On divisors of semigroups of order-preserving mappings of a finite chain

V. H. Fernandes · **M. V. Volkov**

Abstract Let a finite semilattice S be a chain under its natural order. We show that if a semigroup T divides a semigroup of full order preserving transformations of a finite chain, then so does any semidirect product $S \rtimes T$.

Keywords Semigroup of order preserving transformations · Divisor · Semidirect product

All semigroups in this note are supposed to be finite. Recall that a *divisor* of a semigroup S is a homomorphic image of a subsemigroup of S.

A partial transformation s of the chain $X_n = \{1 < 2 < \cdots < n\}$ is said to be *order-preserving* if $x \leq y$ implies $xs \leq ys$ for all $x, y \in \mathrm{Dom}(s)$. We denote by $\mathcal{O}_n$ the semigroup of all full order preserving transformations of X_n.

The semigroups $\mathcal{O}_n$ have been considered in the literature since the 1960s and their algebraic and combinatorial properties are well understood by now. For instance, we presently know a nice presentation for these semigroups [4], a complete description of their congruences [5, 13] and their endomorphisms [8], precise formulas for the number of idempotents in $\mathcal{O}_n$ [12] and for its rank and idempotent rank [10], etc.

Subsemigroups of the semigroups $\mathcal{O}_n$ have been studied to a lesser extent. Still, it is known that the class of all such subsemigroups is decidable, that is, there exists an algorithm that, given a finite semigroup S, decides whether or not S embeds into $\mathcal{O}_n$ for some n [9, 14, 17].

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In contrast, the analogous decision problem about divisors of the semigroups $\mathcal{O}_n$ remains open for more than 20 years (this problem was explicitly stated by J.-E. Pin at a semigroup conference in Szeged in 1987). Only some partial results are known, see, e.g., [2, 3, 11, 15, 16]. Here we recall two results by the first author to which the present note relates. Let O stand for the class of all divisors of $\mathcal{O}_n$, $n = 1, 2, \ldots$. In [6] it is proved that O contains the semigroup $\mathcal{POI}_n$ of all partial injective order preserving transformations of the chain X_n for each $n = 1, 2, \ldots$. In [7] this has been strengthened by showing that O contains all semidirect products of the form $S \rtimes \mathcal{POI}_n$ where S is a semilattice being a chain under the natural order $\leq$ defined by $s \leq t$ if and only if $st = s$, for $s, t \in S$ (further we refer to such a semilattice simply as a chain). The aim of this note is to prove the following generalization of the latter result:

Theorem. *Every semidirect product of a chain by a semigroup from* O *again belongs to* O.

Surprisingly, the proof of this general statement turns out to be much simpler than the proof of its special case in [7].

First, recall the notion of semidirect product. Let V and W be two semigroups and let φ be a left action of W on V, i.e. a monoid homomorphism $\varphi : W^1 \to \mathrm{End}^{\mathrm{r}}(V)$, with $\mathrm{End}^{\mathrm{r}}(V)$ being the (left-right) dual of the monoid $\mathrm{End}(V)$ of all endomorphisms of V. For $w \in W^1$ and $v \in V$, we denote by ${}^{w}v$ the element $(v)(w)\varphi \in V$. The *semidirect product* $V \rtimes W$ with respect to this action is the set $V \times W$ with multiplication given by $(v, w)(v', w') = (v\, {}^{w}v', ww')$.

Let S be a semilattice. Denote by S^{I} the semilattice obtained from S by adjoining a new identity I (even if S already has an identity).

Lemma. *Let* $S \rtimes T$ *be a semidirect product of a semilattice* S *by a semigroup* T*. Then* $S \rtimes T$ *is embeddable into the direct product* $\mathrm{End}^{\mathrm{r}}(S^{\mathrm{I}}) \times T$.

