On an isotropic differential inclusion

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***Abstract*—Differential inclusions arise in successful models proposed to describe the microstructures of elastic crystals. In this paper we are interested in the existence of Lipschitz maps $u : \Omega \to \mathbb{R}^2$ satisfying the inclusion**

$$\begin{cases} Du \in E, & \textbf{a.e. in } \Omega \\ u = \varphi, & \textbf{on } \partial\Omega \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^2$ and E is a compact subset of $\mathbb{R}^{2\times 2}$, which is isotropic, that is to say, invariant under rotations. We will show an existence result under suitable hypotheses on the boundary datum φ.

***Index Terms*—Differential inclusion, isotropic set, singular values, rank one convexity.**

I. Introduction

In the last twenty years successful models for studying the behaviour of crystal lattices undergoing solid-solid phase transitions have been studied. In such models it is assumed that the elements of crystal lattices have certain preferable affine deformations; this is true for example for martensite or for quartz crystals (see [1], [11]). This physical problem motivates the mathematical question of the existence of solutions to Dirichlet problems related to differential inclusions such as $Du \in E$ a.e. in Ω, where Ω is a domain of $\mathbb{R}^n$ and $E \subset \mathbb{R}^{n\times n}$ is a compact set.

Two abstract theories to establish the existence of solutions of general differential inclusion problems are due to Dacorogna and Marcellini (see [5], [7]), whose result is based on Baire's category theorem, and Müller and Šverák [12], [13], who use ideas of convex integration by Gromov. In these two theories special notions of convexity are used. More precisely, the rank one convex hull of the set E, plays an important role. We say that a set $E \subseteq \mathbb{R}^{n\times n}$ is rank one convex if

$$A, B \in E,\ \mathrm{rk}(A-B) = 1 \ \Rightarrow\ tA + (1-t)B \in E, \forall t \in [0,1].$$

Given a set $E \subseteq \mathbb{R}^{n\times n}$ its rank one convex hull, denoted by $\mathrm{Rco}\, E$, is the smallest rank one convex set that contains E. We point out that we are following the notation used by Dacorogna and Marcellini in [7]; the rank one convex hull is denoted by $lc(E)$ by Müller and Šverák in [12]. The following characterization of $\mathrm{Rco}\, E$ holds

$$\mathrm{Rco}\, E = \bigcup_{i=0}^{\infty} \mathrm{R}_i \mathrm{co}\, E\,,$$

where $\mathrm{R}_0\mathrm{co}\, E = E$ and

$$\mathrm{R}_{i+1}\mathrm{co}\, E =$$

$$\{tA + (1-t)B, t \in [0,1], A, B \in \mathrm{R}_i\mathrm{co}\, E, \mathrm{rk}(A-B) = 1\}.$$

Provided certain approximation properties hold, if the gradient of the boundary datum φ belongs to the interior of $\mathrm{Rco}\, E$, then there exists a solution $u \in \varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ to $Du \in E$ a.e. in Ω. However, the approximation properties are different in each of the two theories.

Using these abstract theorems various interesting problems related to the existence of microstructures have been solved, such as the two well problem, where $E = \mathcal{SO}(2)A \cup \mathcal{SO}(2)B$, where A and B are two fixed $\mathbb{R}^{2\times 2}$ matrices (see [6], [7], [9], [11], [12]).

In this article we study the case where the set E is an arbitrary $\mathbb{R}^{2\times 2}$ isotropic set, that is, invariant under rotations. More precisely, we assume that E is a compact subset of $\mathbb{R}^{2\times 2}$ such that $AEB \subseteq E$ for every A, B in the orthogonal group $\mathcal{O}(2)$. Let Ω be an open bounded subset of $\mathbb{R}^2$. We investigate the existence of weakly differentiable maps $u : \Omega \to \mathbb{R}^2$ that satisfy

$$\begin{cases} Du \in E, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Since E is isotropic it can be written as

$$E = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in K\}\,, \tag{2}$$

for some compact set $K \subset \{(x,y) \in \mathbb{R}^2 : 0 \le x \le y\}$, where we have denoted by $0 \le \lambda_1(\xi) \le \lambda_2(\xi)$ the singular values of the matrix ξ, that is, the eigenvalues of the matrix $\sqrt{\xi\xi^t}$, which are

$$\lambda_1(\xi) = \frac{1}{2}\left[\sqrt{\|\xi\|^2 + 2|\det(\xi)|} - \sqrt{\|\xi\|^2 - 2|\det(\xi)|}\right]$$

$$\lambda_2(\xi) = \frac{1}{2}\left[\sqrt{\|\xi\|^2 + 2|\det(\xi)|} + \sqrt{\|\xi\|^2 - 2|\det(\xi)|}\right]\,.$$

