Another Look at the Exact and Near-exact Distributions for the Most Common Likelihood Ratio Test Statistics Used in Multivariate Analysis

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(Invited paper for a special issue of *Journal of Statistical Theory and Practice*, guest-edited by Prof. C. R. Rao)

Abstract

In this paper we will show how, using an expansion of a Logbeta distribution as an infinite mixture of Gamma distributions we are able to obtain near-exact distributions for the negative logarithm of the likelihood ratio test statistics used in Multivariate Analysis to test the independence of several sets of variables, the equality of several mean vectors, sphericity and the equality of several variance-covariance matrices as finite mixtures of Generalized Near-Integer Gamma distributions. These near-exact distributions will match as many of the exact moments as we wish and we will be able to have an *a priori* upper-bound for the difference between their c.d.f. and the exact c.d.f.. These near-exact distributions also display very good performance, with an agreement with the exact distribution which may virtually be taken as far as we wish and which it is not possible to obtain with the usual asymptotic distributions.

AMS Subject Classification: 62H10, 62E20, 62H05, 62H15.

Key Words : mixtures of Gamma distributions, independence test, equality of mean vectors test, sphericity test, test of equality of covariance matrices, Generalized Near-Integer Gamma distribution.

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1. Introduction

Marques et al. (2010) have shown that it is possible to obtain a common structure for both the exact and the near-exact distributions of the l.r.t. (likelihood ratio test) statistics used in Multivariate analysis under multivariate elliptically contoured or left orthogonal-invariant distributions to test

(i) the independence of several sets of variables
(ii) the equality of several mean vectors
(iii) sphericity
(iv) the equality of several variance-covariance matrices,

as well as for any l.r.t. statistic which may be obtained as the product of any number of independent such l.r.t. statistics.

In this paper we will first show how, based on the results in Tricomi & Erdélyi (1951), we may write any Logbeta distribution, that is the distribution of the negative logarithm of a Beta distributed r.v. (random variable) as an infinite mixture of Gamma distributions. Then we will show how, based on this result, we may obtain near-exact distributions for any of the l.r.t. statistics above, and indeed for any l.r.t. statistic which may be built as the product of any number of independent such l.r.t. statistics (Coelho & Marques, 2009), in the form of finite mixtures of GNIG (Generalized Near-Integer Gamma) distributions (Coelho, 2004), which may, by construction, match as many of the exact moments as we wish and for which we will be able to have *a priori* upper-bounds on the difference between their c.d.f.'s and the exact c.d.f..

2. The Logbeta distribution as an infinite mixture of Gammas

From the two first expressions in section 5 of Tricomi & Erdélyi (1951) and also expressions (11) and (14) in the same paper, we may write

$$\frac{\Gamma(a-it)}{\Gamma(a+b-it)} = \sum_{j=0}^{\infty} p_j(b)(a-it)^{-b-j} \tag{1}$$

where

$$p_j(b) = \frac{1}{j}\sum_{m=0}^{j-1}\left(\frac{\Gamma(1-b-m)}{\Gamma(-b-j)(j-m+1)!} + (-1)^{j+m} b^{j-m+1}\right) p_m(b)\,, \qquad j=1,2,\dots\,, \tag{2}$$

with $p_0(b)=1$.

Then, since the c.f. (characteristic function) of $Y=-\log X$, where $X \sim Beta(a,b)$, is given by

$$\Phi_Y(t)=\frac{\Gamma(a+b)}{\Gamma(a)}\frac{\Gamma(a-it)}{(a+b-it)}\,, \tag{3}$$

using (1), we may write

$$\Phi_Y(t)=\sum_{j=0}^{\infty}\underbrace{\frac{\Gamma(a+b)}{\Gamma(a)}\frac{p_j(b)}{a^{b+j}}}_{p_j^*(a,b)}\;a^{b+j}(a-it)^{-(b+j)}\,, \tag{4}$$

which is the c.f. of an infinite mixture of $\Gamma(b+j,a)$ distributions, with weights $p_j^*(a,b)$, with $p_j(b)$ given by (2).

