Generalized synchronization in linearly coupled time periodic systems

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Abstract: We consider the synchronization of a network of linearly coupled and not necessarily identical oscillators. We present an approach to the existence of the synchronization manifold which is based on some results developed by R. Smith for the study of periodic solutions of ODEs. Our framework allows the study of a large class of systems and does not assume that the systems are small perturbations of linear systems. Moreover, it provides a practical way to compute estimations on the parameters of the system for which generalized synchronization occurs. Additionally, we give a new proof of the main result of R. Smith on invariant manifolds using Wazewski's principle. Several examples of application are presented.

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1 Introduction

The major purpose of this paper is to show that some results obtained by R. Smith in [7], [8] to study the periodic solutions of systems of ordinary differential equations may be exploited in the framework of synchronization theory. The main result we present is a sufficient condition for the synchronization

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of a network of m linearly coupled and not necessarily identical oscillators. Here the term oscillator is used in a quite loose sense, and we think of 'oscillator' and 'system' as interchangeable terms. Each oscillator of the network is represented by a first order n-dimensional time periodic system of ordinary differential equations of the form $x_i' = f_i(x_i, t),\ \ x_i \in \mathbb{R}^n,\ \ i = 1, \cdots, m$, and the state equation of the coupled systems is the following:

$$\begin{cases} x_1' = f_1(x_1, t) + \sum_{i=1}^m D_{1,i} x_i \\ \vdots \\ x_m' = f_m(x_m, t) + \sum_{i=1}^m D_{m,i} x_i \end{cases},$$

where the matrix $D_{i,j} \in M_{n\times n}(\mathbb{R})$ describes the coupling between the oscillators i and j. Introducing the square matrix

$$D = \begin{pmatrix} D_{1,1} & \dots & D_{1,m} \\ \vdots & & \vdots \\ D_{m,1} & \dots & D_{n,m} \end{pmatrix}$$

and setting $x = (x_1, \dots, x_m)^T \in \mathbb{R}^{nm}$, $F(x) = (f_1(x_1), \dots, f_m(x_m))^T$, we can rewrite the system above as

$$x' = F(x, t) + Dx. \tag{1}$$

The case of a network of m identical oscillators is the more present in the literature (see [1], [12], [6] and references therein.) In this case, $f_1 = f_2 = \cdots = f_m$, and it is said that system (1) synchronizes if there exists a global invariant attractor for its solutions $x(t) = (x_1, (t), \cdots, x_m(t))^T$ which is contained the n-dimensional diagonal in $\mathbb{R}^{nm}$, defined by $x_1 = x_2 = \cdots, = x_m$. This implies that every solution of (1) satisfies

$$\lim_{t\to+\infty} ||x_i(t) - x_j(t)|| = 0,$$

for every $i, j = 1, \cdots, m$, meaning that all the oscillators behave asymptotically in the same manner. Therefore, we can determine in a trivial way the asymptotic behavior of the state vector x by asymptotic behavior of anyone of its component sub-vectors $x_k,\ \ k = 1, \cdots, m$. This is the property which one wants to retain in passing from the identical case to the more general setting, in which the diagonal is no longer invariant. Accordingly, we shall say that there is generalized synchronization for system (1) if there exists an n-dimensional time periodic manifold $\mathcal{A}_t$ that attracts the orbits of (1) and which, for any fixed j, is a graph of a function of x_j and t. As a consequence, like in the identical case, from the existence of the synchronization manifold

$\mathcal{A}_t$ one can get a functional dependence of the form $x_i = \psi_i(x_j, t)$ between any two state sub-vectors x_i and x_j such that

$$\lim_{t \to +\infty} ||x_i(t) - \psi_i(x_j(t), t)|| = 0$$

along the solutions of (1). Thus, the asymptotic behavior of the network may be determined from the behavior of anyone of its oscillators. The above discussion follows closely the one presented in [6]. Other approaches can be found in the literature. For example, in [4] the dependence between x_i and x_j is not required to be one-to-one and in [2] $\mathcal{A}_t$ is a required to be a graph over the diagonal.

If the attracting property of $\mathcal{A}_t$ is limited to the solutions of (1) which are bounded in future, we talk of generalized bounded synchronization. This is the property we will consider in our main result, Theorem 2.3, which gives a sufficient condition for the generalized bounded synchronization of system (1). Note that in the important case of dissipative systems generalized synchronization and bounded generalized synchronization coincide.

As discussed above, in the case of a network of identical oscillators the natural candidate to synchronization manifold is the diagonal subspace. The methods used to prove its attractiveness make use, essentially, of Lyapunov functions ([12]) or of Lyapunov exponents ([6]). However, when the oscillators are not identical, the very existence of the synchronization manifold becomes an issue. The survey [2] presents several results on the existence of the synchronization manifold based on the classical theory on existence of invariant manifolds. In particular, the systems are seen as perturbations of linear systems. In fact, it is assumed that the linear part of the systems is given by coupling matrices whose eigenvalues go to $-\infty$ as the coupling parameters goes to $+\infty$, and therefore, for the large coupling parameters for which the invariant manifold is obtained, the linear part dominates the nonlinear terms.

This assumption is not needed in the general framework we propose here to get the existence of $\mathcal{A}_t$ (see Example 2 in Section 5.) Our results rely on a nice theory about the existence of invariant manifolds obtained in the eighties by R. Smith to prove Massera's-type theorems for a specific class of nonlinear time periodic differential equations [8]. This class of differential equation satisfies certain hypotheses which we we recall at the end of Section 2 in condition (**H**). If this condition is satisfied, then it is possible to single out certain solutions of the differential equation, called amenable solutions, and show that the union of their values at any fixed time t forms an n-dimensional manifold. This manifold is our candidate to synchronization manifold. Our main result then follows from Lemma 3.6 which establishes that the amenable manifold actually attracts the bounded solutions of system (1). R. Smith's theory was already used in [5] to prove the existence of invariant one-dimensional manifolds in systems of equations with a cylindrical phase space, such as the planar pendulum or systems of coupled pendula.

The paper is organized as follows: In Section 2 we precise further the setting in which we work, give the definition of generalized synchronization and state our main result.

In Section 3 we present a new proof of Smith's result. Our proof has a more explicit geometrical flavor than the original one and is obtained by making use of Wazewski's principle. Besides making the paper more self contained, we think that it may be of some interest in itself.

In Section 4, we discuss some sufficient conditions for (**H**). These conditions are derived from the ones presented in [8] and emphasize the practical interest of our approach to synchronization. In particular, they make it possible to deal with several systems and coupling schemes presented in the literature.

In Section 5 we present some examples of applications. More precisely, in Example 1 we consider a system of two, two-way coupled, n-dimensional systems ([2]), in Example 3 a network formed by two different Lorentz systems coupled by means of a driven-response scheme ([6]), and in Example 4 an array of fully connected coupled oscillators ([12]). The purpose of these three examples is to show that using our approach it is not too hard to obtain estimates on the parameters for which generalized synchronization occur. Finally, Example 2 shows that our method can be used to prove generalized synchronization in systems that can not be seen as perturbations of linear systems. In general, it seems that these systems cannot be dealt with as easily via the classical invariant manifold theory.

2 Assumptions and main result

Throughout this paper we will assume that the function $F : \mathbb{R}^{nm} \times \mathbb{R} \to \mathbb{R}^{nm}$ in system (1) is continuous in the (x, t) variables, locally Lipschitz continuous in the x variable, and T-periodic in t. This ensures that there is existence and uniqueness of solutions for system (1) and consequently the solutions vary continuously with the initial conditions.

