots, p-1\}, \quad \beta_{k+1} \in X_{k+1}, \quad \gamma_{k-1} \in X_{k-1} \tag{2.1}$$

The incidence mapping allows to define a topology on the set of cells using the following notion of adherence:

Definition 2.1. We shall say a $(k-l)$-cell α is adherent to a k-cell β (and write $\alpha \prec \beta$) if $\alpha = \beta$ or if there exists a sequence of cells $\alpha_{k-l} = \alpha, \alpha_{k-l+1}, \alpha_{k-l+2}, \dots, \alpha_k = \beta$ each incident to the next one:

$[\alpha_k : \alpha_{k-1}] \cdot [\alpha_{k-1} : \alpha_{k-2}] \cdot \ldots \cdot [\alpha_{k-l+1} : \alpha_{k-l}] \neq 0$. Conversely, we shall say the k-cell β "contains" the $(k-l)$-cell α if α is adherent to β.

We define the k-chain space $C_k(X, \mathbb{Z})$ to be the free abelian group generated by X_k. A k-chain $c \in C_k(X, \mathbb{Z})$ is obtained when we assign an integer weight $c(\alpha)$ to each k-cell α, where each k-chain takes nonzero value only on a finite number of k-cells. In particular any k-cell $\beta \in X_k$ can be represented by the k-chain c_β that maps β to the weight 1 and any other k-cell γ to the weight 0.

The incidence mapping allows to define a boundary operator:

$$\partial_k : C_k(X, \mathbb{Z}) \to C_{k-1}(X, \mathbb{Z})$$

by $(\partial_k c)(\alpha) = \sum_{\beta \in X_k} [\beta : \alpha] \cdot c(\beta)$. In this situation the boundary of a k-cell γ is defined to be the $(k-1)$-chain

$$\partial_k c_\gamma = \sum_{\alpha \in X_{k-1}} [\gamma : \alpha] \cdot c_\alpha \tag{2.2}$$

The boundary of $\gamma \in X_k$ may be considered as the set of its incident cells α_{k-1} each with positive or negative weight depending on the compatibility of its orientation and that of γ. The finiteness condition imposed for the incidence mapping ensures that $\partial_k c$ is a chain (any $(k-1)$-cell has zero weight except for a finite number of them), and condition (2.1) imposed for the incidence mapping is equivalent to $\partial_k \circ \partial_{k+1} = 0$

$$\partial_k \partial_{k+1} c_\beta = \partial_k \left(\sum_{\alpha \in X_k} [\beta : \alpha] c_\alpha \right) = \sum_{\gamma \in X_{k-1}} \sum_{\alpha \in X_k} [\beta : \alpha] \cdot [\alpha : \gamma] \cdot c_\gamma = 0$$

therefore we have a chain complex $\partial_k : C_k(X, \mathbb{Z}) \to C_{k-1}(X, \mathbb{Z})$ that defines homology groups $\ker \partial_k / \operatorname{Im} \partial_{k+1}$.

In a similar manner (real) k-cochains (or k-forms) $\omega \in \Omega^k(X)$ can be defined to be functions $\omega : \beta \in X_k \mapsto \omega(\beta) \in \mathbb{R}$ on X_k. There is a natural duality product $\langle c, \omega \rangle = \sum_{\alpha \in X_k} c(\alpha) \cdot \omega(\alpha)$ between k-chains and k-cochains. This product allows to define the differential of a k-form ω as the $(k+1)$-form $\mathrm{d}\omega$ given by the rule

$$\langle c, \mathrm{d}\omega \rangle = \langle \partial c, \omega \rangle \tag{2.3}$$

If the duality product between cochains and chains is interpreted as the integration of forms on oriented domains, this definition is essentially Stokes' Theorem for integration of the differential of a k-form on a domain and the integration of the k-form on its boundary.

2.1. Main example: The discrete Euclidean space

One of the simplest examples of oriented p-dimensional discrete manifold is the case of the p-dimensional discrete Euclidean space (or "cartesian lattice", if we ignore the oriented cell complex structure). The vertices of this cell complex shall be the lattice $X_0 = (2\mathbb{Z})^p \subset \mathbb{R}^p$. We introduce this factor 2 for convenience so that any vertex can be indexed with a p-tuple $\alpha = (\alpha_1, \ldots, \alpha_p)$ with even $\alpha_1, \ldots, \alpha_p \in 2\mathbb{Z}$. Odd integers $j \in 2\mathbb{Z} + 1$ should be understood as representing intervals rather than points. In this framework, the discrete Euclidean space is given by cells $\alpha \in X = \mathbb{Z}^p = \mathbb{Z}e_1 \times \mathbb{Z}e_2 \times \ldots \times \mathbb{Z}e_p$ where $e_1, \ldots, e_p$ represent the generators of the grid, and where any element $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}^p$ with k odd entries and $p-k$ even entries represents a k-cell. For an element $\alpha = (\alpha_1, \ldots, \alpha_p) \in X$, we shall denote α^{odd} the ordered sequence of positions $(\alpha_1^{\text{odd}} < \ldots < \alpha_k^{\text{odd}}) = (m_1 < \ldots < m_k)$ corresponding to odd coordinates α_{m_i}, and similarly α^{even} represents the ordered sequence corresponding to even coordinates. The dimension mapping is defined as $\dim \alpha = \sharp \alpha^{\text{odd}}$ (Another common choice in the literature is the use of half-integers to represent intervals and integers to represent vertices).

As a model, each 0-cell can be seen as the point of $\mathbb{R}^p$ with the given coordinates, all of them even integers, each 1-cell α with one odd entry α_i can be seen as the line segment parallel to the e_i axis given by $(\alpha_1, \ldots, \alpha_i + \lambda, \ldots, \alpha_p)_{\lambda \in [-1,1]}$, each 2-cell β with two odd indexes β_i, β_j as the square parallel to the e_i, e_j plane given by $(\beta_1, \ldots, \beta_i + \lambda, \ldots, \beta_j + \mu, \ldots, \beta_p)_{\lambda, \mu \in [-1,1]}$ etc. In this model we interpret each k-cell endowed with the orientation given by its canonical basis. If $e_1, \ldots, e_p$ is the canonical basis of $\mathbb{R}^p$ and if $\alpha = (\alpha_1, \ldots, \alpha_p)$ is a k-cell with odd entries at positions $\alpha^{\text{odd}} = (m_1 < m_2 < \ldots < m_k)$, then we say $\text{span}_\alpha = \text{span}(e_{m_1}, \ldots, e_{m_k}) \subseteq \mathbb{R}^p$ is the director subspace, $(e_{m_1}, \ldots, e_{m_k})$ is the canonical basis and $\text{vol}_\alpha = e_{m_1} \wedge \ldots \wedge e_{m_k} \in \Lambda^k \mathbb{R}^p$ is the canonical orientation associated to the cell α.

For any $(k+1)$-cell $\beta = (\beta_1, \ldots, \beta_p)$ with odd entries at positions $\beta^{\text{odd}} = (n_1 < n_2 < \ldots < n_{k+1})$ we say that a k-cell α is incident to β if and only if β and α differ by an unitary vector of $\mathbb{R}^p$ (that is, $\alpha - \beta = \pm e_n$, where the vector $\alpha - \beta$ is the "outward pointing vector associated to both cells", and the index n must then correspond with one of the positions in β^{odd}). The incidence mapping is defined as:

$$\beta^{\text{odd}} = (n_1 < \ldots < n_{k+1}), \qquad \begin{cases} [\beta\colon \alpha] = s \cdot (-1)^{j-1}, & \forall s \in \{+1, -1\}, \alpha = \beta + s \cdot e_{n_j} \\ [\beta\colon \alpha] = 0, & \forall \alpha \text{ such that } \alpha - \beta \neq \pm e_{n_j} \end{cases} \tag{2.4}$$

In this situation a k-cell α is incident to a $(k+1)$-cell β iff the distance vector $\alpha - \beta$ is unitary, and it is incident with compatible orientation iff the orientation $e_{m_1} \wedge \ldots \wedge e_{m_k}$ associated to α coincides with the one induced by the orientation $e_{n_1} \wedge \ldots \wedge e_{n_{k+1}}$ of the cell β and the outward pointing vector $\alpha - \beta$:

$$\begin{aligned} &\alpha \in X_k,\ \beta \in X_{k+1},\ \|\alpha - \beta\| \quad = 1 \Rightarrow [\beta\colon \alpha] \cdot (\alpha - \beta) \wedge \text{vol}_\alpha = \text{vol}_\beta \\ &\text{else } [\beta\colon \alpha] = 0 \end{aligned} \tag{2.5}$$

For the $(k+1)$-cell β, its boundary may be computed following (2.2) as:

$$\partial c_\beta = \sum_{j=1}^{k+1} \sum_{s=\pm 1} (-1)^{j-1} \cdot s \cdot c_{\beta + s \cdot e_{n_j}}, \qquad \beta^{\text{odd}} = (n_1 < n_2 < \ldots < n_{k+1}) \tag{2.6}$$

which extends by $\mathbb{Z}$-linearity to the whole space of $(k+1)$-chains $C_{k+1}(X, \mathbb{Z})$.

The differential of a k-form can be written, following (2.3) and (2.6), as:

$$(\mathrm{d}\omega_k)(\beta) = \sum_{j=1}^{k+1} \sum_{s=\pm 1} (-1)^{j-1} \cdot s \cdot \omega_k(\beta + s \cdot e_{n_j}), \qquad \beta \in X_{k+1}, \qquad \beta^{\text{odd}} = (n_1 < n_2 < \ldots < n_{k+1}) \tag{2.7}$$

Lemma 2.2. *With the definition given in (2.5) the mapping $[\,:\,]$ on the discrete Euclidean space $X = \mathbb{Z}^p$ is an incidence mapping, that is, each $(k+1)$-cell has only a finite number of incident cells, all with dimension k, and $\sum_\alpha [\beta\colon \alpha] \cdot [\alpha\colon \gamma] = 0$ for any $(k+1)$-cell β and $(k-1)$-cell γ.*

Proof . The first part of the statement is trivial from the definition. We shall only prove that $[\,:\,]$ satisfies (2.1). For $\beta \in X_{k+1}$ such that $\beta^{\text{odd}} = (\beta_1^{\text{odd}} < \ldots < \beta_{k+1}^{\text{odd}})$ we have:

$$\partial \circ \partial c_\beta = \sum_{\gamma \in X_{k-1} \prec \alpha \prec \beta} [\beta\colon \alpha] \cdot [\alpha\colon \gamma] \cdot c_\gamma$$

Any $\gamma \in X_{k-1} \prec \beta$ has the form $\gamma = \beta + s e_{\beta_i^{\text{odd}}} + r e_{\beta_j^{\text{odd}}}$ (where $i \neq j$ and $r, s \in \{\pm 1\}$) and there are exactly two k-cells incident with γ and β, namely $\alpha = \beta + s e_{\beta_i^{\text{odd}}}$ and $\bar{\alpha} = \beta + r e_{\beta_j^{\text{odd}}}$. Therefore the c_γ-component in $\partial \circ \partial c_\beta$ is $[\beta\colon \alpha] \cdot [\alpha\colon \gamma] + [\beta\colon \bar{\alpha}] \cdot [\bar{\alpha}\colon \gamma]$. To prove that this vanishes we may use (2.5) to obtain:

$$\text{vol}_\beta = [\beta\colon \alpha](\alpha - \beta) \wedge \text{vol}_\alpha = [\beta\colon \alpha][\alpha\colon \gamma](\alpha - \beta) \wedge (\gamma - \alpha) \wedge \text{vol}_\gamma = [\beta\colon \alpha][\alpha\colon \gamma](s e_{\beta_i^{\text{odd}}}) \wedge (r e_{\beta_j^{\text{odd}}}) \wedge \text{vol}_\gamma$$
$$\text{vol}_\beta = [\beta\colon \bar{\alpha}](\bar{\alpha} - \beta) \wedge \text{vol}_\alpha = [\beta\colon \bar{\alpha}][\bar{\alpha}\colon \gamma](\bar{\alpha} - \beta) \wedge (\gamma - \bar{\alpha}) \wedge \text{vol}_\gamma = [\beta\colon \bar{\alpha}][\bar{\alpha}\colon \gamma](r e_{\beta_j^{\text{odd}}}) \wedge (s e_{\beta_i^{\text{odd}}}) \wedge \text{vol}_\gamma$$

as both expressions must coincide we conclude that $[\beta\colon \alpha][\alpha\colon \gamma] = -[\beta\colon \bar{\alpha}][\bar{\alpha}\colon \gamma]$, and $\partial\partial c_\beta = 0$, that is, $\sum_\alpha [\beta\colon \alpha] \cdot [\alpha\colon \gamma] = 0$. □

With this result we may conclude that $\partial \circ \partial = 0$ and $\mathrm{d} \circ \mathrm{d} = 0$.

2.2. Symmetry group of the discrete Euclidean space:

We turn our attention now to the group of symmetries of this discrete space and how it behaves with respect to the dimension and incidence mappings. The discrete Euclidean space admits a group of symmetries, the Euclidean group.

$$\mathrm{Eucl}(p,\mathbb{Z}) = \left\{ \varphi = \begin{pmatrix} \vec{\varphi} & b \\ 0 & 1 \end{pmatrix} \in M_{(p+1)\times(p+1)}(\mathbb{Z})\colon \vec{\varphi} \in M_{p\times p}(\mathbb{Z}), \quad \vec{\varphi}^t \cdot \vec{\varphi} = \mathrm{Id}_p, \quad b \in M_{p\times 1}(2\mathbb{Z}) \right\} \tag{2.8}$$

It must be noted that following $\vec{\varphi}^t \cdot \vec{\varphi} = \mathrm{Id}_p$ the columns in matrix $\vec{\varphi}$ must be unitary and linearly independent, therefore column i of matrix $\vec{\varphi}$ is a vector $s_i \cdot e_{\tau(i)}$ where $(s_1, \ldots, s_p)$ is a collection of signs ± 1 and τ is a permutation of the elements $\{1, \ldots, p\}$ depending on $\vec{\varphi}$ (which is the linear component of the affine morphism φ). Each matrix $\vec{\varphi}$ is obtained in a unique way as product $P_\tau \cdot D_s$ of a permutation matrix P_τ (where $\tau \in \mathrm{Bij}(\{1, \ldots, p\})$ and P_τ is the matrix with columns $e_{\tau(1)}, \ldots, e_{\tau(p)}$) and a diagonal matrix D_s with entries $s_1, \ldots, s_p \in \{\pm 1\}$ at the diagonal. Elements of the Euclidean group $\varphi = \begin{pmatrix} P_\tau \cdot D_s & b \\ 0 & 1 \end{pmatrix}$ correspond one to one with ternaries (τ, s, b) where $\tau \in \mathrm{Bij}(\{1, \ldots, p\})$, $s \in \mathrm{Map}(\{1, \ldots, p\}, \{\pm 1\})$, $b \in M_{p\times 1}(2\mathbb{Z})$. It can be easily seen that $D_s \cdot P_{\bar{\tau}} = P_{\bar{\tau}} \cdot D_{s\circ\bar{\tau}}$, also $D_s \cdot D_{\bar{s}} = D_{s\cdot\bar{s}}$, and $P_\tau \cdot P_{\bar{\tau}} = P_{\tau\circ\bar{\tau}}$. Therefore composition of movements is achieved by the following product:

$$\begin{pmatrix} P_\tau \cdot D_s & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} P_{\bar{\tau}} \cdot D_{\bar{s}} & \bar{b} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} P_{\tau\circ\bar{\tau}} \cdot D_{(s\circ\bar{\tau})\cdot\bar{s}} & b + P_\tau \cdot D_s \cdot \bar{b} \\ 0 & 1 \end{pmatrix}$$

which might also be written as $(\tau, s, b) \cdot (\bar{\tau}, \bar{s}, \bar{b}) = (\tau \circ \bar{\tau}, (s \circ \bar{\tau}) \cdot \bar{s}, b + P_\tau \cdot D_s \cdot \bar{b})$.

There is an action of $\varphi \in \mathrm{Eucl}(p, \mathbb{Z})$ on $X = \mathbb{Z}^p$ and on the exterior algebra $\Lambda^k \mathbb{R}^p$. Any element $\varphi = \begin{pmatrix} \vec{\varphi} & b \\ 0 & 1 \end{pmatrix} = (\tau, s, b)$ takes a cell $\alpha = (\alpha_1, \ldots, \alpha_p)$ and an element $\mathrm{vol}_k = e_{n_1} \wedge \ldots \wedge e_{n_k} \in \Lambda^k \mathbb{R}^p$ to the new cell $\varphi(\alpha) = (\alpha'_1, \ldots, \alpha'_p)$ and $\varphi \cdot \mathrm{vol}_k$ given by:

$$\begin{pmatrix} \alpha' \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{\varphi} & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ 1 \end{pmatrix}, \qquad \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{pmatrix}, \ \alpha' = \begin{pmatrix} \alpha'_1 \\ \vdots \\ \alpha'_p \end{pmatrix}, \quad \varphi \cdot \mathrm{vol}_k = \vec{\varphi}(e_{n_1}) \wedge \ldots \wedge \vec{\varphi}(e_{n_k})$$

Hence $(\varphi(\alpha))_i = \alpha'_i = \alpha_{\tau^{-1}(i)} \cdot s_{\tau^{-1}(i)} + b_i$. As $s \in \{\pm 1\}$ and b_i is even, we may conclude that entry i of $(\tau, s, b) \cdot \alpha$ is odd if and only if $i = \tau(j)$ for some $j \in \alpha^{\mathrm{odd}}$. Hence $(\varphi(\alpha))^{\mathrm{odd}}$ is the set $\tau(\alpha_1^{\mathrm{odd}}), \ldots, \tau(\alpha_k^{\mathrm{odd}})$ rearranged into its natural order. We conclude that the director subspace of the cell $\varphi(\alpha)$ is the image by $\vec{\varphi}$ of the director subspace of α, and both have the same dimension (same number of odd entries, perhaps at different positions):

$$\dim \varphi(\alpha) = \dim \alpha, \quad \mathrm{span}_{\varphi(\alpha)} = \vec{\varphi}(\mathrm{span}_\alpha)$$

If $\mathrm{vol}_\alpha = e_{\alpha_1^{\mathrm{odd}}} \wedge \ldots \wedge e_{\alpha_k^{\mathrm{odd}}}$ is transformed by φ, we get:

$$\varphi \cdot \mathrm{vol}_\alpha = (s_{\alpha_1^{\mathrm{odd}}} e_{\tau(\alpha_1^{\mathrm{odd}})}) \wedge \ldots \wedge (s_{\alpha_k^{\mathrm{odd}}} e_{\tau(\alpha_k^{\mathrm{odd}})}) = \pm \mathrm{vol}_{\varphi(\alpha)}$$

