Near-exact distributions for the likelihood ratio test statistic of the multi-sample block-matrix sphericity test

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Summary. The multi-sample block-matrix sphericity test and its particular cases have wide applications in testing the error strucutre in several multivariate linear models. However, the practical implementation of this test has been hindered by difficulties in handling the exact distribution of the associated statistic and the non-availability in the literature of well-fit asymptotic distributions. We use a decomposition of the null hypothesis into conditionally independent hypotheses in order to induce a factorization of the likelihood ratio test (l.r.t.) statistic. We then use the induced factorization of the characteristic function of the logarithm of the l.r.t. statistic to obtain very well-fit but highly manageable near-exact distributions for the l.r.t. statistic of this test and its particular cases. These near-exact distributions will allow for the easy computation of well-fit near-exact quantiles and p-values, enabling this way a more frequent practical use of these tests. A measure of proximity between distributions, based on the corresponding characteristic functions, is used to assess the performance of the near-exact distributions.

Keywords: multi-sample block-matrix sphericity test, near-exact distributions, mixtures, asymptotic distributions

1. Introduction

The multi-sample block-matrix (MS-BM) sphericity test plays a key role in tests of homocedasticity in multivariate analysis and repeated measures designs, where the validity of several other tests rest on the assumption of sphericity. His role is the equivalent to the one that the multi-sample (MS) sphericity test has in univariate analysis. The MS sphericity test should be routinely used in checking for homocedasticity in MANOVA and Multivariate Discriminant Analysis models, as well as in Multivariate Regression or Canonical Analysis models which include indicator variables for the levels of one or more categorical variables, what turn out to be models of Multivariate Analysis of Covariance.

However, both the MS sphericity and the MS-BM sphericity tests are seldom carried out because of the difficulties found in handling and computing the exact distribution and quantiles of the associated test statistics. The development of good and easily computable approximations to the exact distribution of these statistics is thus a much desirable objective and therefore the development of near-exact distributions is a very desirable goal, moreover since there are no asymptotic distributions available in the literature for the MS-BM sphericity test statistic.

If the fact that the test statistics addressed in this paper are usually used under the assumption of multivariate normality may be seen as a somehow severe limitation, we should be aware of the results presented in Anderson et all. (1986), Anderson and Fang (1990) and Anderson (2003), which combined with the decomposition of the overall null hypothesis presented in the next Section and the concomitant factorization of the overall test statistic, show that the distributions studied and developed in this paper still remain valid under the null hypothesis when we assume any elliptically contoured distribution for the underlying populations.

The exact distribution of the MS-BM sphericity test statistic is almost intractable in practical terms and asymptotic approximations are not known for the distribution of this statistic what justifies the need for very accurate approximations. Thus, our purpose is to develop near-exact approximations, (Coelho, 2004; Coelho and Marques, 2008; Marques and Coelho, 2008b), that may render possible to use this test and its particular cases in a practical way.

In a simple way we may say that near-exact distributions are asymptotic distributions built using an whole different concept. These near-exact distributions are built in such a way that the major part of the exact c.f. (characteristic function) of the statistic is left unchanged and the remaining part is replaced by an asymptotic function, so that:

i) if we denote by $\Phi^*(t)$ the part of the exact c.f. of the statistic that is replaced by $\Phi^{**}(\gamma;t)$, where, for simplicity of notation, γ denotes any and every parameter in the distribution of that statistic, we have

$$\lim_{\gamma\to\infty}\int_{-\infty}^{+\infty}\left|\frac{\Phi^{**}(\gamma;t)-\Phi^*(t)}{t}\right|dt=0\,,$$

or equivalently,

$$\lim_{\gamma\to\infty}\Phi^{**}(\gamma;t)=\Phi^*(t)\,,$$

with this replacement yielding what we will call the near-exact c.f., in such a way that,

ii) the near-exact distribution, obtained by inversion of the near-exact c.f., corresponds to a known and manageable distribution, from which the computation of p-values and quantiles is rendered easy.

The aim of this paper is thus to illustrate the development of near-exact distributions in a rather complex situation, where the development of accurate enough traditional asymptotic distributions is very hard, if at all possible, therefore rendering the development of near-exact distributions almost required in order to enable the practical application of the test.

From the decomposition of the null hypothesis into three hypotheses we derive expressions for the likelihood ratio test (l.r.t.) statistic and for its h-moment. Well-fitting near-exact distributions based on this decomposition are developed for the modified l.r.t. statistic. Asymptotic approximations are known for some particular cases of this test (see Moschopoulos (1988), Chao and Gupta (1991)), however we will show that these are not precise enough even when large samples are considered, mainly if compared with the near-exact approximations developed in this paper.

To assess the quality of the approximations we will use a measure of proximity between the exact distribution and the approximating distributions, based on the proximity of the corresponding caracteristic functions.

2. The test statistic and its moments

Let us consider q independent samples taken from the p-variate normal populations $N_p(\underline{\mu}_j\,,\Sigma_j)$, $j=1,\dots,q$. Let the j-th sample have dimension N_j $(j=1,\dots,q)$. We are interested in testing the null hypothesis

$$H_0:\ \Sigma_1=\Sigma_2=\ldots=\Sigma_q=\begin{pmatrix}\Delta & 0 & \ldots & 0\\ 0 & \Delta & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \Delta\end{pmatrix}\Big(=I_k\otimes\Delta\Big)\,,\quad (\Delta\text{ unspecified}) \tag{1}$$

where the k matrices Δ are $p^*\times p^*$, with $p=kp^*$. The null hypothesis in (1) may be decomposed into a sequence of three null hypotheses, more precisely,

$$H_0=H_{0c|(0b|0a)}\circ H_{0b|0a}\circ H_{0a} \tag{2}$$

where

$$H_{0a}:\ \Sigma_1=\Sigma_2=\ldots=\Sigma_q(=\Sigma)\,,\quad (\ \Sigma\text{ unspecified}) \tag{3}$$

is the null hypothesis for testing the equality of q covariance matrices of dimension $p\times p$, with

$$\Sigma=\begin{pmatrix}\Sigma_{11} & \Sigma_{12} & \ldots & \Sigma_{1k}\\ \Sigma_{21} & \Sigma_{22} & \ldots & \Sigma_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ \Sigma_{k1} & \Sigma_{k2} & \ldots & \Sigma_{kk}\end{pmatrix}, \tag{4}$$

then

$$\begin{aligned}H_{0b|0a}:\quad & \Sigma_{ij}=0\ \text{ for }\ i\neq j\,,\quad (i,j=1,\ldots,k)\\ & \text{assuming that }\ \Sigma_1=\Sigma_2=\ldots=\Sigma_q(=\Sigma)\end{aligned} \tag{5}$$

is the null hypothesis for testing the independence of k groups of variables and

$$\begin{aligned}H_{0c|(0b|0a)}:\quad & \Sigma_{11}=\Sigma_{22}=\ldots=\Sigma_{kk}(=\Delta)\quad,\quad (\Delta\text{ unspecified})\\ & \text{assuming }\ H_{0a}\ \text{ and }\ H_{0b|0a}\end{aligned} \tag{6}$$

is the null hypothesis for testing the equality of k covariance matrices with dimension $p^*\times p^*$. Using this decomposition we have that the modified l.r.t. statistic to test (1) is the product of the l.r.t. statistics used to test (3), (5) and (6) (see Lemma 10.3.1 of Anderson (2003)). Thus, the modified l.r.t. statistic to test H_0 in (1) is

$$\lambda^* = \lambda^*_{c|(b|a)}\lambda^*_{b|a}\lambda^*_a \tag{7}$$

$$= \underbrace{\frac{(kn^*)^{kn^*p^*/2}}{\prod_{j=1}^{k}(n^*)^{p^*n^*/2}}\frac{\prod_{j=1}^{k}|A_{jj}|^{n^*/2}}{|A^*|^{kn^*/2}}}_{\lambda^*_{c|(b|a)}}\ \underbrace{\frac{|A|^{n^*/2}}{\prod_{j=1}^{k}|A_{jj}|^{n^*/2}}}_{\lambda^*_{b|a}}\ \underbrace{\frac{(n^*)^{n^*p/2}}{\prod_{j=1}^{q}(n_j)^{pn_j/2}}\frac{\prod_{j=1}^{q}|A_j|^{n_j/2}}{|A|^{n^*/2}}}_{\lambda^*_a} \tag{8}$$

$$= \frac{(kn^*)^{n^*p/2}}{\prod_{j=1}^{q}(n_j)^{pn_j/2}}\frac{\prod_{j=1}^{q}|A_j|^{n_j/2}}{|A^*|^{kn^*/2}}\ , \tag{9}$$

where λ_a^*, $\lambda_{b|a}^*$ and $\lambda_{c|(b|a)}^*$ are the modified l.r.t. statistics to test respectively the null hypotheses in (3), (5) and (6) (see Secs. 10.2 and 9.2 of Anderson (2003) and Secs. 8.2 and 11.2 of Muirhead (1982)), and where A_j is the matrix of corrected sums of squares and products formed from the j-th sample, $A = A_1 + \ldots + A_q$, $n_j = N_j - 1$ is the number of degrees of freedom of the Wishart distribution of A_j and $n^* = n_1 + \ldots + n_j$ is the number of degrees of freedom of the Wishart distribution of A and A_{jj} is the j-th diagonal block of order p^* of A $(j = 1, \ldots, k)$; and also the number of degrees of freedom of the Wishart distribution of each A_{jj} and where kn^* is the number of degrees of freedom of the Wishart distribution of $A^* = A_{11} + A_{22} + \ldots + A_{kk}$.