Thanks to the properties of the singular values (see [10]), problem (1) can be rewritten in the following equivalent way:

$$\begin{cases} \|Du(x)\|^2 = a^2 + b^2 & \text{a.e. in } \Omega, (a,b) \in K, \\ |\det Du(x)| = ab & \text{a.e. in } \Omega, (a,b) \in K, \\ u(x) = \varphi(x) & x \in \partial\Omega\,. \end{cases}$$

In the case where K consists of a unique point these two equations are the vectorial eikonal equation and the equation of prescribed absolute value of the Jacobian determinant.

The main result of our article is the following

Theorem 1.1: Let $E := \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in K\}$ where $K \subset \{(x,y) \in \mathbb{R}^2 : 0 < x \le y\}$ is a compact set. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let $\varphi \in C^1_{piec}(\overline{\Omega}, \mathbb{R}^2)$

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be such that $D\varphi \in \operatorname{int}\, \mathrm{Rco}E$ in Ω. Then there exists a map $u \in \varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ such that $Du \in E$ a.e. in Ω.

This result was first obtained by Croce in [4] using the theory developed by Dacorogna and Marcellini and a refinement due to Dacorogna and Pisante [8]. In this article we treat the same problem using the theory by Müller and Šverák which leads to different technical difficulties. We point out that in the case where K consists of a unique point and $K \subset \mathbb{R}^n$, $n \geq 2$ the same existence result was obtained by Dacorogna and Marcellini in [7].

We will use the following characterisation of the rank one convex hull of E due to Croce [3], [4]. Letting

$$f_\theta(x, y) := xy + \theta(y - x),\ x > 0,\ y > 0,\ \theta \geq 0 \qquad (3)$$

the following result holds.

Theorem 1.2: Let K be a compact set satisfying

$$K \subset \left\{(x, y) \in \mathbb{R}^2 : 0 < x \leq y\right\} \qquad (4)$$

and let

$$E = \left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in K\right\}. \qquad (5)$$

Then $\mathrm{Rco}\, E$ is the set of $\mathbb{R}^{2\times 2}$ matrices ξ such that

$$f_\theta(\lambda_1(\xi), \lambda_2(\xi)) \leq \max_{(a,b)\in K} f_\theta(a, b),\ \forall\, \theta \in [0, \max_{(a,b)\in K} b].$$

Moreover, $\operatorname{int} \mathrm{Rco}\, E$ is the set of $\mathbb{R}^{2\times 2}$ matrices ξ such that

$$f_\theta(\lambda_1(\xi), \lambda_2(\xi)) < \max_{(a,b)\in K} f_\theta(a, b),\ \forall\, \theta \in [0, \max_{(a,b)\in K} b].$$

II. In-Approximation

To show theorem 1.1 we will use an existence result due to Müller and Šverák [12] which requires the following in-approximation property.

Definition 2.1: (In-approximation) Let E be a compact subset of $\mathbb{R}^{m\times n}$. We say that a sequence of sets $\{U_i\}$ is an in-approximation of E if

1) the sets U_i are open and contained in a fixed ball;
2) $U_i \subseteq \mathrm{Rco}U_{i+1}$;
3) if $\xi_n \in U_n$ and $\xi_n \to \xi$, as $n \to \infty$, then $\xi \in E$.

In this section we will show that the set E, defined by (5) and (4), admits an in-approximation. Since a characterization of the rank one convex hull of an open isotropic set is not available, we will construct closed sets V_n from which we will obtain the open sets U_n of the in-approximation.