Hence there is a value $\mathrm{sgn}(\varphi, \alpha) \in \{\pm 1\}$ which determines if $\vec{\varphi}\colon \mathrm{span}_\alpha \to \mathrm{span}_{\varphi(\alpha)}$ respects the natural orientation chosen in these spaces, defined by:

$$\varphi \cdot \mathrm{vol}_\alpha = \mathrm{sgn}(\varphi, \alpha) \cdot \mathrm{vol}_{\varphi(\alpha)}$$

The mapping $\varphi \in \mathrm{Eucl}(p, \mathbb{Z}) \mapsto (\varphi \cdot) \in \mathrm{Aut}(\Lambda^k \mathbb{R}^p)$ is an action of the Euclidean group on the exterior algebra, that is $(\varphi \circ \psi) \cdot \mathrm{vol}_k = \varphi \cdot (\psi \cdot \mathrm{vol}_k)$, and this leads to:

$$\mathrm{sgn}(\varphi \circ \psi, \alpha) = \mathrm{sgn}(\varphi, \psi \cdot \alpha) \cdot \mathrm{sgn}(\psi, \alpha), \quad \mathrm{sgn}(\mathrm{Id}, \alpha) = 1, \quad \mathrm{sgn}(\varphi^{-1}, \varphi \cdot \alpha) = \mathrm{sgn}(\varphi, \alpha) \tag{2.9}$$

As φ is an isometry on $\mathbb{R}^p$ it is clear that $\|\varphi(\alpha) - \varphi(\beta)\| = 1$ if and only if $\|\alpha - \beta\| = 1$. Therefore we conclude:

$$[\beta\colon \alpha] = 0 \Leftrightarrow [\varphi(\beta)\colon \varphi(\alpha)] = 0, \qquad \alpha \prec \beta \Leftrightarrow \varphi(\alpha) \prec \varphi(\beta)$$

All the structures seem to be preserved by the action of φ. However, as we shall see there is the possibility that two incident cells with compatible orientation are transformed to incident cells with opposite orientations. Using (2.5) and the fact that $\varphi(\alpha) - \varphi(\beta) = \vec{\varphi}(\alpha - \beta)$ we have for any two incident cells $(\alpha \in X_{k-1}) \prec (\beta \in X_k)$:

$$\begin{aligned}\operatorname{sgn}(\varphi, \beta) \cdot \mathrm{vol}_{\varphi(\beta)} = \varphi \cdot \mathrm{vol}_\beta = \varphi \cdot ([\beta\colon \alpha](\alpha - \beta) \wedge \mathrm{vol}_\alpha) = [\beta\colon \alpha](\varphi(\alpha) - \varphi(\beta)) \wedge \varphi \cdot \mathrm{vol}_\alpha = \\ = [\beta\colon \alpha]\operatorname{sgn}(\varphi, \alpha)(\varphi(\alpha) - \varphi(\beta)) \wedge \mathrm{vol}_{\varphi(\alpha)}\end{aligned}$$

If we combine this result with $[\varphi(\beta)\colon \varphi(\alpha)](\varphi(\alpha) - \varphi(\beta)) \wedge \mathrm{vol}_{\varphi(\alpha)} = \mathrm{vol}_{\varphi(\beta)}$ given by (2.5) we conclude:

$$[\varphi(\beta)\colon \varphi(\alpha)] = \operatorname{sgn}(\varphi, \beta) \cdot \operatorname{sgn}(\varphi, \alpha) \cdot [\beta\colon \alpha]$$

and the incidence mapping for cells is not preserved by Euclidean movements, not even in the case $\det(\vec{\varphi}) = 1$.

If we want an action of $\varphi \in \mathrm{Eucl}(p, \mathbb{Z})$ on $C_k(X, \mathbb{Z})$ preserving the boundary operator on chains we may not simply define $\varphi \cdot c_\beta = c_{\varphi \cdot \beta}$. The sign $\operatorname{sgn}(\varphi, \beta)$ must be introduced. There is an action of $\varphi \in \mathrm{Eucl}(p, \mathbb{Z})$ on $C_k(X, \mathbb{Z})$ defined by:

$$\varphi \cdot c_\alpha = \operatorname{sgn}(\varphi, \alpha) \cdot c_{\varphi(\alpha)}$$

extended by $\mathbb{Z}$-linearity to $C_k(X, \mathbb{Z})$.

With this definition and the covariance formula for the incidence mapping $[\,:\,]$ one gets for any $\varphi \in \mathrm{Eucl}(p, \mathbb{Z})$:

$$\begin{aligned}\partial(\varphi \cdot c_\gamma) = \partial(\operatorname{sgn}(\varphi, \gamma) c_{\varphi(\gamma)}) = \operatorname{sgn}(\varphi, \gamma) \sum_{\varphi(\alpha)} [\varphi(\gamma)\colon \varphi(\alpha)] c_{\varphi(\alpha)} = \\ = \sum_\alpha \operatorname{sgn}(\varphi, \gamma)[\varphi(\gamma)\colon \varphi(\alpha)]\operatorname{sgn}(\varphi, \alpha)\varphi \cdot c_\alpha = \varphi \cdot (\sum_\alpha [\gamma\colon \alpha] c_\alpha) = \varphi \cdot \partial c_\gamma\end{aligned}$$

Consequently, the boundary morphism is covariant: $\varphi \cdot \partial c = \partial(\varphi \cdot c)$ for any movement $\varphi \in \mathrm{Eucl}(p, \mathbb{Z})$ and any chain $c \in C_k(X, \mathbb{Z})$.

If we define the action on the cochains $\omega \in \Omega^k(X)$ so that $\langle c, \omega \rangle = \langle \varphi \cdot c, \varphi \cdot \omega \rangle$, which can be done by setting $(\varphi \cdot \omega)(\alpha) = \langle c_\alpha, \varphi \cdot \omega \rangle = \langle \varphi^{-1} \cdot c_\alpha, \omega \rangle = \operatorname{sgn}(\varphi^{-1}, \alpha) \cdot \langle c_{\varphi^{-1}(\alpha)}, \omega \rangle = \operatorname{sgn}(\varphi^{-1}, \alpha) \cdot \omega(\varphi^{-1}(\alpha))$, the differential of cochains is also preserved by our action:

$$(\varphi \cdot \omega)(\alpha) = \operatorname{sgn}(\varphi^{-1}, \alpha) \cdot \omega(\varphi^{-1}(\alpha)) \qquad \mathrm{d}(\varphi \cdot \omega) = \varphi \cdot (\mathrm{d}\omega)$$

The Euclidean group acts transitively on the set of k-cells, preserving the boundary and coboundary operators of chains and cochains.

The following technical result concerning the action of $\mathrm{Eucl}(X, \mathbb{Z})$ on cells that share a vertex shall be used in section 4:

Lemma 2.3. *Consider the following* 0*-cell* $v^0 \in X_0$*,* $(p-1)$*-cells* $\alpha^k \in X_{p-1}$ *(for* $k = 1, \ldots, p$*) and* p*-cell* $\beta^0 \in X_p$ *of the discrete Euclidean space* X*:*

$$v^0 = (0, \ldots, 0), \quad \alpha^k = (\overbrace{0, -1, \ldots, -1}^{k}, \overbrace{1, \ldots, 1}^{p-k}), \quad \beta^0 = (1, \ldots, 1)$$

For any $(p-1)$*-cell and* p*-cell* $\alpha \in X_{p-1}$*,* $\beta \in X_p$ *with a common vertex* $v \in X_0$ *(that is,* $v \prec \alpha$*,* $v \prec \beta$*) there exists* $\varphi \in \mathrm{Eucl}(p, \mathbb{Z})$ *such that:*

$$\varphi(v) = v^0, \quad \varphi(\alpha) = \alpha^k, \quad \varphi(\beta) = \beta^0$$

Moreover, for any $\varphi \in \mathrm{Eucl}(p, \mathbb{Z})$ *satisfying* $\varphi(\alpha) = \alpha^k$*,* $\varphi(\beta) = \beta^0$ *there holds:*

$$k = \sharp\{j\colon \alpha_j \neq \beta_j\}, \quad \operatorname{sgn}(\varphi, \alpha) \cdot \operatorname{sgn}(\varphi, \beta) = (-1)^{i-1} \cdot s_i, \ \textit{where } i = \alpha^{\mathrm{even}}, \quad s_i = \beta_i - \alpha_i$$

PROOF . If we have $v \prec \alpha$, $v \prec \beta$, then:

$$v = (v_1, \ldots, v_p), \quad \alpha = (v_1 + r_1, \ldots, v_p + r_p), \quad \beta = (v_1 + s_1, \ldots, v_p + s_p)$$
$$s_1, \ldots, s_p, r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_p \in \{\pm 1\}, \quad r_i = 0 \text{ for } i = \alpha^{\text{even}}$$

Consider $\varphi_1 \in \mathrm{Eucl}(p, \mathbb{Z})$ given as a translation by vector $b = (-v_1, \ldots, -v_p)$. Then $\vec{\varphi}_1 = \mathrm{Id}$ is the identity on span_α and span_β and:

$$\varphi_1 \cdot v = (0, \ldots, 0) = v^0, \quad \varphi_1 \cdot \alpha = (r_1, \ldots, r_p), \quad \varphi_1 \cdot \beta = (s_1, \ldots, s_p), \quad \mathrm{sgn}(\varphi_1, \alpha) \cdot \mathrm{sgn}(\varphi_1, \beta) = 1$$

Consider now φ_2 given by $\vec{\varphi}_2 = D_s$ and $b = (0, \ldots, 0)$. Its linear component $\vec{\varphi}_2$ may change orientation in $\mathrm{span}_{\varphi_1(\alpha)} = \mathrm{span}(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_p)$ and also in $\mathrm{span}_{\varphi_1(\beta)} = \mathrm{span}(e_1, \ldots, e_p)$. There holds $\mathrm{sgn}(\varphi_2, \varphi_1 \cdot \alpha) = s_1 \cdot \ldots \cdot s_{i-1} \cdot s_{i+1} \cdot \ldots \cdot s_p$ and $\mathrm{sgn}(\varphi_2, \varphi_1 \cdot \beta) = \det D_s = s_1 \cdot \ldots \cdot s_p$. Then:

$$\varphi_2 \cdot \varphi_1 \cdot v = v^0, \quad \varphi_2 \cdot \varphi_1 \cdot \alpha = (r_1 s_1, \ldots, r_p s_p), \quad \varphi_2 \cdot \varphi_1 \cdot \beta = (1, \ldots, 1) = \beta^0, \quad \mathrm{sgn}(\varphi_2, \varphi_1 \cdot \alpha) \cdot \mathrm{sgn}(\varphi_2, \varphi_1 \cdot \beta) = s_i$$

Consider now φ_3 where $\vec{\varphi}_3$ is the permutation matrix P taking e_i to e_1 and $e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_p$ to $e_2, \ldots, e_p$ (keeping its order). In this case $\mathrm{sgn}(\varphi_3, \varphi_2 \cdot \varphi_1 \cdot \alpha) = 1$ (because φ_3 respects the ordering when the vectors $e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_p$ are taken to $e_2, \ldots, e_p$) and $\mathrm{sgn}(\varphi_3, \varphi_2 \cdot \varphi_1 \cdot \beta) = \det P = (-1)^{i-1}$, therefore:

$$\varphi_3 \cdot \varphi_2 \cdot \varphi_1 \cdot v = v^0, \quad \varphi_3 \cdot \varphi_2 \cdot \varphi_1 \cdot \alpha = (0, s_1 r_1, \ldots, s_{i-1} r_{i-1}, s_{i+1} r_{i+1}, \ldots, s_p r_p), \quad \varphi_3 \cdot \varphi_2 \cdot \varphi_1 \cdot \beta = \beta^0$$
$$\mathrm{sgn}(\varphi_3, \varphi_2 \cdot \varphi_1 \cdot \beta) \cdot \mathrm{sgn}(\varphi_3, \varphi_2 \cdot \varphi_1 \cdot \alpha) = (-1)^{i-1}$$

We called $k = \sharp\{j : \alpha_j \neq \beta_j\} = \sharp\{j : s_j r_j \neq 1\}$. Finally consider φ_4 given by the permutation matrix that takes e_1 to e_1 and reorders $e_2, \ldots, e_p$ to $e_{\tau(2)}, \ldots, e_{\tau(p)}$ so that

$$\varphi_4 \cdot v^0 = v^0, \quad \varphi_4 \cdot (0, s_1 r_1, \ldots, s_{i-1} r_{i-1}, s_{i+1} r_{i+1}, \ldots, s_p r_p) = (\overbrace{0, -1, \ldots, -1}^{k}, \overbrace{1, \ldots, 1}^{p-k}) = \alpha^k, \quad \varphi_4 \cdot \beta^0 = \beta^0$$

In this case $\mathrm{sgn}(\varphi_4, (0, s_1 r_1, \ldots, s_{i-1} r_{i-1}, s_{i+1} r_{i+1}, \ldots, s_p r_p)) = \mathrm{sgn}(\varphi_4, \beta^0)$ because our permutation affects only the vectors $e_2, \ldots, e_p$, which are both on the respective director spaces of the $(p-1)$-cell $\varphi_3 \cdot \varphi_2 \cdot \varphi_1 \cdot \alpha$ and of the p-cell β^0, keeping e_1 invariant. Therefore:

$$\mathrm{sgn}(\varphi_4, \varphi_3 \cdot \varphi_2 \cdot \varphi_1 \cdot \alpha) \cdot \mathrm{sgn}(\varphi_4, \varphi_3 \cdot \varphi_2 \cdot \varphi_1 \cdot \beta) = 1$$

Hence $\varphi = \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ satisfies the requirements: $\varphi(v) = v^0$, $\varphi(\alpha) = \alpha^k$, $\varphi(\beta) = \beta^0$, with k and $\mathrm{sgn}(\varphi, \alpha) \cdot \mathrm{sgn}(\varphi, \beta)$ as indicated.

There remains to prove that any other $\bar{\varphi}$ that takes α to some α^j, β to β^0 has $j = k$ and $\mathrm{sgn}(\bar{\varphi}, \alpha) \cdot \mathrm{sgn}(\bar{\varphi}, \beta) = (-1)^{i-1} \cdot s_i$. Taking into account the behavior of $\mathrm{sgn}(\varphi, \cdot)$ with respect to composition, we only need to consider $\psi = \bar{\varphi} \circ \varphi^{-1}$ and prove that for any ψ taking α^k to α^j and β^0 to β^0 there holds $k = j$ and $\mathrm{sgn}(\psi, \alpha^k) \cdot \mathrm{sgn}(\psi, \beta^0) = 1$.

Proving $k = j$ is trivial as ψ is an isometry and $\|\alpha^k - \beta^0\|^2 = 1 + 4(k-1)$, $\|\alpha^j - \beta^0\|^2 = 1 + 4(j-1)$. Proving the second part is easy: as $\psi(\alpha^k) = \alpha^k$ and $\psi(\beta^0) = \beta^0$ we conclude that $\vec{\psi}(\beta^0 - \alpha^k) = \beta^0 - \alpha^k$, and as the only odd entry in $\beta^0 - \alpha^k$ is the leading term 1, we may conclude (remember that $\vec{\psi}$ is the composition of a permutation matrix with a diagonal matrix defined by entries $s_i \in \{\pm 1\}$) that $\vec{\psi}(e_1) = e_1$. We repeat now the same argument as for φ_4: the remaining component of $\vec{\psi}$ is a linear automorphism $\vec{\psi}$ of $\langle e_2, \ldots, e_p \rangle = \mathrm{span}_{\alpha^k} = \mathrm{span}_{\alpha^j}$. The sign $\mathrm{sgn}(\psi, \alpha^k)$ and $\mathrm{sgn}(\psi, \beta^0)$ is in both cases the same value, the determinant of $\vec{\psi}$. Therefore $\mathrm{sgn}(\psi, \alpha^k) \cdot \mathrm{sgn}(\psi, \beta^0) = 1$, as we wanted to prove. □

3. Variational problems on discrete manifolds

Consider the discrete p-dimensional Euclidean space $X = X_0 \sqcup X_1 \sqcup \ldots \sqcup X_p$ as introduced in the previous section. We shall consider all possible configurations of this discrete space into a given configuration space.

Definition 3.1. Given a p-dimensional oriented abstract cellular complex X, we call configuration bundle of vertices any bundle $\pi_0 \colon Y_0 \to X_0$. By this we mean a projection where the fiber $(Y_0)_v$ on each vertex $v \in X_0$ is a nonempty differentiable manifold. Sections $y \in \Gamma(X_0, Y_0)$ of the bundle shall be called configurations of vertices. The bundle $\pi \colon VY_0 \to Y_0$ whose fiber at a configuration $q_v \in Y_0$ of the vertex $v \in X_0$ is the tangent space of the fiber $T_{q_v}(Y_0)_v$ shall be called the vertical bundle associated to Y_0. In a similar way a configuration bundle of k-cells is any bundle $\pi_k \colon Y_k \to X_k$, a configuration of k-cells a section y_k of this bundle, and the vertical bundle VY_k associated to Y_k has fiber $T_{y_k(\beta)}(Y_k)_\beta$ at any point $y_k(\beta) \in Y_k$, configuration of the k-cell $\beta \in X_k$.

For a n-dimensional manifold Q ("the configuration space") we shall call the trivial bundle $Y_0 = X_0 \times Q$ the configuration bundle associated to Q. A configuration of X into Q is any section of this bundle $y \in \Gamma(X_0, Y_0)$. Therefore a configuration is defined if we give a mapping $y \colon v \in X_0 \mapsto y(v) = (v, q_v) \in Y_0 = X_0 \times Q$, that is, if we give a set of points $q_v \in Q$ indexed by vertices $v \in X_0$ of the cell complex.

In the general case each configuration of vertices $y \in \Gamma(X_0, Y_0)$ is given as a set of configurations $q_v \in (Y_0)_v$, one for each vertex $v \in X_0$. If we consider $\Gamma(X_0, Y_0)$ as a (infinite dimensional) manifold, we may introduce its tangent space at a point $y = (q_v)_{v \in X_0} \in \Gamma(X_0, Y_0)$ as the product of the tangent spaces of the fibers $\prod_{v \in X_0} T_{q_v}(Y_0)_v$. Any one-parameter deformation of the configuration y can be written as $y(t) = (q_v(t))_{v \in X_0}$ and its first order approximation $\delta y = (\partial q_v / \partial t)_{v \in X_0}(t = 0)$ is given by a set of tangent vectors $\delta q_v \in T_{q_v}(Y_0)_v$, one for each vertex, that is, a section $\delta y \in \Gamma(X_0, VY_0)$ over the section $y = (q_v) \in \Gamma(X_0, Y_0)$.