From the expressions for the h-th moment of each of the statistics λ_a^*, $\lambda_{c|(b|a)}^*$, $\lambda_{b|a}^*$ and given the independence of these three statistics under H_0 in (2) (see Appendix A for further details), the h-th null moment of λ^* is

$$\begin{aligned}
E\left[(\lambda^*)^h\right] &= E\left[\left(\lambda_{c|(b|a)}^*\right)^h\right] \times E\left[\left(\lambda_{b|a}^*\right)^h\right] \times E\left[(\lambda_a^*)^h\right] \\
&= \underbrace{\frac{(kn^*)^{kn^*p^*h/2}}{\prod_{j=1}^{k}(n^*)^{p^*n^*h/2}} \frac{\Gamma_{p^*}\left(\frac{kn^*}{2}\right)}{\Gamma_{p^*}\left(\frac{kn^*}{2}(1+h)\right)} \prod_{j=1}^{k} \frac{\Gamma_{p^*}\left(\frac{n^*}{2}(1+h)\right)}{\Gamma_{p^*}\left(\frac{n^*}{2}\right)}}_{E\left[\left(\lambda_{c|(b|a)}^*\right)^h\right]} \\
&\quad \times \underbrace{\frac{\Gamma_p(\frac{1}{2}n^* + \frac{1}{2}hn^*)}{\Gamma_p(\frac{1}{2}n^*)} \prod_{i=1}^{k} \frac{\Gamma_{p^*}\left(\frac{n^*}{2}\right)}{\Gamma_{p^*}\left(\frac{n^*}{2}(1+h)\right)}}_{E\left[\left(\lambda_{b|a}^*\right)^h\right]} \\
&\quad \times \underbrace{\frac{(n^*)^{n^*ph/2}}{\prod_{j=1}^{q} n_j^{pn_jh/2}} \frac{\Gamma_p\left(\frac{n^*}{2}\right)}{\Gamma_p\left(\frac{n^*}{2}(1+h)\right)} \prod_{j=1}^{q} \frac{\Gamma_p\left(\frac{n_j}{2}(1+h)\right)}{\Gamma_p\left(\frac{n_j}{2}\right)}}_{E\left[(\lambda_a^*)^h\right]} \qquad (10) \\
&= \frac{(kn^*)^{n^*ph/2}}{\prod_{j=1}^{q} n_j^{pn_jh/2}} \frac{\Gamma_{p^*}\left(\frac{kn^*}{2}\right)}{\Gamma_{p^*}\left(\frac{kn^*}{2}(1+h)\right)} \prod_{j=1}^{q} \frac{\Gamma_p\left(\frac{n_j}{2}(1+h)\right)}{\Gamma_p\left(\frac{n_j}{2}\right)} \qquad (11) \\
&\qquad\qquad \left(h > \frac{p-1}{\min(n_j)} - 1\right),
\end{aligned}$$

where $\Gamma_p(\cdot)$ represents the p-multivariate Gamma function (see Anderson (2003)).

3. Particular cases

As particular cases of the MS-BM sphericity test we have, for $q = 1$ the one-sample block-matrix (OS-BM) sphericity test, for $q = 1$ and $p^* = 1$ the usual (one-sample (OS)) sphericity

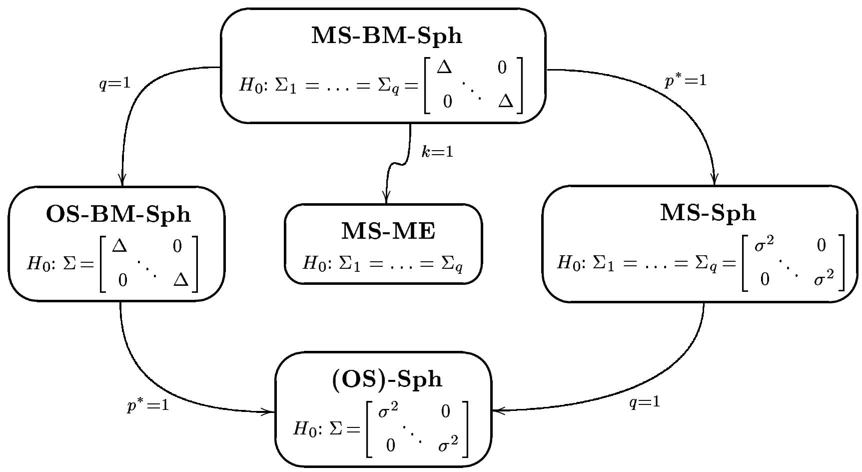

Fig. 1. Particular cases of the multi-sample block-matrix sphericity test
BM: block-matrix; **MS**: multi-sample; **OS**: one-sample;
ME: matrix equality; **Sph**: sphericity

test, for $k = 1$ the equality of several covariance matrices (multi-sample matrix equality (MS-ME)) test and for $p^* = 1$ the multi-sample (MS) sphericity test, which for $q = 1$ turns into the (one-sample) sphericity test. These particular cases of the MS-BM sphericity test as well as their relations may be better seen in Figure 1.

Chao and Gupta (1991) derived the l.r.t. criterion to test $H_{0a|(0b|0a)} \circ H_{0b|0a}$, assuming H_{0a} true. We should stress that they do not test H_{0a}. They rather assume that the covariance matrices, $\Sigma_1, \ldots, \Sigma_q$, are equal and then they use a pooled sample covariance matrix to perform the other two tests. This test is very similar to the particular case of the MS-BM test when we consider $q = 1$. The approximation presented by Chao and Gupta (1991), obtained using Box (1949) method, may thus be used as an asymptotic approximation for the modified l.r.t. statistic of the OS-BM sphericity test. In our numerical studies we will compare their approximation with the near-exact approximations developed in this paper. This particular case of the MS-BM sphericity test was also studied by Cardeño and Nagar (2001). In their paper they obtain the exact null distribution of the l.r.t. statistic for $k = 2$, using Meijer G-functions, what renders quantile computations too heavy even for small values of p^*, reinforcing the need for good manageable approximations.

Moschopoulos (1988) develops an asymptotic approximation for the MS sphericity test statistic by applying Box (1949) method. However, the author says that he was not able to assess the accuracy of his approximation because exact quantiles for the distribution of the statistic are not available. Nagar and Sánchez (2004) present exact values for the quantiles for the MS sphericity test statistic only for the bivariate case, that is for $p = 2$.

In section 6 we will use a measure of proximity between distributions, based on the distance between c.f.'s, to assess the performance of the asymptotic distributions of Moschopoulos (1988), Nagar and Sánchez (2004) and our near-exact distributions. In that section we will also compute near-exact quantiles for the MS sphericity test in the bivariate case, to compare these with the exact quantiles of Nagar and Sánchez (2004). Although we do not compute any other near-exact quantiles, we should stress that we can also obtain near-exact quantiles for the MS-BM sphericity test statistic or any of its particular cases, for any value of p.

Near-exact distributions for the usual (OS) sphericity test statistic are available in Marques and Coelho (2008a) where the near-exact approximations are compared with Box (1949) approximation and also with the sadlle-point approximations presented by Butler et al. (1993). Near-exact distributions are also available for the l.r.t. statistic of the MS-ME test in Coelho and Marques (2007).

4. Factorizations of the characteristic functions

4.1. The factorization of the caracteristic function of $W = -\log\lambda^*$

Since in (11) the Gamma functions are defined for any strictly complex h we may write the c.f. of $W = -\log\lambda^*$ as

$$\begin{aligned}\Phi_W(t) &= E\left[e^{iWt}\right] = E\left[(\lambda^*)^{-it}\right] \\ &= \frac{(kn^*)^{-n^*pit/2}}{\prod_{j=1}^{q} n_j^{-pn_jit/2}} \frac{\Gamma_{p^*}\left(\frac{kn^*}{2}\right)}{\Gamma_{p^*}\left(\frac{kn^*}{2}(1-it)\right)} \prod_{j=1}^{q} \frac{\Gamma_p\left(\frac{n_j}{2}(1-it)\right)}{\Gamma_p\left(\frac{n_j}{2}\right)} \quad . \end{aligned} \tag{12}$$