Definition 2.2: Let $\varepsilon_n = \frac{1}{n}$ and let r_n be a decreasing sequence such that $0 \leq r_n < \varepsilon_n$. For $(a, b) \in K$ we define the sets,

$$R^n_{(a,b)} = \{(x, y) \in \mathbb{R}^2 : a + \varepsilon_n - r_n \leq x \leq a + \varepsilon_n, \quad \frac{ab - \varepsilon_n}{a + \varepsilon_n} - r_n \leq y \leq \frac{ab - \varepsilon_n}{a + \varepsilon_n}\}$$

and $V_n := \left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in K_n\right\}$, where $K_n = \bigcup_{(a,b)\in K} R^n_{(a,b)}$.

Proposition 2.3: The function $f_\theta(x, y)$ defined in (3) satisfies the following properties:

i) f_θ is strictly increasing in y, for every $x > 0$ and $\theta \geq 0$;

ii) f_θ is strictly increasing in x, for every $y > \theta$ and is strictly decreasing in x, for every $y < \theta$;

iii) $f_\theta(\cdot, \theta)$ is constant, for every $\theta \geq 0$;

iv) setting

$$\begin{aligned}\alpha_n^{(a,b)}(\theta) &= f_\theta\left(a + \varepsilon_n - r_n, \tfrac{ab-\varepsilon_n}{a+\varepsilon_n}\right)\\ \beta_n^{(a,b)}(\theta) &= f_\theta\left(a + \varepsilon_n, \tfrac{ab-\varepsilon_n}{a+\varepsilon_n}\right)\end{aligned}$$

one has

$$\max_{(x,y)\in R^n_{(a,b)}} f_\theta(x, y) = \max\left\{\alpha_n^{(a,b)}(\theta), \beta_n^{(a,b)}(\theta)\right\}$$

$$= \begin{cases} \beta_n^{(a,b)}(\theta),\ \theta \in [0, \max\limits_{(x,y)\in R^n_{(a,b)}} y] \\ \alpha_n^{(a,b)}(\theta),\ \theta \geq \max\limits_{(x,y)\in R^n_{(a,b)}} y\,; \end{cases}$$

v) for every $\theta \in [\max_{(x,y)\in R^n_{(a,b)}} y, \max_{(x,y)\in R^{n+1}_{(a,b)}} y]$ the following inequality holds:

$$\alpha_n^{(a,b)}(\theta) < \beta_{n+1}^{(a,b)}(\theta);$$

for every $\theta \geq 0$ the following inequality holds:

$$\beta_n^{(a,b)}(\theta) < \beta_{n+1}^{(a,b)}(\theta);$$

vi) assume that $\max_{(x,y)\in R^{n+1}_{(a,b)}} y < \max_{(x,y)\in R^{n+1}_{(a',b')}} y$, then for every $\theta \in [\max_{(x,y)\in R^{n+1}_{(a,b)}} y, \max_{(x,y)\in R^{n+1}_{(a',b')}} y]$ we have

$$\alpha_n^{(a,b)}(\theta) < \alpha_{n+1}^{(a',b')}(\theta);$$

vii) for every $\theta \in [0, \max_{(x,y)\in R^{n+1}_{(a,b)}} y]$ the following inequality holds:

$$\max_{(x,y)\in R^n_{(a,b)}} f_\theta(x, y) < \max_{(x,y)\in R^{n+1}_{(a,b)}} f_\theta(x, y).$$

Proof: The first three properties are clear and the fourth one follows from $i)$, $ii)$ and $iii)$. The second inequality in $v)$ follows immediately from the fact that

$$\frac{ab - \varepsilon_n}{a + \varepsilon_n} < \frac{ab - \varepsilon_{n+1}}{a + \varepsilon_{n+1}}. \qquad (6)$$

Due to the linearity in θ, it suffices to show the remaining inequalities in $v)$ and $vi)$ for θ belonging to the boundaries of the respective intervals. This is achieved using (6), $i)$ and $iii)$. Finally, $vii)$ is a consequence of $iv)$ and $v)$. ∎

Lemma 2.4: Let $b_M = \max_{(a,b)\in K} b$ and $a_M = \max_{(a,b_M)\in K} a$. Then

$$\max_{(x,y)\in K_n} y = \frac{a_M b_M - \varepsilon_n}{a_M + \varepsilon_n},$$

for all sufficiently large n.