Definition 3.2. For any section $y \colon X_0 \to Y_0$ the fibred product of $\pi \colon VY_0 \to Y_0$ with $y \colon X_0 \to Y_0$ defines a vector bundle $y^* VY_0 \to X_0$, that we call the bundle of infinitesimal variations of y, whose fiber at any $v \in X_0$ is the vector space $T_{y(v)} Y_0$. We shall call any section $\delta y \in \Gamma(X_0, y^* VY_0)$ of this bundle a infinitesimal variation of the configuration y.

Given a configuration bundle of k-cells $\pi_k \colon Y_k \to X_k$ and any configuration $y_k \in \Gamma(X_k, Y_k)$ the fibred product of $\pi \colon VY_k \to Y_k$ with $y_k \colon X_k \to Y_k$ shall be denoted $y_k^* VY_k$ and called bundle of infinitesimal variations of y_k. Its fiber at any k-cell β is the tangent space $T_{y_k(\beta)}(Y_k)_\beta$ at $y_k(\beta)$ of the fiber $(Y_k)_\beta$. A section δy_k of the bundle $y_k^* VY_k$ shall be called infinitesimal variation of y_k.

For any configuration bundle of vertices $\pi_0 \colon Y_0 \to X_0$ one may construct induced configuration bundles of edges $\pi_1 \colon Y_1 \to X_1$, and k-cells in general $\pi_k \colon Y_k \to X_k$ as follows. For any k-cell $\beta \in X_k$ we may consider the vertices $v \in X_0$ adherent to β. The bundle of configurations of k-cells Y_k induced by Y_0 is defined to be the bundle whose fiber $(Y_k)_\beta$ at any k-cell $\beta \in X_k$ is the product $\prod\limits_{v \in X_0 \prec \beta} (Y_0)_v$. Given a configuration bundle of vertices Y_0 and the induced bundles Y_k, any configuration of vertices $y \in \Gamma(X_0, Y_0)$ allows to define a configuration of k-cells defined by $y_k \colon \beta \in X_k \mapsto (y(v))_{v \prec \beta} \in (Y_k)_\beta = \prod_{v \prec \beta} (Y_0)_v$. There are obviously sections of Y_k that are not extensions of any $y \in \Gamma(X_0, Y_0)$. In order that a section y_k of Y_k is induced by some section y of Y_0 it is necessary and sufficient that for any two k-cells $\beta, \beta' \in X_k$ that share a common vertex $v \in X_0$ the v-components of $y_k(\beta) \in (Y_k)_\beta$ and $y_k(\beta') \in (Y_k)_{\beta'}$ coincide. These sections shall be called holonomic sections (they play a similar role to holonomic sections on jet bundles).

Consider the space of infinitesimal variations for each of the bundles of configurations $Y_k \to X_k$ induced by Y_0, where following definition 3.2 any section $y_k \in \Gamma(X_k, Y_k)$ has its own space of infinitesimal variations $\Gamma(X_k, y_k^* VY_k)$. In the same way as Y_0 induces bundles Y_k, the vector bundle $y^* VY_0 \to X_0$ has induced vector bundles $(y^* VY_0)_k \to X_k$ that can be naturally identified with $y_k^* VY_k$ when we use the identity:

$$T_{y_k(\beta)} \left(\prod_{v \prec \beta} (Y_0)_v \right) = \prod_{v \prec \beta} T_{y(v)}(Y_0)_v, \qquad \beta \in X_k, \quad y_k(\beta) = (y(v))_{v \prec \beta}$$

Any infinitesimal variation $\delta y \in \Gamma(X_0, y^* VY_0)$ of $y \in \Gamma(X_0, Y_0)$ naturally extends to a section $(\delta y)_k \in \Gamma(X_k, (y^* VY_0)_k) = \Gamma(X_k, y_k^* VY_k)$, which is an infinitesimal variation of $y_k \in \Gamma(X_k, Y_k)$. In a similar way, any vertical vector field $D \in \Gamma(Y_0, VY_0)$ on the bundle Y_0 naturally extends to a vertical vector field $D_k \in \Gamma(Y_k, VY_k)$ on the bundle Y_k.

Example 3.3. *In the case of the discrete Euclidean space, vertices adherent to the k-cell β are $\alpha \in X_0 = (2\mathbb{Z})^p$ such that $\max(|\beta_i - \alpha_i|) = 1$. Therefore:*

$$(Y_k)_\beta = \prod_{s_1,\ldots,s_k \in \{\pm 1\}} (Y_0)_{\beta_{s_1 \ldots s_k}}, \qquad \beta \in X_k, \quad \beta_{s_1 \ldots s_k} = \beta + s_1 e_{\beta_1^{\text{odd}}} + \ldots + s_k e_{\beta_k^{\text{odd}}} \in X_0$$

where the expressions $\beta_{s_1 \ldots s_k} \in X_0$ represent precisely the set of all vertices adherent to β. In the case of a trivial configuration space $Y_0 = X_0 \times Q$ defined by a manifold Q and the discrete Euclidean space $X = \mathbb{Z}^p$, a point of Y_1 is given by an edge $\beta \in X_1$ and a configuration on Q for each of the two vertices $\beta_- = \beta - e_{\beta_1^{\text{odd}}}, \beta_+ = \beta + e_{\beta_1^{\text{odd}}} \in X_0$ adherent to β. In a similar way, a point of Y_k is given by a k-cell $\beta \in X_k$ and a configuration on Q for each of the 2^k vertices adherent to β. The extension y_1 of a section $y \in \Gamma(X_0, Y_0)$ is defined as $y_1 \colon \beta \in X_1 \mapsto (y(\beta_-), y(\beta_+)) \in (Y_1)_\beta = (Y_0)_{\beta_-} \times (Y_0)_{\beta_+}$ where $\beta_-, \beta_+ \in X_0$ are the two vertices adherent to $\beta \in X_1$. In a similar way, the configuration y allows to define a configuration of k-cells defined by $y_k \colon \beta \in X_k \mapsto (y(\beta_{s_1 \ldots s_k})) \in (Y_k)_\beta = \prod_{s_1,\ldots,s_k \in \{\pm 1\}} (Y_0)_{\beta_{s_1 \ldots s_k}}$.

With these structures in mind, a variational problem can be formulated as finding configurations $y \in \Gamma(X_0, Y_0)$ that are critical (in some sense) for the functional defined by a given Lagrangian density, in the sense we shall state in the following.

Definition 3.4. Given a configuration bundle of vertices of a discrete p-dimensional manifold X, that is, a bundle $\pi_0 \colon Y_0 \to X_0$, we call discrete Lagrangian density any function $L \colon Y_p \to \mathbb{R}$ on the associated bundle of configurations of p-cells.

A Lagrangian density can be determined when we give a real number for any given p-cell β and configurations q_v for each of its adherent vertices:

$$(\beta, (q_v)_{v \prec \beta}) \in Y_p \mapsto L(\beta, (q_v)_{v \prec \beta}) \in \mathbb{R}$$

Most of the problems modelled on the discrete Euclidean space $\mathbb{Z}^p$ will be given by a trivial bundle $Y_0 = X_0 \times Q$ and Lagrangian densities that do not depend on the particular p-cell, but only on the configuration of the vertices. These Lagrangians are determined when we give a function:

$$L \colon \prod_{s_1,\ldots,s_p \in \{\pm 1\}} Q \to \mathbb{R}$$

In dimension 2, these Lagrangians shall be given by funcions $L(q_{--}, q_{+-}, q_{-+}, q_{++}) \colon Q_{--} \times Q_{+-} \times Q_{-+} \times Q_{++} \to \mathbb{R}$, that define a Lagrangian density $L \colon Y_2 \to \mathbb{R}$ taking a given 2-cell $\beta = (i_1, i_2)$, and configurations $(q_{\beta_{--}}, q_{\beta_{+-}}, q_{\beta_{-+}}, q_{\beta_{++}})$ of its adherent vertices $\beta_{--} = (i_1 - 1, i_2 - 1)$, $\beta_{+-} = (i_1 + 1, i_2 - 1)$, $\beta_{-+} = (i_1 - 1, i_2 + 1)$, $\beta_{++} = (i_1 + 1, i_2 + 1)$ to the real number $L(q_{\beta_{--}}, q_{\beta_{+-}}, q_{\beta_{-+}}, q_{\beta_{++}})$.

In the case that $Q = \mathbb{R}^n$ and X is the discrete p-dimensional Euclidean space, any mapping $y \colon \mathbb{R}^p \to \mathbb{R}^n$ induces a section $y \circ i \colon X_0 \to Y_0$ of the trivial bundle $Y_0 = X_0 \times Q$, when we consider the natural immersion $i \colon X_0 \hookrightarrow \mathbb{R}^p$ defined by $(\alpha_1, \ldots, \alpha_p) \in X_0 \mapsto (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p$. Therefore any discrete Lagrangian density $L \colon Y_p \to \mathbb{R}$ on X allows to define a non-continuous functional $\mathbb{L} \colon \mathcal{C}^\infty(\mathbb{R}^p, \mathbb{R}^n) \mapsto \mathbb{R}$ defined as $\sum_{\beta \in X_p} L\left((y \circ i)_p(\beta)\right)$. This functional makes sense only when the (infinite) sum can be done and is concentrated at points of the $(2\mathbb{Z})^p$ grid, leading to a (non-smooth) variational problem on the space $\mathcal{C}^\infty(\mathbb{R}^p, \mathbb{R}^n)$.

Conversely, if we have a first order variational problem in the space of differentiable mappings $y(x) \colon \mathbb{R}^p \to \mathbb{R}^n$, that is, a function $L(x_\nu, y_i, y_{i,\nu})$ $(1 \le \nu \le p,\ 1 \le i \le n)$ and consider the corresponding functional $\int L(x, y(x), \mathrm{d}y(x))\mathrm{d}x$, there are several ways of determining a discrete version of the problem (see [13, 33, 34] for example), some of them substitute the values y_i and $y_{i,\nu}$ by some expressions depending on $(q_v)_{v \prec \beta}$ that serve as good interpolations of the values y_i and its derivatives. Another methods [34] take as value $L(\beta, (q_v)_{v \prec \beta})$ for the discrete Lagrangian at some configuration $(q_v)_{v \prec \beta}$ the actual value of the functional for some mapping $y(x) \colon D_\beta \to Q$ that serves as a model for the configuration $(q_v)_{v \prec \beta}$ of the abstract cell $\beta \in X_p$. We mean:

$$L(\beta, (q_v)_{v \prec \beta}) := \int_{D_\beta} L(x_\nu, y_i(x), \partial y_i(x)/\partial x_\nu)$$

for some domain $D_\beta \subset \mathbb{R}^p$ and differentiable function $y(x)\colon D_\beta \to \mathbb{R}^n$ depending on the configuration $(q_v)_{v\prec\beta}$. This is basically the choice we make to discretize a continuous Lagrangian density, by means of the following result:

Lemma 3.5. *Given a p-cell $\beta = (\beta_1, \ldots, \beta_p)$ of the p-dimensional discrete Euclidean space and a configuration $(q_v)_{v\prec\beta} = (q_{s_1\ldots s_p})_{s_1,\ldots,s_p\in\{\pm1\}}$ of β on the space $Q = \mathbb{R}^n$, there exists a unique mapping $y(x)\colon D_\beta = \prod_{i=1}^{p}[\beta_i - 1, \beta_i + 1] \subseteq \mathbb{R}^p \to \mathbb{R}^n$ that is affine separately on each of the variables $x_1, \ldots, x_p$ such that for each vertex $\beta_{r_1\ldots r_p} \in X_0$ adherent to β holds:*

$$y(i(\beta_{r_1\ldots r_p})) = q_{r_1\ldots r_p}, \qquad r_1, \ldots, r_p \in \{\pm1\}$$

where $i\colon (2\mathbb{Z})^p \hookrightarrow \mathbb{R}^p$ is the natural immersion of the $(2\mathbb{Z})^p$ grid into $\mathbb{R}^p$.

The equations of $y(x)$ are:

$$y(x_1, \ldots, x_p) = \Delta^0 q + \sum_i \Delta_i^1 q\cdot(x_i - \beta_i) + \sum_{i<j} \Delta_{ij}^2 q\cdot(x_i-\beta_i)(x_j-\beta_j) + \ldots + \Delta_{1\ldots p}^p q\cdot(x_1-\beta_1)\ldots(x_p-\beta_p) \quad (3.1)$$

where

$$\Delta^0 q = \frac{1}{2^p}\sum_{s_1,\ldots,s_p\in\{\pm1\}} q_{s_1\ldots s_p}, \qquad \Delta_{i_1\ldots i_k}^k q = \frac{1}{2^p}\sum_{s_1,\ldots,s_p\in\{\pm1\}} s_{i_1}\cdot\ldots s_{i_k}\cdot q_{s_1\ldots s_p}$$

Proof . If we consider the point $i(\beta_{r_1\ldots r_p}) \in D_\beta$ (where $r_1, \ldots, r_p \in \{\pm1\}$ are fixed), whose coordinates satisfy $x_i - \beta_i = r_i$, we want to prove that $y(x_1, \ldots, x_p) = q_{r_1\ldots r_p}$. When we use (3.1) and compute $y(x_1, \ldots, x_p)$ we get:

$$y(x_1, \ldots, x_p) = \sum_{s_1,\ldots,s_p\in\{\pm1\}} \frac{1}{2^p} c_{s_1\ldots s_p} q_{s_1\ldots s_p}$$

Where $c_{s_1\ldots s_p} = 1 + \sum_{i_1} r_{i_1}s_{i_1} + \sum_{i_1<i_2} r_{i_1}r_{i_2}s_{i_1}s_{i_2} + \sum_{i_1<i_2<i_3} \ldots + r_1\cdot\ldots\cdot r_p\cdot s_1\cdot\ldots\cdot s_p$

We want to prove that $c_{s_1\ldots s_p}$ is 2^p when $s_1\ldots s_p$ coincides with $r_1, \ldots r_p$ and 0 in any other case. If we call $\alpha_i = s_i\cdot r_i$ we want to prove that:

$$1 + \sum_{i_1}^{p}\alpha_{i_1} + \sum_{i_1<i_2}^{p}\alpha_{i_1}\cdot\alpha_{i_2} + \sum\ldots + \alpha_1\cdot\ldots\cdot\alpha_p = \begin{cases} 2^p \text{ for } \alpha = (1, \ldots, 1) \\ 0 \text{ for any other } \alpha\in\{\pm1\}^p \end{cases}$$

This can be seen by induction on p. For $p = 1$, it is clear that $1 + \alpha$ is 2 when $\alpha = 1$ and 0 in the case $\alpha = -1$. For any $(\alpha_1, \ldots, \alpha_p) \in \{\pm1\}^p$ it can be seen:

$$\begin{aligned} &1 + \sum_{i_1}^{p}\alpha_{i_1} + \sum_{i_1<i_2}^{p}\alpha_{i_1}\cdot\alpha_{i_2} + \sum\ldots + \alpha_1\cdot\ldots\alpha_p = \\ &= \left(1 + \sum_{i_1}^{p-1}\alpha_{i_1} + \sum_{i_1<i_2}^{p-1}\alpha_{i_1}\cdot\alpha_{i_2} + \sum\ldots + \alpha_1\cdot\ldots\cdot\alpha_{p-1}\right) + \\ &+ \alpha_p\left(1 + \sum_{i_1}^{p-1}\alpha_{i_1} + \sum_{i_1<i_2}^{p-1}\alpha_{i_1}\cdot\alpha_{i_2} + \sum\ldots + \alpha_1\cdot\ldots\cdot\alpha_{p-1}\right) \end{aligned}$$

Whenever $(\alpha_1, \ldots, \alpha_{p-1}) \in \{\pm1\}^{p-1}$ is different from $(1, \ldots, 1)$ this is $0 + \alpha_p\cdot 0 = 0$, by induction, and whenever $\alpha_p = -1$, we are adding and substracting the same quantity, leading also to zero. In the case $\alpha_1 = \ldots = \alpha_{p-1} = \alpha_p = 1$ the result, by induction, would be $2^{p-1} + 2^{p-1} = 2^p$.

For the uniqueness, it must be noted that if $\bar{y}(x)$ is affine on each separate variable, then it must have the form:

$$\bar{y}(x) = d^0 + \sum_{i=1}^{p} d_i^1(x_i - \beta_i) + \sum_{i<j=1}^{p} d_{ij}^2(x_i - \beta_i)(x_j - \beta_j) + \ldots + d_{1\ldots p}^p(x_1 - \beta_1) \cdot \ldots \cdot (x_p - \beta_p), \quad d_{i_1\ldots i_k}^k \in \mathbb{R}$$

If the values of $\bar{y}(x)$ at the vertices are the chosen ones, they coincide with those given by our function $y(x)$, so $y(x) - \bar{y}(x) = 0$ whenever $x_i - \beta_i = \pm 1$. Therefore for any choice of $(r_1, \ldots, r_p) \in \{\pm 1\}^p$ holds:

$$(d^0 - \Delta^0 q) + \sum_{i=1}^{p}(d_i^1 - \Delta_i^1 q)r_i + \sum_{i<j=1}^{p}(d_{ij}^2 - \Delta_{ij}^2 q)r_i r_j + \ldots + (d_{1\ldots p}^p - \Delta_{1\ldots p}^p q)r_1 \cdot \ldots \cdot r_p = 0$$

