ENDOMORPHISMS OF THE SEMIGROUP OF ORDER-PRESERVING MAPPINGS

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ABSTRACT. We characterize the endomorphisms of the semigroup of all order-preserving mappings on a finite chain. We show that there are three types of endomorphism: automorphisms, constants, and a certain type of endomorphism with two idempotents in the image.

1. INTRODUCTION & THE MAIN THEOREM

A mapping $f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ is called *order-preserving* if $(i)f \leqslant (j)f$ whenever $i \leqslant j$. We write functions to the right of their argument and compose them from left to right. The semigroup of all order-preserving mappings from $\{1, 2, \ldots, n\}$ to itself under composition of functions is denoted O_n. The semigroup O_n has been extensively studied by many authors since the 1960s. The identity of O_n is denoted by 1_n. An *endomorphism* ϕ of a semigroup S is a mapping $\phi : S \longrightarrow S$ such that $(x)\phi(y)\phi = (xy)\phi$ for all $x, y \in S$. We will denote the semigroup of endomorphisms of S by $\mathrm{End}(S)$. A bijective endomorphism is called an *automorphism*. In this note we completely describe the endomorphisms of O_n for all $n \in \mathbb{N}$ and specify the number of these endomorphisms.

In 1962, Aĭzenštat [1] gave a presentation for O_n from which it can be deduced that the only non-trivial automorphism of O_n where $n > 1$ is that given by conjugation by the permutation $\sigma = (1\,n)(2\,n-1)\cdots(\lfloor n/2\rfloor\;\lceil n/2\rceil + 1)$. In other words, if ϕ is a non-identity automorphism of O_n and $n > 1$, then $(s)\phi = \sigma^{-1}s\sigma$ for all $s \in O_n$. We will write f^σ to denote $\sigma^{-1}f\sigma$.

The endomorphisms of O_n are described in the following theorem.

Theorem 1.1. *Let $\phi : O_n \to O_n$ be any mapping. Then ϕ is an endomorphism of O_n if and only if one of the following holds:*

(a) *ϕ is an automorphism;*
(b) *there exist idempotents $e, f \in O_n$ with $e \neq f$ and $ef = fe = f$ such that $1_n\phi = e$ and $(O_n \setminus \{1_n\})\phi = f$;*
(c) *ϕ is a constant mapping with idempotent value.*

Corollary 1.2. *If $n > 1$, then* $|\mathrm{End}(O_n)| = 2 + \sum_{i=0}^{n-1} \binom{n+i}{2i+1} F_{2i+2}$.

The remainder of the note is dedicated to proving Theorem 1.1 and its corollary. To do so we require the following notions. The *image* of an element $f \in O_n$ is denoted by $\mathrm{im}(f)$ and the *kernel* of f is the equivalence relation $\{\,(x, y) \in \{1, \ldots, n\} \times \{1, \ldots, n\} : xf = yf\,\}$ denoted by $\ker(f)$. The *rank* of an element $f \in O_n$ is $|\mathrm{im}(f)|$ and denoted $\mathrm{rank}(f)$. Some of the important properties of O_n that we require later are: it is regular, its Green's relations are described by

$$\begin{array}{lll} f\mathcal{L}g & \text{if and only if} & \text{im}(f) = \text{im}(g) \\ f\mathcal{R}g & \text{if and only if} & \ker(f) = \ker(g) \\ f\mathcal{D}g & \text{if and only if} & \text{rank}(f) = \text{rank}(g) \\ f\mathcal{H}g & \text{if and only if} & f = g, \end{array}$$

and the number of idempotent elements is the $2n$ th Fibonacci number F_{2n}. Further information regarding O_n can be found in [3] and [4] and regarding semigroups, in general, can be found in [5]. We denote the $\mathcal{D}$-class of those elements in O_n with rank k by D_k.

It is well-known that I is an ideal of O_n if and only if $I = \{f \in O_n : \text{rank}(f) \leqslant k\}$ for some $1 \leqslant k \leqslant n$; for a proof see [6]. In 1962, Aĭzenštat [2] showed that the non-trivial congruences of O_n are exactly those where the only non-singleton class is I_k for some $1 \leqslant k \leqslant n$. Another proof of this can be found in [8].

2. Proof of Theorem 1.1

It is straightforward to verify that the mappings described in Theorem 1.1 are endomorphisms of O_n. So, it remains to prove that there are no further endomorphisms. Throughout the remainder of the note we will assume that $n > 1$.

Let ϕ be an endomorphism of O_n. If ϕ is an automorphism of O_n or a constant mapping, then ϕ satisfies (a) or (c) in Theorem 1.1. So, we may assume that ϕ is not an automorphism and not constant. From the comments in the introduction, there exists $1 \leqslant k \leqslant n-1$ such that the unique non-singleton kernel class of ϕ is I_k. If $k = n-1$, then ϕ is of type (b) from Theorem 1.1. Hence we may assume that $1 \leqslant k \leqslant n-2$. In the following lemmas we will prove that this is not possible and so conclude the proof.

