The distribution of the product of powers of independent Uniform random variables

a simple but useful tool to address and better understand the structure of some distributions

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Abstract

We begin with some simple questions. What is the distribution of the product of given powers of independent continuous uniform random variables? Is this distribution useful? Are there some test statistics with this distribution? Is this distribution somehow related to the distribution of the product of other random variables? Is this distribution already known in some context? This short paper will give the answers to the above questions. It will be seen that the answer to the last four questions above is: yes! We will show how this distribution may help us to shed some new light on other well known distributions and also how it may help us in approaching in a much simpler way some distributions usually considered to be rather complicated.

Key words: product of independent Beta random variables, Wilks Lambda statistic, Generalized Integer Gamma distribution.

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1 Introduction

This paper will provide a different view of some rather well known distributions and some relations among them. Such new insights, besides shedding some new light on these distributions may also help us understand the problems we face when trying to handle closely related distributions which are far less manageable than the ones studied in this paper and the reasons why that happens.

The distributions we are interested in studying are the distributions of r.v.'s (random variables) of the form

$$V = \prod_{j=1}^{p} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{\delta_{ij}}} \,, \quad \text{for } \delta_{ij} > 0 \,, \tag{1}$$

where the U_{ij}'s are i.i.d. $Unif(0,1)$ r.v.'s, for $j = 1, \ldots, p$; $i = 1, \ldots, r_j$.

In terms of notation, we will use $\mathbb{N}$ to denote the set of positive integers, $\mathbb{R}$ to denote the set of reals, $\mathbb{R}^+$ to denote the set of positive reals and $\mathbb{C}$ to denote the set of complex numbers. We will also use "$\sim$" to denote "is distributed as" and "$\stackrel{st}{\sim}$" to denote "has the same distribution as" or "is stochastically equivalent to".

2 Several distributions which may be represented as the distribution of the product of powers of independent Uniform r.v.'s

In this section we will show how particular choices of δ_{ij} will lead to V in (1) having interesting and useful distributions, whose p.d.f.'s (probability density functions) and c.d.f.'s (cumulative distribution functions) have concise and manageable representations.

2.1 The general case – the product of powers of Uniforms as an exponentiated generalized integer gamma (EGIG) distribution and as the distribution of the product of Pareto r.v.'s

If all the δ_{ij}'s are distinct, V in (1) has an exponentiated generalized integer gamma (EGIG) distribution of depth $\sum_{j=1}^{p} r_j$ (see Appendix B) with all shape parameters equal to 1 and rate parameters δ_{ij}, with p.d.f. and c.d.f. respectively given by

$$f_V(v) = \left(\prod_{j=1}^{p} \prod_{i=1}^{r_j} \delta_{ij} \right) \sum_{j=1}^{p} \sum_{i=1}^{r_j} c_{ij} v^{\delta_{ij}-1}$$

and

$$F_V(v) = \left(\prod_{j=1}^{p} \prod_{i=1}^{r_j} \delta_{ij} \right) \sum_{j=1}^{p} \sum_{i=1}^{r_j} c_{ij} \frac{v^{\delta_{ij}}}{\delta_{ij}}$$

with

$$c_{ij} = \prod_{\substack{h=1\\(h\neq j \vee \ell\neq i)}}^{p} \prod_{\ell=1}^{r_h} \frac{1}{\delta_{\ell h} - \delta_{ij}}\,.$$

For $\delta_{ij} = \delta_j$ for all $i = 1, \ldots, r_j$, the distribution of V is an EGIG distribution of depth p with shape parameters r_j and rate parameters δ_j, assuming that all the δ_j's are different for $j \in \{1, \ldots, p\}$. In case there are some equalities among the δ_j's, we only have to group the corresponding U_{ij}'s in (1) or (2) under the same product in i, this way adding the corresponding r_j's and thus reducing the depth of the distribution, which will be equal to the number of different δ_j's.