We may conclude that $d_{i_1\ldots i_k}^k = \Delta_{i_1\ldots i_k}^k q$ if we prove that the system of 2^p equations

$$z^0 + \sum_{i_1=1}^{p} z_{i_1}^1 r_{i_1} + \sum_{i_1<i_2=1}^{p} z_{i_1 i_2}^2 r_{i_1} r_{i_2} + \ldots + z_{1\ldots p}^p r_1 \cdot \ldots \cdot r_p = 0, \qquad \forall r_1, \ldots, r_p = \pm 1 \tag{3.2}$$

has only the solution $z_{i_1\ldots i_k}^k = 0$, which we prove again by induction. In the case $p = 1$ it is obvious:

$$z^0 + z_1^1 = 0, \quad z^0 - z_1^1 = 0 \Rightarrow z^0 = 0, \quad z_1^1 = 0$$

For any other p, equations (3.2) for $r_p = +1$, $r_p = -1$ lead to:

$$\left.\begin{aligned}
&\left(z^0 + \sum_{i_1=1}^{p-1} z_{i_1}^1 r_{i_1} + \sum_{i_1<i_2=1}^{p-1} z_{i_1 i_2}^2 r_{i_1} r_{i_2} + \ldots + z_{1\ldots p-1}^{p-1} r_1 \cdot \ldots \cdot r_{p-1}\right) + \\
&+ \left(z_p^1 + \sum_{i_1=1}^{p-1} z_{i_1 p}^2 r_{i_1} + \sum_{i_1<i_2=1}^{p-1} z_{i_1 i_2 p}^3 r_{i_1} r_{i_2} + \ldots + z_{1\ldots(p-1)p}^p r_1 \cdot \ldots \cdot r_{p-1}\right) = 0 \\
&\left(z^0 + \sum_{i_1=1}^{p-1} z_{i_1}^1 r_{i_1} + \sum_{i_1<i_2=1}^{p-1} z_{i_1 i_2}^2 r_{i_1} r_{i_2} + \ldots + z_{1\ldots p-1}^{p-1} r_1 \cdot \ldots \cdot r_{p-1}\right) + \\
&+ (-1) \cdot \left(z_p^1 + \sum_{i_1=1}^{p-1} z_{i_1 p}^2 r_{i_1} + \sum_{i_1<i_2=1}^{p-1} z_{i_1 i_2 p}^3 r_{i_1} r_{i_2} + \ldots + z_{1\ldots(p-1)p}^p r_1 \cdot \ldots \cdot r_{p-1}\right) = 0
\end{aligned}\right\} \quad \forall r_1, \ldots, r_{p-1} = \pm 1$$

which mean:

$$\left.\begin{aligned}
&\left(z^0 + \sum_{i_1=1}^{p-1} z_{i_1}^1 r_{i_1} + \sum_{i_1<i_2=1}^{p-1} z_{i_1 i_2}^2 r_{i_1} r_{i_2} + \ldots + z_{1\ldots p-1}^{p-1} r_1 \cdot \ldots \cdot r_{p-1}\right) = 0 \\
&\left(z_p^1 + \sum_{i_1=1}^{p-1} z_{i_1 p}^2 r_{i_1} + \sum_{i_1<i_2=1}^{p-1} z_{i_1 i_2 p}^3 r_{i_1} r_{i_2} + \ldots + z_{1\ldots(p-1)p}^p r_1 \cdot \ldots \cdot r_{p-1}\right) = 0
\end{aligned}\right\} \quad \forall r_1, \ldots, r_{p-1} = \pm 1$$

and by induction hypothesis, $z_{i_1\ldots i_k}^k = 0$, $z_{i_1\ldots i_k p}^{k+1} = 0$ for any $i_1 < \ldots < i_k \in \{1, \ldots, p-1\}$. $\square$

It must be noted that for two neighboring p-cells β, $\tilde{\beta} = \beta + 2e_j$ with the common $(p-1)$-cell $\alpha = \beta + e_j$, the domains D_β and $D_{\tilde{\beta}}$ intersect as $\prod_{i=1}^{p}[\beta_i - 1, \beta_i + 1] \cap (x_j = \beta_j + 1)$. For two configurations $(\beta, (q_v)_{v \prec \beta})$ and $(\tilde{\beta}, (\tilde{q}_v)_{v \prec \tilde{\beta}})$ we may consider the unique mapping $y(x)$ affine on each variable and taking the vertices $v \prec \beta$ to q_v, and the unique mapping $\tilde{y}(x)$ affine on each variable taking the vertices $v \prec \tilde{\beta}$ to $\tilde{q}_v$. The restriction of $y(x)$ and $\tilde{y}(x)$ to $D_\beta \cap D_{\tilde{\beta}}$ is in both cases an affine function, and if $(q_v)_{v \prec \beta}$, $(\tilde{q}_v)_{v \prec \tilde{\beta}}$ were chosen so that $q_v = \tilde{q}_v$ for each common vertex $v \prec \beta, v \prec \tilde{\beta}$, then both restrictions coincide on all the vertices adherent to $D_\beta \cap D_{\tilde{\beta}}$. Using the uniqueness indicated in the previous lemma for the $(p-1)$-dimensional case (where $(x_j = \beta_j + 1) \subset \mathbb{R}^p$ can be seen as $\mathbb{R}^{p-1}$), we may conclude that $y(x)$ coincides with $\tilde{y}(x)$ in the intersection. Therefore any configuration of vertices $y_0 \colon X_0 \to Y_0 = X_0 \times Q$ determines a configurations of p-cells y_p, and each $y_p(\beta)$ defines a mapping $y_\beta(x) \colon D_\beta \to \mathbb{R}^n$, all compatible among themselves, that allow to univocally construct $y(x) \colon \mathbb{R}^p \to \mathbb{R}^n$ continuous, affine on each separate variable on each domain $D_\beta = \prod_{i=1}^p [\beta_i - 1, \beta_i + 1] \subseteq \mathbb{R}^p$ and such that $y(i(v)) = y_0(v)$ for any vertex v. This space of continuous mappings $y(x) \colon \mathbb{R}^p \to \mathbb{R}^n$ that are affine on each separate variable on each domain D_β is in one to one correspondence with the space of configurations $y \in \Gamma(X_0, Y_0)$.

Example 3.6. *The movement of a rectilinear spring is given by a function $y(s,t) \colon \mathbb{R}^2 \to \mathbb{R}$. The action that defines the dynamics of a rectilinear spring is given by the Lagrangian density:*

$$L \colon y(s,t) \mapsto \frac{1}{2} m \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2} k \left(\frac{\partial y}{\partial s}\right)^2$$

A discretized version of this situation arises when we consider discrete configurations as mappings $y \colon X_0 \to \mathbb{R}$, were X_0 is the space of vertices of the bidimensional discrete euclidean space. These are sections of the bundle $X_0 \times \mathbb{R} \to X_0$.

If we consider a configuration at $\beta \in X_2$, $(\beta, q_{--}, q_{+-}, q_{-+}, q_{++}) \in X_2 \times \mathbb{R}^4 = Y_2$, the associated mapping on the domain $D_\beta \subseteq \mathbb{R}^2$ is:

$$\begin{aligned} y(s,t) &= \Delta^0 q + (s - \beta_1)\Delta^1_1 q + (t - \beta_2)\Delta^1_2 q + (s - \beta_1)(t - \beta_2)\Delta^2_{12} q \\ \Delta^0 q &= \frac{1}{4}(q_{--} + q_{+-} + q_{-+} + q_{++}), \quad \Delta^1_1 q = \frac{1}{4}(-q_{--} + q_{+-} - q_{-+} + q_{++}), \\ \Delta^1_2 q &= \frac{1}{4}(-q_{--} - q_{+-} + q_{-+} + q_{++}), \quad \Delta^2_{12} q = \frac{1}{4}(q_{--} - q_{+-} - q_{-+} + q_{++}), \end{aligned}$$

We saw in the previous theorem that $y(s,t)$ is affine on each separate variable s, t and at the points $(\beta_1 \pm 1, \beta_2 \pm 1)$ the function takes the values $q_{\pm\pm}$.

Taking the integral of $L(y(s,t))$ on the domain $D_\beta = [\beta_1 - 1, \beta_1 + 1] \times [\beta_2 - 1, \beta_2 + 1]$ we get a discrete Lagrangian density:

$$L(\beta, q_{--}, q_{+-}, q_{-+}, q_{++}) = 2m(\Delta^1_2 q)^2 - 2k(\Delta^1_1 q)^2 + \frac{2}{3}(-k + m)(\Delta^2_{12} q)^2$$

This would be a discrete version of the Lagrangian when we interpret the domain associated to β as D_β (which has diameter 2, reflected in the multiplication by a factor 4). We may interpret $\Delta^1_1 q$ as a discrete version of $\frac{\partial y}{\partial s}$, $\Delta^1_2 q$ as a discrete version of $\frac{\partial y}{\partial t}$ and $\Delta^2_{12} q$ as a discrete version of the second derivative operator $\frac{\partial^2 y}{\partial s \partial t}$.

Another method to obtain a discrete Lagrangian L would be the substitution of $\partial y/\partial s$ by the constant value $\left(\frac{q_{++}+q_{+-}}{2} - \frac{q_{-+}+q_{--}}{2}\right)/2$ (analogous for $\partial y/\partial t$). Here the inner factor $\frac{1}{2}$ reflects the consideration of the mean value to approximate y on the edge $(\beta_1 + 1, \beta_2)$ and on the edge $(\beta_1 - 1, \beta_2)$, the difference of both terms may be interpreted as the computation of the increment of y between both edges, and the second $\frac{1}{2}$ factor is due to the fact that s increments its value in two units when we go from $(\beta_1 - 1, \beta_2)$ to $(\beta_1 + 1, \beta_2)$. That is, L could be obtained by direct substitution of $\partial y/\partial s$ by $\Delta^1_1 q$ and of $\partial y/\partial t$ by $\Delta^1_2 q$ (so $y(s,t)$ would not depend on s, t). Integration of this constant value over the domain D_β (which has volume 4) would lead to a different discrete version of the continuous Lagrangian density:

$$L(\beta, q_{--}, q_{+-}, q_{-+}, q_{++}) = 2m(\Delta^1_2 q)^2 - 2k(\Delta^1_1 q)^2$$

Which might be a simpler expression to work with but some information is lost if we want to relate the discrete and smooth variational problems. Both expressions coincide when $\Delta^2_{12} q$ vanishes, which happens only *when $y(s,t)$ is affine on s,t, that is, when the four vertices $(\beta_{--},q_{--}),(\beta_{+-},q_{+-}),(\beta_{-+},q_{-+}),(\beta_{++},q_{++})$ determine a parallelogram.*

There is still the possibility of a simpler expression, substituting $\partial y/\partial s$ by the constant value $\frac{1}{2}(q_{+-}-q_{--})$ and $\partial y/\partial t$ by $\frac{1}{2}(q_{++}-q_{+-})$ and integrate on D_β to derive the discrete Lagrangian from the continuous one (as can be found in [26] except for the correcting factor $1/2$ due to the diameter 2 of our domains), ignoring then the configuration q_{-+} for each face, and obtaining.

$$L(\beta,q_{--},q_{+-},q_{-+},q_{++})=2m\left(\frac{q_{+-}-q_{--}}{2}\right)^2-2k\left(\frac{q_{++}-q_{+-}}{2}\right)^2=\frac{1}{2}m(q_{+-}-q_{--})^2-\frac{1}{2}k(q_{++}-q_{+-})^2$$

This expression also coincides with the previous ones if the four vertices determine a parallelogram.

After deriving the discrete Lagrangian in any of the above indicated ways, we might study the discrete action and the associated variational problem: A configuration $y=(q_{i,j})_{i,j\in 2\mathbb{Z}}$ extends to a configuration $y_2\colon X_2\to Y_2$, that transforms any 2-cell $\beta=(i,j)$ into $(q_{i-1,j-1},q_{i+1,j-1},q_{i-1,j+1},q_{i+1,j+1})\in Q\times Q\times Q\times Q$. Each of these configurations of the 2-cells β allows to compute $L(y_2(\beta))$, and the original action $\int(1/2)m(\partial y/\partial t)^2-(1/2)k(\partial y/\partial s)^2$ may be substituted in the discrete formalism by $\sum L(y_2(\beta))$.

3.1. Critical sections:

Given a Lagrangian density $L\colon Y_p\to\mathbb{R}$, any configuration $y\colon X_0\to Y_0$ allows to define a p-cochain (discrete p-form) on X by $y_p^*L\colon \beta\in X_p\mapsto L(y_p(\beta))\in\mathbb{R}$. For any p-chain $c\in C_p(X,\mathbb{Z})$ one may try to determine a configuration $y\in\Gamma(X_0,Y_0)$ for which $\langle c,y_p^*L\rangle$ is minimal. However, this minimality condition for arbitrary variations of y is usually too strong. A weaker condition appears if we want to determine the minimum only among a subset of "admissible" variations of y, for example if we consider minimality among those sections whose values coincide with those of y at any vertex of the boundary of c. We shall make this point clear with the following definitions.

Definition 3.7. Given a k-chain $c\in C_k(X,\mathbb{Z})$ we call support of c the finite set of vertices $v\in X_0$ such that $v\prec\beta$ for some k-cell β with $c(\beta)\neq 0$, we call frontier of the chain c the support of the boundary ∂c of c, and interior of the chain c the set of vertices that are in the support of c and not in its frontier. We denote these objects by $\operatorname{supp}(c)$, $\operatorname{fr}(c)$ and $\operatorname{int}(c)$ respectively:

$$\begin{aligned}\operatorname{supp}(c)&=\{v\in X_0:\ \exists\beta\in X_k, v\prec\beta, c(\beta)\neq 0\},\qquad c\in C_k(X,\mathbb{Z})\\ \operatorname{fr}(c)&=\operatorname{supp}(\partial c)\\ \operatorname{int}(c)&=\operatorname{supp}(c)\setminus\operatorname{supp}(\partial c)\end{aligned}$$

For each chain its support, frontier and interior are finite sets of vertices.

For any p-chain $c\in C_p(X,\mathbb{Z})$ and Lagrangian density $L\colon Y_p\to\mathbb{R}$ it is clear that $\langle c,y_p^*L\rangle=\sum_\beta c(\beta)\cdot(y_p^*L)(\beta)$ only depends on y through the values of y_p^*L on p-cells β where $c(\beta)\neq 0$, and this only depends on the values of y at vertices $v\in\operatorname{supp}(c)$.

For a choice of $y\in\Gamma(X_0,Y_0)$ and a choice of $c\in C_p(X,\mathbb{Z})$ we may study the behavior of $\langle c,\bar y_p^*L\rangle$ for arbitrary configurations $\bar y\in\Gamma(X_0,Y_0)$ that differ from y only at vertices $v\in\operatorname{int}(c)$. That is, we may consider the space:

$$\Gamma_{y,c}(X_0,Y_0)=\{\bar y\colon X_0\to Y_0:\ \bar y(v)=y(v),\quad \forall v\notin\operatorname{int}(c)\}$$

where the domain can be identified with $\Gamma(\operatorname{int}(c),Y_0)=\prod_{v\in\operatorname{int}(c)}(Y_0)_v$, a finite-dimensional manifold, whose tangential space at the point $\bar y$ is

$$T_{\bar y}(\Gamma_{y,c}(X_0,Y_0))=\{\delta y\in\Gamma(X_0,\bar y^*VY_0):\ \delta y(v)=0\,\forall v\notin\operatorname{int}(c)\}=\Gamma(\operatorname{int}(c),\bar y^*VY_0)\subseteq\Gamma(X_0,\bar y^*VY_0)$$

Definition 3.8. We say a configuration of vertices $y\in\Gamma(X_0,Y_0)$ is critical for the discrete variational problem defined by the Lagrangian density $L\colon Y_p\to\mathbb{R}$ and domain $c\in C_p(X,\mathbb{Z})$ if it is a critical point for the function:

$$\begin{array}{rccc}\mathbb{L}_{y,c}\colon & \Gamma_{y,c}(X_0,Y_0) & \to & \mathbb{R}\\ & \bar y & \mapsto & \langle c,\bar y_p^*L\rangle\end{array}$$

that is, if $\mathrm{d}_y\mathbb{L}_{y,c} = 0$.

We say that y is critical for the variational problem defined by $L\colon Y_p \to \mathbb{R}$ if it is critical for any domain $c \in C_p(X,\mathbb{Z})$.

Critical configurations $y \in \Gamma(X_0, Y_0)$ for a given domain $c \in C_p(X,\mathbb{Z})$ are then characterized by the vanishing of $\mathrm{d}_y\mathbb{L}_{y,c}$ on the finite dimensional space $T_y\left(\Gamma_{y,c}(X_0,Y_0)\right) = \Gamma(\mathrm{int}(c), y^*VY_0)$.

This justifies the introduction of the linear operator:

$$\begin{array}{rccc} \mathrm{d}_y\mathbb{L}\colon & \Gamma(X_0, y^*VY_0) & \to & \Omega^p(X) \\ & \delta y & \mapsto & \langle \mathrm{d}L \circ y_p, (\delta y)_p \rangle \end{array} \tag{3.3}$$

where $\langle \mathrm{d}L \circ y_p, (\delta y)_p \rangle$ represents the cochain that takes at $\beta \in X_p$ the value $\langle \mathrm{d}_{y_p(\beta)}L, (\delta y)_p(\beta)\rangle$, given by duality product of $\mathrm{d}_{y_p(\beta)}L \in T^*_{y_p(\beta)}(Y_p)$ and $(\delta y)_p(\beta) \in T_{y_p(\beta)}(Y_p)$

Lemma 3.9. *A configuration $y \in \Gamma(X_0, Y_0)$ is critical for the variational problem given by the Lagrangian L and domain $c \in C_p(X,\mathbb{Z})$ if and only if:*

$$\langle c, (\mathrm{d}_y\mathbb{L})(\delta y)\rangle = 0, \quad \forall \delta y \in \Gamma(\mathrm{int}(c), y^*VY_0)$$

A configuration $y \in \Gamma(X_0, Y_0)$ is critical for the variational problem defined by L if and only if:

$$\langle c, (\mathrm{d}_y\mathbb{L})(\delta y)\rangle = 0, \quad \forall c \in C_p(X,\mathbb{Z}),\ \forall \delta y \in \Gamma(\mathrm{int}(c), y^*VY_0)$$

Proof . If one considers an infinitesimal variation $\delta y \in \Gamma(\mathrm{int}(c), y^*VY_0)$ whose value at some fixed vertex $v \in \mathrm{int}(c)$ is chosen arbitrarily $\delta_v y \in (y^*VY_0)_v = T_{y(v)}(Y_0)_v$, and which vanishes at any other vertex, it is straightforward from the definition of $\mathbb{L}_{y,c}$ and the chain rule for the differential of $L \circ \bar{y}_p$ that:

$$\left(\mathrm{d}_y\mathbb{L}_{y,c}\right)(\delta y) = \sum_{v \prec \beta} c(\beta)\cdot \langle (\mathrm{d}_{y_p(\beta)}L)_v, \delta_v y\rangle = \sum_{v \prec \beta} c(\beta)\cdot\langle \mathrm{d}_{y_p(\beta)}L, (\delta y)_p(\beta)\rangle$$

therefore $(\mathrm{d}_y\mathbb{L}_{y,c})(\delta y) = \langle c, (\mathrm{d}_y\mathbb{L})(\delta y)\rangle$. As the chosen infinitesimal variations δy generate the finite-dimensional space $\Gamma(\mathrm{int}(c), y^*VY_0)$ we conclude that this formula holds for any chain $c \in C_p(X,\mathbb{Z})$ and any infinitesimal variation $\delta y \in \Gamma(\mathrm{int}(c), y^*VY_0)$, so the result follows from the definition of criticality. ☐

Definition 3.10. We call bundle of admissible infinitesimal variations $\mathrm{Var} \to Y_0$ on the bundle of configurations $\pi\colon X_0 \to Y_0$ any vector sub-bundle of VY_0. We say a configuration y_0 is critical for the Lagrangian $L\colon Y_0 \to \mathbb{R}$ and admissible variations Var if $\langle c, (\mathrm{d}_y\mathbb{L})(\delta y)\rangle$ vanishes for any chain c and any admissible infinitesimal variation $\delta y \in \Gamma(\mathrm{int}(c), y^*\mathrm{Var})$.