However, in order to obtain near-exact distributions for W we will use the expression for the caracteristic function of W induced by the decomposition in (10). Thus, for $N_j = N$ and $n_j = n = N-1$ ($j = 1, \ldots, q$) with $n^* = nq$, the caracteristic function of W is also given by

$$\Phi_W(t) = \underbrace{\frac{(n^*)^{-n^*pit/2}}{\prod_{j=1}^{q} n^{-pnit/2}} \frac{\Gamma_p\left(\frac{n^*}{2}\right)}{\Gamma_p\left(\frac{n^*}{2}(1-it)\right)} \prod_{j=1}^{q} \frac{\Gamma_p\left(\frac{n}{2}(1-it)\right)}{\Gamma_p\left(\frac{n}{2}\right)}}_{\Phi_{W_a}(t)} \tag{13}$$

$$\times \underbrace{\frac{\Gamma_p(\frac{1}{2}n^* - \frac{1}{2}itn^*)}{\Gamma_p(\frac{1}{2}n^*)} \prod_{i=1}^{k} \frac{\Gamma_{p^*}\left(\frac{n^*}{2}\right)}{\Gamma_{p^*}\left(\frac{n^*}{2}(1-it)\right)}}_{\Phi_{W_{b|a}}(t)} \tag{14}$$

$$\times \underbrace{\frac{(kn^*)^{-kn^*p^*it/2}}{\prod_{j=1}^{k} (n^*)^{-p^*n^*it/2}} \frac{\Gamma_{p^*}\left(\frac{kn^*}{2}\right)}{\Gamma_{p^*}\left(\frac{kn^*}{2}(1-it)\right)} \prod_{j=1}^{k} \frac{\Gamma_{p^*}\left(\frac{n^*}{2}(1-it)\right)}{\Gamma_{p^*}\left(\frac{n^*}{2}\right)}}_{\Phi_{W_{c|(b|a)}}(t)} \tag{15}$$

where $\Phi_{W_a}(t)$, $\Phi_{W_{b|a}}(t)$ and $\Phi_{W_{c|(b|a)}}(t)$ are respectively the caracteristic functions of $W_a = -\log\lambda_a^*$, $W_{b|a} = -\log\lambda_{b|a}^*$ and $W_{c|(b|a)} = -\log\lambda_{c|(b|a)}^*$ in expression (8).

4.1.1. The characteristic functions of $W_a = -\log\lambda_a^*$ and $W_{c|(b|a)} = -\log\lambda_{c|(b|a)}^*$

Coelho and Marques (2007) have shown that $\Phi_{W_a}(t)$ in (13) can be factorized in the following way

$$\Phi_{W_a}(t) = \Phi_{W_{a,1}}(t) \times \Phi_{W_{a,2}}(t)$$

where

$$\Phi_{W_{a,1}}(t) = \prod_{j=1}^{p-1} \left(\frac{n-j}{n}\right)^{r_{1,j}} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-r_{1,j}} \tag{16}$$

is the c.f. of the sum of $p-1$ independent Gamma r.v.'s, that is a Generalized Integer Gamma (GIG) distribution of depth $p-1$ (see Coelho (1998) and Appendix B for further details on the GIG distribution) with integer shape parameters $r_{1,j}$ given by (51) through (53) in Appendix C, and

$$\begin{aligned} \Phi_{W_{a,2}}(t) \quad = \quad & \prod_{j=1}^{\lfloor p/2 \rfloor} \prod_{i=1}^{q} \frac{\Gamma(a_{1;j}+b_{1;ij})}{\Gamma(a_{1;j}+b^*_{1;ij})} \frac{\Gamma(a_{1;j}+b^*_{1;ij}-n\mathrm{i}t)}{\Gamma(a_{1;j}+b_{1;ij}-n\mathrm{i}t)} \\ & \times \left(\prod_{i=1}^{q} \frac{\Gamma(a_{1;p}+b_{1;ip})}{\Gamma(a_{1;p}+b^*_{1;ip})} \frac{\Gamma\left(a_{1;p}+b^*_{1;ip}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(a_{1;p}+b_{1;ip}-\frac{n}{2}\mathrm{i}t\right)}\right)^{p \perp\!\!\!\perp 2} \end{aligned} \tag{17}$$

with

$$a_{1;j} = n+1-2j\ , \qquad b_{1;ij} = 2j-1+\frac{i-2j}{q}\ , \qquad b^*_{1;ij} = \lfloor b_{1;ij} \rfloor \tag{18}$$

and

$$a_{1;p} = \frac{n+1-p}{2}\ , \qquad b_{1;ip} = \frac{pq-q-p+2i-1}{2q}\ , \qquad b^*_{1;ip} = \lfloor b_{1;ip} \rfloor\ , \tag{19}$$

is the c.f. of the sum of $\lfloor p/2 \rfloor \times q$ independent Logbeta r.v.'s multiplied by n and other $q \times (p \perp\!\!\!\perp 2)$ independent Logbeta r.v.'s multiplied by $n/2$, where $p \perp\!\!\!\perp 2$ is the remainder of the integer division of p by 2.

We may also obtain a factorization for the c.f. of $-\log \lambda^*_{c|(b|a)}$ using the results in Coelho and Marques (2007). The c.f. $\Phi_{W_{c|(b|a)}}(t)$ in (15) may be written as

$$\Phi_{W_{c|(b|a)}}(t) = \Phi_{W_{c|(b|a),1}}(t) \times \Phi_{W_{c|(b|a),2}}(t)$$

where

$$\begin{aligned} \Phi_{W_{c|(b|a),1}}(t) \quad & = \quad \prod_{j=1}^{p^*-1} \left(\frac{nq-j}{nq}\right)^{r_{2,j}} \left(\frac{nq-j}{nq} - \mathrm{i}t\right)^{-r_{2,j}} \\ & = \quad \prod_{j=1}^{p^*-1} \left(\frac{n-\frac{j}{q}}{n}\right)^{r_{2,j}} \left(\frac{n-\frac{j}{q}}{n} - \mathrm{i}t\right)^{-r_{2,j}} \end{aligned} \tag{20}$$

is the c.f. of the sum of p^*-1 independent Gamma r.v.'s, that is a GIG distribution of depth p^*-1 with integers shape parameters $r_{2,j}$ obtained from the $r_{1,j}$ defined in expressions (51) through (53) in Appendix C, replacing q by k, p by p^* and n by nq, and yet

$$\begin{aligned} \Phi_{W_{c|(b|a),2}}(t) \quad = \quad & \prod_{j=1}^{\lfloor p^*/2 \rfloor} \prod_{i=1}^{k} \frac{\Gamma(a_{2;j}+b_{2;ij})}{\Gamma(a_{2;j}+b^*_{2;ij})} \frac{\Gamma(a_{2;j}+b^*_{2;ij}-nq\mathrm{i}t)}{\Gamma(a_{2;j}+b_{2;ij}-nq\mathrm{i}t)} \\ & \times \left(\prod_{i=1}^{k} \frac{\Gamma(a_{2;p}+b_{2;ip})}{\Gamma(a_{2;p}+b^*_{2;ip})} \frac{\Gamma\left(a_{2;p}+b^*_{2;ip}-\frac{nq}{2}\mathrm{i}t\right)}{\Gamma\left(a_{2;p}+b_{2;ip}-\frac{nq}{2}\mathrm{i}t\right)}\right)^{p^* \perp\!\!\!\perp 2} \end{aligned} \tag{21}$$

with

$$a_{2;j} = nq + 1 - 2j \ , \qquad b_{2;ij} = 2j - 1 + \frac{i-2j}{k} \ , \qquad b^*_{2;ij} = \lfloor b_{2;ij} \rfloor \ , \tag{22}$$

and

$$a_{2;p} = \frac{nq+1-p^*}{2} \ , \qquad b_{2;ip} = \frac{p^*k - k - p^* + 2i - 1}{2k} \ , \qquad b^*_{2;ip} = \lfloor b_{2;ip} \rfloor \ , \tag{23}$$

is the c.f. of the sum of $\lfloor p^*/2 \rfloor \times k$ independent Logbeta r.v.'s multiplied by nq and other $k \times (p^* \perp\!\!\!\perp 2)$ independent Logbeta r.v.'s multiplied by $nq/2$.

4.1.2. The characteristic function of $W_{b|a} = -\log \lambda^*_{b|a}$

Coelho (2004) has shown that $\Phi_{W_{b|a}}(t)$ in (14) may be given by

$$\begin{aligned}
\Phi_{W_{b|a}}(t) &= \prod_{j=1}^{p-2} \left(\frac{nq-p+j}{2} \right)^{z_j} \times \left(\frac{nq-p+j}{2} - \mathrm{i}t\frac{nq}{2} \right)^{-z_j} \\
&\quad \times \left\{ \frac{\Gamma\left(\frac{nq}{2}\right)\Gamma\left(\frac{nq}{2}-\frac{1}{2}-\frac{nq}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{nq}{2}-\frac{nq}{2}\mathrm{i}t\right)\Gamma\left(\frac{nq}{2}-\frac{1}{2}\right)} \right\}^{\lfloor m^*/2 \rfloor} \\
&= \underbrace{\prod_{j=2}^{p-1} \left(\frac{n-\frac{j}{q}}{n} \right)^{z_{p-j}} \times \left(\frac{n-\frac{j}{q}}{n} - \mathrm{i}t \right)^{-z_{p-j}}}_{\Phi_{W_{b|a,1}}(t)} \qquad (24) \\
&\quad \times \underbrace{\left\{ \frac{\Gamma\left(\frac{nq}{2}\right)\Gamma\left(\frac{nq}{2}-\frac{1}{2}-\frac{nq}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{nq}{2}-\frac{nq}{2}\mathrm{i}t\right)\Gamma\left(\frac{nq}{2}-\frac{1}{2}\right)} \right\}^{\lfloor m^*/2 \rfloor}}_{\Phi_{W_{b|a,2}}(t)} \qquad (25)
\end{aligned}$$

with

$$m^* = \begin{cases} 0 & p^* \ \mathit{even} \\ k & p^* \ \mathit{odd} \ , \end{cases} \tag{26}$$

and integer shape parameters z_j equal to the shape parameters r^*_j given in expression (33) of Coelho (2004). The c.f. $\Phi_{W_{b|a,1}}(t)$ in (24) corresponds to the sum of $p-2$ independent r.v.'s with Gamma distribution, that is a GIG distribution of depth $p-2$ and integers shape parameters z_j, and $\Phi_{W_{b|a,2}}(t)$ in (25) corresponds to the sum of $\lfloor m^*/2 \rfloor$ independent r.v.'s with LogBeta distribution multiplied by $nq/2$.