Note that ϕ has only singleton kernel classes on $O_n \backslash I_k$ and so ϕ must be injective on $O_n \backslash I_k$. The unique element of $I_k\phi$ is an idempotent. Throughout the remainder of the note we will denote this idempotent by f.

Lemma 2.1. *Let $g, h \in D_i$ where $k+1 \leqslant i \leqslant n-1$. Then $g\mathcal{R}h$ if and only if $g\phi\mathcal{R}h\phi$. Likewise, $g\mathcal{L}h$ if and only if $g\phi\mathcal{L}h\phi$.*

Proof. We prove the theorem only for Green's $\mathcal{R}$-relation, the proof for Green's $\mathcal{L}$-relation is analogous.

($\Rightarrow$) Since ϕ is a homomorphism, this implication is immediate.

($\Leftarrow$) Let $g\phi\mathcal{R}h\phi$. As I_i is a regular subsemigroup of O_n, it follows that $I_i\phi$ is a regular subsemigroup of O_n. Thus, from [5, Proposition 2.4.2], $\mathcal{R}^{I_i\phi} = \mathcal{R} \cap (I_i\phi \times I_i\phi)$. Thus $g\phi\mathcal{R}^{I_i\phi}h\phi$ and so there exist $a, b \in I_i$ with $h\phi = g\phi \cdot a\phi$ and $g\phi = h\phi \cdot b\phi$.

If $\text{rank}(ga) \leqslant k$, then $(ga)\phi = f$ and so $h\phi = f$, a contradiction. Hence $\text{rank}(ga) > k$ and likewise $\text{rank}(hb) > k$. It follows, since ϕ is injective on $O_n \setminus I_k$, that $h = ga$ and $g = hb$. In other words, $g\mathcal{R}h$. □

Lemma 2.2. *$D_{k+1}\phi \subseteq D_l$ where* $\text{rank}(f) < l < k+1$.

Proof. Since ϕ is injective on $D_{k+1} \cup D_{k+2} \cup \cdots \cup D_n$ and ϕ preserves $\leqslant_{\mathcal{D}}$, it follows that $l \leqslant k+1$. As a consequence, $\text{rank}(f) \leqslant k$ and $f\phi = f$.

Assume that $l = k+1$. Then $D_{k+1}\phi = D_{k+1}$, since $\phi|_{D_{k+1}}$ is injective. Let $d \in D_{k+1}$ with $fd \neq f$. Note that $d\phi^{-1}$ is a set containing a single element and it is contained in D_{k+1}. Hence

$$f \neq f \cdot d = f\phi \cdot d = (f \cdot d\phi^{-1})\phi = f,$$

a contradiction. Therefore $l < k+1$, as required.

Assume that $\text{rank}(f) = l$ and let $g \in D_{k+1}$. Then $g\phi \mathcal{D} f$. Since $g\phi \cdot f = f \cdot g\phi = f$, it follows that $\text{im}(f) = \text{im}(g\phi)$ and $\ker(f) = \ker(g\phi)$. Thus $f \mathcal{H} g\phi$ and so $f\phi = f = g\phi$, contradicting the assumption that f and g are in different kernel classes of ϕ. □

Lemma 2.3. $k \geqslant \lfloor n/2 \rfloor$.

Proof. By Lemma 2.1, the number of $\mathcal{L}$-classes in $D_{k+1}\phi$ is $\binom{n}{k+1}$. But, by Lemma 2.2, $D_{k+1}\phi \subseteq D_l$ where $l < k+1$ and the number of $\mathcal{L}$-classes in D_l is $\binom{n}{l}$. Hence $\binom{n}{k+1} \leqslant \binom{n}{l}$ and so $k \geqslant \lfloor n/2 \rfloor$. □

The following lemma is straightforward but we include a proof for the sake of completeness.

Lemma 2.4. *There exist idempotents $e_1, \ldots, e_{n-1} \in D_{k+1}$ such that either $e_i e_j \in I_k$ or $e_j e_i \in I_k$ for all $1 \leqslant i < j \leqslant n-1$.*

Proof. Let $g_i, h_i \in O_n$ be defined by

$$(j)g_i = \begin{cases} i & j \in \{i, i+1, \ldots, i+n-k-1\} \\ j & j < i \text{ or } j > i+n-k-1 \end{cases}$$

for all $1 \leqslant i \leqslant k+1$ and

$$(j)h_i = \begin{cases} i & j \in \{i, i+1\} \\ i+2 & j \in \{i+2, i+3, \ldots, i+n-k\} \\ j & j < i \text{ or } j > i+n-k \end{cases}$$

for all $1 \leqslant i \leqslant k$. Then $E = \{g_1, \ldots, g_{k+1}, h_1, \ldots, h_k\}$ are all idempotents in D_{k+1} satisfying conditions (i) and (ii). If $k = n-2$, then $g_i = h_i$ for all $1 \leqslant i \leqslant k$, and so E contains $n-1$ elements. If $k < n-2$, then $|E| = 2k+1$. From Lemma 2.3, we have that $|E| \geqslant n-1$, as required. □