Typical examples of bundles of admissible infinitesimal variations Var shall be the whole VY_0, or the bundle with fibre VY_0 on certain vertices and 0 on other vertices (this “fixed boundary” vertices are not allowed to vary), or a bundle that coincides with VY_0 on certain vertices and some sub-space of $V_{y(v)}Y_0$ on other vertices (this “restricted boundary” vertices are allowed to vary along some sub-manifold $S_v \subset (Y_0)_v$ of the fiber): this situation corresponds to variational problems with holonomic constraints.

To characterize critical configurations by means of difference equations we need a deeper comprehension of mappings like $\mathrm{d}_y\mathbb{L}$, that take sections of a vector bundle y^*VY to p-forms. This is the objective of next section.

4. Vector bundles and difference operators on the discrete Euclidean space

We recall from the differentiable theory that in the presence of a connection on a differentiable vector bundle $E \to X$ over an oriented p-dimensional manifold X, any first order differential operator $F\colon \Gamma(X,E) \mapsto \Omega^p(X)$ taking vector fields $D \in \Gamma(X,E)$ to densities $F(D) \in \Omega^p(X)$ can be decomposed so that:

$$F(D) = \mathrm{d}(\omega_F(D)) + L_F(D)\cdot \mathrm{vol}_X$$

where $\omega_F\colon \Gamma(X,E) \to \Omega^{p-1}(X)$ is a linear operator taking vector fields to $(p-1)$-forms, $L_F\colon \Gamma(X,E) \to \mathcal{C}^\infty(X)$ is a bundle morphism (the linear component of F) and vol_X is a fixed volume element on X.

In this section we shall consider any vector bundle $E_0 \to X_0$ on the set X_0 of vertices of the p-dimensional discrete Euclidean space, whose fibers will be vector spaces E_v. The extension of E_0 to the space of k-cells defines new vector bundles $E_k \to X_k$. Any element $e_\alpha \in E_k$ is given by a k-cell $\alpha \in X_k$ and an element $e_{\alpha,v} \in E_v$ for each $v \prec \alpha$. When we consider its dual bundle E_k^*, taking into account that a k-cell only has a finite number of adherent vertices, $(\prod_{v\prec\alpha} E_v)^* = \prod_{v\prec\alpha} E_v^*$, and elements $f_\alpha \in E_k^*$ are determined by $\alpha \in X_k$ and $f_{\alpha,v} \in E_v^*$ for each $v \prec \alpha$.

A section $D \in \Gamma(X_k, E_k)$ is given when we choose $e_\alpha \in (E_k)_\alpha$ for each $\alpha \in X_k$, and this amounts to giving $(e_{\alpha,v})_{(v\in X_0)\prec(\alpha\in X_k)}$ with $e_{\alpha,v} \in E_v$. In a similar way a section $F \in \Gamma(X_k, E_k^*)$ is given when we choose $(f_{\alpha,v})_{(v\in X_0)\prec(\alpha\in X_k)}$ with $f_{\alpha,v} \in E_v^*$.

Definition 4.1. We call first order difference operator from E_0 to the space $\Omega^k(X)$ any section $F \in \Gamma(X_k, E_k^*)$. A first order difference operator allows to define a mapping taking any vector field $D \in \Gamma(X_0, E_0)$ to the k-form $F(D) \in \Omega^k(X)$ whose value at the k-cell $\alpha \in X_k$ is:

$$(F(D))(\alpha) = \langle F(\alpha), D_k(\alpha)\rangle = \sum_{v\prec\alpha} f_{\alpha,v}(e_v), \quad F = (f_{\alpha,v})_{v\prec\alpha},\, D = (e_v)_{v\in X_0}$$

where $D_k \in \Gamma(X_k, E_k)$ stands for the natural extension of $D \in \Gamma(X_0, E_0)$ and $\langle\,,\,\rangle$ is the duality product between E_k^* and E_k.

Consider the group $\mathrm{Aut}(E_0)$ of automorphisms of E_0. By this we mean any vector bundle automorphism $\varphi\colon E_0 \to E_0$ fibred over $\varphi_X \in \mathrm{Eucl}(p,\mathbb{Z})$. Giving $\varphi \in \mathrm{Aut}(E_0)$ is equivalent to giving $\varphi_X \in \mathrm{Eucl}(p,\mathbb{Z})$ and vector space isomorphisms $\varphi_v\colon E_v \to E_{\varphi_X\cdot v}$ for each $v \in X_0$. Elements $\varphi = (\varphi_X, (\varphi_v)_{v\in X_0}) \in \mathrm{Aut}(E_0)$ act on X_k, on $\mathrm{C}_k(X,\mathbb{Z})$, on $\Omega^k(X)$, on E_k and on E_k^* as follows:

$$\begin{aligned}
\varphi\cdot\alpha &= \varphi_X(\alpha), \quad \forall\alpha \in X_k\\
\varphi\cdot c_\alpha &= \mathrm{sgn}(\varphi_X,\alpha)\cdot c_{\varphi\cdot\alpha}, \quad \forall\alpha\in X_k, \quad c_\alpha \in \mathrm{C}_k(X,\mathbb{Z})\\
(\varphi\cdot\omega_k)(c) &= \omega_k(\varphi^{-1}\cdot c), \quad \forall\omega_k \in \Omega^k(X), \forall c \in C_k(X,\mathbb{Z})\\
\varphi\cdot e_v &= \varphi_v(e_v) \in E_{\varphi\cdot v}, \quad \forall e_v \in E_v \subset E_0\\
\varphi\cdot e_\alpha &= (\varphi\cdot e_{\alpha,v})_{v\prec\alpha} \in (E_k)_{\varphi\cdot\alpha}, \quad \forall e_\alpha = (e_{\alpha,v})_{v\prec\alpha} \in (E_k)_\alpha, \quad e_{\alpha,v}\in E_v\\
\varphi\cdot f_\alpha &= \mathrm{sgn}(\varphi_X,\alpha)\cdot(f_{\alpha,v}\circ\varphi_v^{-1})_{v\prec\alpha} \in (E_k^*)_{\varphi\cdot\alpha}, \quad \forall f_\alpha = (f_{\alpha,v})_{v\prec\alpha} \in (E_k^*)_\alpha, \quad f_{\alpha,v}\in E_v^*
\end{aligned}$$

Recall the definition of the operator $D \in \Gamma(X_0,E_0) \mapsto F(D) \in \Omega^k(X)$ and how φ_X acts on chains and cochains (in particular on $F(D)$). For the action on E_k^* the inclusion of $\mathrm{sgn}(\varphi_X,\alpha)$ might seem unnecessary, but it is natural to do so if one wants to define actions on $\Gamma(X_k,E_k)$ and on $\Gamma(X_k,E_k^*)$:

$$\begin{aligned}
\varphi\cdot F &= (\varphi\cdot f_{\varphi^{-1}\cdot\alpha})_{\alpha\in X_k}, \quad \forall F = (f_\alpha)_{\alpha\in X_k} \in \Gamma(X_k,E_k^*)\\
\varphi\cdot D &= (\varphi\cdot e_{\varphi^{-1}\cdot v})_{v\in X_0}, \quad \forall D = (e_v)_{v\in X_0} \in \Gamma(X_0,E_0)
\end{aligned}$$

in such a way that:

$$\varphi\cdot(F(D)) = (\varphi\cdot F)(\varphi\cdot D), \quad \forall F\in\Gamma(X_k,E_k^*),\, D\in\Gamma(X_0,E_0),\, F(D)\in\Omega^p(X)$$

In a more explicit way:

$$F = (f_{\alpha,v})_{(v\in X_0)\prec(\alpha\in X_k)} \Rightarrow \varphi\cdot F = \left(\mathrm{sgn}(\varphi_X^{-1},\alpha)\cdot f_{\varphi^{-1}\cdot\alpha,\varphi^{-1}\cdot v}\circ\varphi^{-1}_{\varphi^{-1}\cdot v}\right)_{(v\in X_0)\prec(\alpha\in X_k)}, \quad \forall F\in\Gamma(X_k,E_k^*) \tag{4.1}$$

Definition 4.2. For any 0-cochain $g \in \Omega^0(X) = \mathrm{Map}(X_0,\mathbb{R})$, we call $g\mathrm{vol}_X \in \Omega^p(X)$ the p-form defined by:

$$(g\mathrm{vol}_X)(\alpha) = \sum_{v\prec\alpha} g(v), \qquad \alpha \in X_p$$

As $\mathrm{sgn}(\varphi, v) = 1$ for any vertex and any $\varphi \in \mathrm{Eucl}(p, \mathbb{Z})$, it is clear that $(\varphi \cdot g)(v) = g(\varphi^{-1} \cdot v)$ and that $(\varphi \cdot (g\mathrm{vol}_X))(\alpha) = \mathrm{sgn}(\varphi, \alpha)((\varphi \cdot g)\mathrm{vol}_X)(\alpha)$. Covariance of $g \in \Omega^0(X)$ and covariance of $g\mathrm{vol}_X \in \Omega^p(X)$ for some $\varphi \in \mathrm{Eucl}(p, \mathbb{Z})$ are different concepts.

Proposition 4.3. *Let X be the p-dimensional discrete Euclidean space, $E_0 \to X_0$ a vector bundle and $x_1, \dots, x_p$ real numbers. The mapping $\Omega\colon \Gamma(X_p, E_p^*) \to \Gamma(X_{p-1}, E_{p-1}^*)$ that takes $F = (f_{\beta,v})_{(v \in X_0) \prec (\beta \in X_p)} \in \Gamma(X_p, E_p^*)$ to $\omega_F \in \Gamma(X_{p-1}, E_{p-1}^*)$ given by:*

$$(\omega_F)_{\alpha,v} = \sum_{v \prec \beta \in X_p} x_{\alpha,\beta} \cdot f_{\beta,v}, \quad \forall (v \in X_0) \prec (\alpha \in X_{p-1})$$
$$\text{where } x_{\alpha,\beta} = (-1)^{i-1} \cdot s_i \cdot x_k \quad \text{for } i = \alpha^{\mathrm{even}}, \quad s_i = \beta_i - \alpha_i \in \{\pm 1\}, \quad k = \sharp\{j \colon \alpha_j \neq \beta_j\} \tag{4.2}$$

is covariant for any $\varphi \in \mathrm{Aut}(E_0)$, that is: $\varphi \cdot \omega_F = \omega_{\varphi \cdot F}$.

Moreover, if the projection $\varphi \in \mathrm{Aut}(E_0) \to \varphi_X \in \mathrm{Eucl}(p, \mathbb{Z})$ is surjective (which holds in the case that all fibers are isomorph) then any linear covariant mapping $\Gamma(X_p, E_p^*) \to \Gamma(X_{p-1}, E_{p-1}^*)$ is given by (4.2) for some choice of $x_1, \dots, x_p \in \mathbb{R}$.

Proof . Let us prove first the covariance of expression (4.2). Consider any $F = (f_{\beta,v})_{(v \in X_0) \prec (\beta \in X_p)}$ and any $\varphi = (\varphi_X, (\varphi_v)_{v \in X_0}) \in \mathrm{Aut}(E_0)$. We know by (4.1):

$$\varphi \cdot F = \left(\mathrm{sgn}(\varphi_X^{-1}, \beta) \cdot f_{\varphi^{-1} \cdot \beta, \varphi^{-1} \cdot v} \circ \varphi^{-1}_{\varphi^{-1} \cdot v} \right)_{(v \in X_0) \prec (\beta \in X_p)}$$

and applying Ω given by (4.2):

$$(\omega_{\varphi \cdot F})_{\alpha,v} = \sum_{v \prec \beta \in X_p} x_{\alpha,\beta} \cdot \mathrm{sgn}(\varphi_X^{-1}, \beta) \cdot f_{\varphi^{-1} \cdot \beta, \varphi^{-1} \cdot v} \circ \varphi^{-1}_{\varphi^{-1} \cdot v}$$

On the other hand ω_F is a $(p-1)$-form whose components are those in (4.2) and when transformed by φ, following (4.1) and calling $\beta = \varphi^{-1} \cdot \bar{\beta}$ one gets:

$$(\varphi \cdot (\omega_F))_{\alpha,v} = \mathrm{sgn}(\varphi_X^{-1}, \alpha) \cdot \sum_{\varphi^{-1} \cdot v \prec \varphi^{-1} \cdot \bar{\beta} \in X_p} x_{\varphi^{-1} \cdot \alpha, \varphi^{-1} \cdot \bar{\beta}} \cdot f_{\varphi^{-1} \cdot \bar{\beta}, \varphi^{-1} \cdot v} \circ \varphi^{-1}_{\varphi^{-1} \cdot v}$$

Therefore to prove $\varphi \cdot (\omega_F) = \omega_{\varphi \cdot F}$ for any $F \in \Gamma(X_p, E_p^*)$ and any $\varphi \in \mathrm{Aut}(E_0)$ is equivalent to prove that $x_{\alpha,\beta} = \mathrm{sgn}(\varphi_X^{-1}, \alpha) \cdot \mathrm{sgn}(\varphi_X^{-1}, \beta) \cdot x_{\varphi^{-1} \cdot \alpha, \varphi^{-1} \cdot \beta}$ for any $\varphi^{-1} \in \mathrm{Aut}(E_0)$ and any $\alpha \in X_{p-1}$ and $\beta \in X_p$ with a common vertex $v \prec \alpha$, $v \prec \beta$.

Now Lemma 2.3 shows that the definition of $x_{\alpha,\beta}$ as given in (4.2) can be written as:

$$x_{\alpha^k, \beta^0} = x_k, \quad x_{\alpha,\beta} = \mathrm{sgn}(\varphi_X, \alpha) \cdot \mathrm{sgn}(\varphi_X, \beta) \cdot x_{\alpha^k, \beta^0}, \qquad \forall \varphi_X \in \mathrm{Eucl}(p, \mathbb{Z}) \text{ s.t. } \varphi_X \cdot \alpha = \alpha^k,\ \varphi_X \cdot \beta = \beta^0$$

Moreover, no pair $\alpha^k \in X_{p-1}$, $\beta^0 \in X_p$ with a common vertex can be transformed into a different pair α^j, β^0 $(j \neq k)$. This alternative description of $x_{\alpha,\beta}$ together with the properties (2.9) of $\mathrm{sgn}(\varphi, \cdot)$ shows that $x_{\alpha,\beta} = \mathrm{sgn}(\psi, \alpha) \cdot \mathrm{sgn}(\psi, \beta) x_{\psi \cdot \alpha, \psi \cdot \beta}$ for any $\alpha \in X_{p-1}$, $\beta \in X_p$ and $\psi \in \mathrm{Eucl}(p, \mathbb{Z})$, finishing then our proof of the covariance of formula (4.2).

Now we want to prove that any covariant Ω has necessarily the form given by (4.2). First we want to prove that with our covariance assumption the component $\omega_{\alpha,v}$ of $\omega_F \in \Gamma(X_{p-1}, E_{p-1}^*)$ does not depend on the components $f_{\beta,\bar{v}}$ of $F \in \Gamma(X_p, E_p^*)$ for $\bar{v} \neq v$. Let us fix $v \in X_0$. For any choice of F whose (β, v)-components vanish, when we consider $\varphi \in \mathrm{Aut}(E_0)$ over $\varphi_X = \mathrm{Id} \in \mathrm{Eucl}(p, \mathbb{Z})$ and such that $\varphi_v = -\mathrm{Id}$, $\varphi_{\bar{v}} = \mathrm{Id}$ (for any $\bar{v} \neq v$), we have by formula (4.1) applied to $F \in \Gamma(X_p, E_p^*)$ that $\varphi \cdot F = F$, because $f_{\beta,\bar{v}} \circ \mathrm{Id} = f_{\beta,\bar{v}} \in (E_0)^*_{\bar{v}}$ and $0_{\beta,v} \circ (-\mathrm{Id}) = 0_{\beta,v} \in (E_0)^*_v$. Hence $\omega_{\varphi \cdot F} = \omega_F$. On the other hand following (4.1) applied to $\omega_F \in \Gamma(X_{p-1}, E_{p-1}^*)$, and that automorphism $\varphi \in \mathrm{Aut}(E_0^*)$ one has:

$$(\varphi \cdot \omega_F)_{\alpha,\bar{v}} = \begin{cases} (\omega_F)_{\alpha,\bar{v}}, & \forall \bar{v} \neq v \\ -(\omega_F)_{\alpha,v}, & \bar{v} = v \end{cases}$$

As $\omega_F = \omega_{\varphi\cdot F}$ must coincide with $\varphi\cdot\omega_F$, we conclude that $(\omega_F)_{\alpha,v}$ vanishes when all the (β,v)-components of F vanish.

Therefore the (α,v)-component of ω_F depends linearly on the $f_{\beta,\bar v}$ components of F, and only on those with $\bar v = v$. There exist then linear morphisms $\bar\Omega^{v\prec\beta}_{v\prec\alpha}\colon (E_0)^*_v \to (E_0)^*_v$ that allow to write the (α,v)-component of ω_F as:

$$(\omega_F)_{\alpha,v} = \sum_{v\prec\beta} \bar\Omega^{v\prec\beta}_{v\prec\alpha}(f_{\beta,v})$$

Taking the transpose morphisms, we conclude that there are morphisms $\Omega^{v\prec\beta}_{v\prec\alpha}\colon (E_0)_v \to (E_0)_v$ such that

$$(\omega_F)_{\alpha,v} = \sum_{v\prec\beta} f_{\beta,v}\circ\Omega^{v\prec\beta}_{v\prec\alpha}$$

Moreover, when we consider $F = (f_{\beta,v}) \in \Gamma(X_p, E_p^*)$ with only one non-vanishing component $f_{\beta,v}$ and any automorphism over $\mathrm{Id} \in \mathrm{Eucl}(p,\mathbb{Z})$ whose component at $(E_0)_v$ is φ_v, we may compute the (α,v)-component of $\omega_{\varphi\cdot F} = \varphi\cdot\omega_F$ and deduce:

$$f_{\beta,v}\circ\varphi_v^{-1}\circ\Omega^{v\prec\beta}_{v\prec\alpha} = f_{\beta,v}\circ\Omega^{v\prec\beta}_{v\prec\alpha}\circ\varphi_v^{-1}$$

which implies that $\Omega^{v\prec\beta}_{v\prec\alpha}$ commutes with any automorphism $\varphi_v\colon (E_0)_v \to (E_0)_v$ therefore $\Omega^{v\prec\beta}_{v\prec\alpha} = x_{\alpha,v,\beta}\cdot\mathrm{Id}_v$ for some real numbers $x_{\alpha,v,\beta}$. The morphism $\Omega\colon F\in\Gamma(X_p,E_p^*) \to \omega_F \in \Gamma(X_{p-1},E^*_{p-1})$ has the form:

$$\Omega\colon F = (f_{\beta,v})_{v\prec\beta} \mapsto \omega_F = \left(\sum_{v\prec\beta} x_{\alpha,v,\beta}\cdot f_{\beta,v}\right)_{v\prec\alpha}$$

In the same manner as before, if Ω has this form and is covariant then:

$$x_{\alpha,v,\beta} = \mathrm{sgn}(\varphi_X,\alpha)\cdot\mathrm{sgn}(\varphi_X,\beta)\cdot x_{\varphi\cdot\alpha,\varphi\cdot v,\varphi\cdot\beta} \qquad \forall\varphi\in\mathrm{Aut}(E_0)$$

As $\varphi_X \in \mathrm{Eucl}(p,\mathbb{Z})$ is arbitrary when we consider all $\varphi\in\mathrm{Aut}(E_0)$, using Lemma 2.3 again we may conclude that $x_{\alpha,v,\beta}$ has the form given by (4.2) with $x_k = x_{\alpha^k,v^0,\beta^0}$ which completes our proof. □

Proposition 4.4. *Let X be the p-dimensional discrete Euclidean space, $E_0 \to X_0$ a vector bundle and z a real number. The mapping $L\colon \Gamma(X_p,E_p^*) \to \Gamma(X_0,E_0^*)$ that takes $F = (f_{\beta,v})_{(v\in X_0)\prec(\beta\in X_p)} \in \Gamma(X_p,E_p^*)$ to $L_F\in\Gamma(X_0,E_0^*)$ given by:*

$$(L_F)_v = \sum_{v\prec\beta\in X_p} z\cdot f_{\beta,v}, \quad \forall(v\in X_0) \tag{4.3}$$

is covariant for any $\varphi\in\mathrm{Aut}(E_0)$, that is: $\varphi\cdot(L_F\mathrm{vol}_X) = L_{\varphi\cdot F}\mathrm{vol}_X$.