4.2. A convenient factorization of the c.f. of $W = -\log \lambda^*$

Towards the use of the procedure outlined in Coelho and Marques (2008) and using the previous factorizations of the c.f.'s in subsections 4.1.1 and 4.1.2, we may rewrite the c.f. of $W = -\log \lambda^*$ as expressed in Theorem 2, where we show that the distribution of W may be seen as the sum of a r.v. with a Generalize Integer Gamma (GIG) distribution of depth $2p - \left\lfloor \frac{p-1}{q} \right\rfloor - 2$ with a number of independent r.v.'s with Logbeta distributions multiplied by different parameters.

THEOREM 2. *The c.f. of $W = -\log \lambda^*$ may be written as*

$$\begin{aligned}
\Phi_W(t) &= \underbrace{\prod_{j=1}^{p-1} \left(\frac{n-j}{n}\right)^{r_j^{++}} \left(\frac{n-j}{n} - \mathrm{i}t\right)^{-r_j^{++}}}_{\Phi_1(t)} \\
&\quad \times \underbrace{\prod_{\substack{j=1 \\ j\neq q,\dots,\alpha q}}^{p-1} \left(\frac{n-\frac{j}{q}}{n}\right)^{r_j^{+}} \left(\frac{n-\frac{j}{q}}{n} - \mathrm{i}t\right)^{-r_j^{+}}}_{\Phi_2(t)} \qquad (27)\\
&\quad \times \underbrace{\Phi_{W_{a,2}}(t) \times \Phi_{W_{b|a,2}}(t) \times \Phi_{W_{c|(b|a),2}}(t)}_{\Phi_3(t)}
\end{aligned}$$

where $\alpha = \left\lfloor \frac{p-1}{q} \right\rfloor$,

$$r_j^{+} = \begin{cases} r_{2,j} & j = 1 \\ r_{2,j} + z_{p-j} & j = 2, \dots, p^* - 1 \\ z_{p-j} & j = p^*, \dots, p-1 \end{cases} \qquad (28)$$

and

$$r_j^{++} = \begin{cases} r_{1,j} + r^{+}_{q\times j} & j = 1, \dots, \alpha \\ r_{1,j} & j = \alpha+1, \dots, p-1 \ , \end{cases} \qquad (29)$$

with $r_{1,j}$ given by expressions (51) *through* (53) *in Appendix C, $r_{2,j}$ also given by the same expressions by replacing q by k, p by p^* and n by nq and z_j given by expression (33) in Coelho (2004), and where $\Phi_{W_{a,2}}(t)$, $\Phi_{W_{b|a,2}}(t)$ and $\Phi_{W_{c|(b|a),2}}(t)$ are given respectively by* (17)*,* (25) *and* (21) *above.*

PROOF. We only have to group together in $\Phi_1(t)$ all the Gamma r.v.'s in $\Phi_{W_{a,1}}(t)$ in (16) and all the Gamma r.v.'s in $\Phi_{W_{b|a,1}}(t)$ and $\Phi_{W_{c|(b|a),1}}(t)$ in (24) and (20) whose rate parameters have integer values for j/q and then group together in $\Phi_2(t)$ all the remaining Gamma r.v.'s in $\Phi_{W_{b|a,1}}(t)$ and $\Phi_{W_{c|(b|a),1}}(t)$ whose rate parameters have non-integer values of j/q. □

The c.f. given by the product $\Phi_1(t) \times \Phi_2(t)$ in (27) corresponds to the sum of

$$(p-1) + (p-1-\alpha) = 2p - \alpha - 2$$

independent r.v.'s with Gamma distribution, that is, the c.f. of a GIG distribution with depth $2p-\alpha-2$ and the c.f. $\Phi_3(t)$ in (27) corresponds to the sum of

$$\lfloor p/2 \rfloor \times q + q \times (p \perp\!\!\!\perp 2) + \lfloor m^*/2 \rfloor + \lfloor p^*/2 \rfloor \times k + k \times (p^* \perp\!\!\!\perp 2)$$

independent Logbeta distributions multiplied by different parameters.

We may stress that although the above result works in all cases, when p^* is even we have $m^* = 0$ and then we have the exact distribution for the Wilks Lambda statistic $\lambda^*_{b|a}$, whose logarithm has a GIG distribution of depth $p-2$, this result may be simplified as shown in the next Corollary.

COROLLARY 2.1 *When p^* is even the c.f. $\Phi_W(t)$ is given by*

$$\begin{aligned}\Phi_W(t) &= \underbrace{\prod_{j=1}^{p-1}\left(\frac{n-j}{n}\right)^{r_j^{++}}\left(\frac{n-j}{n}-\mathrm{i}t\right)^{-r_j^{++}}}_{\Phi_1(t)} \\ &\times \underbrace{\prod_{\substack{j=1\\ j\neq q,\ldots,\alpha q}}^{p-1}\left(\frac{n-\frac{j}{q}}{n}\right)^{r_j^{+}}\left(\frac{n-\frac{j}{q}}{n}-\mathrm{it}\right)^{-r_j^{+}}}_{\Phi_2(t)} \\ &\times \underbrace{\Phi_{W_{a,2}}(t)\ \times\ \Phi_{W_{c|(b|a),2}}(t)}_{\Phi_3(t)}\end{aligned} \tag{30}$$

with r_j^{++} given in (29)*, r_j^{+} in* (28) *and with $\Phi_{W_{a,2}}(t)$ and $\Phi_{W_{c|(b|a),2}}(t)$ given in* (17) *and* (21) *respectively.*

PROOF. We just have to note that when p^* is even $m^* = 0$ and $\Phi_{W_{b|a,2}}(t)$ in (25) vanishes. □

5. Near-exact distributions for W and λ^*

The near-exact distributions we will be dealing with in this paper will have c.f.'s of the form

$$\underbrace{\Phi_1(t)\ \Phi_2(t)}_{\text{GIG distribution}}\ \Phi_3^*(t)\,, \tag{31}$$

where $\Phi_1(t)$ and $\Phi_2(t)$ are the same as in (27) or (30) above, while $\Phi_3^*(t)$ may be either the c.f. of a single Gamma distribution or of a mixture of two or three Gamma distributions, depending on the number of exact moments we want to match. The c.f. $\Phi_3^*(t)$ will indeed have, accordingly, the same 2, 4 or 6 first derivatives (with respect to t at $t=0$) as the part of the exact c.f. of W that will be replaced, that is, $\Phi_3(t)$ in (27) or (30). In other words, we will have

$$\frac{d^j}{dt^j}\,\Phi_3^*(t)\bigg|_{t=0} = \frac{d^j}{dt^j}\,\Phi_3(t)\bigg|_{t=0}\,, \qquad j=1,\ldots,h \tag{32}$$

for $h=2,4$ or 6, according to the case of $\Phi_3^*(t)$ being the c.f. of a single Gamma distribution, or the c.f. of a mixture of 2 or 3 Gamma distributions with the same rate parameter, that is,

$$\Phi_3^*(t) = \sum_{k=1}^{h/2} p_k\,\lambda^{s_k}\,(\lambda-\mathrm{i}t)^{-s_k}\,, \tag{33}$$

with weights $p_k > 0$ $(k=1,\ldots,h/2)$ and $\sum_{k=1}^{h/2} p_k = 1$.

While if $\Phi_3^*(t)$ is the c.f. of a single Gamma distribution, equating the two first derivatives of $\Phi_3(t)$ at $t=0$, there is a simple analytical solution for the problem of equating moments, with the rate and shape parameters of $\Phi_3^*(t)$ being given by

$$\lambda = \frac{m_1}{m_2-m_1^2} \qquad \text{and} \qquad s_1 = \frac{m_1}{m_2-m_1^2}\,,$$

where

$$m_1 = \frac{1}{i}\,\frac{d}{dt}\,\Phi_3(t)\bigg|_{t=0} \qquad \text{and} \qquad m_2 = -\,\frac{d^2}{dt^2}\,\Phi_3(t)\bigg|_{t=0}\,,$$

if $\Phi_3^*(t)$ is the c.f. of a mixture of two Gamma distributions it is possible to prove (through quite long and tedious calculations) that there is always one unique analytic real solution, or rather, a pair of conjugate real solutions, with the values for the two shape parameters and corresponding weights interchanged. If $\Phi_3^*(t)$ is the c.f. of a mixture of three Gamma distributions, it is believed that there is also always one only real solution with all positive parameters, or rather, a six-tuple of conjugate solutions, although this is not easy to prove analytically. Anyway, for the cases where $\Phi_3^*(t)$ is the c.f. of a mixture of 2 or 3 Gamma distributions we advocate the numerical solution of the system of equations (32) (respectively for $h = 4$ and $h = 6$).