Proof: If $(a, b) \in K$ satisfies $a \leq a_M$ then it is easy to see that

$$\frac{ab - \varepsilon_n}{a + \varepsilon_n} \leq \frac{a_M b_M - \varepsilon_n}{a_M + \varepsilon_n}.$$

It remains to show the above inequality for points $(a, b) \in K$ such that $a \geq a_M$ and $b < b_M$. We argue by contradiction and assume there exists a sequence $(a_n, b_n) \in K$ and $\varepsilon_{n'}$ a subsequence of ε_n such that

$$\frac{a_M b_M - \varepsilon_{n'}}{a_M + \varepsilon_{n'}} \leq \frac{a_n b_n - \varepsilon_{n'}}{a_n + \varepsilon_{n'}} .$$

Since K is compact and $(a_n, b_n) \in K$, up to a subsequence $(a_n, b_n) \to (a, b) \in K$, so passing to the limit in the above inequality we obtain $b \geq b_M$, contradicting $b < b_M$. ■

We will now prove the following proposition.

Proposition 2.5: The sets V_n introduced in Definition 2.2 satisfy $V_n \subseteq \text{int Rco}V_{n+1}$.

Proof: Due to the compactness of K and to the definition of the sets $R^n_{(a,b)}$, standard arguments show that the set K_n is compact. Therefore, by Theorem 1.2, it suffices to prove that for every $\theta \in [0, \max_{(x,y)\in K^{n+1}} y]$

$$\max_{(x,y)\in K_n} f_\theta(x, y) < \max_{(x,y)\in K_{n+1}} f_\theta(x, y).$$

Let $\theta \in [0, \max_{(x,y)\in K^{n+1}} y]$. Then there exists $(a, b) \in K$ (depending on θ) such that

$$\max_{(x,y)\in K_n} f_\theta(x, y) = \max_{(x,y)\in R^n_{(a,b)}} f_\theta(x, y).$$

Recall that, by Lemma 2.4,

$$\max_{(x,y)\in K^{n+1}} y = \frac{a_M b_M - \varepsilon_{n+1}}{a_M + \varepsilon_{n+1}} \geq \frac{ab - \varepsilon_{n+1}}{a + \varepsilon_{n+1}}.$$

If $0 \leq \theta \leq \frac{ab-\varepsilon_{n+1}}{a+\varepsilon_{n+1}}$ the result follows by property $vii)$ of Proposition 2.3.

If $\theta \in \left[\frac{ab-\varepsilon_{n+1}}{a+\varepsilon_{n+1}}, \frac{a_M b_M-\varepsilon_{n+1}}{a_M+\varepsilon_{n+1}}\right]$ we have, by $iv)$ and $vi)$ of Proposition 2.3,

$$\begin{aligned}
\max_{(x,y)\in K_n} f_\theta(x, y) &= \max_{(x,y)\in R^n_{(a,b)}} f_\theta(x, y) \\
&= \alpha^{(a,b)}_n(\theta) \\
&< \alpha^{(a_M,b_M)}_{n+1}(\theta) \\
&\leq \max_{(x,y)\in R^{n+1}_{(a_M,b_M)}} f_\theta(x, y) \\
&\leq \max_{(x,y)\in K_{n+1}} f_\theta(x, y).
\end{aligned}$$

■

Theorem 2.6: Let E be given by (5) and (4). Then E admits an in-approximation.

Proof: Let V_n be the sets considered in the previous proposition and define the sets $U_n = \text{int}V_n$.

Step 1) U_n are open by definition and it is clear that the sequence $\{U_n\}$ is uniformly bounded.

Step 2) We will prove the second condition of the definition of in-approximation. Due to Proposition 2.5, we have $U_n \subset V_n \subseteq \text{int Rco}\, V_{n+1}$. We will show that, for every n,

$$\text{int Rco}V_n \subseteq \text{Rco}\, U_n, \tag{7}$$

this will imply that $U_n \subset \text{Rco}\, U_{n+1}$, as required.