Proof of Theorem 1.1. Let $g \in O_n$ be arbitrary. Then $\text{im}(f) \subseteq \text{im}(g\phi)$ as $f \cdot g\phi = f$. Let $e_1, \ldots, e_{n-1}$ be the idempotents from Lemma 2.4. If $i \neq j$, then $e_i e_j \in I_k$ or $e_j e_i \in I_k$. Hence $e_i\phi \cdot e_j\phi = f$ or $e_j\phi \cdot e_i\phi = f$. Thus every element in $\text{im}(e_i\phi) \cap \text{im}(e_j\phi)$ is fixed by f (as $e_i\phi$ and $e_j\phi$ are idempotents). It follows that

$$\text{im}(e_i\phi) \cap \text{im}(e_j\phi) = \text{im}(f),$$

for all $i \neq j$.

Let $E_i = \text{im}(e_i\phi) \setminus \text{im}(f)$ for all i. Then $E_1, \ldots, E_{n-1}$ are pairwise disjoint. Since $e_1\phi, \ldots, e_{n-1}\phi \in D_{k+1}\phi$ it follows from Lemma 2.2 that $|E_1| = \cdots = |E_{n-1}| \geqslant 1$. It follows that $|E_1 \cup \cdots \cup E_{n-1}| \geqslant n-1$. Therefore $|E_i| = 1$ for all i and $|\text{im}(f)| = 1$. In other words, $D_{k+1}\phi \subseteq D_2$ and, again, $\text{im}(f) \subseteq \text{im}(g\phi)$ for all $g \in D_{k+1}$. Hence $D_{k+1}\phi$ contains at most $n-1$ different $\mathcal{L}$-classes, and so, by Lemma 2.1, D_{k+1} has at most $n-1$ different $\mathcal{L}$-classes. Thus $k+1 = n$, a contradiction, and so every endomorphism of O_n is of type (a), (b), or (c). □

Proof of Corollary 1.2. We must prove that

$$|\operatorname{End}(O_n)| = 2 + \sum_{i=0}^{n-1} \binom{n+i}{2i+1} F_{2i+2}.$$

If X is a subset of O_n, then denote by $E(X)$ the set of all idempotents in X. It was shown in [7, Corollary 4.4] and [4, Theorem 2.3] that

$$|E(D_{i+1})| = \binom{n+i}{2i+1}$$

and

$$|E(O_n)| = F_{2n}$$

where F_{2n} is the $2n$ th Fibonacci number.

Let $e \in E(O_n)$ be arbitrary and let $S(e) = \{ f \in E(O_n) \ : \ ef = fe = f \}$. Then the numbers of endomorphisms of O_n of types (b) and (c) where $1_n\phi = e$ are $|S(e)| - 1$ and 1, respectively.

Let $e \in D_{i+1}$ where $0 \leqslant i \leqslant n-1$. Then we will prove that $|S(e)| = F_{2i+2}$.

Let $O_{\mathrm{im}(e)}$ be the semigroup of order-preserving mappings on $\mathrm{im}(e)$ and let $\Psi : S(e) \longrightarrow O_{\mathrm{im}(e)}$ be defined so that $(f)\Psi$ is the restriction $f|_{\mathrm{im}(e)}$ of f to $\mathrm{im}(e)$. If $f \in S(e)$, then $fe = f$ and so $\mathrm{im}(f) \subseteq \mathrm{im}(e)$. Hence Ψ is well-defined. Moreover $f|_{\mathrm{im}(e)}$ fixes $\mathrm{im}(f)$ pointwise, and so $f|_{\mathrm{im}(e)} \in E(O_{\mathrm{im}(e)})$.

We will prove that Ψ is a bijection from $S(e)$ to $E(O_{\mathrm{im}(e)})$. If $f \in E(O_{\mathrm{im}(e)})$, then $e \cdot ef = ef$ and $ef \cdot e = ef$ as $\mathrm{im}(f) \subseteq \mathrm{im}(e)$. It follows that $ef \in S(e)$ and $(ef)\Psi = (ef)|_{\mathrm{im}(e)} = f|_{\mathrm{im}(e)} = f$. That is, $\mathrm{im}(\Psi) = E(O_{\mathrm{im}(e)})$.

If $f, g \in S(e)$ such that $f|_{\mathrm{im}(e)} = g|_{\mathrm{im}(e)}$, then $f = ef = e \cdot f|_{\mathrm{im}(e)} = e \cdot g|_{\mathrm{im}(e)} = eg = g$, and so Ψ is injective. So, $|S(e)| = |E(O_{\mathrm{im}(e)})|$ and from above $|E(O_{\mathrm{im}(e)})| = F_{2|\mathrm{im}(e)|} = F_{2i+2}$.

Therefore there are

$$|E(D_{i+1})|(|S(e)| - 1) = \binom{n+i}{2i+1}(F_{2i+2} - 1) \tag{1}$$

endomorphisms of type (b) where $1_n\phi \in D_{i+1}$.

There are two automorphisms and F_{2n} constant endomorphisms. Summing these two values and (1) over all i we obtain the required value. □

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