We denote by $x(t; t_0, x_0) := (x_1(t; t_0, x_0), \ldots, x_m(t; t_0, x_0))^T \in \mathbb{R}^{nm}$, with $x_i(t; t_0, x_0) \in \mathbb{R}^n$, the solution of system (1) which satisfies the initial condition $x(t_0; t_0, x_0) = x_0 \in \mathbb{R}^{nm}$..

Definition 2.1 *We shall say that a n-dimensional submanifold M of R^{nm}*

is diagonal-like, if the projection $\Pi_i : M \subset \mathbb{R}^{nm} \to \mathbb{R}^n$, $\Pi_i(x) = x_i$ *is an homeomorphism, for each* $i = 1, \ldots, m$.

Observe that if M is an n-dimensional diagonal-like submanifold then each $x = (x_1, \ldots, x_m) \in M$ is completely determined once we know one of the x_i, $i = 1, \ldots, m$.

Definition 2.2 *We shall say that there is generalized (bounded generalized) synchronization for system* (1) *if for each* $t \in \mathbb{R}$ *there is an* n*-dimensional diagonal-like submanifold* $\mathcal{A}_t \subset \mathbb{R}^{nm}$, *periodic in* t, *that is an attracting manifold for every (bounded in the future) solution, i.e.*

$$dist(\mathcal{A}_t, x(t; t_0, x_0)) \to 0$$

when $t \to +\infty$ *for every* (t_0, x_0) *(for every* (t_0, x_0) *which corresponds to a solution bounded in the future.) We will call such manifold* $\mathcal{A}_t$ *a synchronization manifold.*

It follows that if there is generalized synchronization for system (1), then we can obtain the asymptotic behavior of the full system from the asymptotic behavior of either of its n dimensional state vectors x_i.

In many applications there is an absorbing set for the system of coupled oscillators. In this case all the solutions of (1) are bounded in the future, and the two definitions considered above are equivalent.

When all the oscillators are identical to each other, i.e. $f_1 = f_2 = \ldots = f_m$ it is usually assumed that the oscillators are decoupled in the diagonal, i.e.

$$\sum_{j=1}^{m} D_{i,j} = 0, \tag{2}$$

(this is the case studied in [12]). In this conditions the subspace

$$\Delta = \{x = (x_1, \cdots, x_m) \in \mathbb{R}^{nm} : x_i \in \mathbb{R}^n \text{ and } x_i = x_j \quad \forall i, j\}$$

is an invariant manifold for (1) and we could expect to obtain generalized synchronization with $\mathcal{A}_t = \Delta$, $\forall t \in \mathbb{R}$. In this particular case we say that we have identical synchronization.

As already mentioned in the previous section, in the case of generalized synchronization, the existence of a candidate to synchronization manifold may be obtained from a general result given by R. Smith in [8] for a certain class of differential equations. In our setting, this class satisfies the following assumption:

(**H**) *there exist constants* $\lambda > 0$, $\epsilon > 0$, *and a constant real symmetric matrix* P *with precisely* n *negative eigenvalues, such that*

$$(x-y)^T P[F(x,t) - F(y,t) + (D+\lambda I)(x-y)] \le -\epsilon \|x-y\|^2,$$

for all $x, y \in \mathbb{R}^{nm}$ *and* $t \in \mathbb{R}$.

The result by Smith concerns certain solutions of (1), called amenable solutions. We recall that a solution $x(\cdot)$ of (1) is amenable if the integral

$$\int_{-\infty}^{t_0} e^{2\lambda t} \|x(t)\|^2 dt$$

converges. Note that any solution that is bounded in the past is amenable. For each $t \in \mathbb{R}$ we define the amenable set

$$\mathcal{A}_t = \{x(t) \, : \, x(\cdot) \text{ is an amenable solution of } (1)\}.$$

In [8] it is proved that if (**H**) holds and all solutions of (1) are defined in future, then for each $t \in \mathbb{R}$ the amenable set $\mathcal{A}_t$ is an n dimensional manifold. More precisely, in [8] it is proved that $\mathcal{A}_t$ is the graph of a globally Lipschitz continuous function whose domain is the n dimensional subspace V_- spanned by the eigenvectors of P associated to negative eigenvalues. The proof of this fact splits into several steps, whose geometrical content may be summarized as follows: the amenable manifold is obtained as a limit of a sequence of graphs $\mathcal{G}_n$ of globally equi-Lipschitz continuous functions defined on V_-. Each set $\mathcal{G}_n$ is defined as $\mathcal{G}_n := x(t; t_n, \bar{x}(t_n) + V_-)$, where $\bar{x}(\cdot)$ is a fixed amenable solution of (1) and $t_n \to -\infty$.

However, as we will see in the next section, the geometry of the flow of system (1) when (**H**) holds naturally suggests an approach to the existence of the amenable manifold based on Wazewski's retract method ([10], [11]). This is a method used in the theory of differential equations to prove the existence of solutions which remain in a given set in the future (or in the past.) We will recall the statement of Wazewski's principle in Section 3. To apply this method the boundary points of the set must satisfy a 'strict egress condition' and the set of the strict egress point must not be a retract of the whole set. Condition (**H**) will allow to define a set which satisfy both these conditions. Moreover, the solutions remaining in the past in such set will be the amenable ones.

We are now in a position to state the main result of this paper:

Theorem 2.3 *Assume that all the solutions of system* (1) *are defined in* $\mathbb{R}$. *If system* (1) *satisfies* (***H***) *and has at least one amenable solution, then,* *after at most one linear change of coordinates, there is bounded generalized synchronization. Moreover, the synchronization manifold $\mathcal{A}_t$ is the amenable set.*

In Corollary 3.7 we give conditions on P under which generalized synchronization occurs in the original coordinates x_i.

Some comments to the statement of Theorem 2.3 are in order. We start by noticing that the assumption about the existence of an amenable solution of system (1) is fulfilled if the system has either an equilibrium point or a nontrivial periodic solution. Actually, as a consequence of a) in Lemma 3.6, for the existence of an amenable solution it is sufficient that there exists a trajectory of (1) that is bounded in the future.

Moreover, two hypotheses made in the statement of Theorem 2.3 may be relaxed.

The first of such assumptions is the one made about the structure of system (1). In fact, the restriction that the components f_i of $F(x)$ depend only of x_i was made here only to fit our setting of linearly coupled oscillators, but plays no role in the proof of the existence of the amenable manifold. It follows that our approach is not limited to linearly coupled systems, but may be used to tackle nonlinear couplings.

The second hypothesis that can be weakened is the one about the domain of the solutions of (1). In fact, the amenable manifold was obtained by Smith assuming that each solution of (1) is defined in an interval of the form $(\theta, +\infty)$, which is a weaker condition than the one we considered throughout this paper. However, our less general setting permits to give an alternative proof of Smith's result which exploit the link between condition (**H**) and Wazewski's topological principle. We believe that, besides making more self contained the paper, our proof may be of some independent interest. Finally, we note that from a practical point of view our framework is in many cases equivalent to Smith's. In fact, the sufficient conditions for (**H**) to hold, presented in Section 4 and used in many applications, assume that F is globally Lipschitz continuous on x. Obviously, in such a case the solutions of (1) are defined in $\mathbb{R}$.

3 Existence of the synchronization manifold and proof of the main result

In this section we prove our main result, namely Theorem 2.3. The proof will immediately follows from several lemmas. The first one, Lemma 3.3, collects some basics facts proved in [8] for the amenable set and of which we make use. In Lemma 3.4 we use Wazewski's theorem to prove that $\mathcal{A}_t$ is a n dimensional manifold. In Lemma 3.5 we show that after a change of coordinates this manifold is diagonal like. Finally, in Lemma 3.6 we prove that the manifold of the amenable solutions is an attracting manifold for the bounded solutions of system (1).

We start by discussing some geometrical features of condition (**H**) which lead in a natural way to consider an application of Wazewski's topological principle. For the reader's sake, we will recall below the statement of Wazewski's theorem.