Let $\xi \in \text{int Rco}\, V_n$. We will prove that $\xi \in \text{Rco}\tilde{V}_n$, where $\tilde{V}_n$ is the set defined by

$$\tilde{V}_n = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \tilde{K}_n\},$$

for a certain compact set $\tilde{K}_n \subset \text{int}K_n$. By continuity of the function $\xi \to (\lambda_1(\xi), \lambda_2(\xi))$ this will entail that $\tilde{V}_n \subseteq \text{int}V_n$, therefore

$$\xi \in \text{Rco}\tilde{V}_n \subseteq \text{Rco int}V_n = \text{Rco}\, U_n ,$$

and we will have proved (7). For simplicity of notation we set $(\lambda_1(\xi), \lambda_2(\xi)) = (x, y)$. Our aim is thus to find a compact set $\tilde{K}_n \subset \text{int}K_n$ such that, for every $\theta \in [0, \max_{(a,b)\in\tilde{K}_n} b]$, the following inequality holds

$$f_\theta(x, y) \leq \max_{(a,b)\in\tilde{K}_n} f_\theta(a, b). \tag{8}$$

For $\lambda > 0$ define $\tilde{K}^\lambda_n = \bigcup_{(a,b)\in K} R^{n,\lambda}_{(a,b)}$ where

$$\begin{aligned}
R^{n,\lambda}_{(a,b)} = \{(x, y) \in \mathbb{R}^2 : & \, a + \varepsilon_n - r_n + \lambda \leq x \leq a + \varepsilon_n - \lambda, \\
& \frac{ab - \varepsilon_n}{a + \varepsilon_n} - r_n + \lambda \leq y \leq \frac{ab - \varepsilon_n}{a + \varepsilon_n} - \lambda\}.
\end{aligned}$$

It follows that $R^{n,\lambda}_{(a,b)} \subset \text{int}R^n_{(a,b)} \subset \text{int}K_n$ and so $\tilde{K}^\lambda_n \subset \text{int}K_n$. Since $\xi \in \text{int Rco}\, V_n$, we have that $f_\theta(x, y) < \max_{(a,b)\in K_n} f_\theta(a, b)$, so to show (8) it suffices to prove that, as $\lambda \to 0^+$,

$$\max_{(a,b)\in\tilde{K}^\lambda_n} f_\theta(a, b) \to \max_{(a,b)\in K_n} f_\theta(a, b), \tag{9}$$

uniformly with respect to θ. Notice that, as in the case of K_n,

$$\max_{(x,y)\in\tilde{K}^\lambda_n} f_\theta(x, y) = \sup_{(a,b)\in K} \max\left\{\alpha^{(a,b)}_{n,\lambda}(\theta), \beta^{(a,b)}_{n,\lambda}(\theta)\right\}$$

where

$$\alpha^{(a,b)}_{n,\lambda}(\theta) = f_\theta\left(a + \varepsilon_n - r_n + \lambda, \frac{ab - \varepsilon_n}{a + \varepsilon_n} - \lambda\right)$$

$$\beta^{(a,b)}_{n,\lambda}(\theta) = f_\theta\left(a + \varepsilon_n - \lambda, \frac{ab - \varepsilon_n}{a + \varepsilon_n} - \lambda\right).$$

Hence (9) will follow from the fact that, as $\lambda \to 0^+$,

$$\begin{aligned}
\sup_{(a,b)\in K} \alpha^{(a,b)}_{n,\lambda}(\theta) &\to \sup_{(a,b)\in K} \alpha^{(a,b)}_n(\theta) \\
\sup_{(a,b)\in K} \beta^{(a,b)}_{n,\lambda}(\theta) &\to \sup_{(a,b)\in K} \beta^{(a,b)}_n(\theta),
\end{aligned} \tag{10}$$

uniformly in θ. As

$$\begin{aligned}
|\alpha^{(a,b)}_{n,\lambda}(\theta) - \alpha^{(a,b)}_n(\theta)| &\leq \lambda\left(\max_{(a,b)\in K} a + 1\right) + \lambda\frac{ab + \varepsilon_n}{a + \varepsilon_n} \\
&\quad +\lambda^2 + 2\lambda\theta \\
&\leq \lambda\left(\max_{(a,b)\in K} a + 1\right) + \lambda^2 + 2\lambda\theta \\
&\quad +\lambda \max\{1, \max_{(a,b)\in K} b\}
\end{aligned}$$

and this last expression tends to 0, uniformly with respect to θ and to (a, b), we conclude the first statement of (10). A similar argument yields the second one.