The distribution of

$$V_j = \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{\delta_j}} \tag{2}$$

where the U_{ij}'s are i.i.d. $Unif(0,1)$ r.v.'s, is a simple one, since, using the relation in section A.1 in Appendix A, we know that

$$-\log\, U_{ij}^{\frac{1}{\delta_j}} \sim Exp(\delta_j)\,,$$

so that

$$-\log\, V_j = \sum_{i=1}^{r_j} -\log\, U_{ij}^{\frac{1}{\delta_j}} \sim \Gamma(r_j, \delta_j)\,,$$

that is, $-\log V_j$ is an r.v. with p.d.f.

$$f_{-\log V_j}(w) = \frac{\delta_j^{r_j}}{(r_j-1)!}\, e^{-\delta_j w}\, w^{r_j-1}\,, \quad (w>0)$$

so that V_j has p.d.f.

$$f_{V_j}(v) = \frac{\delta_j^{r_j}}{(r_j-1)!}\, v^{\delta_j-1}\,(-\log\, v)^{r_j-1}\,, \quad 0<v<1\,.$$

Of course, if $\delta_j = 1$, this reduces to the distribution of the product of r_j i.i.d. $Unif(0,1)$ r.v.'s, with p.d.f.

$$f_{V_j}(v) = \frac{1}{(r_j-1)!}\,(-\log\, v)^{r_j-1}\,, \quad 0<v<1\,.$$

The c.d.f. of V_j is then easy to obtain.

If in (1) we take $r_j = 1$ and $\delta_{ij} = \delta_j < 0$ $(j = 1, \ldots, p)$, the r.v.'s $\theta_j\, U_{1j}^{\frac{1}{\delta_j}}$ will have $Pareto(\theta_j, \delta_j)$ distributions (see Appendix A). Thus, in this case

$$V = \prod_{j=1}^{p} \theta_j\, U_{1j}^{\frac{1}{\delta_j}}$$

will have the distribution of the product of p independent r.v.'s with $Pareto(\theta_j, \delta_j)$ distributions.

But then, the r.v.

$$V^* = \left(\prod_{j=1}^{p} \theta_j\right) V^{-1} = \prod_{j=1}^{p} U_{1j}^{-\frac{1}{\delta_j}}$$

will have an EGIG distribution of depth p (if all δ_j's are different), with all shape parameters equal to 1 and rate parameters $(-\delta_j)$, so that V will have p.d.f. and c.d.f. respectively given by (see Appendix B),

$$\begin{aligned} f_V(v) &= f^{\text{EGIG}}\left(\left(\prod_{j=1}^{p} \theta_j\right) v^{-1}; 1, -\delta_j; p\right) \frac{1}{v^2} \prod_{j=1}^{p} \theta_j \\ &= f^{\text{GIG}}\left(\log\, v - \sum_{j=1}^{p} \log\, \theta_j; 1, -\delta_j; p\right) \frac{1}{v^3} \prod_{j=1}^{p} \theta_j\ , \qquad (1 < v < \infty) \end{aligned}$$

and

$$\begin{aligned} F_V(v) &= F^{\text{EGIG}}\left(\left(\prod_{j=1}^{p} \theta_j\right) v^{-1}; 1, -\delta_j; p\right) \\ &= 1 - F^{\text{GIG}}\left(\log\, v - \sum_{j=1}^{p} \log\, \theta_j; 1, -\delta_j; p\right)\ , \qquad (1 < v < \infty)\,. \end{aligned}$$

We may note that in this case we have

$$E\left(V^h\right) = \prod_{j=1}^{p} \theta_j^h\, \frac{-\delta_j}{-\delta_j - h} \qquad \left(h < -\max_{1\le j\le p} \delta_j\right).$$

If some of the δ_j's are equal, we only have to group the corresponding U_{ij}'s under the same product in i, this way reducing the depth of the EGIG or

GIG distribution, which will then be equal to the number of different δ_j's. At the same time we will have to increase the corresponding shape parameters so that they reflect for each δ_j the number of times it appears repeated. More precisely, in this case V^* would be written as

$$V^* = \prod_{j=1}^{p^*} \prod_{i=1}^{r_j} U_{ij}^{-\frac{1}{\delta_j}}$$

where $p^* \le p$ is the number of different δ_j's and as such is the depth of the EGIG distribution, and r_j is the number of times that δ_j appears repeated. Then, in this case the r_j's will be the shape parameters in the EGIG distribution of V, which will have p.d.f. and c.d.f. respectively given by