Let $V(x) := x^T Px$. Then, it is easy to see that the inequality in (**H**) is equivalent to the following:

$$\frac{d}{dt}\{e^{2\lambda t}V(x(t)-y(t))\} \leq -e^{2\lambda t}\epsilon||x(t)-y(t)||^2 \tag{3}$$

for any pair $x(\cdot)$, $y(\cdot)$ of solutions of (1) and for any $t \in \mathbb{R}$. Note that an immediate consequence of (3) is that the function $t \to e^{2\lambda t}V(x(t)-y(t))$ is strictly decreasing in its domain. Therefore, if we define the cone

$$C := \{x \in \mathbb{R}^{nm} : V(x) < 0\}$$

and consider any solution $x(\cdot)$ of (1) we have that the time dependent set

$$x(t) + \overline{C} \subset \mathbb{R}^{nm}, \quad t \in \mathbb{R},$$

where $\overline{C}$ denotes the closure of C, attracts in future all the solutions of (1) that start outside it. In fact, such solutions move through the leaves

$$\mathcal{L}^t_\alpha := \{x \in \mathbb{R}^{nm} \ : \ V(x(t)-x) = \alpha, \quad \alpha \in \mathbb{R}\}$$

of the foliations $\mathcal{L}^t := \cup_\alpha \mathcal{L}^t_\alpha$ of $\mathbb{R}^{nm}$ in such a way that α decreases for increasing time. As a consequence, these solutions tend to approach the boundary $x(t) + \partial C$ of the cone $x(t) + C$. In particular, we note that if $x(\cdot)$ and $y(\cdot)$ are solutions of (1) satisfying $V(x(\theta) - y(\theta)) = 0$ for some $\theta \in \mathbb{R}$ (i.e. $y(\theta) \in x(\theta) + \partial C$ or, equivalently, $x(\theta) \in y(\theta) + \partial C$) then $V(x(t) - y(t)) < 0, \quad \forall t \in (\theta, +\infty)$, (i.e. $y(t) \in x(t) + C$ or, equivalently, $x(t) \in y(t) + C, \ \forall t \in (\theta, +\infty)$, and $V(x(t) - y(t)) > 0, \quad \forall t \in (-\infty, \theta)$, (i.e. $y(t) \notin x(t) + C$ or, equivalently, $x(t) \notin y(t) + C, \ \forall t \in (-\infty, \theta)$.

By the discussion above, it follows that we may consider the inequality in (**H**) as a dissipation condition.

Let us recall now the statement of Wazewski's topological principle. We start by introducing the proper setting. Let $f : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k, \ (x,t) \to f(x,t)$

be a continuous function which is locally Lipschitz continuous in the first variable. For $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^k$ consider the Cauchy problem

$$\begin{cases} y' = f(y,t) \\ y(t_0) = y_0 \end{cases} \tag{4}$$

We denote by $y(t;t_0,y_0)$ the unique solution of (4) and by $(\alpha(t_0,y_0),\omega(t_0,y_0)) \subset \mathbb{R}$ its maximal interval of definition. Let $\Omega \subset \mathbb{R}^k \times \mathbb{R}$ be an open set.

Definition 3.1 *A point $(y_0,t_0) \in \partial\Omega$ is called an ingress point for $y' = f(y,t)$ if there exists $\epsilon > 0$ such that $(t, y(t;t_0,y_0)) \in \Omega$ for every $t \in (t_0, t_0+\epsilon]$. Moreover if $(t, y(t;t_0,y_0)) \notin \overline{\Omega}$ for any $t \in (t_0-\epsilon, t_0)$ then (y_0,t_0) is called a strict ingress point for $y' = f(y,t)$.*

We denote by Ω_i and Ω_{si}, respectively, the set of ingress points and the set of strict ingress points. Of course, $\Omega_{si} \subset \Omega_i \subset \partial\Omega$. Finally, recall that if X is a topological space and $A \subset X$ is a subspace, we say that A is a retract of X if there exists a continuous map $r : X \to A$ such that $r(x) = x$ for any $x \in A$. The map r is called a retraction.

We are now ready to state the main topological result that will be used in this section.

Theorem 3.2 (Wazewski's principle) *Assume that $\Omega_i = \Omega_{si}$. Let $S \subset \Omega \cup \Omega_i$ such that $S \cap \Omega_i$ is a retract of Ω_i and $S \cap \Omega_i$ is not a retract of S. Then, there exists $(y_0,t_0) \in S \cap \Omega$ such that the corresponding solution of* (4) *satisfies $(y(t;t_0,y_0),t) \in \Omega$ for any $t \in (\alpha(t_0,y_0), t_0]$.*

As a final step before presenting our results, let us summarize some facts established in [8] for the amenable set $\mathcal{A}_t$. These facts are a straight consequence of assumption (**H**). In what follows we denote by V_- and V_+ the subspaces of $\mathbb{R}^{nm}$ spanned, respectively, by the eigenvectors of P corresponding to negative, respectively positive, eigenvalues. These subspaces, of dimensions, respectively, n and $nm-n$, are orthogonal and complementary, that is $\mathbb{R}^{nm} = V_- \perp V_+$. We denote by $\mathcal{P}_-$ the orthogonal projection of $\mathbb{R}^{nm}$ onto V_-.

Lemma 3.3 *Assuming* (1) *and the existence of an amenable solution $\bar{x}(\cdot)$ of* (1)*, the following holds:*
i) a solution $y(\cdot)$ of (1) different from $\bar{x}(\cdot)$ is amenable iff $V(x(t)-y(t)) < 0$ for any $t \in \mathbb{R}$.
ii) $\mathcal{P}_-$ is an homeomorphism between $\mathcal{A}_t$ and $\mathcal{P}_-(\mathcal{A}_t) \subset V_-$. Moreover, $\mathcal{A}_t$ is the graph of a globally Lipschitz continuous function.

From a geometrical point of view, $i)$ of the previous lemma implies that

$$\mathcal{A}_t \setminus \{\bar{x}(t)\} \subset \bar{x}(t) + C$$

for any $t \in \mathbb{R}$.

We are finally in a position to give our first result:

Lemma 3.4 *Assume that all the solutions of system* (1) *are defined in* $\mathbb{R}$. *If* ($\boldsymbol{H}$) *holds and there is at least one amenable solution* $\bar{x}(\cdot)$, *then for each* $t_0 \in \mathbb{R}$ *the restriction* $\mathcal{P}_- : \mathcal{A}_{t_0} \to V_-$ *is an homeomorphism between* $\mathcal{A}_{t_0}$ *and* V_-.

Proof.

In order to enter the setting of Wazewski's topological principle, we define the open set

$$\Omega := \{(x,t) \in \mathbb{R}^{nm} \times \mathbb{R} : x \in \bar{x}(t) + C\}$$

in the extended phase space. Then, by the discussion in the beginning of this paragraph about the geometrical meaning of (**H**), it follows that

$$\Omega_i = \Omega_{si} = \{(x,t) \in \mathbb{R}^{nm} \times \mathbb{R} : x \neq \bar{x}(t),\ \ x \in \bar{x}(t) + \partial C\}.$$

Fix $t_0 \in \mathbb{R}$ and $\xi \in V_-$ such that $\xi \neq \mathcal{P}_-(\bar{x}(t_0))$. We set $\Omega_{t_0} := \bar{x}(t_0) + C$, and define the set

$$S_{t_0} := \mathcal{P}_-^{-1}\xi \cap \overline{\Omega}_{t_0} = (\xi + V_+) \cap \overline{\Omega}_{t_0} =$$

$$= \{x \in \mathbb{R}^{nm} : x = \xi + x_+,\ \ x_+ \in V_+ \ \text{ and } \ V(\xi + x_+ - \bar{x}(t_0)) \leq 0\}.$$

The set $S = (S_{t_0}, t_0) \subset \Omega \cup \Omega_i$ is diffeomorphic to the unit disk $D^{nm-n} \subset \mathbb{R}^{nm-n}$ and $S \cap \Omega_i = (\partial S_{t_0}, t_0)$ is diffeomorphic to $S^{nm-n-1} = \partial D^{nm-n}$. As it is well known that S^{nm-n-1} is not a retract D^{nm-n}, we conclude that $S \cap \Omega_i$ is not a retract of $S \cap \Omega$. To apply Wazewski's theorem we need to show that $S \cap \Omega_i$ is a retract of Ω_i. We first observe that the retraction of $(-\infty, +\infty)$ onto $\{t_0\}$ induces a retraction r_1 of Ω_i onto the set $(\partial\Omega_{t_0} \setminus \bar{x}(t_0), t_0)$, which is the slice of Ω_i with the hyperplane $t = t_0$ in $\mathbb{R}^{nm} \times \mathbb{R}$.