Step 3) Given $(x_n, y_n) \in K_n$ there exists $(a_n, b_n) \in K$ such that $(x_n, y_n) \in R^n_{(a_n,b_n)}$. As K is compact, up to a subsequence, $(a_n, b_n) \to (a, b) \in K$, so, by the inequalities that define $R^n_{(a_n,b_n)}$, $(x_n, y_n) \to (a, b)$.

Let us now show the third condition of the definition of in-approximation. Assume that $\xi_n \in U_n$ and that $\xi_n \to \xi$. Since $\xi_n \in V_n$, $(\lambda_1(\xi_n), \lambda_2(\xi_n)) \in K_n$ so, by the above reasoning, $(\lambda_1(\xi_n), \lambda_2(\xi_n))$ converges to a point $(a, b) \in K$. On the other hand, by continuity, $\lambda_i(\xi_n) \to \lambda_i(\xi), i = 1, 2$, and therefore $(\lambda_1(\xi), \lambda_2(\xi)) = (a, b) \in K$. Thus $\xi \in E$. ■

III. Existence Theorem

We are going to prove Theorem 1.1. We will assume that the boundary datum φ is $C^1_{piec}(\overline{\Omega}, \mathbb{R}^2)$, that is to say, $\varphi \in W^{1,\infty}(\Omega, \mathbb{R}^2)$, there exist open sets $\omega_i \subset \Omega$ such that $\varphi \in C^1(\overline{\omega_i}, \mathbb{R}^2)$ and $\Omega \setminus \bigcup_i \omega_i$ is a set of Lebesgue measure zero.

We will use the following abstract theorem of Müller and Šverák [12]) to prove Theorem 1.1.

Theorem 3.1: Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set and let E be a compact set which admits an in-approximation by the open sets U_i. Let $\varphi : \Omega \to \mathbb{R}^m$ be a C^1 function such that $D\varphi \in U_1$ in Ω. Then there exists $u \in \varphi + W^{1,\infty}_0(\Omega, \mathbb{R}^m)$ such that $Du \in E$ a.e. in Ω.

We begin by considering the case where φ is an affine function and to this effect we will need the following proposition.

Proposition 3.2: Let E be the set defined by (5) and (4) and let $\xi \in \text{int Rco}\, E$. Then there exists an in-approximation sequence U_n for E such that $\xi \in U_1$.

Proof: Consider the sequence of sets V_n defined in the previous section. We will show that there exists $N = N(\xi) \in \mathbb{N}$ such that

$$\xi \in \text{int Rco}\, V_N. \tag{11}$$

For simplicity of notation set $(\lambda_1(\xi), \lambda_2(\xi)) = (x, y)$. We must show that if

$$f_\theta(x, y) < \max_{(a,b)\in K} f_\theta(a, b),\ \forall\, \theta \in [0, \max_{(a,b)\in K} b]$$

then there exists $N = N(x, y)$ such that, letting K_n be the sequence of sets in Definition 2.2,

$$f_\theta(x, y) < \max_{(a,b)\in K_N} f_\theta(a, b),\ \forall\, \theta \in [0, \max_{(a,b)\in K_N} b]. \tag{12}$$

Since, by construction of K_n, $\max_{(a,b)\in K_n} b < \max_{(a,b)\in K} b$, it suffices to prove that

$$\max_{(a,b)\in K_n} f_\theta(a, b) \to \max_{(a,b)\in K} f_\theta(a, b), \quad n \to +\infty, \tag{13}$$

uniformly with respect to $\theta \in [0, \max_{(a,b)\in K} b]$. By Proposition 2.3, $iv)$

$$\max_{(a,b)\in K_n} f_\theta(a, b) = \sup_{(a,b)\in K} \max\{\alpha_n^{(a,b)}(\theta), \beta_n^{(a,b)}(\theta)\}$$

and

$$|\sup_{(a,b)\in K} \max\{\alpha_n^{(a,b)}(\theta), \beta_n^{(a,b)}(\theta)\} - \max_{(a,b)\in K} f_\theta(a, b)|$$
$$\leq \sup_{(a,b)\in K} |\max\{\alpha_n^{(a,b)}(\theta), \beta_n^{(a,b)}(\theta)\} - f_\theta(a, b)|.$$