$$\begin{aligned} f_V(v) &= f^{\text{EGIG}}\left(\left(\prod_{j=1}^{p} \theta_j\right) v^{-1}; r_j, -\delta_j; p^*\right) \frac{1}{v^2} \prod_{j=1}^{p} \theta_j \\ &= f^{\text{GIG}}\left(\log\, v - \sum_{j=1}^{p} \log\, \theta_j; r_j, -\delta_j; p^*\right) \frac{1}{v^3} \prod_{j=1}^{p} \theta_j\ , \qquad (1 < v < \infty) \end{aligned}$$

and

$$\begin{aligned} F_V(v) &= F^{\text{EGIG}}\left(\left(\prod_{j=1}^{p} \theta_j\right) v^{-1}; r_j, -\delta_j; p^*\right) \\ &= 1 - F^{\text{GIG}}\left(\log\, v - \sum_{j=1}^{p} \log\, \theta_j; r_j, -\delta_j; p^*\right)\ , \qquad (1 < v < \infty)\,. \end{aligned}$$

Note that these results agree with those given in Arnold (1983) for the cases in which the δ_j's are either all equal or all different.

2.2 When the product of powers of Uniforms is a Beta or a product of Betas

By looking at the results in section A.2 in Appendix A, we may easily see that if in (2) we take $\delta_{ij} = a_j + i - 1$ $(a_j > 0)$, then,

$$V_j = \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{a_j+i-1}} \ \sim\ Beta(a_j, r_j)$$

and as such, in this case,

$$V=\prod_{j=1}^{p}\prod_{i=1}^{r_j} U_{ij}^{\frac{1}{a_j+i-1}} \overset{st}{\sim} \prod_{j=1}^{p} Y_j\,, \quad \begin{array}{l}\text{where } Y_j\sim Beta(a_j,r_j)\\ \text{are } p \text{ independent r.v.'s } (r_j\in I\!N)\end{array} \tag{3}$$

Further, from section A.3 in Appendix A, if in (1) we take $\delta_{ij}=a_j+i/k-1/k$, we will have

$$V=\prod_{j=1}^{p}\prod_{i=1}^{r_j} U_{ij}^{\frac{1}{a_j+i/k-1/k}} \overset{st}{\sim} \prod_{j=1}^{p}\prod_{i=1}^{k} Y_{ji}\,, \quad \begin{array}{l}\text{where } Y_{ji}\sim Beta\left(a_j+\frac{i-1}{k},\frac{r_j}{k}\right)\\ \text{are all independent r.v.'s.}\end{array} \tag{4}$$

If all the a_j+i-1 in (3) and all the $a_j+i/2-1/2$ in (4) are different for $j=1,\dots,p$, $i=1,\dots,r_j$, the distribution of V in both cases will be an EGIG distribution of depth $\sum_{j=1}^{p} r_j$, with shape parameters all equal to 1 and rate parameters a_j+i-1 in the first case and $a_j+i/2-1/2$ in the second case, with p.d.f. and c.d.f. given respectively by (B.3) and (B.4) in Appendix B. In case some of the a_j+i-1 in (3) or some of the $a_j+i/2-1/2$ in (4) are equal, then the depth of the EGIG distribution has to be reduced accordingly and the corresponding shape parameters increased accordingly.

As a particular case of the distribution in (4), we have, the distribution, for even p, of

$$\prod_{j=1}^{p} Y_j\,, \quad \begin{array}{l}\text{where } Y_j\sim Beta\left(a+\frac{j}{2},\frac{b}{2}\right), \quad (a>0, b\in I\!N)\,,\\ \text{are } p \text{ independent r.v.'s}\end{array} \tag{5}$$

which is then the distribution of

$$V=\prod_{j=1}^{p/2}\prod_{i=1}^{b} U_{ij}^{\frac{1}{a+\frac{2j-1}{2}+\frac{i}{2}-\frac{1}{2}}} \tag{6}$$

where $a+\frac{2j-1}{2}+\frac{i}{2}-\frac{1}{2}$ varies between $a+\frac{1}{2}$ and $a+\frac{p+b}{2}-1$, with $a+\frac{j}{2}$ occurring r_j times ($j=1,\dots,p+b-2$), with

$$r_j=\begin{cases} h_j, & j=1,2\\ h_j+r_{j-2}, & j=3,\dots,p+b-2\,,\end{cases} \tag{7}$$

where

$$h_j = \begin{cases} 1, & j = 1, \dots, \min(p, b) \\ 0, & j = 1 + \min(p, b), \dots, \max(p, b) \\ -1, & j = 1 + \max(p, b), \dots, p + b - 2\,, \end{cases} \tag{8}$$

so that we may write

$$V = \prod_{j=1}^{p+b-2} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{a+\frac{j}{2}}}\,, \tag{9}$$

which shows that the exact distribution of V is in this case an EGIG distribution of depth $p+b-2$ with shape parameters r_j and rate parameters $a+\frac{j}{2}$ $(j = 1, \dots, p+b-2)$.