Our next step is to retract

$$\partial\Omega_{t_0} \setminus \bar{x}(t_0) = \{x \in \mathbb{R}^{nm} :\ x \neq \bar{x}(t_0) \ \text{ and } \ V(x - \bar{x}(t_0)) = 0\}$$

onto the set

$$T = \{x \in \mathbb{R}^{nm} :\ \ V(x - \bar{x}(t_0)) = 0,\ \ V(\mathcal{P}_-(x - \bar{x}(t_0))) = V(\xi - \mathcal{P}_-(\bar{x}(t_0)))\}.$$

The retraction $r_2 : \partial\Omega_{t_0} \setminus \bar{x}(t_0) \to T$ is given by

$$r_2(x) := \bar{x}(t_0) + \frac{V(\xi - \mathcal{P}_-\bar{x}(t_0))}{V(\mathcal{P}_-(x - \bar{x}(t_0)))}(x - \bar{x}(t_0)).$$

Finally, we let $\mathcal{P}_+ := I - \mathcal{P}_-$, and observe that the set T can be defined also by the equalities

$$V(\mathcal{P}_-(x-\bar{x}(t_0))) = V(\xi-\mathcal{P}_-(\bar{x}(t_0)), \ \ V(\mathcal{P}_+(x-\bar{x}(t_0))) = -V(\xi-\mathcal{P}_-(\bar{x}(t_0)).$$

The first equality defines a set which is diffeomorphic to the sphere $S^{n-1} \subset V_-$, whereas the second equality defines a set which is diffeomorphic to the sphere $S^{nm-n-1} \subset V_+$. As a consequence, T has a product structure and is diffeomorphic to $S^{n-1} \times S^{nm-n-1}$. The retraction $r_3 : T \to S_{t_0} \cap \partial\Omega_{t_0}$ is obtained by collapsing the first factor to its point ξ, namely:

$$r_3(x) := \xi + \mathcal{P}_+(x).$$

Summing up our steps, if we denote by $i : \mathbb{R}^{nm} \to \mathbb{R}^{nm} \times \mathbb{R}$ the inclusion $i(x) = (x, t_0)$, then $i \circ r_3 \circ r_2 \circ i^{-1} \circ r_1$ is a retraction from Ω_i to $S \cap \Omega_i$. Then, by Wazewski's theorem, and since we are assuming that all the solutions are defined up to $-\infty$, there exists a point $(x_{t_0}, t_0) \in S \cap \Omega$ and a solution $x(\cdot; t_0, x_0)$ of (1) such that $(x(t; t_0, x_{t_0}), t) \in \Omega$ for any $t \in (-\infty, t_0]$. Clearly $\mathcal{P}_-(x_0) = \xi$ and by Lemma 3.3, $x(\cdot, t_0, x_0)$ is amenable. As $\xi \neq \mathcal{P}_-(\bar{x}(t_0))$ was arbitrary in V_- and, of course, $\bar{x}(t_0) \in \mathcal{A}_{t_0}$, we conclude that $\mathcal{P}_-(\mathcal{A}_{t_0}) = V_-$. Then our thesis follows from *ii*) of Lemma 3.3. $\square$

Our next result shows that after a linear change of coordinates we can always obtain a system for which the amenable manifold is diagonal-like.

Lemma 3.5 *Assume that all the solutions of system* (1) *are defined in* $\mathbb{R}$. *Moreover, assume condition* (***H***) *and that there is at least one amenable solution* $\bar{x}(\cdot)$. *Then there exists a change of coordinates* $\tilde{x} = Bx$, *where* B *is a non-singular matrix, that transforms system* (1) *into a system of the form*

$$\tilde{x}' = \tilde{F}(\tilde{x}, t) + \tilde{D}\tilde{x} \tag{5}$$

that has a diagonal-like amenable manifold $\tilde{\mathcal{A}}_t = B\mathcal{A}_t$.

Proof.

The proof consists of three main steps.

Step 1: We shall show that there are n-dimensional complementary subspaces $W_i \subset \mathbb{R}^{nm}, i = 1, \cdots m$, such that the following property holds: if for $j = 1, \cdots, m-1$ we let

$$L_j := W_1 \oplus W_2 \oplus \cdots \oplus \hat{W}_j \oplus \cdots \oplus W_m,$$

where the hat indicates the subspace which is omitted in the direct sum, then $V_{|L_j}$ is positive definite. Geometrically, this property means that $L_j \cap \overline{C} = 0$ (recall that the cone C is defined by $C := \{x \in \mathbb{R}^{nm} : V(x) < 0\}$).

Let $\{v_1, v_2, \ldots, v_{mn}\}$ be an orthogonal set of eigenvectors of P such that the first $nm - n$ vectors span V_+ and satisfy the relation $V(v_h) = 1$, $h = 1, \ldots, nm - n$, and the last n vectors span V_- and are such that $V(v_h) = -1$, $h = nm - n + 1, \ldots, nm$. We split the set $\{v_1, \ldots, v_{nm-n}\}$ into the $m - 1$ disjoint sets $S_i := \{v_{(i-1)n+1}, \cdots, v_{(i-1)n+n}\}$, $i = 1, \cdots, m - 1$ and define W_i as the subspace spanned by S_i for each $i = 1, \cdots, m - 1$. Finally, let W_m be the subspace spanned by the vectors

$$b_h = 2(v_h + v_{n+h} + v_{2n+h} + \ldots + v_{(m-2)n+h}) + v_{(m-1)n+h}, \quad h = 1, \ldots, n.$$

It is straightforward to show that W_m is n-dimensional and that $W_i \cap W_m = 0$ for $i = 1, \ldots, m - 1$. In what follows, being the other cases similar, we assume $j = 1$ and prove that $V_{|L_1}$ is positive definite. Consider $w \in L_1 = \hat{W}_1 \oplus W_2 \oplus \cdots \oplus W_m$. Then

$$w = \sum_{k=n+1}^{nm-n} \alpha_k v_k + \sum_{k=1}^{n} \beta_k b_k =$$

$$= \sum_{k=1}^{n} 2\beta_k v_k + \sum_{i=1}^{m-2} \sum_{k=1}^{n} (\alpha_{in+k} + 2\beta_k) v_{in+k} + \sum_{k=1}^{n} \beta_k v_{(m-1)n+k},$$

and thus

$$V(w) = \sum_{k=1}^{n} 4\beta_k^2 + \sum_{i=1}^{m-2} \sum_{k=1}^{n} (\alpha_{in+k} + 2\beta_k)^2 - \sum_{k=1}^{n} \beta_k^2 > 0.$$

Step 2: Given the decomposition of $\mathbb{R}^{nm}$ constructed in Step 1, we fix an index $i \in \{1, \cdots, m\}$ and consider the projection

$$\Pi_i : \mathbb{R}^{nm} = W_1 \oplus W_2 \oplus \cdots \oplus W_m \to W_i.$$

We will show that for any $w_i \in W_i$, if $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^{nm}$ is a sequence such that $V(x_k) \leq 0$ and satisfying $||\Pi_i x_k - w_i|| \to 0$ when $k \to \infty$, then $\|(Id - \Pi_i)x_k\|$ is bounded.