As already remarked in Marques and Coelho (2008a) and Coelho and Marques (2007), the replacement of $\Phi_3(t)$ by $\Phi_3^*(t)$, that is, the replacement of a sum of independent Logbeta random variables (multiplied by a constant) by a single Gamma distribution or a mixture of two or three Gamma distributions, matching the first 2, 4 or 6 exact moments is a much adequate decision, since, as it is shown in Coelho et all. (2006), a single Logbeta distribution may be represented under the form of an infinite mixture of GIG distributions, and, as such, a sum of independent Logbeta random variables may thus be represented under the form of an infinite mixture of sums of GIG distributions, which are themselves GIG distributions, while, on the other hand, the GIG distribution may itself be seen as a mixture of Gamma distributions Coelho (2007).

This amounts to be able to write the near-exact c.f. of the logarithm of the l.r.t. statistic for the MS-BM sphericity test in the form in (31) where $\Phi_3^*(t)$ is either the c.f. of a Gamma distribution or the c.f. of a mixture of 2 or 3 Gamma distributions, being thus the near-exact distributions obtained in this way, correspondingly a Generalized Near-Integer Gamma (GNIG) distribution (see Coelho (2004) and Appendix B) of depth $2(p-1)-\alpha+1$ with $\alpha = \left\lfloor \frac{p-1}{q} \right\rfloor$, or a mixture of two or three GNIG distributions of the same depth, which have very manageable expressions, allowing this way for an easy computation of very accurate near-exact quantiles.

Theorem 3. *If we replace $\Phi_3(t)$ in (27) or (30) by $\Phi_3^*(t)$ in (33) we obtain as near-exact distributions for W a GNIG distribution or a mixture of two or three GNIG distributions of depth $2(p-1)-\alpha+1$ with $\alpha = \left\lfloor \frac{p-1}{q} \right\rfloor$ and for $h = 2, 4, 6$ with p.d.f. (using the notation in Appendix B)*

$$\sum_{\nu=1}^{h/2} p_\nu\, f^{\,GNIG}\Bigg(w\big|r_1^{++},\dots,r_{p-1}^{++},\underbrace{r_1^{+},\dots,r_{p-1}^{+}}_{\text{except } r_j^+ \text{ for } j=q,\dots,\alpha q},s_\nu; \frac{n-1}{n},\dots,\frac{n-p+1}{n},\underbrace{\frac{n-1/q}{n},\dots,\frac{n-(p-1)/q}{n}}_{\text{except } \frac{n-j/q}{n} \text{ for } j=q,\dots,\alpha q},\lambda\Bigg) \qquad (34)$$

and c.d.f.

$$\sum_{\nu=1}^{h/2} p_\nu \, F^{\,GNIG}\Big(w|r_1^{++},\dots,r_{p-1}^{++},\underbrace{r_1^{+},\dots,r_{p-1}^{+}}_{\text{except } r_j^+ \text{ for } j=q,\dots,\alpha q},s_\nu; \frac{n-1}{n},\dots,\frac{n-p+1}{n},\underbrace{\frac{n-1/q}{n},\dots,\frac{n-(p-1)/q}{n}}_{\text{except } \frac{n-j/q}{n} \text{ for } j=q,\dots,\alpha q},\lambda\Big) \qquad (35)$$

where r_j^+ *and* r_j^{++} $(j=1,\dots,p-1)$ *are given respectively by* (28) *and* (29), *and where for* $h=2$

$$\lambda=\frac{m_1}{m_2-m_1^2} \qquad \text{and} \qquad s_1=\frac{m_1^2}{m_2-m_1^2} \qquad (36)$$

with

$$m_j=\mathrm{i}^{-j}\,\frac{\partial^j}{\partial t^j}\,\Phi_3(t)\Big|_{t=0}\,, \qquad j=1,2\,,$$

and for $h=4$ *or* $h=6$ *(according to the case of* $\Phi_3^*(t)$ *being the c.f. of a single Gamma distribution, or the c.f. of a mixture of 2 or 3 Gamma distributions with the same rate parameter) the values of* p_ν, s_ν *and* λ *are obtained from the numerical solution of the system of equations in* (32), *that is*

$$\frac{d^j}{dt^j}\,\Phi_3^*(t)\Big|_{t=0}=\frac{d^j}{dt^j}\,\Phi_3(t)\Big|_{t=0}\,, \qquad j=1,\dots,h$$

with

$$p_{h/2}=1-\sum_{k=1}^{h/2-1}p_k\,.$$

Proof. In this proof we will consider only the case of $h=6$, since the cases $h=2$ and $h=4$ are derived in a similar way.

If in the c.f. of W in (27) we replace $\Phi_3(t)$ by

$$\Phi_3^*(t)=\sum_{k=1}^{3}p_k\,\lambda^{s_k}\,(\lambda-\mathrm{i}t)^{-s_k}\,,$$

we obtain

$$\begin{aligned}\Phi_W(t)\;&\approx\;\Phi_1(t)\times\Phi_2(t)\times\underbrace{\sum_{k=1}^{3}p_k\,\lambda^{s_k}\,(\lambda-\mathrm{i}t)^{-s_k}}_{\Phi_3^*(t)}\\&\approx\;\sum_{k=1}^{3}p_k\,\underbrace{\underbrace{\Phi_1(t)\times\Phi_2(t)}_{\text{GIG distribution}}\times\underbrace{\lambda^{s_k}\,(\lambda-\mathrm{i}t)^{-s_k}}_{\text{Gamma distribution}}}_{\text{GNIG distribution}}\end{aligned}$$

that is the c.f. of the mixture of three GNIG distributions of depth $2(p-1)-\alpha+1$ with $\alpha = \left\lfloor \frac{p-1}{q} \right\rfloor$ with c.d.f. given by (35). The parameters p_ν, s_ν and λ are defined in such a way that

$$\frac{d^j}{dt^j}\,\Phi_3^*(t)\bigg|_{t=0} = \frac{d^j}{dt^j}\,\Phi_3(t)\bigg|_{t=0}\,, \qquad j=1,\ldots,6\,,$$

what gives rise to a near-exact distribution that matches the first six exact moments of W. □

Corolally 3.1 *Near-exact p.d.f.'s and c.d.f.'s for the l.r.t. statistic λ^* in (9) may be obtained (using the notation in Appendix B) in the form*

$$f_{\lambda^*}(\ell) \approx \sum_{\nu=1}^{h/2} p_\nu\, f^{\,GNIG}\Big(-\log\ell \,|\, r_1^{++},\ldots,r_{p-1}^{++},\; \underbrace{r_1^{+},\ldots,r_{p-1}^{+}}_{\text{except } r_j^{+}\;,\; j=q,\ldots,\alpha q},\; s_\nu;\; \frac{n-1}{n},\ldots,\frac{n-p+1}{n},\underbrace{\frac{n-1/q}{n},\ldots,\frac{n-(p-1)/q}{n}}_{\text{except } \frac{n-j/q}{n}\;,\; j=q,\ldots,\alpha q},\lambda\Big)\frac{1}{\ell} \qquad (37)$$

and

$$F_{\lambda^*}(\ell) \approx 1-\sum_{\nu=1}^{h/2} p_\nu\, F^{\,GNIG}\Big(-\log\ell \,|\, r_1^{++},\ldots,r_{p-1}^{++},\; \underbrace{r_1^{+},\ldots,r_{p-1}^{+}}_{\text{except } r_j^{+}\;,\; j=q,\ldots,\alpha q},\; s_\nu;\; \frac{n-1}{n},\ldots,\frac{n-p+1}{n},\underbrace{\frac{n-1/q}{n},\ldots,\frac{n-(p-1)/q}{n}}_{\text{except } \frac{n-j/q}{n}\;,\; j=q,\ldots,\alpha q},\lambda\Big) \qquad (38)$$

for $h=2$, $h=4$ and $h=6$, where the parameters are the same as in Theorem 3, and $0<\ell<1$ represents the running value of the statistic $\lambda^* = e^{-W}$.

Proof. Since the near-exact distributions in Theorem 3 were developed for the random variable $W = -\log\lambda^*$, in order to obtain the corresponding near-exact distributions for λ^*, we only need to bear in mind the relation

$$F_{\lambda^*}(\ell) = 1 - F_W(-\log\ell)$$

where $F_{\lambda^*}(\cdot)$ is the c.d.f. of λ^* and $F_W(\cdot)$ the c.d.f. of W. □

Some authors use different versions of this statistic. For example, instead of the modified l.r.t. statistic use could have used $(\lambda^*)^{N/n}$. However, we may note that we can easily obtain both the distribution and quantiles of different powers of λ^* from the ones for λ^*.