Proof. For $s \in S$ and $t \in T$, consider the mapping $\alpha_{s,t} : S^{\mathrm{I}} \to S^{\mathrm{I}}$ defined by

$$(x)\alpha_{s,t} = \begin{cases} s\, {}^{t}x & \text{if } x \in S \\ s & \text{if } x = \mathrm{I}\,, \end{cases}$$

which is, clearly, an endomorphism of the semilattice S^{I}. Furthermore, it is a routine matter to check that the mapping $S \rtimes T \to \mathrm{End}^{\mathrm{r}}(S^{\mathrm{I}}) \times T$ defined by $(s, t) \mapsto (\alpha_{s,t}, t)$ is an injective homomorphism of semigroups, as required. □

Proof of the Theorem. Let S be a chain with n elements and let T be a divisor of $\mathcal{O}_m$ for some m. By the Lemma, every semidirect product $S \rtimes T$ is embeddable into the direct product $\mathrm{End}^{\mathrm{r}}(S^{\mathrm{I}}) \times T$. Clearly, S^{I} is again a chain and the semigroup $\mathrm{End}(S^{\mathrm{I}})$ is isomorphic to $\mathcal{O}_{n+1}$. It is known (see [11, Theorem 2.4]), that the dual $\mathcal{O}^{\mathrm{r}}_{n+1}$ of $\mathcal{O}_{n+1}$ is isomorphic to a subsemigroup of $\mathcal{O}_{n+2}$. We conclude that $S \rtimes T$ embeds into $\mathcal{O}_{n+2} \times T$. Since T divides $\mathcal{O}_m$, the latter product divides the semigroup $\mathcal{O}_{n+2} \times \mathcal{O}_m$ that can be easily identified with a subsemigroup in $\mathcal{O}_{n+m+2}$, see [11, Proposition 4.1.9]. □

In particular, we have:

Corollary. *If* S *is a chain with* n *elements, then any semidirect product* $S \rtimes \mathcal{O}_m$ *is embeddable into* $\mathcal{O}_{n+m+2}$. □

From [1, Theorem 4] it follows that there exist a semilattice S and semigroup $T \in \mathsf{O}$ such that a semidirect product $S \rtimes T$ does not belong to O. Here we present an explicit 12-element example of such a product.

Observe that if V is a semigroup and W is a subsemigroup of $\mathrm{End}^{\mathrm{r}}(V)$, then we obtain a semidirect product $V \rtimes W$ by considering the natural left action of W on V: ${}^{\alpha}v = (v)\alpha$ for $v \in V$ and $\alpha \in W$.

Let $D_4 = \{0, a, b, 1\}$ be the semilattice with zero 0, identity 1 and such that $ab = 0$. Then (the diamond) D_4 is the least non-tree semilattice. Let $\alpha = \begin{pmatrix} 0 & a & b & 1 \\ a & a & a & b \end{pmatrix} \in \mathrm{End}(D_4)$. We notice that α preserves the linear order $0 < a < b < 1$ on D_4. Let W be the submonoid of $\mathrm{End}^{\mathrm{r}}(D_4)$ generated by α. Then $\alpha^2 = \begin{pmatrix} 0 & a & b & 1 \\ a & a & a & a \end{pmatrix}$ and so $W = \{1, \alpha, \alpha^2\}$ is a semigroup of order-preserving transformations of the chain $0 < a < b < 1$.

Consider the natural left action of W on D_4: ${}^{w}d = (d)w$ for $d \in D_4$ and $w \in W$ and form the semidirect product $D_4 \rtimes W$ with respect to this action. Recall that for each element s of a finite semigroup s^{ω} stands for the unique idempotent being a power of s. By taking $x = (a, 1), y = (1, \alpha) \in D_4 \rtimes W$, we obtain

$$x^{\omega}y(xy^2)^{\omega} = (a, \alpha^2) \neq (0, \alpha^2) = x^{\omega}(xy^2)^{\omega}.$$

However it is known [3] that the equality $x^{\omega}y(xy^2)^{\omega} = x^{\omega}(xy^2)^{\omega}$ must hold for every choice of x and y from a semigroup in O. We conclude that the semidirect product $D_4 \rtimes W$ of the 4-element semilattice D_4 by the 3-element subsemigroup W of $\mathcal{O}_4$ cannot belong to O.

To the best of our knowledge, all known examples of semidirect products $S \rtimes T \notin \mathsf{O}$, with S a semilattice and $T \in \mathsf{O}$, are such that S is not a tree. Supported by this evidence, we conclude with the following:

Conjecture. *Every semidirect product of a tree by a semigroup from* O *again belongs to* O*.*

The above example shows that this conjecture (if it is true) constitutes the maximum possible generalization of our theorem.

Acknowledgements. The authors gratefully acknowledge support of FCT and PIDDAC, within the project PTDC/MAT/69514/2006 of CAUL. The second author acknowledges support from the Federal Education Agency of Russia, grant 2.1.1/3537, and from the Russian Foundation for Basic Research, grant 09-01-12142.

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