To prove this fact we argue by contradiction. Consider a sequence $\{x_k\}_{k\in\mathbb{N}}$ in the above conditions but with $\|(Id - \Pi_i)x_k\| \to +\infty$ when $k \to +\infty$. Defining the sequence

$$\tilde{x}_k := \frac{x_k}{||(Id - \Pi_i)x_k||} = \frac{(Id - \Pi_i)x_k}{||(Id - \Pi_i)x_k||} + \frac{\Pi_i x_k}{||(Id - \Pi_i)x_k||},$$

we may assume without loss of generality that $\tilde{x}_k \to \tilde{x}$, where $\tilde{x} = \tilde{w}_1 + \cdots + \tilde{w}_{i-1} + \tilde{w}_i + \cdots + \tilde{w}_m, \quad \tilde{w}_i \in W_i$, and $||\tilde{x}|| = 1$. Since $0 \neq \tilde{x} \in L_i$ it must be $V(\tilde{x}) > 0$, which is in contradiction with $0 \geq V(x_k) \to V(\tilde{x})$.

Step 3: Using the decomposition of $\mathbb{R}^{nm}$ constructed in Step 1, and using Step 2, we shall show that for each $i = 1, \cdots, m$ the projection Π_i is an homeomorphism between $\mathcal{A}_t$ and W_i.

We first note that Π_i restricted to $\mathcal{A}_t$ is injective. In fact, if we consider $x_1, x_2 \in \mathcal{A}_t$ with $x_1 \neq x_2$ and such that $\Pi_i(x_1) = \Pi_i(x_2)$, then $0 \neq x_1 - x_2 \in L_i$ and therefore $V(x_1 - x_2) > 0$. On the other hand, since $x_1, x_2 \in \mathcal{A}_t$, it should be $V(x_1 - x_2) < 0$ and we get a contradiction. Since Π_i is also a continuous map between the n manifold $\mathcal{A}_t$ and W_i, it follows that Π_i is an homeomorphism between $\mathcal{A}_t$ and the open subset $\Pi_i(\mathcal{A}_t)$ of W_i.

It remains to show that $\Pi_{i|\mathcal{A}_t}$ is onto. Consider $\xi \in \partial\Pi_i(\mathcal{A}_t)$ and let $x_k \in \mathcal{A}_t$ be a sequence such that $\Pi_i(x_k) \to \xi$. By the properties of $\mathcal{A}_t$ we know that $V(x_k - \bar{x}(t)) \leq 0$. Moreover, $\Pi_i(x_k - \bar{x}(t)) \to \bar{w}_i := \xi - \Pi_i(\bar{x}(t))$. Then, Step 2 imply that $(Id - \Pi_i)(x_k - \bar{x}(t))$ is bounded, so that $(Id - \Pi_i)(x_k)$ is also bounded. Since $\Pi_i(x_k)$ is bounded, we conclude that x_k is also bounded. Then, without loss of generality, we may assume that $\mathcal{A}_t \ni x_k \to x_0$. Since $\mathcal{A}_t$ is closed in $\mathbb{R}^{nm}$, we have that $x_0 \in \mathcal{A}_t$ and by the continuity of Π_i we get that $\Pi_i(x_k) \to \Pi_i(x_0) = \xi \in \Pi_i(\mathcal{A}_t)$. As a consequence $\Pi_i(\mathcal{A}_t) = W_i$ and $\Pi_{i|\mathcal{A}_t}$ is an homeomorphism between $\mathcal{A}_t$ and W_i.

To conclude our proof, we observe that if $\tilde{x}_i$ are any coordinates in W_i then we can take $\tilde{x} := (\tilde{x}_1, \cdots, \tilde{x}_m)$ as coordinates in $\mathbb{R}^{nm}$ and there exists a change of coordinates of the form $\tilde{x} = Bx$, where B is a nonsingular square matrix of order nm, that gives a one-to-one correspondence between the solutions of (1) and the ones of (5). In particular, the amenable solutions of (5) are precisely the solutions $\tilde{x}(t) = Bx(t)$, where $x(t)$ is amenable solution of (1). We conclude that $\tilde{\mathcal{A}}_t = B\mathcal{A}_t$ and that $\tilde{\Pi}_i : \tilde{\mathcal{A}}_t \to \mathbb{R}^n$, $\tilde{\Pi}_i(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) = \tilde{x}_i$ is an homeomorphism, so that the manifold $\tilde{\mathcal{A}}_t$ is diagonal-like. $\square$

Our last lemma describes the attracting property of $\mathcal{A}_t$.

Lemma 3.6 *Suppose that* (1) *satisfies* (***H***) *and there is at least one bounded solution* $\bar{x}(.)$ *in the future, then:*
a) There is at least one amenable solution, in particular $\mathcal{A}_t \neq \emptyset$ *for all* $t \in \mathbb{R}$.
b) The ω*-limit of* $\{\bar{x}(kT + t)\}_{k\in\mathbb{N}}$ *is a subset of* $\mathcal{A}_t$*, for every* $t \in \mathbb{R}$.
c) $dist(\mathcal{A}_t, \bar{x}(t)) \to 0$ *when* $t \to +\infty$.

Proof.

a) Since the sequence $\{\bar{x}(kT)\}_{k\geq 0}$ is bounded, its ω-limit set, A, is compact and invariant for the Poincaré stroboscopic map $P^T : x_0 \to x(T; 0, x_0)$.

Let $y(t)$ be a solution of (1) such that $y(0) \in A$. Since $y(t)$ is inside the compact set $\{x(t;0,A)/t \in [0,T]\}$ it follows that it is bounded and hence is amenable.

b) Notice that the ω-limit of $\{\bar{x}(kT+t)\}_{k\in\mathbb{N}}$ is $x(t,0,A)$. By the proof of the last item, $A \subset \mathcal{A}_0$ and therefore $x(t,0,A) \subset x(t,0,\mathcal{A}_0) = \mathcal{A}_t$.

c) Suppose by contradiction that there is a sequence $t_k \to +\infty$, such that $dist(\mathcal{A}_{t_k}, \bar{x}(t_k)) > \epsilon > 0$. Let $t_k = l_k + h_k T$, with $l_k \in [0,T[$ and $h_k \in \mathbb{Z}$. Since $\{l_k\}_{k\in\mathbb{N}}$ and $\{\bar{x}(t_k)\}_{k\in\mathbb{N}}$ are bounded, we can suppose that $l_k \to l$ and $\bar{x}(t_k) \to p$.

Since $\bar{x}(\cdot)$ is bounded in the future and is a solution of (1), it follows that $\bar{x}'(\cdot)$ is also bounded in the future, and for a sufficiently large k we get

$$\begin{aligned}\|\bar{x}(h_kT+l)-p\| &\le \|\bar{x}(t_k-l_k+l)-\bar{x}(t_k)\| + \|\bar{x}(t_k)-p\| \\ &\le \max_{t\in[0,+\infty[} \|\bar{x}'(t)\|\,\|l_k-l\| + \|\bar{x}(t_k)-p\| \to 0,\end{aligned}$$

when $k \to +\infty$. Hence, $\bar{x}(h_kT+l) \to p$ and by property b) we conclude that $p \in \mathcal{A}_l$. On the other hand,

$$\begin{aligned}0 < \epsilon < dist(\mathcal{A}_{t_k}, \bar{x}(t_k)) &= dist(\mathcal{A}_{l_k}, \bar{x}(t_k)) \\ < \|x(l_k;l,p)-\bar{x}(t_k)\| &< \|x(l_k;l,p)-p\| + \|p-\bar{x}(t_k)\| \to 0,\end{aligned}$$

that is a contradiction. □

Just collecting the previous Lemmas, we get our main result.