Moreover, if the projection morphism $\varphi\in\mathrm{Aut}(E_0)\mapsto\varphi_X\in\mathrm{Eucl}(p,\mathbb{Z})$ is surjective, then any linear mapping $\Gamma(X_p,E_p^*)\to\Gamma(X_0,E_0^*)$ covariant in this sense, is given by (4.3) for some choice of $z\in\mathbb{R}$.

Proof . This is a simpler version of the previous proposition, and the proof is a straightforward adaptation of the previous one for this case, where now for any pair $(v\in X_0)\prec(\beta\in X_p)$ there is a movement that takes v to v^0 and β to β^0. The covariance condition is not meant for L_F, but rather for F and $L_F\mathrm{vol}_X$, so it can be written in terms of p-forms. Therefore the appearance of $\mathrm{sgn}(\varphi_X,\beta)\cdot\mathrm{sgn}(\varphi_X,\tilde\beta)$ (which is always 1) for p-cells $\beta,\tilde\beta$ sharing a vertex means no problem when handling with the signs. □

Theorem 4.5 (Adjointness formula). *There exists a linear morphism*

$$\begin{array}{ccc} \Gamma(X_p,E_p^*) & \to & \Gamma(X_{p-1},E^*_{p-1})\oplus\Gamma(X_0,E_0^*) \\ F & \mapsto & (\omega_F, L_F) \end{array}$$

that is covariant for any bundle automorphism $\varphi\colon E_0\to E_0$, in the following sense:

$$\omega_{\varphi\cdot F} = \varphi\cdot\omega_F, \qquad L_{\varphi\cdot F}\mathrm{vol}_X = \varphi\cdot(L_F\mathrm{vol}_X)$$

and such that for any $D \in \Gamma(X_0, E_0)$ holds the adjointness formula:

$$F(D) = \mathrm{d}(\omega_F(D)) + L_F(D)\mathrm{vol}_X \tag{4.4}$$

This decomposition is given by formulas (4.2), (4.3) with

$$z = \frac{1}{2^p}, \quad x_k = \frac{-1}{2^p \cdot k}\left(\frac{\binom{p}{0} + \ldots + \binom{p}{p-k}}{\binom{p}{p-k}}\right), \quad k = 1, \ldots, p \tag{4.5}$$

and is unique if all fibers of E_0 are isomorph.

Proof . Following the previous results and the covariance, formulae (4.2) and (4.3) for some $z, x_1, \ldots, x_p \in \mathbb{R}$ are possible choices to define the morphism $F \mapsto (\omega_F, L_F)$ (and are the only choices if all fibers of E_0 are isomorph). Given $F = (f_{\beta,v})_{(v\in X_0)\prec(\beta\in X_p)}$, $D = (e_v)_{v\in X_0}$, if we compute the values at some p-cell $\tilde\beta \in X_p$, we have:

$$\begin{aligned}
(L_F(D)\mathrm{vol}_X)(\tilde\beta) &= \sum_{v\prec\tilde\beta} L_F(D)(v) = \sum_{v\prec\tilde\beta}\sum_{v\prec\beta} z\cdot f_{\beta,v}(e_v)\\
(\omega_F(D))(\alpha) &= \sum_{v\prec\alpha}\sum_{v\prec\beta} x_{\alpha,\beta}\cdot f_{\beta,v}(e_v), \qquad \alpha \in X_{p-1}\\
(\mathrm{d}(\omega_F(D)))(\tilde\beta) &= \sum_{\alpha\prec\tilde\beta}[\tilde\beta\colon \alpha]\,(\omega_F(D))(\alpha) = \sum_{\alpha\prec\tilde\beta}\sum_{v\prec\alpha}\sum_{v\prec\beta}[\tilde\beta\colon\alpha]x_{\alpha,\beta}\cdot f_{\beta,v}(e_v)\\
(F(D))(\tilde\beta) &= \sum_{v\prec\tilde\beta} f_{\tilde\beta,v}(e_v)
\end{aligned}$$

where $x_{\alpha,\beta} = (-1)^{i-1}\cdot s_i\cdot x_k$ only depends on $i = \alpha^{\mathrm{even}}$, $s_i = \beta_i - \alpha_i$; $k = \sharp\{j\colon \alpha_j \neq \beta_j\}$.

Considering adjointness formula (4.4) at any $\tilde\beta$, for any choice of F and any choice of D, for the adjointness formula to hold there should exist a choice of $z, x_1, \ldots, x_p \in \mathbb{R}$ such that, for any fixed $\tilde\beta \in X_p$, any $\beta \in X_p$ and any $v \prec \beta$, $v \prec \tilde\beta$ holds:

$$\begin{aligned}
z + \sum_{v\prec\alpha\prec\tilde\beta}[\tilde\beta\colon\alpha]x_{\alpha,\beta} &= 1, \quad \text{if } \tilde\beta = \beta\\
z + \sum_{v\prec\alpha\prec\tilde\beta}[\tilde\beta\colon\alpha]x_{\alpha,\beta} &= 0, \quad \text{if } \tilde\beta \neq \beta
\end{aligned} \tag{4.6}$$

We want to prove that this system of linear equations (one equation for each $v \in X_0$ and each $\beta, \tilde\beta$ with $v \prec \beta$, $v \prec \tilde\beta$) has only one solution $(z, x_1, \ldots, x_p)$.

The sign of $\varphi \in \mathrm{Eucl}(p, \mathbb{Z})$ at different p-cells does not depend on the cell: $\mathrm{sgn}(\varphi, \tilde\beta) = \det\vec\varphi = \mathrm{sgn}(\varphi, \beta)$. As we know:

$$[\beta\colon\alpha] = \mathrm{sgn}(\varphi,\beta)\cdot\mathrm{sgn}(\varphi,\alpha)\cdot[\varphi\cdot\beta\colon\varphi\cdot\alpha], \qquad x_{\alpha,\beta} = \mathrm{sgn}(\varphi,\beta)\cdot\mathrm{sgn}(\varphi,\alpha)\cdot x_{\varphi\cdot\beta,\varphi\cdot\alpha}$$

therefore:

$$[\tilde\beta\colon\alpha]x_{\alpha,\beta} = [\varphi\cdot\tilde\beta\colon\varphi\cdot\alpha]x_{\varphi\cdot\alpha,\varphi\cdot\beta}, \quad \forall\varphi\in\mathrm{Eucl}(p,\mathbb{Z})$$

Hence we do not need to verify our equations for every $\beta, \tilde\beta$ with a common vertex. For the pair of cells $\varphi\cdot\beta, \varphi\cdot\tilde\beta$ the equations are the same. It suffices to prove the existence and uniqueness for a solution of the equations (4.6) appearing in the case when $\tilde\beta = \beta^0 = (1, \ldots, 1)$, $v = v^0 = (0, \ldots, 0)$, $\beta = \beta^k = (-1, \ldots, -1, 1, \ldots, 1) = \beta^0 - 2e_1 - 2e_2 - \ldots - 2e_k$ (Where $k = 0, \ldots, p$).

In this case, $(p-1)$-cells α with $v^0 \prec \alpha \prec \tilde\beta$ are $\alpha^{0,j} = (1, \ldots, 1, 0, 1, \ldots, 1) = \beta^0 - e_j$ (where $j = 1, \ldots, p$). Simple computations give then:

$$[\tilde\beta\colon\alpha^{0,j}] = (-1)^j,$$

$$x_{\alpha^{0,j},\beta^k} = \begin{cases} (-1)\cdot(-1)^{j-1}\cdot x_k \text{ if } j \le k\,, \\ 1\cdot(-1)^{j-1}\cdot x_{k+1} \text{ if } j > k \end{cases} \qquad [\tilde\beta\colon\alpha^{0,j}]\cdot x_{\alpha^{0,j},\beta^k} = \begin{cases} x_k \text{ if } j \le k \\ -x_{k+1} \text{ if } j > k \end{cases}$$

Therefore we may reduce our system of equations (4.6) to:

$$\begin{cases} z - px_1 = 1, \\ z + k \cdot x_k - (p-k) \cdot x_{k+1} = 0, \qquad k = 1, \dots, p-1 \\ z + px_p = 0 \end{cases}$$

We try now to determine if this system has unique solution $z, x_1, \dots, x_p$. Indeed the system of equations is:

$$\begin{bmatrix} 1 & -p & 0 & 0 & \dots & 0 \\ 1 & 1 & 1-p & \ddots & & \vdots \\ 1 & 0 & 2 & 2-p & \ddots & \vdots \\ \vdots & \vdots & \ddots & 3 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & -1 \\ 1 & 0 & \dots & \dots & 0 & p \end{bmatrix} \cdot \begin{bmatrix} z \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Consider the matrix associated to this system. If we substitute the first line L_1 of this matrix by $L_1 + pL_2$, we get rid of $-p$ on the second position but obtain $p \cdot (1-p)$ on the third position. If we add again $(p(p-1)/2) \cdot L_3$ we get rid of this entry but obtain $p \cdot (p-1) \cdot (2-p)/2$ on the fourth position. We may continue adding again $(p(p-1)(p-2)/(2 \cdot 3)) \cdot L_4$ on the first line, and so on until we obtain:

$$\begin{bmatrix} \binom{p}{0} & \binom{p}{1} & \binom{p}{2} & \dots & \binom{p}{p} \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -p & 0 & \dots & 0 & 1 \\ 1 & 1 & 1-p & \ddots & \vdots & 0 \\ 1 & 0 & 2 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & \vdots \\ 1 & 0 & \dots & 0 & p & 0 \end{bmatrix} = \begin{bmatrix} 2^p & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1-p & \ddots & \vdots & 0 \\ 1 & 0 & 2 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & \vdots \\ 1 & 0 & \dots & 0 & p & 0 \end{bmatrix}$$

because $\sum_{k=0}^{p} \binom{p}{k} = 2^p$ and $\binom{p}{k} \cdot (k-p) + \binom{p}{k+1} \cdot (k+1) = 0$ for any $k = 0, \dots, p-1$. The system to be solved is then:

$$\begin{bmatrix} 2^p & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1-p & \ddots & & \vdots \\ 1 & 0 & 2 & 2-p & \ddots & \vdots \\ \vdots & \vdots & \ddots & 3 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & -1 \\ 1 & 0 & \dots & \dots & 0 & p \end{bmatrix} \cdot \begin{bmatrix} z \\ x_1 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

that can be solved first for $z = 1/2^p$ and then by a backwards substitution procedure beginning with the determination of x_p, and substituting backwards until determining x_1. In fact, simple computations show that (4.5) is the (unique) solution. □

5. First variation formula. Noether Theorem

In the previous sections we introduced the discrete version of the usual analytical machinery of the variational theory. Recall that, given a configuration bundle $\pi \colon X_0 \to Y_0$, a discrete Lagrangian density is any function $L \colon Y_p \to \mathbb{R}$ on the induced bundle of configurations of p-cells (definition 3.4). This leads to a variational problem, consisting on the determination of critical configurations in the sense of definition 3.8. We want to study the mapping:

$$\begin{array}{cccc} \mathbb{L}\colon & \Gamma(X_0, Y_0) & \to & \Omega^p(X) \\ & y & \mapsto & L \circ y_p \end{array}$$

which induces the linear operator $\mathrm{d}_y\mathbb{L}$ given in (3.3). As we already indicated, it is the vanishing of $\langle c, \mathrm{d}_y\mathbb{L}(\delta y)\rangle$ for any chain $c \in C_p(X,\mathbb{Z})$ and any admissible variation $\delta y \in \Gamma(\mathrm{int}(c), y^*VY_0)$ that determines if $y \in \Gamma(X_0, Y_0)$ is critical for the discrete Lagrangian L.

The mapping $\mathrm{d}_y\mathbb{L}$ has the form $D \mapsto F(D)$ indicated in definition 4.1, if we consider the first order difference operator F from $E_0 = y^*VY$ to $\Omega^p(X)$, given by $F = \mathrm{d}L \circ y_p \in \Gamma(X_p, E_p^*)$.

Definition 5.1. Given a bundle of configurations $\pi\colon X_0 \to Y_0$ and a Lagrangian density $L\colon Y_p \to \mathbb{R}$ we call discrete Euler form and discrete momentum form associated to L the mappings:

$$\begin{array}{rccc} \mathcal{E}_L\colon & \Gamma(X_0,Y_0) & \to & \Gamma(X_0, V^*Y_0) \\ & y & \mapsto & L_{(\mathrm{d}L\circ y_p)} \end{array}, \qquad \begin{array}{rccc} \omega_L\colon & \Gamma(X_0,Y_0) & \to & \Gamma(X_{p-1}, (V^*Y_0)_{p-1}) \\ & y & \mapsto & \omega_{(\mathrm{d}L\circ y_p)} \end{array}$$

respectively, where $L_{(\mathrm{d}L\circ y_p)}$ and $\omega_{(\mathrm{d}L\circ y_p)}$ represent the objects associated to $F = \mathrm{d}L \circ y_p$ in Theorem 4.5, which are determined univocally by the $\mathrm{Eucl}(p,\mathbb{Z})$-covariance condition.

Using (4.2), (4.3) and the solution (4.5) found in theorem 4.5, these objects $\mathcal{E}_L(y) \in \Gamma(X_0, y^*V^*Y_0)$ and $\omega_L(y) \in \Gamma(X_{p-1}, (y^*V^*Y_0)_{p-1})$ can be given by the explicit formulas:

$$\begin{aligned} &(\mathcal{E}_L(y))_v = \sum_{v\prec\beta} \frac{1}{2^p}(y_p^*\mathrm{d}L)_{\beta,v}, \\ &(\omega_L(y))_{\alpha,v} = \sum_{v\prec\beta} x_{\alpha,\beta}\cdot(y_p^*\mathrm{d}L)_{\beta,v}, \qquad x_{\alpha,\beta} = (-1)^i\cdot s_i\cdot\frac{1}{k\cdot 2^p}\cdot\left(\frac{\binom{p}{0}+\ldots+\binom{p}{p-k}}{\binom{p}{p-k}}\right) \end{aligned} \tag{5.1}$$

where i, s_i, k in the last expression depends on α, β by $i = \alpha^{\mathrm{even}} \in \{1,\ldots,p\}$, $s_i = \beta_i - \alpha_i \in \{\pm 1\}$, $k = \sharp\{j\colon \alpha_j \neq \beta_j\} \in \{1,\ldots,p\}$.

These explicit formulas show that $\mathcal{E}_L(y)$ depends on each vertex $v \in X_0$ only on the configurations $y_p(\beta)$ of the p-cells β containing the vertex v. Analogously, $\omega_L(y)$ depends on each $(p-1)$-cell α only on the configurations $y_p(\beta)$ of the faces that share a vertex with α. The dependence on L at some vertex $v \in X_0$ appears only trough the values of $\mathrm{d}_{y_p(\beta)}L$ for p-cells containing the vertex v. The following result is a direct consequence of the definitions, using the adjointness formula theorem:

Theorem 5.2. *For any configuration $y \in \Gamma(X_0,Y_0)$ the difference operators $\mathrm{d}L \circ y_p \in \Gamma(X_p, (y^*V^*Y)_p)$, $\omega_L(y) \in \Gamma(X_{p-1}, (y^*V^*Y)_{p-1})$ and $\mathcal{E}_L(y) \in \Gamma(X_0, y^*V^*Y_0)$ satisfy the following fundamental* ***discrete first variation formula****:*

$$(\mathrm{d}_y\mathbb{L})(\delta y) = (\mathrm{d}L\circ y_p)(\delta y) = \mathrm{d}\left((\omega_L(y))(\delta y)\right) + ((\mathcal{E}_L(y))(\delta y))\cdot \mathrm{vol}_X \qquad \forall \delta y \in \Gamma(X_0, y^*VY_0) \tag{5.2}$$

As for the continuous calculus of variations, first variation formula leads straightforward to the characterization of critical configurations and to the construction of Noether currents associated to any infinitessimal symmetries. This can be done as follows:

Theorem 5.3 (Discrete Euler equations). *A configuration $y \in \Gamma(X_0,Y_0)$ is critical for the Lagrangian $L\colon Y_0 \to \mathbb{R}$ and admissible variations* Var *if and only if:*

$$\mathcal{E}_L(y)|_{y^*\mathrm{Var}} = 0$$

Proof . For any chain $c \in C_p(X,\mathbb{Z})$ and any infinitesimal variation $\delta y \in \Gamma(\mathrm{int}(c), y^*VY_0)$ we may apply (5.2) and integrate on c to get:

$$\begin{aligned} \langle c, (\mathrm{d}_y\mathbb{L})(\delta y)\rangle &= \langle c, \mathrm{d}\left((\omega_L(y))(\delta y)\right)\rangle + \langle c, ((\mathcal{E}_L(y))(\delta y))\cdot\mathrm{vol}_X\rangle = \\ &= \langle \partial c, ((\omega_L(y))(\delta y))\rangle + \langle c, ((\mathcal{E}_L(y))(\delta y))\cdot\mathrm{vol}_X\rangle = \\ &= 0 + \langle c, ((\mathcal{E}_L(y))(\delta y))\cdot\mathrm{vol}_X\rangle \end{aligned}$$

where the equality $\langle \partial c, ((\omega_L(y))(\delta y))\rangle = 0$ holds because whenever $(\partial c)(\alpha) \neq 0$ for some $\alpha \in X_{p-1}$, all the adherent vertices $v \prec \alpha$ must lie in $\mathrm{supp}(\partial c)$, outside of $\mathrm{int}(c)$, thus each $\delta_v y$ must be zero for $v \prec \alpha$ and $(\delta y)_{p-1}(\alpha) = 0$, which tells us that either $(\partial c)(\alpha) = 0$ or $((\omega_L(y))(\delta y))(\alpha) = 0$ and the duality product $\langle \partial c, ((\omega_L(y))(\delta y))\rangle$ vanishes.