Therefore we must show that, as $n \to +\infty$,

$$|\alpha_n^{(a,b)}(\theta) - f_\theta(a, b)| \to 0\,,\ |\beta_n^{(a,b)}(\theta) - f_\theta(a, b)| \to 0,$$

uniformly with respect to θ and to (a, b). We start with the first limit. Letting

$$m_n = \frac{ab - \varepsilon_n}{a + \varepsilon_n} - a - \varepsilon_n + r_n,\ \ q_n = (a + \varepsilon_n - r_n)\, \frac{ab - \varepsilon_n}{a + \varepsilon_n}$$

we have $\alpha_n^{(a,b)}(\theta) = m_n\theta + q_n$. Notice that $q_n - ab \to 0$ and $m_n - b + a \to 0$ uniformly with respect to (a, b). This implies the result. The same reasoning applies to the second limit.

To complete the proof we notice that, letting $U_n = \text{int}\, V_n$, for every fixed $N \in \mathbb{N}$, the sequence

$$\text{Rco}\, U_N, U_{N+1}, U_{N+2}, \dots$$

is an in-approximation of E. Indeed the rank one convex hull of an open set U_N is open and rank one convex. Since, by construction, $U_N \subseteq \text{Rco}\, U_{N+1}$ and $\text{Rco}\, U_N$ is the smallest rank one convex set that contains U_N we conclude that $\text{Rco}\, U_N \subseteq \text{Rco}\, U_{N+1}$. Moreover, if $\xi \in \text{int Rco}E$ then $\xi \in \text{Rco}\, U_N$, by (11) and inclusion (7). ■

Theorem 3.3: Let Ω be an open, bounded subset of $\mathbb{R}^2$ and let E be the set defined by (5) and (4). Let $\xi \in \mathbb{R}^{2\times 2}$ be such that $\xi \in \text{int Rco}\, E$ and let $\varphi : \Omega \to \mathbb{R}^2$ satisfy $D\varphi = \xi$ in Ω. Then there exists $u \in \varphi + W^{1,\infty}_0(\Omega, \mathbb{R}^2)$ such that $Du \in E$.

The proof of Theorem 3.3 follows immediately from the previous proposition and from Theorem 3.1. To obtain our existence result in the general case we will once again make use of Proposition 3.2 together with the following result, proved by Dacorogna and Marcellini in [7] (Corollary 10.15).

Theorem 3.4: Let Ω be an open subset of $\mathbb{R}^n$ and A be an open subset of $\mathbb{R}^{m\times n}$. Let $\varphi \in C^1(\Omega, \mathbb{R}^m) \cap W^{1,\infty}(\Omega, \mathbb{R}^m)$ be such that

$$D\varphi(x) \in A,\ \forall\ x \in \Omega.$$

Then there exists a function $v \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ such that v is piecewise affine in Ω, $v = \varphi$ on $\partial\Omega$ and $Dv \in A$ a.e. in Ω.

We will now prove Theorem 1.1.

Proof: Assume first that $\varphi \in C^1(\bar{\Omega}, \mathbb{R}^2)$. We define the open set A as the set consisting of $\mathbb{R}^{2\times 2}$ matrices ξ such that

$$f_\theta(\lambda_1(\xi), \lambda_2(\xi)) < \max_{(a,b)\in K} f_\theta(a, b),\ \theta \in [0, \max_{(a,b)\in K} b].$$

We apply the previous theorem to φ and A, in order to obtain a map $v \in W^{1,\infty}(\Omega, \mathbb{R}^2)$ such that $v = \varphi$ on $\partial\Omega$, $Dv = c_i$ in Ω_i for some constant $c_i \in A$ and $\bigcup_i \Omega_i = \Omega$. Due to Theorem 3.3 we can solve the problem

$$\begin{cases} Du \in E, & \text{a.e. in } \Omega_i \\ u(x) = v(x), & x \in \partial\Omega_i \end{cases}$$

in each set Ω_i. Denoting by u_i the solution in Ω_i, the map defined by $u = u_i$ in Ω_i belongs to $\varphi + W^{1,\infty}_0(\Omega, \mathbb{R}^2)$ and satisfies $Du \in E$.