Proof of Theorem 2.3

Is is an immediate consequence of Lemma 3.5 and Lemma 3.6. □

The next corollary gives sufficient conditions for the generalized synchronization to occur with respect to the canonical variables $(x_1, \cdots, x_n)$.

Corollary 3.7 *Assume that all the solutions of system* (1) *are defined in* $(-\infty, +\infty)$. *Moreover, assume condition* ($\boldsymbol{H}$) *and that there is at least one amenable solution* $\bar{x}(\cdot)$. *Consider the following block decomposition of* P *in* $n \times n$ *blocks:*

$$P = \begin{pmatrix} P_{1,1} & \cdots & P_{1,m} \\ \vdots & \ddots & \vdots \\ P_{m,1} & \cdots & P_{m,m} \end{pmatrix}.$$

For each $j = 1, \cdots, m$, *denote by* P_j *the* $n(m-1) \times n(m-1)$ *matrix obtained from* P *by deleting the blocks from the* $j-th$ *row and from the* $j-th$ *column. If for each* $j = 1, \cdots, m$, *the matrix* P_j *is positive definite, then there is bounded generalized synchronization in the original coordinates.*

Proof.

Observe that in this case, in the proof of Lemma 3.5, we can choose W_j as the subspace spanned by the subset of the canonical basis of $\mathbb{R}^{nm}$ given by $\{e_{(j-1)n+1}, \ldots, e_{jn}\}$. $\square$

Finally, we will use our approach to deal with the case in which all the oscillators are identical.

Corollary 3.8 *Assume that system (1) is such that $f := f_1 = f_2 = \ldots = f_m$, and that (2) holds. Assume also that all the solutions of system* (1) *are defined in $\mathbb{R}$ and that condition* (***H***) *is satisfied. Moreover, consider the block decomposition of the matrix P defined in Corollary 3.7 and assume that the symmetric matrix*

$$Q := \sum_{i,j=1}^{m} P_{i,j}$$

is negative definite. If the system $u' = f(u,t)$ in $\mathbb{R}^n$ has at least one amenable solution, then there is identical bounded synchronization for system (1).

Proof.

By assumption, there exists an amenable solution $\bar{u}(t)$ of the system $u' = f(u,t)$. This solution corresponds to an amenable solution $\bar{x}(t) = (\bar{u}(t), \bar{u}(t), \cdots, \bar{u}(t))$ of the full system in the diagonal Δ. Then, applying Theorem 2.3 we have bounded synchronization for system (1). Moreover, since Q is negative definite, Δ is included in the cone $\bar{x}(t) + C$ and by $i)$ of Lemma 3.3 we conclude that Δ is the amenable manifold. $\square$

4 Sufficient conditions for (H)

Suppose that there exists a $\lambda > 0$ such that D does not have eigenvalues with real part equal to $-\lambda$ and it has precisely n eigenvalues with real part strictly larger than $-\lambda$. In this case, $D + \lambda I$ has precisely n eigenvalues with positive real part, and the Lyapunov equation

$$(D + \lambda I)^T P + P(D + \lambda I) = -I \tag{6}$$

has only one solution P if and only if (see [3])

$$\sigma(D + \lambda I) \cap \overline{\sigma(-D - \lambda I)} = \emptyset. \tag{7}$$

Since there are a finite number of eigenvalues, we can easily choose λ so that (7) holds. Let P be the solution of the Lyapunov equation for such λ. From (6) we obtain

$$(D+\lambda I)^T P^T + P^T(D+\lambda I) = -I^T = -I,$$

and from the uniqueness of the solution of this equation we conclude that P is symmetric. Moreover, from the general inertia theorem (see [3]) we have that P has n negative and $nm-n$ positive eigenvalues. The next theorem asserts that in certain condition equation (1) satisfies (**H**) with such P.

Theorem 4.1 *Given λ satisfying* (7), *let P be the corresponding the solution of the Lyapunov equation* (6). *If there exists an $\epsilon > 0$ such that*

$$(x-y)^T P[F(x,t) - F(y,t)] \leq (1/2 - \epsilon)\|x-y\|^2, \tag{8}$$

then equation (1) *satisfies* (***H***) *for such λ, ϵ, and P.*

Proof.

Notice that

$$(x-y)^T P[F(x,t) - F(y,t) + (D+\lambda I)(x-y)] =$$

$$= \frac{1}{2}(x-y)^T[(D+\lambda I)^T P + P(D+\lambda I)](x-y) + (x-y)^T P[F(x,t) - F(y,t)]$$

$$\leq -\epsilon\|x-y\|^2.$$

□

Remark 4.1 *Sometimes in the applications the function F is globally K-Lipschitz in the variable x, i.e. there exists a constant $K > 0$ such that*

$$\|F(x_1,t) - F(x_2,t)\| \leq K\|x_1 - x_2\|,$$

for every $x_1,\ x_2 \in \mathbb{R}^{nm}$ and $t \in \mathbb{R}$. In this case, inequality (8) is obviously satisfied if

$$K < \frac{1}{2\|P\|}.$$

Remark 4.2 *Let $\lambda > 0$ be such that equation* (7) *holds and let P be the corresponding solution of the Lyapunov equation* (6). *Then, the inequality*

$$(x-y)^T P(D+\lambda I)(x-y) < 0$$

holds for all $x,\ y \in \mathbb{R}^{nm}$. Turning to the setting of identical synchronization, we assume that the matrix D satisfies condition (2). *In this case, if we restrict the previous inequality to the diagonal Δ, we have that the matrix Q defined in Corollary 3.8 is negative definite. Therefore, if we use the sufficient conditions stated above, the assumption about Q made in Corollary 3.8 is fulfilled.*

5 Applications

In this section we shall give some examples in which we apply our results. Several coupling schemes will be considered. We shall give criteria for generalized synchronization in terms of some conditions on the parameters of the system (eigenvalues of D, coupling strength, etc). In general, given a particular system, we can write it in the form (1) in many ways, obtaining several different conditions on the parameters for the existence of synchronization. However, in this section the stress is put on how our approach may work in practice, rather than in obtaining sharp results. Therefore we shall consider only some settings which are more friendly in terms of computations. In particular, we choose examples where the computations involved can be easily made by hand. However, we think that a computer algebra system may be a very effective tool for the practical application of our method to more concrete systems.

Example 1.

Consider a system of two n-dimensional equations which are two-way coupled

$$\left\{ \begin{array}{l} x_1' = f_1(x_1,t) + c(x_2 - x_1) \\ x_2' = f_2(x_2,t) + c(x_1 - x_2) \end{array} \right. , \tag{9}$$

where $c > 0$ is a parameter, called the coupling coefficient, that measures the coupling strength.