6. Numerical studies

In order to evaluate the quality of the near-exact approximations developed in this work we will use a measure of proximity between c.f.'s which is also a measure of proximity between c.d.f.'s. This measure is,

$$\Delta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\Phi_W(t) - \Phi(\gamma;t)}{t} \right| \mathrm{d}t\,, \tag{39}$$

where $\Phi_W(t)$ represents the exact c.f. of the negative logarithm of the modified l.r.t. statistic and $\Phi(\gamma;t)$ represents an approximate c.f. for the same statistic, where γ intends to represent any and every parameter in the distribution corresponding to $\Phi(\gamma;t)$. Taking S for the support of W, we have,

$$\max_{w\in S} |F_W(w) - F^*(w)| \,\leq\, \Delta\,, \tag{40}$$

where $F_W(\cdot)$ represents the exact c.d.f. of W and $F^*(\cdot)$ represents the c.d.f. corresponding to $\Phi(\gamma;t)$. We should note that

$$\lim_{\gamma\to\infty} \Delta = 0 \iff W_\gamma \xrightarrow{d} W\,, \tag{41}$$

where γ may represent either the sample size, the number of variables or matrices and blocks involved in the MS-MB sphericity test and where W_γ represents the r.v. with distribution with c.f. $\Phi(\gamma;t)$.

Indeed the relation in (40) may be derived directly from inversion formulas and Δ may be seen as based on the Berry-Esseen upper bound on $|F_Y(y)-F^*(y)|$ (Berry, 1941; Esseen, 1945; Loève, 1977, Chap. VI, Sec. 21; Hwang, 1998) which may, for any $b > 1/(2\pi)$ and any $T > 0$, be written as

$$\max_{w\in S} |F_W(w) - F^*(w)| \,\leq\, b \int_{-T}^{T} \left| \frac{\Phi_W(t) - \Phi(\gamma;t)}{t} \right| dt + C(b)\,\frac{M}{T} \tag{42}$$

where $M = max_{w\in S} f^*(w)$ and $C(b)$ is a positive constant that only depends of b. If in (42) above we take $T \to \infty$ then we will have Δ, since then we may take $b = 1/(2\pi)$. The measure Δ was already used by Grilo and Coelho (2007), Marques and Coelho (2008a), Coelho and Marques (2007) to study the accuracy of near-exact approximations.

In a first stage we intend to assess the performance of the near-exact approximations developed in this paper by computing the values of the measure Δ between the exact distribution of $W = -\log \lambda^*$ and the three proposed near-exact approximations. In the calculations we use the exact c.f. in (12) and the near-exact c.f.'s corresponding to the near-exact distributions in Theorem 3 and given by (31) and (33) for $h = 2, 4$ and 6. We will denote respectively by $GNIG$, $M2GNIG$ and $M3GNIG$ the near-exact distributions corresponding to $h = 2, 4$ and 6 in (33) and Theorem 3.

In Tables 1 through 4 we compute values of Δ for increasing values of p^*, q, n and k, respectively. We may observe that the values of Δ decrease in all cases, that is, all three near-exact distributions show a marked asymptotic behavior not only for increasing sample sizes but also for increasing values of the number of variables, number of blocks and number of samples involved.

Together with these good asymptotic properties near-exact distributions also present very accurate results for small sample sizes.

Table 1. Values of Δ for the near-exact distributions for $W = -\log\lambda^*$

p^*	k	q	n	GNIG	M2GNIG	M3GNIG
3	3	3	11	1.8×10^{-5}	5.1×10^{-8}	2.4×10^{-10}
6	3	3	20	1.4×10^{-6}	6.3×10^{-10}	4.1×10^{-13}
9	3	3	29	6.8×10^{-7}	1.9×10^{-10}	7.7×10^{-14}
15	3	3	47	1.8×10^{-7}	2.0×10^{-11}	3.1×10^{-15}

Table 2. Values of Δ for the near-exact distributions for $W = -\log\lambda^*$

p^*	k	q	n	GNIG	M2GNIG	M3GNIG
3	3	6	11	2.0×10^{-6}	9.3×10^{-10}	5.0×10^{-13}
3	3	9	11	1.2×10^{-6}	4.2×10^{-10}	1.8×10^{-13}
3	3	12	11	9.3×10^{-7}	2.7×10^{-10}	1.1×10^{-13}
3	3	15	11	7.5×10^{-7}	1.9×10^{-10}	6.5×10^{-14}

Table 3. Values of Δ for the near-exact distributions for $W = -\log\lambda^*$

p^*	k	q	n	GNIG	M2GNIG	M3GNIG
5	3	3	17	3.6×10^{-6}	3.3×10^{-9}	4.6×10^{-12}
5	3	3	30	2.2×10^{-6}	1.2×10^{-9}	8.8×10^{-13}
5	3	3	50	8.7×10^{-7}	2.1×10^{-10}	7.9×10^{-14}
5	3	3	100	2.2×10^{-7}	1.7×10^{-11}	4.1×10^{-15}
5	3	3	200	5.7×10^{-8}	1.5×10^{-12}	3.0×10^{-16}

Table 4. Values of Δ for the near-exact distributions for $W = -\log\lambda^*$

p^*	k	q	n	GNIG	M2GNIG	M3GNIG
5	3	3	17	3.6×10^{-6}	3.3×10^{-9}	4.6×10^{-12}
5	4	3	22	1.3×10^{-6}	5.6×10^{-10}	3.5×10^{-13}
5	5	3	27	8.3×10^{-7}	2.6×10^{-10}	1.1×10^{-13}
5	6	3	32	5.0×10^{-7}	1.1×10^{-10}	3.5×10^{-14}
5	7	3	37	3.9×10^{-7}	7.4×10^{-11}	2.0×10^{-14}
5	8	3	42	2.5×10^{-7}	3.5×10^{-11}	7.1×10^{-15}

Table 5. Exact and near-exact α-quantiles of $(\lambda^*)^{1/n}$ for $\alpha = 0.01$

q	n	$exact$	$GNIG$	$M2GNIG$	$M3GNIG$
2	3	0.0276701	0.0276014	0.0276708	0.0276702
	4	0.088825	0.088761	0.088826	0.088825
	5	0.160494	0.160447	0.160494	0.160494
	15	0.586606	0.586603	0.586606	0.586606
	30	0.771972	0.771972	0.771972	0.771972
6	3	0.000455436	0.000457393	0.000455414	0.000455436
	4	0.00531188	0.00531981	0.00531183	0.00531188
	5	0.0187298	0.0187424	0.0187298	0.0187298
	15	0.309048	0.309042	0.309042	0.309048
	30	0.564702	0.564704	0.564702	0.564702

Nagar and Sánchez (2004) present Tables for the quantiles of $(\lambda^*)^{1/n}$ the statistic used for the MS sphericity test, a particular case of the MS-MB sphericity test for $p^* = 1$. The quantiles presented are obtained only for the bivariate case ($p = 2$) and for $q = 2, \ldots, 6$. In Tables 5 and 6 we compute the near-exact quantiles, for the cases $q = 2$ and $q = 6$, using the near-exact distributions developed in this paper. We have only considered the cases $q = 2$ and $q = 6$ since we think these to be sufficient to show the good precision of the new near-exact approximations, however we must remark that we can easily obtain near-exact quantiles for larger values of p and q.

We can observe from Tables 5 through 7 that the near-exact quantiles are very close to the exact quantiles; the number of decimal places equal to the exact ones being at leats four, for the $GNIG$ distribution and increasing for larger values of n. The quantiles provided by the $M3GNIG$ distribution are, except in three cases, equal to the exact ones, but even in these cases they match the first 6 decimal places for the two first cases and the first 8 for the third case.

Moschopoulos (1988) presents, for the case $p^* = 1$, that is, for the MS sphericity test statistic, an asymptotic approximation for the distribution of $-m\log(\lambda^*)^{2/n^*}$, following the method of Box (1949), in the form of a mixture of χ^2 distributions. We may easily derive from this asymptotic approximation an asymptotic approximation for the c.f. of $W = -\log\lambda^*$, for the MS shpericity test statistic, under the form of a mixture of two Gamma distributions that we will denote by Box_1 and that we will use to compare with the near-exact approximations proposed. From Table 8 we may observe that near-exact distributions continue to present an excellent behavior for all values of p, q and n, always presenting smaller values of Δ than the asymptotic distribution of Moschopoulos (1988). The near-exact distributions also show an excellent behavior for small sample sizes. The approximation given by Moschopoulos only shows a slight improvement for increasing values of n.