It is interesting to note that the representation for V in (9) clearly indicates the role of the parameters $a+\frac{j}{2}$ as the powers of the $Unif(0,1)$ r.v.'s, corresponding to the $p+b-2$ first parameters of the Beta r.v.'s appearing in the product-of-betas representation of V.

We may also note that just by writing the first parameters of the Beta r.v.'s in (5) in reverse order, we may see that we have for V in (9),

$$V \overset{st}{\sim} \prod_{j=1}^{p} Y_j\,, \quad \text{where } Y_j \sim Beta\left(a + \tfrac{p+1}{2} - \tfrac{j}{2}, \tfrac{b}{2}\right) \text{ are independent r.v.'s}\,,$$

so that, given the symmetry of the r_j's in (7), with $r_j = r_{p+b-1-j}$, we have for V in (9),

$$V = \prod_{j=1}^{p+b-2} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{a+\frac{j}{2}}} = \prod_{j=1}^{p+b-2} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{a+\frac{p+b-1}{2}-\frac{j}{2}}}\,.$$

For ease of notation for the r_j's and the δ_j's, in the next section we will actually prefer the second of these representations..

2.3 The exact distribution of the Wilks Λ statistic

If we assume

$$\underline{X} = [\underline{X}_1', \underline{X}_2', \dots, \underline{X}_m']' \sim N_p(\underline{\mu}, \Sigma)$$

where

$$\underline{\mu} = \left[\underline{\mu}_1', \underline{\mu}_2', \ldots, \underline{\mu}_m'\right]' \quad \text{and} \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \ldots & \Sigma_{1m} \\ \Sigma_{21} & \Sigma_{22} & \ldots & \Sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m1} & \Sigma_{m2} & \ldots & \Sigma_{mm} \end{bmatrix}$$

with

$$\underline{\mu}_k = E(\underline{X}_k) \quad \text{and} \quad \Sigma_{kk} = Var(\underline{X}_k)\,, \quad \Sigma_{kk'} = Cov(\underline{X}_k, \underline{X}_{k'})\,, \;\; k, k' \in \{1, \ldots, m\}\,,$$

and we wish to test the null hypothesis

$$H_0 : \Sigma_{kk'} = 0 \;\; \text{for all } k \neq k' \in \{1, \ldots, m\}\,, \tag{10}$$

that is the null hypothesis of independence of the m sets of variables $\underline{X}_k$ ($k = 1, \ldots, m$), the l.r.t. (likelihood ratio test) statistic is

$$\Lambda = \frac{|S|}{\prod_{k=1}^{m} |S_{kk}|}$$

where S is the MLE of Σ and S_{kk} the MLE of Σ_{kk} ($k = 1, \ldots, m$).

Let us suppose that we have a sample of size $n + 1$ from the distribution of $\underline{X}$ and suppose that each $\underline{X}_k$ has p_k variables and let us suppose that at most one of these p_k's is odd. Without any loss of generality, let it be p_m.

For $m = 2$ we know that, taking $p = p_1 + p_2$,

$$\Lambda \stackrel{st}{\sim} \prod_{j=1}^{p_1} Y_j\,, \quad \begin{array}{l} \text{with } Y_j \sim Beta\left(\frac{n-p+j}{2}, \frac{p_2}{2}\right) \\ \text{or } Y_j \sim Beta\left(\frac{n-1-j}{2}, \frac{p_2}{2}\right), \;\text{all independent} \end{array}$$

so that we may write, from (5) and (9), for r_j defined as in (7) and (8), with p and b replaced respectively by p_1 and p_2,

$$\Lambda = \prod_{j=1}^{p_1+p_2-2} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{(n-p+j)/2}} = \prod_{j=1}^{p_1+p_2-2} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{(n-1-j)/2}}\,,$$

which shows that the exact distribution of Λ is, in this case, an EGIG distribution of depth $p_1 + p_2 - 2$, with shape parameters r_j, given by (7) and (8), and rate parameters $\frac{n-1-j}{2}$ $(j=1,\ldots,p_1+p_2-2)$, with p.d.f. and c.d.f. given by (B.3) and (B.4) in Appendix B.