3. The exact distribution of the l.r.t. statistics to test independence and equality of mean vectors as infinite mixtures of GNIG distributions

From (3) and (4) above we may write,

$$\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}-\frac{n}{2}\mathrm{i}t\right)}=\sum_{\ell=0}^{\infty}p_\ell^*\!\left(\tfrac{n-2}{2},\tfrac{1}{2}\right)\left(\frac{n-2}{n}\right)^{\frac{1}{2}+\ell}\left(\frac{n-2}{n}-\mathrm{i}t\right)^{-\left(\frac{1}{2}+\ell\right)}$$

with

$$p_\ell^*\!\left(\tfrac{n-2}{2},\tfrac{1}{2}\right)=\frac{\Gamma\left(\frac{n-1}{2}\right)p_\ell\!\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)\left(\frac{n-2}{2}\right)^{\frac{1}{2}+\ell}}\,,\qquad \ell=0,1,\ldots \tag{5}$$

where $p_\ell\!\left(\frac{1}{2}\right)$ $(\ell=1,2,\ldots)$ are given by (2) above, with $p_0\!\left(\frac{1}{2}\right)=1$.

But then, for any $k^*\in I\!N$, we may write

$$\begin{aligned}\left(\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}-\frac{n}{2}\mathrm{i}t\right)}\right)^{k^*}&=\left(\sum_{\ell=0}^{\infty}p_\ell^*\!\left(\tfrac{n-2}{2},\tfrac{1}{2}\right)\left(\frac{n-2}{n}\right)^{\ell+\frac{1}{2}}\left(\frac{n-2}{n}-\mathrm{i}t\right)^{-\left(\ell+\frac{1}{2}\right)}\right)^{k^*}\\&=\sum_{\ell=0}^{\infty}\pi_\ell\left(\frac{n-2}{n}\right)^{\ell+\frac{k^*}{2}}\left(\frac{n-2}{n}-\mathrm{i}t\right)^{-\left(\ell+\frac{k^*}{2}\right)}\end{aligned}$$

with

$$\pi_\ell=\sum_{\text{all }L_\ell}\prod_{\ell^*\in L_\ell}p_{\ell^*}^*\!\left(\tfrac{n-2}{2},\tfrac{1}{2}\right), \tag{6}$$

where L_ℓ is a set of k^* values $\ell^* \in I\!N_0$ such that their sum is equal to ℓ.

But then, from expression (A.1) in Marques et al. (2010), we may write the c.f. of W_1, the negative logarithm of the l.r.t. statistic to test the independence of m sets of variables with an elliptically contoured or left orthogonal-invariant distribution, under the null hypothesis of independence of the m sets of variables, for a sample of size n, and where $m^*(\leq m)$ of the sets have an odd number of variables, as

$$\begin{aligned}
\Phi_{W_1}(t) &= \left\{\prod_{j=2}^{p}\left(\frac{n-j}{n}\right)^{r_j}\left(\frac{n-j}{n}-\mathrm{i}t\right)^{-r_j}\right\}\left(\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}-\frac{n}{2}\mathrm{i}t\right)}\right)^{k^*}\\
&= \left\{\prod_{j=2}^{p}\left(\frac{n-j}{n}\right)^{r_j}\left(\frac{n-j}{n}-\mathrm{i}t\right)^{-r_j}\right\}\\
&\qquad\qquad \times\left\{\sum_{\ell=0}^{\infty}\pi_\ell\left(\frac{n-2}{n}\right)^{\frac{k^*}{2}+\ell}\left(\frac{n-2}{n}-\mathrm{i}t\right)^{-\left(\frac{k^*}{2}+\ell\right)}\right\}\\
&= \sum_{\ell=0}^{\infty}\pi_\ell\left\{\prod_{j=2}^{p}\left(\frac{n-j}{n}\right)^{r_j^*}\left(\frac{n-j}{n}-\mathrm{i}t\right)^{-r_j^*}\right\},
\end{aligned}$$

where $k^* = \left\lfloor \frac{m^*}{2} \right\rfloor$, $p = \sum_{k=1}^{m} p_k$, where p_k is the number of variables in the k-th set ($k = 1, \ldots, m$), and

$$r_j^* = \begin{cases} \frac{k^*}{2}+\ell & j=2 \\ r_j & j=3,\ldots,p \end{cases} \tag{7}$$

where the r_j are given by (A.2) and (A.3) in Marques et al. (2010). This expression for the c.f. of W_1 shows that the exact distribution of W_1 is an infinite mixture of GNIG distributions of depth $p-1$, with weights π_ℓ ($\ell = 0, 1, \ldots$), each GNIG distribution having shape parameters r_j^* and rate parameters $\frac{n-j}{n}$ ($j = 2, \ldots, p$) (see Marques et al. (2010) and Coelho (2004) for further details on the GNIG distribution, its p.d.f. and c.d.f.).