If we know $\mathcal{E}_L(y)|_{y^*\mathrm{Var}} = 0$, then $(\mathcal{E}_L(y))_v(\delta_v y) = 0$ for any choice of $v \in X_0$ and $\delta_v y \in \mathrm{Var}_{y(v)} = (y^*\mathrm{Var})(v)$. Therefore $\langle c, (\mathrm{d}_y\mathbb{L})(\delta y)\rangle = 0$ for any $c \in C_p(X, \mathbb{Z})$ and $\delta y \in \Gamma(\mathrm{int}(c), y^*\mathrm{Var})$, and y is a critical configuration.

Conversely, if we know that y is critical, we may consider any $\delta y \in \Gamma(X_0, y^*VY_0)$ whose value is 0 everywhere except for the vertex $v \in X_0$, where $(\delta y)(v) = \delta_v y \in V_{y(v)}Y_0$ is chosen to be arbitrary in $\mathrm{Var}_{y(v)}$ and consider also the chain $c = \sum_{v\prec\beta} c_\beta \in C_p(X, \mathbb{Z})$ ("the sphere with center v"). It is obvious that $v \in \mathrm{supp}(c)$, $v \notin \mathrm{supp}(\partial c)$, therefore $\delta y \in \Gamma(\mathrm{int}(c), y^*\mathrm{Var})$. Applying our previous result:

$$\langle c, (\mathrm{d}_y\mathbb{L})(\delta y)\rangle = \langle c, ((\mathcal{E}_L(y))(\delta y)) \cdot \mathrm{vol}_X\rangle$$

We know that $(\mathcal{E}_L(y))(\delta y) \in \Omega^0(X)$ vanishes at any vertex except at v. Therefore

$$\langle c, ((\mathcal{E}_L(y))(\delta y)) \cdot \mathrm{vol}_X\rangle = \sum_{v\prec\beta} ((\mathcal{E}_L(y))(\delta y) \cdot \mathrm{vol}_X)(\beta) = \sum_{v\prec\beta}\sum_{\bar v\prec\beta} (\mathcal{E}_L(y))_{\bar v}(\delta y(\bar v)) = 2^p \cdot (\mathcal{E}_L(y))_v(\delta_v y)$$

where the 2^p factor represents the number of p-cells containing the vertex $v \in X_0$. This proves that for any critical section holds $(\mathcal{E}_L(y))_v(\delta_v y) = 0$ for any choice of $v \in X_0$ and $\delta_v y \in \mathrm{Var}_{y(v)} = (y^*\mathrm{Var})(v)$, as we wanted. □

In a similar way to the continuous case, first variation formula for the discrete Lagrangian leads to a momentum map taking any infinitesimal symmetry of the problem to a $(p-1)$-form that is closed on critical sections.

Definition 5.4. Given a configuration bundle $\pi\colon Y_0 \to X_0$, and a discrete Lagrangian density $L\colon Y_p \to \mathbb{R}$, we call infinitesimal symmetry of the Lagrangian any vector field $D \in \mathfrak{X}(Y_0)$ whose extension $D_p \in \mathfrak{X}(Y_p)$ to the bundle $Y_p \to X_p$ verifies:

$$\langle \mathrm{d}L, D_p\rangle = 0 \in \mathcal{C}^\infty(Y_p)$$

It must be noted that the restriction to $y_p \in \Gamma(X_p, Y_p)$ of the function $\langle \mathrm{d}L, D_p\rangle \in \mathcal{C}^\infty(Y_p)$ determines a p-form $(y_p)^*\langle \mathrm{d}L, D_p\rangle \in \mathrm{Map}(X_p, \mathbb{R}) = \Omega^p(X)$ which coincides with $\langle \mathrm{d}_y\mathbb{L}, y^*D\rangle$ where $y^*D = D \circ y \in \Gamma(X_0, y^*VY_0)$.

Theorem 5.5 (Discrete Noether theorem). *Let $L\colon Y_p \to \mathbb{R}$ be a discrete Lagrangian density and $D \in \mathfrak{X}(Y_0)$ be an infinitesimal symmetry of L. If $y \in \Gamma(X_0, Y_0)$ is a critical section for the variational problem with Lagrangian L and admissible variations* Var *and if $y^*D = D \circ y \in \Gamma(X_0, y^*VY_0)$ is admissible (that is, $(D \circ y)(v) \in \mathrm{Var}_{y(v)}$), then $((\omega_L(y))(y^*D))$ is a closed discrete $(p-1)$-form.*

Proof . As D is a symmetry, we know that $\langle \mathrm{d}L, D_p\rangle = 0$, which restricted to points $y_p(\beta) \in Y_p$ tells us that $(\mathrm{d}_y\mathbb{L})(y^*D) = 0$. Moreover, y is critical for admissible variations Var and $y^*D \in \Gamma(X_0, y^*\mathrm{Var})$, so we have $\mathcal{E}_L(y)(y^*D) = 0$. The result is then a direct consequence of (5.2). □

The following two examples illustrate the meaning of the previous results and its relation with the work of other authors:

Example 5.6. *In the* 1*-dimensional case a variational problem is given by a Lagrangian density $L\colon X_1 \times Q \times Q \to \mathbb{R}$. That is, we have a sequence of functions $L_{1+2i}(q_-, q_+)\colon Q \times Q \to \mathbb{R}$, one for each edge $1 + 2i \in X_1$ limited by the vertices $2i, 2i + 2 \in X_0$. The $T^*_{q_-}Q$ and $T^*_{q_+}Q$ components of $\mathrm{d}_{(q_-,q_+)}L_{2i+1} \in T^*_{q_-}Q \oplus T^*_{q_+}Q$ are accordingly represented by $(\mathrm{d}L)^-$ and $(\mathrm{d}L)^+$. In this case the values z, x_1 indicated in theorem 4.5 are $z = 1/2$, $x_1 = -1/2$ and for a given configuration $y = (q_{2i})$ of vertices the associated Euler form and momentum form are:*

$$\begin{aligned}(\mathcal{E}_L(y))_v &= \frac{1}{2}\left(\mathrm{d}_{(q_{v-2},q_v)}L_{v-1}\right)^+ + \frac{1}{2}\left(\mathrm{d}_{(q_v,q_{v+2})}L_{v+1}\right)^- \in T^*_{q_v}Q \\ (\omega_L(y))_v &= \frac{1}{2}\left(\mathrm{d}_{(q_{v-2},q_v)}L_{v-1}\right)^+ + \frac{-1}{2}\left(\mathrm{d}_{(q_v,q_{v+2})}L_{v+1}\right)^- \in T^*_{q_v}Q\end{aligned} \qquad \begin{pmatrix} v \in 2\mathbb{Z} = X_0 = X_{p-1} \\ v-1, v+1 \in X_1 = X_p\end{pmatrix}$$

The discrete first variation formula (which can be proven by direct computation) only states that:

$$\left(\mathrm{d}_{(q_v,q_{v+2})}L_{v+1}\right)(\delta_v y,\delta_{v+2}y) = (\omega_L(y))_{v+2}(\delta_{v+2}y) - (\omega_L(y))_v(\delta_v y) + \left[(\mathcal{E}_L(y))_v(\delta_v y) + (\mathcal{E}_L(y))_{v+2}(\delta_{v+2}y)\right]$$

and Noether's theorem states that, given $y \in \Gamma(X_0, Y_0)$ critical and $\delta y = D \circ y$ defined by some symmetry $D \in \mathfrak{X}(Y)$ of the Lagrangian density, there holds $(\omega_L(y))_v(\delta_v y) = (\omega_L(y))_{v+2}(\delta_{v+2}y)$ for any vertex $v \in X_0 = 2\mathbb{Z}$. It must be indicated that, for critical configurations $(\mathcal{E}_L(y))_v(\delta_v y) = 0$ and we might take

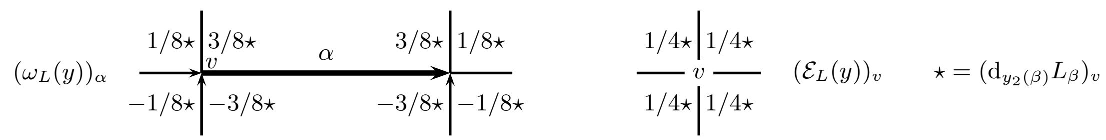
$$(\omega_L(y))_v(\delta_v y) = (\omega_L(y) + \mathcal{E}_L(y))_v(\delta_v y) = \left(\mathrm{d}_{(q_{v-2},q_v)}L_{v-1}\right)^+(\delta_v y)$$

which is the expression that can be found overall in the literature as conserved quantity in one discrete independent variable (see [24] or [33] for example, or [6] in a different context).

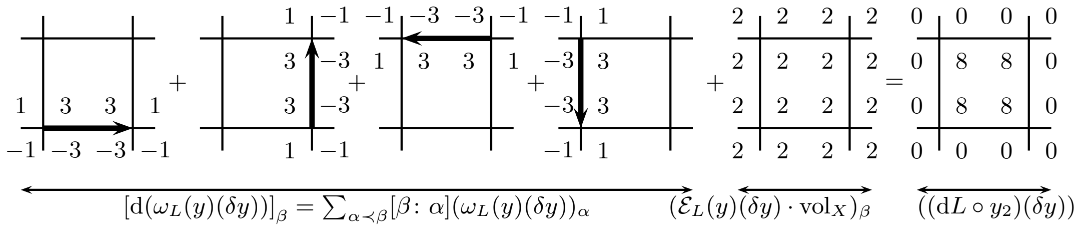
Example 5.7. *In the 2-dimensional case a variational problem is given by a Lagrangian density $L\colon X_2\times Q\times Q\times Q\times Q \to \mathbb{R}$. That is, we have a family of functions $L_{(1+2i,1+2j)}(q_{--},q_{+-},q_{-+},q_{++})\colon Q\times Q\times Q\times Q \to \mathbb{R}$, one for each face $(1+2i,1+2j) \in X_2$ limited by the edges $(2i,1+2j),(2+2i,1+2j),(1+2i,2+2j),(1+2i,2+2j) \in X_1$, with adherent vertices $(2i,2j),(2+2i,2j),(2i,2+2j),(2+2i,2+2j) \in X_0$. The $T^*_{q_{--}}Q$, $T^*_{q_{+-}}Q$, $T^*_{q_{-+}}Q$ and $T^*_{q_{++}}Q$ components of $\mathrm{d}_{(q_{--},q_{+-},q_{-+},q_{++})}L_{(1+2i,1+2j)} \in T^*_{q_{--}}Q \oplus T^*_{q_{+-}}Q \oplus T^*_{q_{-+}}Q \oplus T^*_{q_{++}}Q$ are accordingly represented by $(\mathrm{d}L)^{--}$, $(\mathrm{d}L)^{+-}$, $(\mathrm{d}L)^{-+}$ and $(\mathrm{d}L)^{++}$. In this case the values z, x_1, x_2 indicated in theorem 4.5 are $z = 1/2$, $x_1 = -3/8$, $x_2 = -1/8$ and for a given configuration $y = (q_{(2i,2j)})$ of vertices the associated Euler form and momentum form (for brevity we give only the component at an horizontal edge $\alpha = (1+2i,2j) \in X_1$ and its initial vertex $v = (2i,2j) \in X_0$, hence these expressions are elements of $T^*_{q_v}Q$) are:*

$$\begin{aligned}
(\mathcal{E}_L(y))_v &= \frac{1}{4}\left(\mathrm{d}_{(q_v,q_{v+(2,0)},q_{v+(0,2)},q_{v+(2,2)})}L_{v+(1,1)}\right)^{--} + \frac{1}{4}\left(\mathrm{d}_{(q_{v+(-2,0)},q_v,q_{v+(-2,2)},q_{v+(0,2)})}L_{v+(-1,1)}\right)^{+-} + \\
&+ \frac{1}{4}\left(\mathrm{d}_{(q_{v+(0,-2)},q_{v+(2,-2)},q_v,q_{v+(2,0)})}L_{v+(1,-1)}\right)^{-+} + \frac{1}{4}\left(\mathrm{d}_{(q_{v+(-2,-2)},q_{v+(0,-2)},q_{v+(-2,0)},q_v)}L_{v+(-1,-1)}\right)^{++} \\
(\omega_L(y))_{\alpha,v} &= \frac{3}{8}\left(\mathrm{d}_{(q_v,q_{v+(2,0)},q_{v+(0,2)},q_{v+(2,2)})}L_{v+(1,1)}\right)^{--} + \frac{1}{8}\left(\mathrm{d}_{(q_{v+(-2,0)},q_v,q_{v+(-2,2)},q_{v+(0,2)})}L_{v+(-1,1)}\right)^{+-} + \\
&- \frac{3}{8}\left(\mathrm{d}_{(q_{v+(0,-2)},q_{v+(2,-2)},q_v,q_{v+(2,0)})}L_{v+(1,-1)}\right)^{-+} - \frac{1}{8}\left(\mathrm{d}_{(q_{v+(-2,-2)},q_{v+(0,-2)},q_{v+(-2,0)},q_v)}L_{v+(-1,-1)}\right)^{++}
\end{aligned}$$

all remaining components of the momentum form can be computed using the symmetries in $\mathrm{Eucl}(2,\mathbb{Z})$, *as indicated in the following diagram*



In this case, first variation formula states that the differential of the Lagrangian at a given face $\beta \in X_2$ (which has 4 components, one at each adherent vertex) is just the sum of Euler form for each of the vertices with the momentum forms corresponding to each of the four oriented edges that form the boundary of the face, which can be interpreted by the following diagram (where each number appearing at a face β of the 2-dimensional grid, next to a vertex v, represents the appearance of $\frac{1}{8}(\mathrm{d}_{y_2(\beta)}L_\beta)_v(\delta_v y)$ for the corresponding vertex and face)



In this context it is worth mentioning that, to our knowledge, while some authors derive conservation laws –local or global– in integral form (see [23, 26] and compare its results with those indicated in section 6), the only authors that, similar to our work, consider conservation laws in differential form (in terms of discrete forms and its differentials, though not always in an explicit way) for several discrete variables are Hydon and Mansfield [15, 16, 25]. The results indicated in these references are centered in the theory of the discrete variational complex, while in [15, 16] the derivation of conservation laws is directly done from the discrete Euler equations using a certain homotopy operator (whose explicit formula in the smooth case is considered in [15] as "very cumbersome", while for the discrete case "At least in principle, it is possible to construct conservation laws systematically using the homotopy operator, but the complexity of the calculations is even more fearsome than for PDEs!") and is not given with an explicit formula. In this context there is not a momentum mapping determining one single conservation law for each symmetry, but rather a family of conserved quantities that can be derived from the discrete Euler-Lagrange equations, and different possibilities arise depending on the choice we make about the nature of the conserved quantity. However many results can be compared with the ones presented here, as the theory developed in both cases deals with functions depending on several discrete independent variables living in the lattice $X = \mathbb{Z}^p$ and several continuous dependent variables, where the translation morphisms $S_k\colon v \in X \mapsto v + e_k \in X$ (symmetries of the lattice) play an essential role. The notion given by Hydon/Mansfield of a 2-dimensional conservation law as being a expression $(S_1 - \mathrm{Id}) \cdot F + (S_2 - \mathrm{Id}) \cdot G = 0$ for some F, G depending on dependent and independent variables can be seen in our context as a condition that certain discrete 1-form is closed. Indeed if ω is a 1-form in our lattice, it can be represented by certain functions $F(i,j)$ and $G(i,j)$ if we denote the components of ω at vertical edges and horizontal edges as $\omega(2i, 2j-1) = -F(i,j)$ and $\omega(2i-1, 2j) = G(i,j)$, respectively. The condition $(\mathrm{d}\omega)(2i-1, 2j-1) = 0$ can be written then as $0 = ((S_1 - \mathrm{Id}) \cdot F + (S_2 - \mathrm{Id}) \cdot G)_{(i,j)} = F(i-1,j) - F(i,j) + G(i,j-1) - G(i,j)$. The article [25] gives conserved quantities from symmetries and a discrete variational principle: a similar notion of conservation law in several discrete variables is introduced, together with the integration by parts procedure (see also [24]) that corresponds with our adjointness formula in one discrete variable. A conservation law is derived if the group action commutes with all discrete translations, though its local expressions are again not explicitly given. Our formalism, more geometric, doesn't impose these conditions on the group action.

6. Physical interpretation.

We shall give an interpretation of our discrete Euler equations and Noether theorem in terms of the components of $\mathrm{d}L$, that, motivated by our example in section 7 and with a little abuse of language, we call contact forces.

Definition 6.1. Given some configuration bundle Y_0 on the discrete p-dimensional Euclidean space $X_0 = (2\mathbb{Z})^p$, a discrete Lagrangian $L\colon Y_p \to \mathbb{R}$, and a configuration of vertices $y \in \Gamma(X_0, Y_0)$, we shall denote the components of $\mathrm{d}L \circ y_p \in \Gamma(X_p, (y^* V^* Y)_p)$ at any p-cell $\beta \in X_p$ by $(\mathrm{d}_{y_p(\beta)} L) = (f^{L,y}_{\beta,v})_{v \prec \beta} \in T^*_{y_p(\beta)}(Y_p)_\beta = \oplus_{v \prec \beta} T^*_{y(v)}(Y_0)_v$ and we shall call contact force pulling from the p-cell β at the vertex $v \prec \beta$, for the configuration y, the element:

$$f^{L,y}_{\beta,v} = (\mathrm{d}_{y_p(\beta)} L)_v \in T^*_{y(v)}(Y_0)_v$$

We may observe from formula (5.1) that Euler form may be written as:

$$(\mathcal{E}_L(y))_v = \frac{1}{2^p} \sum_{v \prec \beta} f^{L,y}_{\beta,v} \tag{6.1}$$

Vanishing of Euler form for any $\delta_v y \in \mathrm{Var}_{y(v)}$ can be interpreted as the condition that the sum of all contact forces pulling at the vertex v is incident to any infinitesimal admissible variation $\delta_v y \in \mathrm{Var}_{y(v)}$. Critical configurations appear when the total contact force pulling at any vertex vanishes when applied to any admissible infinitesimal variation $\partial_v y \in \mathrm{Var}_{y(v)}$ of the vertex.