For $m>2$, it may be shown (Anderson, 2003), that

$$\Lambda=\prod_{k=1}^{m-1}\Lambda_{k,(k+1,\ldots,m)}\,, \tag{11}$$

where $\Lambda_{k,(k+1,\ldots,m)}$ $(k=1,\ldots,m-1)$ is the l.r.t. statistic to test

$$H_{0k}:\ \Sigma_{k\ell}=0,\ \ell=k+1,\ldots,m \qquad (k=1,\ldots,m-1).$$

The $\Lambda_{k,(k+1,\ldots,m)}$'s are independent under H_0 in (10), and they may be represented in the form

$$\Lambda_{k,(k+1,\ldots,m)} \stackrel{st}{\sim} \prod_{j=1}^{p_k} Y_j\,, \quad \begin{array}{l}\text{with } Y_j\sim Beta\left(\frac{n+1-q_k-j}{2},\frac{q_k}{2}\right)\\ \text{or } Y_j\sim Beta\left(\frac{n-q_{k-1}+j}{2},\frac{q_k}{2}\right),\ \text{all independent}\end{array} \tag{12}$$

where $q_k=p_{k+1}+\ldots+p_m$, so that, from (5) and (6) and then (9),

$$\begin{aligned}\Lambda=\prod_{k=1}^{m-1}\prod_{j=1}^{p_k}Y_j &= \prod_{k=1}^{m-1}\prod_{j=1}^{p_k/2}\prod_{i=1}^{q_k}U_{ij}^{\frac{1}{\frac{n-q_{k-1}+2j-1}{2}+\frac{i}{2}-\frac{1}{2}}}\\ &=\prod_{k=1}^{m-1}\prod_{j=1}^{p_k+q_k-2}\prod_{i=1}^{r_{kj}}U_{ij}^{\frac{1}{(n-q_{k-1}+j)/2}}\,,\end{aligned} \tag{13}$$

where the r_{kj}, for $k=1,\ldots,m-1$, are defined in the same manner as the r_j's in (7) and (8) with p and b replaced respectively by p_k and q_k. But it is evident that in (13) there are several repetitions of each $\delta_{kj}=\frac{n-q_{k-1}+j}{2}$, which indeed, for $p=p_1+\ldots+p_m$, range from a minimum value of $\frac{n-p+1}{2}$ through a maximum of $\frac{n-2}{2}$ with $\delta_j=\frac{n-1-j}{2}$ $(j=1,\ldots,p-2)$ occurring r_j times, where

$$r_j=\sum_{k=1}^{m-1}r_{kj} \tag{14}$$

in which, for $k=1,\ldots,m-1$,

$$r_{kj}=\begin{cases}h_{kj} & j=1,2\\ h_{kj}+r_{k,j-2} & j=3,\ldots,p_k+q_k-2\\ 0 & j=p_k+q_k-1,\ldots,p-2\end{cases} \tag{15}$$

with h_{kj} $(k = 1, \ldots, m-1; j = 1, \ldots, p_k + q_k - 2)$ given by (8) with p and b replaced respectively by p_k and q_k. Consequently we may write

$$\Lambda = \prod_{j=1}^{p-2} \prod_{i=1}^{r_j} U_{ij}^{\frac{1}{(n-1-j)/2}} \tag{16}$$

which shows that the exact distribution of Λ is in this case an EGIG distribution of depth $p-2$ with shape parameters r_j and rate parameters $\delta_j = \frac{n-1-j}{2}$ $(j = 1, \ldots, p-2)$, with p.d.f. and c.d.f. given by (B.3) and (B.4) in Appendix B.

Note that this distribution, under H_0 in (10) is still valid if we assume for $\underline{X}$ an underlying elliptically contoured or left orthogonal-invariant distributions (Anderson et al., 1986; Anderson and Fang, 1990; Jensen and Good, 1981; Kariya, 1981).