Let us start by considering the case $f_1 = f_2 := f$. In this setup we can find a Lyapunov function to determine conditions under which the system synchronizes and compare them with the results given by the methods of this paper. This will also help to clarify the notion of generalized synchronization. Notice that the last system could be written in the form (1) with

$$D = \begin{pmatrix} -cI & cI \\ cI & -cI \end{pmatrix}.$$

Since D satisfies condition (2), the manifold $\Delta = \{x_1 = x_2\}$ is invariant. Given a solution $(x_1(t), x_2(t))^T$ of system (9), we consider the function $u(t) = x_1(t) - x_2(t)$. This function satisfies the differential equation

$$u' = f(x_1,t) - f(x_2,t) - 2cu. \tag{10}$$

If we assume that f is globally K-Lipschitz with $K < 2c$, then $E(u) = \|u\|^2$ is a Lyapunov function for equation (10). Indeed, the derivative along a solution satisfies

$$\dot{E}(u) = 2uu' = 2u(f(x_1,t) - f(x_2,t)) - 4c\|u\|^2 \leq 2(K - 2c)\|u\|^2 < 0.$$

We conclude that $\|u(t)\| = \|x_1(t) - x_2(t)\| \to 0$ when $t \to +\infty$, so there is identical synchronization.

Let us consider now the case in which the functions $f_1,\ f_2$ are not necessarily identical. Since the the eigenvalues of D are 0 and $-2c$, both with multiplicities n, accordingly to the last section, we choose $-\lambda \in]-2c, 0[$. Notice that (7) is equivalent to

$$\{\lambda, -2c+\lambda\} \cap \{-\lambda, 2c-\lambda\} = \emptyset,$$

that holds when $\lambda \in]0, 2c[\setminus\{c\}$. Under this condition, equation (6) is easily solved by blocks, yielding

$$P = \begin{pmatrix} -\dfrac{c-\lambda}{2(2c-\lambda)\lambda} I & -\dfrac{c}{2(2c-\lambda)\lambda} I \\ -\dfrac{c}{2(2c-\lambda)\lambda} & -\dfrac{c-\lambda}{2(2c-\lambda)\lambda} I \end{pmatrix}.$$

Since the eigenvalues of P are $\dfrac{1}{2(2c-\lambda)}$ and $-\dfrac{1}{2\lambda}$, we have

$$\|P\| = \max\left\{\frac{1}{2(2c-\lambda)}, \frac{1}{2\lambda}\right\}.$$

By Remark 4.1, (**H**) is satisfied whenever $F = (f_1, f_2)$ is globally K-Lipschitz in the variable x and

$$K < \max_{\lambda \in]0,2c[\setminus\{c\}} \frac{1}{2\|P\|} = \max_{\lambda \in]0,2c[\setminus\{c\}} \min\{2c-\lambda, \lambda\} = c.$$

Moreover, notice that we could obtain the same estimate with $\lambda \in]c, 2c[$. In this case, since $-\dfrac{c-\lambda}{2(2c-\lambda)\lambda} > 0$, the block sub-matrices of P

$$P_{1,1} = P_{2,2} = -\frac{c-\lambda}{2(2c-\lambda)\lambda} I$$

defined in Corollary 3.7 are positive definite. Then, we conclude that the conditions of Corollary 3.7 are satisfied (provided that there is at least one amenable solution) and the generalized bounded synchronization of system (9) occurs with respect to the variables $x_1,\ x_2$.

As a final remark, we observe that, although our method gave a worse estimate on K than the one obtained in the identical case using a Lyapunov function, this estimate is valid in a much more general setting.

Example 2.

The purpose of this example is to show that we can apply the method presented in this paper to a system that is not a small perturbation of a linear system. Consider the following system of two scalar oscillators with a Drive-Response coupling

$$\begin{cases} x_1' = f_1(x_1, t) \\ x_2' = f_2(x_2, t) + (x_1 - x_2), \end{cases} \tag{11}$$

where $x_1,\ x_2 \in \mathbb{R}$ and with the coupling matrix

$$D = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

since the eigenvalues of D are 0 and -1, in order to satisfy condition (7) we must choose $\lambda \in]0, 1[$ and $\lambda \neq 1/2$. In what follows, instead of optimize with respect to λ our estimate for K as in Example 1, we fix $\lambda = 1/4$, it is sufficient for our aim and simplify the computations. With $\lambda = 1/4$, the solution of the Lyapunov equation (6) is

$$P = \frac{2}{3} \begin{pmatrix} -11 & 2 \\ 2 & 1 \end{pmatrix}.$$

Observe that

$$(x-y)^T P[F(x,t) - F(y,t)] = \frac{2}{3}(x-y)^T \begin{pmatrix} -11a & 2b \\ 2a & b \end{pmatrix} (x-y),$$

where, for $x_i \neq y_i,\ \ i = 1, 2,$ we define

$$a = a(x_1, y_1, t) := \frac{f_1(x_1,t) - f_1(y_1,t)}{x_1 - y_1} \text{ and } b = b(x_2, y_2, t) := \frac{f_2(x_2,t) - f_2(y_2,t)}{x_2 - y_2}.$$

Therefore, (8) is satisfied if there is an $\epsilon > 0$ such that

$$(x-y)^T \left[\left(\frac{1}{2} - \epsilon \right) I - \frac{2}{3} \begin{pmatrix} -11a & 2b \\ 2a & b \end{pmatrix} \right] (x-y) \geq 0,$$

and this happens whenever the symmetric part

$$\left(\frac{1}{2} - \epsilon \right) I - \frac{2}{3} \begin{pmatrix} -11a & a+b \\ a+b & b \end{pmatrix}$$

is positive semi-definite for all $x,\ y \in \mathbb{R}^{nm}$ and $t \in \mathbb{R}$. Since the eigenvalues of the last sum are

$$\frac{1}{6}(3 + 22a - 2b \pm 2\sqrt{5}\sqrt{25a^2 + 6ab + b^2}) - \epsilon,$$

the inequality (8) is satisfied if the image of the domain of the functions a and b is contained in the set

$$\mathcal{C} = \{(r, s) \in \mathbb{R}^2 \,:\, 3 + 22r - 2s - 2\sqrt{5}\sqrt{25r^2 + 6rs + s^2} > 6\epsilon\}.$$

Notice that the set $\mathcal{C}$ is unbounded. For example, we have

$$\{(r, s) \in \mathbb{R}^2 \,:\, r = -s,\ r > 0,\ r > (2\epsilon - 1)3/4\} \subset \mathcal{C}.$$

Thus, there are examples where, provided an amenable solution exists and all the solutions are defined in future, we can ensure bounded generalized synchronization with nonlinearities f_1 and f_2 which are not globally K-Lipschitz for any K. In particular, such examples could not be seen as a perturbation of a linear system and the existence of the respective invariant manifold could not be proved via the classical invariant manifold theory. Notice that we can guarantee that all solutions are defined in $\mathbb{R}$ by choosing f_1 and f_2 bounded in $\mathbb{R}$. As to the existence of an amenable solution, we may require that f_1 and f_2 are such that an equilibrium point exists for the coupled system.

Example 3.

This example shows an alternative way to deal with a nonlinearity that is not globally Lipschitz. We consider a systems of two chaotic oscillators, namely two Lorenz Systems. Let us couple these systems with a Driven-Response scheme similar to the one in our last example. When we choose parameters in a range where chaotic behavior take place, leaving the first system free from the coupling we ensure that when the global system synchronize each system follows a chaotic orbit. More precisely, consider the system

$$\begin{cases} x_1' = \sigma_1(y_1 - x_1) \\ y_1' = -y_1 - x_1 z_1 + \rho_1 x_1 \\ z_1' = -\beta_1 z_1 + x_1 y_1 \\ x_2' = \sigma_2(y_2 - x_2) + c(x_1 - x_2) \\ y_2' = -y_2 - x_2 z_2 + \rho_2 x_2 + c(y_1 - y_2) \\ z_2' = -\beta_2 z_2 + x_2 y_2 + c(z_1 - z_2) \end{cases},$$

where $\sigma_1,\ \sigma_2,\ \rho_1,\ \rho_2,\ \beta_1,\ \beta_2$ are the positive parameters of the Lorenz system and $c > 0$ is a coupling parameter. Since the origin is an amenable solution, according to Theorem 2.3 there is bounded generalized synchronization provided that property (**H**) holds. Moreover, as we shall see below, this system has a global absorbing set. Therefore, all its orbits are bounded in the future and there is bounded generalized synchronization iff there is generalized synchronization. Notice that system (5) fits our general framework with

$m = 2$ and $n = 3$, so we expect to obtain generalized synchronization with a synchronization manifold of dimension 3.