Chao and Gupta (1991) present an asymptotic approximation for the OS-BM sphericity test, also based on Box's (1949) method as a mixture of χ^2 distributions. Once again, we may easily derive from this asymptotic approximation an asymptotic approximation for the c.f. of $W = -\log\lambda^*$, for the case $q = 1$, under the form of a mixture of two Gamma distributions, which we will denote by Box_2 and that we will use to compare with the near-exact approximations proposed. From Table 9 we may observe that the asymptotic approximation proposed by Chao and Gupta (1991) shows in every case considered larger values for Δ than any of the near-exact approximations proposed in this paper, even when

Table 6. Exact and near-exact α-quantiles of $(\lambda^*)^{1/n}$ for $\alpha = 0.025$

q	n	$exact$	$GNIG$	$M2GNIG$	$M3GNIG$
2	3	0.0475534	0.0475265	0.0475540	0.0475535
	4	0.127788	0.127767	0.127789	0.127788
	5	0.211137	0.211122	0.211137	0.211137
	15	0.635280	0.635279	0.635280	0.635280
	30	0.802407	0.802407	0.802407	0.802407
6	3	0.000965728	0.000967845	0.000965697	0.000965729
	4	0.00883076	0.00883732	0.00883070	0.00883076
	5	0.0275224	0.0275316	0.0275224	0.0275224
	15	0.346065	0.346068	0.346065	0.346065
	30	0.596657	0.596658	0.596657	0.596657

Table 7. Exact and near-exact α-quantiles of $(\lambda^*)^{1/n}$ for $\alpha = 0.025$

q	n	$exact$	$GNIG$	$M2GNIG$	$M3GNIG$
2	3	0.072533	0.072557	0.072533	0.072533
	4	0.169741	0.169756	0.169741	0.169741
	5	0.261570	0.261578	0.261570	0.261570
	15	0.676126	0.676127	0.676126	0.676126
	30	0.827033	0.827033	0.827033	0.827033
6	3	0.00177173	0.00177326	0.00177170	0.00177173
	4	0.0133169	0.0133206	0.0133169	0.0133169
	5	0.0375702	0.0375748	0.0375701	0.0375702
	15	0.379246	0.379248	0.379246	0.379246
	30	0.623836	0.623836	0.623836	0.623836

Table 8. Values of Δ for the near-exact and asymptotic distributions for $W = -\log \lambda^*$, for $p^* = 1$

$k(=p)$	q	n	$GNIG$	$M2GNIG$	$M3GNIG$	Box_1
10	2	12	3.7×10^{-5}	1.9×10^{-7}	1.7×10^{-9}	1.9×10^{0}
10	5	12	1.6×10^{-6}	6.7×10^{-10}	3.2×10^{-13}	3.1×10^{0}
10	7	12	6.8×10^{-7}	1.5×10^{-10}	3.8×10^{-14}	3.7×10^{0}
10	2	50	2.6×10^{-6}	1.7×10^{-9}	2.4×10^{-12}	6.6×10^{-2}
10	2	100	6.3×10^{-7}	1.5×10^{-10}	1.3×10^{-13}	1.5×10^{-2}
15	2	17	2.1×10^{-5}	6.9×10^{-8}	3.8×10^{-10}	3.7×10^{0}
20	2	22	7.4×10^{-6}	1.1×10^{-8}	2.9×10^{-11}	5.8×10^{0}
50	2	52	8.1×10^{-7}	2.5×10^{-10}	1.2×10^{-13}	19.2×10^{0}

Table 9. Values of Δ for the near-exact and asymptotic distributions for $W = -\log\lambda^*$, for $q = 1$

p^*	k	n	$GNIG$	$M2GNIG$	$M3GNIG$	Box_2
5	2	12	8.8×10^{-7}	2.6×10^{-10}	8.2×10^{-13}	6.1×10^{-2}
2	7	16	1.0×10^{-7}	1.5×10^{-11}	4.5×10^{-15}	1.8×10^{-1}
2	9	20	2.7×10^{-8}	1.9×10^{-12}	2.9×10^{-16}	3.1×10^{-1}
4	2	10	3.1×10^{-6}	2.9×10^{-10}	9.1×10^{-13}	3.3×10^{-2}
4	2	50	2.1×10^{-7}	6.9×10^{-13}	1.8×10^{-14}	3.2×10^{-5}
4	2	100	5.5×10^{-8}	4.3×10^{-14}	3.2×10^{-18}	3.3×10^{-6}

large samples are considered.

The asymptotic properties revealed in Tables 6 through 4 by the near-exact aproximations for the MS-MB sphericity test statistic, can also be observed if we consider its particular cases. In these cases we get even smaller values for the measure Δ which reveal that the approximations are even better.

Numerical studies for the case $p^* = 1$ and $q = 1$, the usual sphericity test, are already available in Marques and Coelho (2008a) and Coelho and Marques (2008), and for $k = 1$, the test of equality of several covariance matrices, studies were also conducted in Coelho and Marques (2007).

7. Conclusions

In this paper it was shown how, beyond any doubt, the decomposition of the null hypothesis H_0 in (1) into simpler conditionally independent hypotheses is indeed an extremely efficient tool for the development of near-exact distributions for elaborate test statistics, namely for the MS-BM sphericity l.r.t. statistic.

The decomposition approach proposed in Coelho and Marques (2008) and used in this paper indeed "not only enables us to build very accurate and manageable near-exact approximations to the exact distribution of the overall test statistics but also concomitantly enables us to easily overcome the problems of controlling statistical errors, in particular the error of the first kind, which arise when we have to test sequentially the partial hypotheses". Also, as the same authors conclude, "Now we may easily compute near-exact quantiles which enable us to carry the overall test in just one step, avoiding this way the problems brought to our attention by Hogg (1961) and avoiding also the need for any corrections of the first kind error level, like Sidak's correction (Sidak, 1967, 1968)".

The near-exact distributions developed are, in every case, more accurate than the asymptotic approximations and they present very good asymptotic properties, not only for increasing sample sizes but also, and opposite to the usual asymptotic distributions, for increasing number of variables, number of Δ matrices and number of samples involved, showing yet an excellent behavior for small sample sizes, cases where standard asymptotic approximations do not perform well.

Given the results in Anderson et all. (1986) and Anderson and Fang (1990) the near-exact distributions developed in this paper are also applicable in the cases where the underlying random vectors have an elliptically contoured distribution.

Finally, the manageability, the accuracy, the asymptotic properties and the wide range of application of near-exact distributions allow for a fresh look over the problem of approximating the distributions of test statistics with exact c.d.f.'s that do not have a closed form representation or are almost impossible to handle.

Given the results obtained, the use of the near-exact distribution $M3GNIG$ in place of the exact distribution may be reliably advised.

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Appendix A

The independence of the statistics $\lambda^*_{c|(b|a)}$, $\lambda^*_{b|a}$ and λ^*_a under H_0

The independence of the three statistics $\lambda^*_{c|(b|a)}$, $\lambda^*_{b|a}$ and λ^*_a in (7) under H_0 in (1) is easy to establish by using a couple of known results. We may start by noticing that λ^*_a is independent of

$$A = A_1 + \ldots + A_q = \sum_{j=1}^{q} A_j$$

(see Lemma 10.4.1 in Anderson (2003, Section 10.4) and the note right after expression (13) in Section 10.4 of the same reference) and thus λ^*_a is independent of both $\lambda^*_{c|(b|a)}$ and $\lambda^*_{b|a}$ since both these statistics are built only from A. The statistic $\lambda^*_{b|a}$ is independent of $A_{11}, A_{22}, \ldots, A_{kk}$, being this fact possible to prove through an extended version of Lemma 10.4.1 of Anderson (2003, Section 10.4) or, in more detail, through the fact that we may write

$$\lambda^*_{b|a} = \prod_{j=1}^{k-1} \lambda^*_{j(b|a)}\,,$$

where

$$\lambda^*_{j(b|a)} = \frac{|\widetilde{A}_j|^{n^*/2}}{|\widetilde{A}_{jj}|^{n^*/2}|\widetilde{A}_{j+1}|^{n^*/2}}$$

with

$$\widetilde{A}_j = \begin{pmatrix} A_{jj} & A_{j,j+1} & \ldots & A_{j,k} \\ A_{j+1,j} & A_{j+1,j+1} & \ldots & A_{j+1,k} \\ \vdots & \vdots & \vdots & \vdots \\ A_{kj} & A_{k,j+1} & \ldots & A_{kk} \end{pmatrix}$$

where the $k-1$ statistics $\lambda^*_{j(b|a)}$ are independent under $H_{(0b|0a)}$ (see for example Theorem 9.3.2, Anderson, 1984) and where under the null hypothesis of independence between the j-th group of variables and the super-group formed by the groups of variables $j+1, \ldots, k$, each statistic $\lambda^*_{j(b|a)}$ is independent of both A_{jj} and $\widetilde{A}_{j+1}$ (see for example Section 8.2 of Kshirsagar, 1972), since we may write

$$(\lambda^*_{j(b|a)})^{2/n^*} = \frac{|\widetilde{A}_j|}{|A_{jj}||\widetilde{A}_{j+1}|}\;, \quad j = 1, \ldots, k$$

where,

$$|\widetilde{A}_j| \;=\; |A_{jj}||\widetilde{A}_{j+1\,.\,j}|$$
$$|\widetilde{A}_{j+1}||A_{jj\,.\,(j+1,\dots,k)}|$$

with

$$A_{jj\,.\,(j+1,\dots,k)} = A_{jj} - \widetilde{A}_{j,(j+1,\dots,k)}\,\widetilde{A}^{-1}_{j+1,j+1}\,\widetilde{A}_{(j+1,\dots,k),j}$$
$$\widetilde{A}_{j+1\,.\,j} = \widetilde{A}_{j+1} - \widetilde{A}_{(j+1,\dots,k),j}\,\widetilde{A}^{-1}_{j+1,j+1}\,\widetilde{A}_{j,(j+1,\dots,k)}$$

for

$$\widetilde{A}_{j,(j+1,\dots,k)} = \Big[\,A_{j,j+1}\,|\,\dots\,|\,A_{jk}\,\Big]$$

and

$$\widetilde{A}_{(j+1,\dots,k),j} = \widetilde{A}^{\,\prime}_{j,(j+1,\dots,k)} \quad \text{(where the prime denotes transpose)}$$

we may write

$$(\lambda^*_{j(b|a)})^{2/n^*} = \frac{|A_{jj\,.\,(j+1,\dots,k)}|}{|A_{jj}|}\,, \quad j=1,\dots,k$$

where, from the above reference, $(\lambda^*_{j(b|a)})^{2/n^*}$ is independent of A_{jj} and thus $\lambda^*_{(b|a)}$ is independent of $A_{11},\dots,A_{kk}$, since $(\lambda^*_{k-1(b|a)})^{2/n^*}$, that may be written as

$$(\lambda^*_{k-1(b|a)})^{2/n^*} = \frac{|A_{k-1,k-1\,.\,k}|}{|A_{k-1,k-1}|} = \frac{|\widetilde{A}_{k\,.\,k-1}|}{|\widetilde{A}_k|} = \frac{|A_{kk} - A_{k,k-1}A^{-1}_{k-1,k-1}A_{k-1,k}|}{A_{kk}}\,,$$

is independent of both $A_{k-1,k-1}$ and A_{kk}.
Then since $\lambda^*_{c|(b|a)}$ in (7) is built only from the A_{jj} $(j=1,\dots,k)$ it is independent of $\lambda^*_{b|a}$.