For the general case, $m \geq 2$, from (11) and (12) and the expression for the h-th moment of a Beta r.v., we have the usual expression for the h-th moment of Wilks Λ statistic given by

$$E\left(\Lambda^h\right) = \prod_{k=1}^{m-1} \prod_{j=1}^{p_k} \frac{\Gamma\left(\frac{n+1-j}{2}\right) \Gamma\left(\frac{n+1-q_k-j}{2} + h\right)}{\Gamma\left(\frac{n+1-q_k-j}{2}\right) \Gamma\left(\frac{n+1-j}{2} + h\right)}, \quad \left(h > -\frac{n+1-p}{2}\right),$$

while from (16) and the results in Appendix A, we have for the case in which at most one of the m sets of variables has an odd number of variables, the h-th moment of Λ can be expressed explicitly as (see also Appendix B for the moments of the EGIG distribution)

$$E\left(\Lambda^h\right) = \prod_{j=1}^{p-2} \prod_{\ell=1}^{r_j} \frac{1}{\frac{2}{n-1-j}h + 1} = \prod_{j=1}^{p-2} \left(\frac{n-1-j}{n-1-j+2h}\right)^{r_j}, \quad \left(h > -\frac{n-p+1}{2}\right)$$

for r_j given by (14) and (15).

3 Conclusions

Interestingly, the distribution of V in (1) yields, for different values of the δ_{ij}'s, several other distributions, including the exact distribution of Wilks Λ statistic, used to test the independence of several sets of variables. The difficulty encountered in writing down a concise and manageable expression for the exact distribution of this statistic when more than one set of variables has an odd number of variables is related with the fact that in this case we will be left with at least one Beta r.v. with a non-integer second parameter, and as such it cannot be represented as a product of powers of $Unif(0,1)$ r.v.'s.

The distribution of the product of powers of independent $Unif(0,1)$ r.v.'s is a useful tool to approach in an unified way the distributions of several random variables, which would be more challenging to deal with if they were not recognized as being amenable to such a representation. The distribution of the product of powers of $Unif(0,1)$ r.v.'s appears as the common link and common elementary structure among these distributions.

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Appendices

A Some simple relations among distributions

Many of the relations presented in this Appendix are suitable to be used as simple problems in an undergraduate course in Statistics or Distribution Theory. We list them to systematize and establish some of the notation used in this paper.

A.1 The relation between Uniform, Beta and Exponential variables

It is very easy to establish, by simple transformation, that if

$$X \sim Unif(0,1)\,,$$

then

$$Y = X^{1/a} \sim Beta(a,1) \quad (a>0)\,,$$

and

$$W = -\log Y = -\frac{1}{a}\log X \;\sim\; Exp(a)\,,$$

and vice-versa, that is, if

$$W \;\sim\; Exp(a)\,,$$

then

$$Y = e^{-W} \;\sim\; Beta(a,1)$$

and

$$X = Y^a = e^{-aW} \;\sim\; Unif(0,1)\,.$$

Also, if we take $\theta > 0$,

$$Z = \theta Y^{-1} = \theta X^{-1/a} \sim Pareto(\theta, a)\,,$$

that is, Z will have p.d.f.

$$f_Z(z) = a\,\frac{\theta^a}{z^{a+1}} \qquad (z > \theta)\,.$$

Concerning the moments of these r.v.'s, we may easily see that

$$\begin{aligned} & E(X^h) = \frac{1}{h+1}\,, \quad (h > -1) \\ \Longrightarrow\; & E(Y^h) = E\left(X^{h/a}\right) = E\left(e^{-hW}\right) = \frac{a}{a+h}\,, \quad (h > -a) \\ \Longrightarrow\; & E(Z^h) = \theta^h\, E(Y^{-h}) = \theta^h\, E(X^{-h/a}) = \theta^h\,\frac{a}{a-h}\,, \quad (h < a) \end{aligned}$$

and

$$E(W^h) = \frac{1}{a^h}\,\underbrace{E\left[(-\log X)^h\right]}_{\Gamma(h+1)} = \frac{\Gamma(h+1)}{a^h}\,, \quad (h > -1)\,,$$

which for $h \in I\!\!N$ is also

$$E(W^h) = \frac{d^h}{dt^h}\,E(Y^{-t})\bigg|_{t=0} = \frac{d^h}{dt^h}\,\frac{a}{a-t}\bigg|_{t=0} = \frac{h!}{a^h}\,.$$

A.2 Some Beta r.v.'s as finite products of independent Beta r.v.'s

The following well known result is easy to establish using the representation of beta random variables in terms of independent gamma variables.