Similarly to the last example, consider the coupling matrix

$$D = \begin{pmatrix} 0 & 0 \\ cI & -cI \end{pmatrix},$$

with eigenvalues $0, -c$, and choose $\lambda = c/4$. With this value of λ, equation (6) can be easily solved, yielding

$$P = \frac{2}{3c} \begin{pmatrix} -11I & 2I \\ 2I & I \end{pmatrix}.$$

Since the eigenvalues of P are $\frac{-5 \pm 2\sqrt{10}}{3c}$, we have $\|P\| = \frac{5+2\sqrt{10}}{3c}$.

In this case F is not globally Lipschitz. However, we can show that all the orbits enter and never leave a suitable compact set, so we can truncate F outside this set and apply the results of the last section to the truncated equation. More precisely, since the first three variables are decoupled from the last three, we consider the standard Lyapunov function

$$E_1(x_1, y_1, z_1) = x_1^2 + y_1^2 + (z_1 - \sigma_1 - \rho_1)^2.$$

and observe that the derivative along a solution of the first three equations is

$$\dot{E}_1 = -2(\sigma_1 x_1^2 + y_1^2 + \beta_1 (z_1 - \frac{\sigma_1 + \rho_1}{2})^2 - \beta_1 \frac{(\sigma_1 + \rho_1)^2}{4}).$$

We conclude that there is an absorbing compact set $\mathcal{E} \subset \mathbb{R}^3$ (depending on $\sigma_1,\ \rho_1,\ \beta_1$) for the solutions of the first Lorenz sub-system which is given by the union of a suitable ellipsoid with its interior and which contains the origin of $\mathbb{R}^3$ in its interior.

Consider now the last three equations of system (5) as a system driven by (x_1, y_1, z_1) and define a second Lyapunov function as

$$E_2(x_2, y_2, z_2) = x_2^2 + y_2^2 + (z_2 - \sigma_2 - \rho_2)^2.$$

The derivative of E_2 along the solutions of system (5) is

$$\dot{E}_2 = -2 \Bigg[\left(\sqrt{\sigma_2 + c}\ x_2 - \frac{c x_1}{2\sqrt{\sigma_2 + c}} \right)^2 - \frac{c^2 x_1^2}{4(\sigma_2 + c)} + \left(\sqrt{1 + c}\ y_2 - \frac{c y_1}{2\sqrt{1 + c}} \right)^2 -$$

$$- \frac{c^2 y_1^2}{4(1 + c)} + \left(\sqrt{\beta_2 + c}\ z_2 - \frac{(\sigma_2 + \rho_2)(\beta_2 + c) + c z_1}{2\sqrt{\beta_2 + c}} \right)^2 -$$

$$-\left(\frac{(\sigma_2+\rho_2)(\beta_2+c)+cz_1}{2\sqrt{\beta_2+c}}\right)^2+c(\sigma_2+\rho_2)z_1\Bigg].$$

Hence, we have

$$\dot{E}_2 < -2c\left[\left(x_2-\frac{c}{2(\sigma_2+c)}x_1\right)^2-\frac{c}{4(\sigma_2+c)}x_1^2+\left(y_2-\frac{c}{2(1+c)}y_1\right)^2-\right.$$

$$-\frac{c}{4(1+c)}y_1^2+\left(z_2-\frac{\sigma_2+\rho_2}{2}-\frac{c}{2(\beta_2+c)}z_1\right)^2-$$

$$\left.-\left(\frac{\sigma_2+\rho_2}{2}\sqrt{\frac{\beta_2+c}{c}}+\frac{1}{2}\sqrt{\frac{c}{\beta_2+c}}z_1\right)^2+(\sigma_2+\rho_2)z_1\right].$$

Note that the dependence on c of the larger factor in the right-hand side of the above inequality is given by some bounded functions of c. It follows that we can choose an absorbing set $\mathcal{K}\subset\mathbb{R}^6$ for system (5) that depends on $\beta_1,\ \sigma_1,\ \rho_1,\ \beta_2,\ \sigma_2,\ \rho_2$ but does not depend on c. Taking into account that all the solutions $(x_1(\cdot),y_1(\cdot),z_1(\cdot))$ of the first sub-system of (5) are absorbed by $\mathcal{E}$, from the above inequality it follows that we can define the absorbing set $\mathcal{K}$ as follows. Let $\mathcal{B}\subset\mathbb{R}^3$ be a sufficiently large ball containing the origin which is an absorbing set for the solutions $(x_2(\cdot),y_2(\cdot),z_2(\cdot))$ of the second Lorenz sub-system for. Then, $\mathcal{K}:=\mathcal{E}\times\mathcal{B}\subset\mathbb{R}^6$. This set depends on $\sigma_1,\ \rho_1,\ \beta_1,\ \sigma_2,\ \rho_2,\ \beta_2$.

If $K=\sup_{x\in\mathcal{K}}\|D_xF\|$, then F is K-Lipschitz in the variable x in $\mathcal{K}$ and K does not depends on c. Consider the truncated function

$$\tilde{F}(x,t)=\begin{cases}F(x,t), & \text{if } x\in\mathcal{K}\\ F(g(x),t), & \text{if } x\notin\mathcal{K}\end{cases},$$

where $g(x)$ is the projection of $x\in\mathbb{R}^6$ on the convex set $\mathcal{K}$. Clearly $\tilde{F}$ is also K-Lipschitz in x in $\mathbb{R}^6$.

By Remark 4.1 we conclude that there is bounded generalized synchronization for $x'=\tilde{F}(x,t)+Dx$ whenever

$$K\leq\frac{1}{2\|P\|}=\frac{3c}{4(5+2\sqrt{10})}. \tag{12}$$

Any orbit of the original system enter and never leave $\mathcal{K}$, and inside $\mathcal{K}$ it coincides with an orbit of the truncated equation and is attracted to $\mathcal{A}_t$. We conclude that there is generalized synchronization for the original system whenever (12) holds.

Example 4.

In [12] several coupling schemes between arrays of systems are presented to which our method can be applied. As an example, consider the case of an equation of the form of (1), with $x_i \in \mathbb{R}^n$, $m \geq 2$ and

$$D = c \begin{pmatrix} (-m+1)I & I & I & \dots & I \\ I & (-m+1)I & I & \dots & I \\ \vdots & & \ddots & & \vdots \\ I & I & \dots & (-m+1)I & I \\ I & I & \dots & I & (-m+1)I \end{pmatrix},$$

where $c > 0$ is a coupling parameter and I is the $n \times n$ identity matrix. This coupling matrix represents a fully connected array of m systems. In [12] there are conditions under which a coupled array of identical systems synchronize under this coupling scheme. The eigenvalues of D are 0 with multiplicity n and $-mc$ with multiplicity $nm - n$, therefore condition (7) is satisfied if we take $-\lambda \in]-mc, 0[$. Under this condition, equation (6) is easily solved, since D is symmetric, yielding $P = -1/2(D + \lambda I)^{-1}$. We obtain

$$\|P\| = \max\left\{\frac{2}{mc - \lambda}, \frac{2}{\lambda}\right\}.$$

By Remark 4.1, condition (**H**) is satisfied whenever

$$K < \max_{\lambda \in]0, mc[} \frac{1}{2\|P\|} = \max_{\lambda \in]0, mc[} \min\left\{(mc - \lambda)/4, \lambda/4\right\} = c/4.$$

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