After this interpretation of Euler equations it is also possible to give a physically meaningful interpretation of Noether's Theorem for certain p-chains associated to sets of p-cells. Consider any finite subset of p-cells on X_p, say $A \subseteq X_p$.

Definition 6.2. For any finite set of p-cells, $A \subset X_p$, we call characteristic chain associated to A and denote by $c_A \in C_p(X, \mathbb{Z})$ the p-chain $c_A = \sum_{\alpha \in A} c_\alpha$, whose values at any cell are:

$$c_A(\beta) = \left(\sum_{\alpha \in A} c_\alpha \right)(\beta) = \begin{cases} 0 \text{ if } \beta \notin A \\ 1 \text{ if } \beta \in A \end{cases}$$

Lemma 6.3. *For any finite set of cells $A \subset X_p$ and any interior vertex $v \in \operatorname{int}(c_A)$ of its characteristic chain there holds $\beta \in A$ for every p-cell β containing the vertex v.*

Proof . We have $v \in \operatorname{int}(c_A)$, therefore $v \in \operatorname{supp}(c_A)$ and hence there exists a p-cell $\tilde{\beta}$ with $v \prec \tilde{\beta}$, which has the form $\tilde{\beta} = v + (s_1, \ldots, s_p)$ with $s_i \in \{\pm 1\}$ and $c_A(\tilde{\beta}) \neq 0$, so $\tilde{\beta} \in A$. Imagine there exists a p-cell $\beta \in X_p$ containing v with $\beta \notin A$: It has the form $\beta = v + (r_1, \ldots, r_p)$ with $r_i \in \{\pm 1\}$. We shall prove that $v \in \operatorname{supp}(\partial c_A)$. In fact, it suffices to consider the following p-cells: $\beta_0 = \tilde{\beta} = v + (s_1, \ldots, s_p)$, $\beta_1 = v + (r_1, s_2, \ldots, s_p)$,... $\beta_{i-1} = v + (r_1, \ldots, r_{i-1}, s_i, \ldots, s_p)$,... $\beta_p = \beta = v + (r_1, \ldots, r_p)$. As the first one is in A and the last one is not in A, we conclude that for some i, $\beta_{i-1} = v + (r_1, \ldots, r_{i-1}, s_i, s_{i+1}, \ldots, s_p) \in A$ and $\beta_i = v + (r_1, \ldots, r_{i-1}, r_i, s_{i+1}, \ldots, s_p) \notin A$. In this case the $(p-1)$-cell $\alpha = v + (r_1, \ldots, r_{i-1}, 0, s_{i+1}, \ldots, s_p) \in A$ is incident only to these two p-cells, and therefore $(\partial c_A)(\alpha) = c_A(\beta_{i-1}) \cdot [\beta_{i-1} \colon \alpha] + c_A(\beta_i) \cdot [\beta_i \colon \alpha] = [\beta_{i-1} \colon \alpha] \in \{\pm 1\}$. As v is adherent to α and $(\partial c_A)(\alpha) \neq 0$, we conclude that $v \in \operatorname{supp}(\partial c_A)$, which is in contradiction with the hypothesis $v \in \operatorname{int}(c_A) = \operatorname{supp}(c_A) \setminus \operatorname{supp}(\partial c_A)$. Therefore the only possibility is that every p-cell containing v belongs to A. $\square$

From (6.1) we may in particular conclude:

$$\sum_{v \prec \beta} (\mathrm{d}_{y_p(\beta)} L)_v = \sum_{v \prec \beta} f^{L,y}_{\beta,v} = \sum_{v \prec \beta} (\mathcal{E}_L(y))_v, \qquad \forall v \in X_0 \tag{6.2}$$

because the number of p-cells containing v is precisely 2^p. Combining this result with the previous lemma and first variation formula of the Lagrangian density leads to:

Theorem 6.4. *Let $A \subset X_p$ be a finite subset of p-cells on X and $c_A \in C_p(X, \mathbb{Z})$ its characteristic p-chain. For any Lagrangian density $L \colon Y_p \to \mathbb{R}$ defined on the configuration bundle $Y_0 \to X_0$, any configuration $y \in \Gamma(X_0, Y_0)$ and any admissible variation $\delta y \in \Gamma(X_0, y^* V Y_0)$ holds:*

$$\langle \partial c_A, \omega_L(y)(\delta y) \rangle = \sum_{v \in \operatorname{fr}(c_A)} \sum_{v \prec \beta \in A} f^{L,y}_{\beta,v}(\delta_v y) - \sum_{v \in \operatorname{fr}(c_A)} \rho(v, A) \cdot (\mathcal{E}_L(y))_v(\delta_v y) \tag{6.3}$$

where $\rho(v, A) = \sharp\{\beta \in X_p : v \prec \beta \in A\}$ represents the number of p-cells belonging to A containing v.

Proof . Taking the integral of first variation formula (5.2) on c_A we get:

$$\langle \partial c_A, (\omega_L(y))(\delta y) \rangle = \langle c_A, (\mathrm{d}L \circ y_p)(\delta y) - (\mathcal{E}_L(y))(\delta y) \cdot \mathrm{vol}_X \rangle = \sum_{\beta \in A} \sum_{v \prec \beta} \left((\mathrm{d}_{y_p(\beta)} L)_v - (\mathcal{E}_L(y))_v \right)(\delta_v y) =$$

$$= \sum_{v \in \operatorname{fr}(c_A)} \sum_{v \prec \beta \in A} \left((\mathrm{d}_{y_p(\beta)} L)_v - (\mathcal{E}_L(y))_v \right)(\delta_v y) + \sum_{v \in \operatorname{int}(c_A)} \sum_{v \prec \beta \in A} \left((\mathrm{d}_{y_p(\beta)} L)_v - (\mathcal{E}_L(y))_v \right)(\delta_v y)$$

where the last equality holds because vertices v such that $v \prec \beta$ for some $\beta \in A$ are precisely those in the support of c_A, which might be either in $\operatorname{int}(c_A)$ or in $\operatorname{fr}(c_A)$. On the other hand, we know by Lemma 6.3 that $v \in \operatorname{int}(c_A), v \prec \beta \Rightarrow \beta \in A$ so:

$$\langle \partial c_A, \omega_L(y)(\delta y) \rangle = \sum_{v \in \operatorname{fr}(c_A)} \sum_{v \prec \beta \in A} \left((\mathrm{d}_{y_p(\beta)} L)_v - (\mathcal{E}_L(y))_v \right)(\delta_v y) + \sum_{v \in \operatorname{int}(c_A)} \sum_{v \prec \beta} \left((\mathrm{d}_{y_p(\beta)} L)_v - (\mathcal{E}_L(y))_v \right)(\delta_v y)$$

Applying formula (6.2) we conclude that the summation for $v \in \operatorname{int}(c_A)$ vanishes, leading to (6.3). $\square$

Remark . In the case that $\delta_v y$ is admissible and $y \in \Gamma(X_0, Y_0)$ is critical, we know that $(\mathcal{E}_L(y))(\delta y) = 0$, therefore (6.3) says:

$$\langle \partial c_A, (\omega_L(y))(\delta y)\rangle = \sum_{v\in \mathrm{fr}(c_A)} \sum_{v\prec\beta\in A} \langle f^{L,y}_{\beta,v}, \delta_v y\rangle \tag{6.4}$$

which shows that for critical configurations $y \in \Gamma(X_0, Y_0)$ and any p-chain c_A given by a finite subset $A \subseteq X_p$, when we consider the Noether current associated to some admissible infinitesimal variation δy, computing the integral of this Noether current $(\omega_L(y))(\delta y)$ on the boundary ∂c_A is nothing more than to compute at each vertex $v \in \mathrm{fr}(c_A)$ of the boundary the $\delta_v y$-component of the contact forces pulling at v from p-chains inside of A and summing them.

In the case that the admissible infinitesimal variation δy is the restriction $D \circ y$ of some infinitesimal symmetry $D \in \mathfrak{X}(Y_0)$ of the problem, for any critical configuration and any finite subset $A \subset X_p$ Noether's Theorem states that $\langle \partial c_A, (\omega_L(y))(\delta y)\rangle = \langle c_A, \mathrm{d}(\omega_L(y))(\delta y)\rangle = 0$ therefore the total δy-component of the contact forces pulling at vertices $v \in \mathrm{fr}(c_A)$ from cells inside A vanishes, which is the physical meaning of Noether's Theorem in this discrete context.

For critical configurations $y \in \Gamma(X_0, Y_0)$ holds $\sum_{v\prec\beta\in A} f^{L,y}_{\beta,v} + \sum_{v\prec\beta\notin A} f^{L,y}_{\beta,v} = 0$ at any vertex $v \in X_0$, therefore we may give analogous results for forces pulling from outside A.

$$\langle \partial c_A, (\omega_L(y))(\delta y)\rangle = -\sum_{v\in \mathrm{fr}(c_A)} \sum_{v\prec\beta\notin A} \langle f^{L,y}_{\beta,v}, \delta_v y\rangle$$

and in the case of an infinitesimal symmetry, Noether's Theorem also states that the total δy-component of the contact forces pulling at vertices $v \in \mathrm{fr}(c_A)$ from cells outside A also vanishes.

7. Example

Let us finish with a simple illustrative example of the theory: Consider the possible configurations of a membrane in $Q = \mathbb{R}^3$. We shall assume that our membrane is composed of several quadrilateral membrane elements indexed by $\beta \in X_2$, where X is the 2-dimensional discrete Euclidean space, and that the configuration of each quadrilateral element on Q is determined by the position of each of its vertices. The configuration of the membrane may then be modelled as a mapping $y \in \Gamma(X_0, Y_0)$, where $Y_0 = X_0 \times Q$ is the configuration space associated to Q. Each configuration $y_2(\beta) \in (Y_2)_\beta = Q_{--} \times Q_{+-} \times Q_{-+} \times Q_{++}$ of a membrane element has an internal energy that can be obtained as a function $L(y_2(\beta)) = L(\beta, q_{--}, q_{+-}, q_{-+}, q_{++})$.

In a simple situation we may consider that this internal energy is generated by a system of springs joining its vertices, in particular, if there were two springs connecting the two diagonals of the quadrilateral, the internal energy of a membrane element $\beta \in X_2$ with configuration $y_2(\beta)$ would be given by:

$$L(y_2(\beta)) = L(\beta, q_{--}, q_{+-}, q_{-+}, q_{++}) = \frac{-1}{2}k\|q_{++} - q_{--}\|^2 + \frac{-1}{2}k\|q_{+-} - q_{-+}\|^2$$

being k Hooke's constant for the springs and $\|\ \|$ the Euclidean norm in $Q = \mathbb{R}^3$.

In this situation, for a given configuration $y_2(\beta) = (\beta, q_{--}, q_{+-}, q_{-+}, q_{++})$ the components of $\mathrm{d}_{y_2(\beta)}L$ (which have a clear interpretation as contact forces of the system) are:

$$(\mathrm{d}_{y_2(\beta)}L)^{++} = k(q_{--} - q_{++}), \quad (\mathrm{d}_{y_2(\beta)}L)^{+-} = k(q_{-+} - q_{+-}),$$
$$(\mathrm{d}_{y_2(\beta)}L)^{-+} = k(q_{+-} - q_{-+}), \quad (\mathrm{d}_{y_2(\beta)}L)^{--} = k(q_{++} - q_{--})$$

where we make the natural identification $T^*_q\mathbb{R}^3 = T_q\mathbb{R}^3 = \mathbb{R}^3$ given by the Euclidean metric in $\mathbb{R}^3$. These represent the contact forces exerted at each vertex by the membrane element configuration $y_2(\beta)$, which is due to the spring that joins each vertex with the opposite one.

As indicated in section 6, if we don't impose boundary conditions, critical configurations $y = (q_v)_{v\in X_0} \in \Gamma(X_0, Y_0)$ are then characterized by the vanishing of the total contact force f_v pulling at a given vertex

$v \in (2\mathbb{Z})^2$:

$$\begin{aligned} f_v &= (\mathrm{d}_{y_2(v+(1,1))}L)^{--} + (\mathrm{d}_{y_2(v+(-1,1))}L)^{+-} + (\mathrm{d}_{y_2(v+(1,-1))}L)^{-+} + (\mathrm{d}_{y_2(v+(-1,-1))}L)^{++} = \\ &= k(q_{v+(2,2)} - q_v) + k(q_{v+(-2,2)} - q_v) + k(q_{v+(2,-2)} - q_v) + k(q_{v+(-2,-2)} - q_v) = 0 \end{aligned}$$

If we impose boundary conditions, say that each vertex v has a configuration lying on some submanifold $S_v \subset \mathbb{R}^3$, then critical configurations $(q_v)_{v\in X_0}$ are characterized by the condition that the total contact force $f_v \in T_{q_v}\mathbb{R}^3$ at each vertex v should be orthogonal to the tangent space $T_{q_v}S_v$.

Let us explore Noether's conservation law. It is obvious that L admits as symmetries the whole 3-dimensional Euclidean group. In fact $\langle \mathrm{d}_{y_2(\beta)}L, D_2\rangle = 0$ for any vector field $D = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{R}^3)$ can be seen as the condition that:

$$k(q_{--} - q_{++})\cdot(a,b,c) + k(q_{-+} - q_{+-})\cdot(a,b,c) + k(q_{+-} - q_{-+})\cdot(a,b,c) + k(q_{++} - q_{--})\cdot(a,b,c) = 0$$

which represents only the equilibrium of any (a,b,c)-component of the forces that all the points of a face induce on the remaining points. In this way, covariance of the lagrangian with respect to translations is some way of stating that Newton's third law of action-reaction is valid for springs.

For the vector field $D_{(x,y,z)} = (\vec{v} \times (x,y,z)) \cdot (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \in \mathfrak{X}(\mathbb{R}^3)$ representing infinitesimal rotations generated by $\vec{v} \in \mathbb{R}^3$, holds $\langle \mathrm{d}_{y_2(\beta)}L, D_2\rangle = 0$ which means:

$$\begin{aligned} &k\det((q_{--} - q_{++}), q_{++}, \vec{v}) + k\det((q_{-+} - q_{+-}), q_{+-}, \vec{v}) + \\ &\quad + k\det((q_{+-} - q_{-+}), q_{-+}, \vec{v}) + k\det((q_{++} - q_{--}), q_{--}, \vec{v}) = 0 \end{aligned}$$

representing thus that the total angular momentum with respect to the origin of the forces that all the points of a face induce on the remaining points vanishes, which is a different form of Newton's third law, in this case for angular momenta rather than linear momenta, which is also valid for our springs. In a similar way, in general, validity of Newton's third law for a mechanical system on a manifold (where in principle forces at different points may not be summed or compared as in Newton's law) can be seen as a symmetry condition for the Lagrangian of the system with respect to some group of diffeomorphisms.

For any of these symmetries and any finite subset of membrane elements $A \subseteq X_2$ we may apply Theorem 6.4 and the remark afterwards to conclude that if $y \in \Gamma(X_0, Y_0)$ is critical, then for any constant vector field $D = (a,b,c)$ the component in the direction D of all the forces pulling at the vertices $v \in \mathrm{fr}(c_A)$ from the membrane elements $\beta \in A$ must vanish. That is, the linear momentum in any direction induced by these forces pulling at the boundary vanishes. In the same manner, using infinitesimal rotations we conclude that the total angular momentum with respect to the origin (also with respect to any other point) of the forces pulling at $v \in \mathrm{fr}(c_A)$ from the membrane elements $\beta \in A$ also vanishes. These are the discrete momentum conservation laws in our formalism.

We don't want to finish without mentioning energy conservation properties, besides the above mentioned momentum conservation. However, for reasons of brevity, all these ideas are left for the reader for completion: Imagine we want to incorporate time and kinetic energy in the previous example. If time is introduced as independent variable (therefore discrete) then the discrete formalism doesn't allow to consider infinitesimal time translations as infinitesimal symmetries, and no discrete energy conservation would be obtained using our discrete Noether theorem. For a discrete energy conservation to appear in this context, time must be a continuous dependent variable.

In the continuous case if we consider the 2-dimensional membrane $\mathcal{M}_{(\alpha,\beta)}$ (2-dimensional manifold with local coordinates α, β), including the kinetic part in the Lagrangian and including the time as dependent variable in our formalism leads to a variational problem where configurations are mappings $\mathcal{M}_{(\alpha,\beta)} \times \mathbb{R}_s \to \mathbb{R}^3_{(x,y,z)} \times \mathbb{R}_t$ that can be represented by equations $(x(\alpha,\beta,s), y(\alpha,\beta,s), z(\alpha,\beta,s), t(\alpha,\beta,s))$. The continuous Lagrangian is the sum of Kinetic and internal energy, which can be expressed in terms of $(x, y, z, t, x_\alpha, y_\alpha, z_\alpha, t_\alpha, x_\beta, y_\beta, z_\beta, t_\beta, x_s, y_s, z_s, t_s)$, and if $D = \partial/\partial t$ is an infinitesimal symmetry, for a given configuration with $t_s = \partial t/\partial s \neq 0$ the corresponding conservation law for the domain $C_{t_0,t_1} = \{(\alpha,\beta,s) \in A \times \mathbb{R} \colon t_0 \le t(\alpha,\beta,s) \le t_1\}$ (where $A \subset \mathcal{M}$ is a compact domain with boundary ∂A) and for its boundary $\partial C_{t_0,t_1} = [A \times (t = t_0)] \cup [A \times (t = t_1)] \cup [\partial A \times (t_0 < t < t_1)]$ gives a work-energy equilibrium property:

that the difference of the kinetic energy of the domain A between instants t_0, t_1 equals the work exerted by external forces at the boundary ∂A in the time interval $[t_0, t_1]$.

In the discrete formulation, mimicking this situation, we may introduce in our example another discrete variable and a time variable as dependent variable, considering configurations of $X_0 = (2\mathbb{Z})^3$ into $\mathbb{R}^4_{(x,y,z,t)}$. In this case, when we give a configuration, each membrane vertex $(2i, 2j)$ has its own associated sequence (parametrized by $2k \in 2\mathbb{Z}$) of times and positions for these times, which slightly differs from the approach adopted in [23], where each finite element (representing a membrane surface element, not a vertex) has its own sequence of times, and from these times one derives for each membrane vertex its own sequence of times ("sequence of nodal times") and positions for those times. In our formalism all dependent variables (no matter if they represent a time or a position) deserve the same treatment and (spatial or temporal) configurations are always determined by the vertices, differing from the mentioned approach where temporal configurations are associated to surface elements rather than vertices. Being the continuous Lagrangian invariant by time translation, the discrete Lagrangian derived in Lemma 3.5 from the continuous one would admit time translation as symmetry (because $(x, y, z, t) \mapsto (x, y, z, t + \mathrm{cte})$ is an affine transformation on each separate variable in $\mathbb{R}^4$) and lead to an exact discrete conservation law which should be interpreted in this formalism as the analogous discrete work-energy equilibrium property.

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