If

$$Y \sim Beta(a,b)\,, \quad a \in \mathbb{R}^+,\, b \in \mathbb{N}\,,$$

then

$$Y \stackrel{st}{\sim} \prod_{j=1}^{b} Y_j\,,$$

where $Y_j \sim Beta(a+j-1,1)$ are b independent r.v.'s.

Alternatively the result can be obtained by using the mgf's or cf 's of the logarithms of the random variables in question, by observing that

$$E(Y^h) = \frac{\Gamma(a+b)}{\Gamma(a)}\,\frac{\Gamma(a+h)}{\Gamma(a+b+h)}\,, \quad h > -a\,,$$

so that the c.f. of the r.v. $W = -\log Y$ is given by

$$\Phi_W(t) = \frac{\Gamma(a+b)}{\Gamma(a)}\,\frac{\Gamma(a-\mathrm{i}t)}{\Gamma(a+b-\mathrm{i}t)}\,, \quad t \in \mathbb{R}\,.$$

Thus, using the relation

$$\frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{i=0}^{n-1}(a+i)\,, \quad a \in \mathbb{C},\ \in \mathbb{N},$$

we may write

$$\Phi_W(t) = \prod_{j=1}^{b}(a+j-1)(a+j-i-\mathrm{i}t)^{-1}\,,$$

which shows that

$$W \stackrel{st}{\sim} \sum_{j=1}^{b} W_j\,, \qquad \text{where } W_j \sim Exp(a+j-1) \text{ are } b \text{ independent r.v.'s.}$$

But then, using the relation in the previous section of this Appendix,

$$Y = e^{-W} \overset{st}{\sim} \prod_{j=1}^{b} Y_j$$

where

$$Y_j = e^{-W_j} \sim Beta(a + j - 1, 1)$$

are b independent r.v.'s.

Further, if

$$Y_j \sim Beta\left(a + \frac{j-1}{k}, \frac{b}{k}\right) , \qquad (j = 1, \ldots, k)$$

are k independent r.v.'s and if we define

$$Y = \prod_{j=1}^{k} Y_j$$

we will have

$$Y^{1/k} \sim Beta(ka, b) .$$

To verify this we may use the Gamma function multiplication formula,

$$\Gamma(kz) = (2\pi)^{\frac{1}{2}(1-k)} k^{kz-\frac{1}{2}} \prod_{j=0}^{k-1} \Gamma\left(z + \frac{j}{k}\right) ,$$

to write the c.f. of $-\log Y$ as

$$\begin{aligned} \Phi_{-\log Y}(t) &= \prod_{j=1}^{k} \Phi_{-\log Y_j}(t) \\ &= \prod_{j=1}^{k} \frac{\Gamma\left(a + \frac{j-1}{k} + \frac{b}{k}\right) \Gamma\left(a + \frac{j-1}{k} - \mathrm{i}t\right)}{\Gamma\left(a + \frac{j-1}{k}\right) \Gamma\left(a + \frac{j-1}{k} + \frac{b}{k} - \mathrm{i}t\right)} \\ &= \frac{\Gamma(ka + b)\, \Gamma(ka - k\mathrm{i}t)}{\Gamma(ka)\, \Gamma(ka + b - k\mathrm{i}t)} \end{aligned}$$

so that Y has the same distribution as the k-th power of a Beta r.v. with parameters ka and b. Consequently we may write, for $b \in I\!N$,

$$Y^{1/k} \overset{st}{\sim} \prod_{j=1}^{b} U_j^{\frac{1}{ka+j-1}}$$

where U_j are i.i.d. $Unif(0,1)$ r.v.'s, or

$$Y \overset{st}{\sim} \prod_{j=1}^{b} U_j^{\frac{1}{a+\frac{j-1}{2}}}\,.$$

B The GIG and EGIG distributions

We will say that the r.v. W has a GIG (Generalized Integer Gamma) distribution of depth p, with shape parameters $r_j \in I\!N$ and rate parameters δ_j ($j = 1, \ldots, p$), if

$$W = \sum_{j=1}^{p} W_j$$

where

$$W_j \sim \Gamma(r_j, \delta_j)\,, \qquad r_j \in I\!N,\ \delta_j > 0,\ j = 1, \ldots, p$$

are p independent r.v.'s, and $\delta_j \neq \delta_{j'}$ for all $j, j' \in \{1, \ldots, p\}$.