Appendix B

The Gamma, GIG (Generalized Integer Gamma) and GNIG (Genealized Near-Integer Gamma) distributions

We will use this Appendix to establish some notation concerning distributions used in the paper, as well as to give the expressions for the p.d.f.'s (probability density functions) and c.d.f.'s (cumulative distribution functions) of the GIG (Generalized Integer Gamma) and GNIG (Generalized Near-Integer Gamma) distributions.

We will say that the r.v. X has a Gamma distribution with rate parameter $\lambda > 0$ and shape parameter $r > 0$, if its p.d.f. may be written as

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)}\,e^{-\lambda x}\,x^{r-1}\,, \qquad (x>0)$$

and we will denote this fact by

$$X \sim \Gamma(r,\lambda)\,.$$

Let

$$X_j \;\sim\; \Gamma(r_j,\lambda_j) \qquad j=1,\dots,p$$

be p independent random variables with Gamma distributions with shape parameters $r_j \in I\!N$ and rate parameters $\lambda_j > 0$, with $\lambda_j \neq \lambda_{j'}$, for all $j \neq j' \in \{1, \dots, p\}$. We will say that then the r.v.

$$Y = \sum_{j=1}^{p} X_j$$

has a GIG (Generalized Integer Gamma) distribution of depth p, with shape parameters r_j and rate parameters λ_j, $(j = 1, \dots, p)$, and we will denote this fact by

$$Y \sim GIG(r_j, \lambda_j;\, p)\,.$$

The p.d.f. and c.d.f. of Y are respectively given by (Coelho, 1998)

$$f^{GIG}(y|r_1, \dots, r_p; \lambda_1, \dots, \lambda_p) = K \sum_{j=1}^{p} P_j(y)\, \mathrm{e}^{-\lambda_j\, y}\,, \qquad (y > 0) \tag{43}$$

and

$$F^{GIG}(y|r_1, \dots, r_j; \lambda_1, \dots, \lambda_p) = 1 - K \sum_{j=1}^{p} P_j^*(y)\, \mathrm{e}^{-\lambda_j\, y}\,, \qquad (y > 0) \tag{44}$$

where

$$K = \prod_{j=1}^{p} \lambda_j^{r_j}\ , \qquad P_j(y) = \sum_{k=1}^{r_j} c_{j,k}\, y^{k-1} \tag{45}$$

and

$$P_j^*(y) = \sum_{k=1}^{r_j} c_{j,k}\, (k-1)! \sum_{i=0}^{k-1} \frac{y^i}{i!\, \lambda_j^{k-i}}$$

with

$$c_{j,r_j} = \frac{1}{(r_j - 1)!} \prod_{\substack{i=1 \\ i \neq j}}^{p} (\lambda_i - \lambda_j)^{-r_i}\ , \qquad j = 1, \dots, p\,, \tag{46}$$

and

$$c_{j,r_j-k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(r_j - k + i - 1)!}{(r_j - k - 1)!}\, R(i, j, p)\, c_{j, r_j - (k-i)}\,, \qquad (k = 1, \dots, r_j - 1;\ j = 1, \dots, p) \tag{47}$$

where

$$R(i, j, p) = \sum_{\substack{k=1 \\ k \neq j}}^{p} r_k\, (\lambda_j - \lambda_k)^{-i} \quad (i = 1, \dots, r_j - 1)\,. \tag{48}$$

The GNIG (Generalized Near-Integer Gamma) distribution of depth $p+1$ (Coelho, 2004) is the distribution of the r.v.

$$Z = Y_1 + Y_2$$

where Y_1 and Y_2 are independent, Y_1 having a GIG distribution of depth p and Y_2 with a Gamma distribution with a non-integer shape parameter r and a rate parameter $\lambda \neq \lambda_j$ $(j = 1, \dots, p)$. The p.d.f. of Z is given by

$$f^{GNIG}(z|r_1,\ldots,r_p,r;\,\lambda_1,\ldots,\lambda_p,\lambda) =$$

$$K\lambda^r \sum_{j=1}^{p} \mathrm{e}^{-\lambda_j z} \sum_{k=1}^{r_j} \left\{ c_{j,k} \frac{\Gamma(k)}{\Gamma(k+r)} z^{k+r-1} {}_1F_1(r,k+r,-(\lambda-\lambda_j)z) \right\}, \qquad (z>0) \tag{49}$$

and the c.d.f. given by

$$F^{GNIG}(z|r_1,\ldots,r_p,r;\,\lambda_1,\ldots,\lambda_p,\lambda) = \frac{\lambda^r\, z^r}{\Gamma(r+1)} {}_1F_1(r,r+1,-\lambda z)$$

$$-K\lambda^r \sum_{j=1}^{p} \mathrm{e}^{-\lambda_j z} \sum_{k=1}^{r_j} c^*_{j,k} \sum_{i=0}^{k-1} \frac{z^{r+i}\lambda_j^i}{\Gamma(r+1+i)} {}_1F_1(r,r+1+i,-(\lambda-\lambda_j)z) \qquad (z>0) \tag{50}$$

where

$$c^*_{j,k} = \frac{c_{j,k}}{\lambda_j^k}\Gamma(k)$$

with $c_{j,k}$ given by (46) through (48) above. In the above expressions ${}_1F_1(a,b;z)$ is the Kummer confluent hypergeometric function. This function has usually very good convergence properties and is nowadays easily handled by a number of software packages.

Appendix C

Shape parameters for the c.f. in (16)

The shape parameters $r_{1,j}$ in (16) are given by

$$r_{1,j} = \begin{cases} r^*_j & \text{for } j=1,\ldots,p-1, \\ & \text{except for } j=p-1-2\alpha_1 \\ r^*_j + (p \perp\!\!\!\perp 2)(\alpha_2-\alpha_1) & \\ \qquad \times\left(q-\frac{p-1}{2}+q\left\lfloor\frac{p}{2q}\right\rfloor\right) & \text{for } j=p-1-2\alpha_1 \end{cases} \tag{51}$$

where

$$\alpha = \left\lfloor \frac{p-1}{q} \right\rfloor, \quad \alpha_1 = \left\lfloor \frac{q-1}{q}\,\frac{p-1}{2} \right\rfloor, \quad \alpha_2 = \left\lfloor \frac{q-1}{q}\,\frac{p+1}{2} \right\rfloor,$$

and

$$r^*_j = \begin{cases} c_j & \text{for} \quad j=1,\ldots,\alpha+1 \\ q\left(\left\lfloor\frac{p}{2}\right\rfloor - \left\lfloor\frac{j}{2}\right\rfloor\right) & \text{for} \quad j=\alpha+2,\ldots,\min(p-2\alpha_1,p-1) \\ & \text{and} \quad j=2+p-2\alpha_1,\ldots,2\left\lfloor\frac{p}{2}\right\rfloor-1, \text{ step } 2 \\ q\left(\left\lfloor\frac{p+1}{2}\right\rfloor - \left\lfloor\frac{j}{2}\right\rfloor\right) & \text{for} \quad j=1+p-2\alpha_1,\ldots,p-1, \text{ step } 2\,, \end{cases} \tag{52}$$

with,

$$c_j = \left\lfloor\frac{q}{2}\right\rfloor \left((j-1)q-2\left((q+1)\perp\!\!\!\perp 2\right)\left\lfloor\frac{j}{2}\right\rfloor\right) + \left\lfloor\frac{q}{2}\right\rfloor \left\lfloor\frac{q+j \perp\!\!\!\perp 2}{2}\right\rfloor \quad \text{for } j=1,\ldots,\alpha$$

and

$$c_{\alpha+1} = -\left(\left\lfloor\frac{p}{2}\right\rfloor - \alpha\left\lfloor\frac{q}{2}\right\rfloor\right)^2 + q\left(\left\lfloor\frac{p}{2}\right\rfloor - \left\lfloor\frac{\alpha+1}{2}\right\rfloor\right)$$
$$+(q \perp\!\!\!\perp 2)\left(\alpha\left\lfloor\frac{p}{2}\right\rfloor + \frac{\alpha \perp\!\!\!\perp 2}{4} - \frac{\alpha^2}{4} - \alpha^2\left\lfloor\frac{q}{2}\right\rfloor\right). \qquad (53)$$

The expressions for these parameters were derived in Coelho and Marques (2007).

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