Now suppose that $\varphi \in C^1_{piec}(\bar{\Omega}, \mathbb{R}^2)$. This means that there exist open sets $\omega_i \subset \Omega$ such that $\varphi \in C^1(\overline{\omega_i}, \mathbb{R}^2)$ and $\Omega \setminus \bigcup_i \omega_i$ is a set of Lebesgue measure zero. By the first case, for each i, there exists $w_i \in \varphi + W^{1,\infty}_0(\omega_i, \mathbb{R}^2)$ such that $Dw_i \in E$ a.e. in ω_i. Thus, the function u defined as w_i in ω_i belongs to $\varphi + W^{1,\infty}_0(\Omega, \mathbb{R}^2)$ and satisfies $Du \in E$, a.e. in Ω. ■

We conclude this article by pointing out that Theorem 1.1 is not far from being optimal in the case where the boundary datum φ is affine. To explain this, we need some further notions of convexity given in [7].

Definition 3.5: A function $f : \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ is polyconvex if there exists $g : \mathbb{R}^5 \to \mathbb{R} \cup \{+\infty\}$ convex such that $f(A) = g(A, \det(A))$.

A measurable function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is quasiconvex if

$$f(A) \leq \frac{1}{|\Omega|} \int_\Omega f(A + D\psi)\, dx$$

for every bounded domain Ω of $\mathbb{R}^2$, for every $A \in \mathbb{R}^{2\times 2}$ and for every $\psi \in W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ ($|\Omega|$ stands for the Lebesgue measure of Ω).

A function $f : \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ is rank one convex if $f(tA + (1-t)B) \leq tf(A) + (1-t)f(B)$ whenever $t \in [0,1]$ and $\text{rk}(A - B) = 1$.

It is well known that, for $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$,

$$f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex.}$$

Definition 3.6: A set $E \subseteq \mathbb{R}^{2\times 2}$ is polyconvex if for all $t_i \geq 0$ with $\sum_{i=1}^{5} t_i = 1$ and all $A_i \in E$ with

$$\sum_{i=1}^{5} t_i \det A_i = \det\left(\sum_{i=1}^{5} t_i A_i\right)$$

then $\sum_{i=1}^{5} t_i A_i \in E$.

The polyconvex hull of a given set E is defined as the smallest polyconvex set that contains E.

Let $E \subset \mathbb{R}^{2\times 2}$. Let $\mathcal{P}$ be the set of polyconvex functions $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ such that $f\lfloor_E \leq 0$. We recall the following characterization of the closure of the polyconvex hull of E

$$\overline{\text{Pco}\, E} = \{\xi \in \mathbb{R}^{2\times 2} : f(\xi) \leq 0, \forall\ f \in \mathcal{P}\}.$$

Now, suppose that u is a solution of

$$\begin{cases} Du \in E, & \text{a.e. in } \Omega \\ u = u_{\xi_0}, & \text{on } \partial\Omega \end{cases}$$

where u_{ξ_0} is an affine function with $Du_{\xi_0} = \xi_0$. Then there exists a map $\psi \in W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ such that $u = u_{\xi_0} + \psi$. Let $f \in \mathcal{P}$. Then f is also quasiconvex and thus

$$f(\xi_0) \leq \frac{1}{|\Omega|} \int_\Omega f(\xi_0 + D\psi)\, dx = \frac{1}{|\Omega|} \int_\Omega f(Du)\, dx \leq 0$$

since $f\lfloor_E \leq 0$. This implies that $\xi_0 \in \overline{\text{Pco}\, E}$. In the case where E is an isotropic compact subset of $\mathbb{R}^{2\times 2}$ is has been shown in [4] and [2] that $\text{Rco}\, E = \overline{\text{Rco}\, E} = \overline{\text{Pco}\, E}$. Therefore $\xi_0 \in \text{Rco}\, E$.

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