The exact p.d.f. and c.d.f. of W are given by (Coelho, 1998),

$$f^{\mathrm{GIG}}(w; r_j, \delta_j; p) \;=\; K \sum_{j=1}^{p} P_j(w)\, \mathrm{e}^{-\delta_j\, w}\,, \qquad (w > 0) \tag{B.1}$$

and

$$F^{\mathrm{GIG}}(w; r_j, \delta_j; p) \;=\; 1 - K \sum_{j=1}^{p} P_j^*(w)\, \mathrm{e}^{-\delta_j\, w}\,, \qquad (w > 0) \tag{B.2}$$

where $K = \prod_{j=1}^{p} \delta_j^{r_j}$,

$$P_j(w) = \sum_{k=1}^{r_j} c_{j,k}\, w^{k-1} \qquad \text{and} \qquad P_j^*(w) = \sum_{k=1}^{r_j} c_{j,k}\, (k-1)! \sum_{i=0}^{k-1} \frac{w^i}{i!\, \delta_j^{k-i}}$$

with

$$c_{j,r_j} = \frac{1}{(r_j-1)!} \prod_{\substack{i=1\\i\neq j}}^{p} (\delta_i - \delta_j)^{-r_i}\,, \qquad j=1,\dots,p\,,$$

and, for $k=1,\dots,r_j-1$; $j=1,\dots,p$,

$$c_{j,r_j-k} = \frac{1}{k}\sum_{i=1}^{k} \frac{(r_j-k+i-1)!}{(r_j-k-1)!}\, R(i,j,p)\, c_{j,r_j-(k-i)}\,,$$

where

$$R(i,j,p) = \sum_{\substack{k=1\\k\neq j}}^{p} r_k\,(\delta_j-\delta_k)^{-i} \qquad (i=1,\dots,r_j-1)\,.$$

We will then say that the r.v. $V=e^{-W}$ has an EGIG (Exponentiated Generalized Integer Gamma) distribution of depth p, with shape parameters r_j and rate parameters δ_j, with p.d.f. and c.d.f. respectively given by

$$f^{\text{EGIG}}(v;r_j,\delta_j;p) = f^{\text{GIG}}(-\log\, v;r_j,\delta_j;p)\,\frac{1}{v} \qquad (0<v<1) \tag{B.3}$$

and

$$F^{\text{EGIG}}(v;r_j,\delta_j;p) = 1 - F^{\text{GIG}}(-\log\, v;r_j,\delta_j;p) \qquad (0<v<1)\,. \tag{B.4}$$

Note that the h-th moment of V is, for $h > -\min\limits_{1\le j\le p} \delta_j$, given by

$$E(V^h) = E\left(e^{-hW}\right) = \prod_{j=1}^{p} E\left(e^{-hW_j}\right) = \prod_{j=1}^{p} \delta_j^{r_j}\,(\delta_j+h)^{-r_j}\,,$$

while the h-th moment of W, for $h\in I\!N$, is given by

$$\begin{aligned} E(W^h) &= E\left[\left(\sum_{j=1}^{p} W_j\right)^h\right] = (-1)^h\,\frac{d^h}{dt^h}\,E(V^t)\Big|_{t=0} \\ &= \sum_{j=1}^{N} \frac{h!}{\prod_{\ell=1}^{p} d_{j\ell}!}\,\prod_{\ell=1}^{p} \frac{\Gamma(r_\ell+d_{j\ell})}{\Gamma(r_\ell)}\,\delta_\ell^{-d_{j\ell}}\,, \end{aligned}$$

with

$$N = \sum_{j=1}^{p} \binom{h-1}{j-1} \binom{p}{j} = \binom{h+p-1}{p-1},$$

and where $d_{j\ell}$ is a non-negative integer which represents the ℓ-th element in the j-th partition of h into p integers ranging from zero to h, so that

$$\frac{\Gamma(r_\ell + d_{j\ell})}{\Gamma(r_\ell)} = \prod_{i=0}^{d_{j\ell}-1} (r_\ell + i)\,,$$

with any empty product evaluated as 1.

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