Using the notation in Appendix B in Marques et al. (2010), the exact p.d.f. and c.d.f. of W_1 and of $\Lambda_1 = e^{-W_1}$, when there are two or more sets with an odd number of variables, that is, when $k^* \geq 1$, are thus given by the following Theorem.

Theorem 3.1 *The exact p.d.f. and c.d.f. of* $W_1 = -\log \Lambda_1$*, where* Λ_1 *is the l.r.t. statistic to test the independence of* m *sets of variables with a joint* *elliptically contoured or left orthogonal-invariant distribution, based on a sample of size n, under the null hypothesis of independence, are respectively*

$$f_{W_1}(w) \;=\; \sum_{\ell=0}^{\infty} \pi_\ell \; f^{GNIG}\Big(w \,|\, r_3^*, \ldots, r_p^*, r_2^*; \frac{n-3}{n}, \ldots, \frac{n-p}{n}, \frac{n-2}{n}; p-1\Big), \qquad w>0$$

and

$$F_{W_1}(w) \;=\; \sum_{\ell=0}^{\infty} \pi_\ell \; F^{GNIG}\Big(w \,|\, r_3^*, \ldots, r_p^*, r_2^*; \frac{n-3}{n}, \ldots, \frac{n-p}{n}, \frac{n-2}{n}; p-1\Big), \qquad w>0\,,$$

while the exact p.d.f. and c.d.f. of Λ_1 are given by

$$f_{\Lambda_1}(z) = \sum_{\ell=0}^{\infty} \pi_\ell \, f^{GNIG}\Big(-\log\, z \,|\, r_3^*, \ldots, r_p^*, r_2^*; \frac{n-3}{n}, \ldots, \frac{n-p}{n}, \frac{n-2}{n}; p-1\Big)\,\frac{1}{z}\,, \qquad 0<z<1$$

and

$$F_{\Lambda_1}(w) = \sum_{\ell=0}^{\infty} \pi_\ell \, \Big(1 - F^{GNIG}\Big(-\log\, z \,|\, r_3^*, \ldots, r_p^*, r_2^*; \frac{n-3}{n}, \ldots, \frac{n-p}{n}, \frac{n-2}{n}; p-1\Big)\Big)\,, \qquad 0<z<1\,,$$

where the π_ℓ are given by (6) and the r_j^* by (7).

When at most one of the sets of variables has an odd number of variables, the exact distribution of W_1 is just a GIG (Generalized Integer Gamma) distribution (Coelho, 1998). See Marques et al. (2010) for details.

But then, since the distribution of the l.r.t. statistic to test the equality of q mean vectors from q p-multivariate elliptically contoured or left orthogonal-invariant distributions, based on q independent random samples, the k-th of which has size n_k ($k = 1, \ldots, q$), is only a particular case of the distribution of Λ_1 above, for $m = 2$, $p_1 = p$, $p_2 = q-1$ and $n = \sum_{k=1}^{q} n_k$. If we call this statistic Λ_2 and take $W_2 = -\log\,\Lambda_2$, the exact c.f. of W_2 may be written as (see Marques et al. (2010) for details),

$$
\begin{aligned}
\Phi_{W_2}(t) &= \left\{ \prod_{j=2}^{p+q-1} \left(\frac{n-j}{n}\right)^{r_j} \left(\frac{n-j}{n}\right)^{-r_j} \right\} \left(\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}-\frac{n}{2}\mathrm{i}t\right)}{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}-\frac{n}{2}\mathrm{i}t\right)} \right)^{k^*} \\
&= \left\{ \prod_{j=2}^{p+q-1} \left(\frac{n-j}{n}\right)^{r_j} \left(\frac{n-j}{n}\right)^{-r_j} \right\} \\
&\qquad \times \left(\sum_{\ell=0}^{\infty} p_\ell^*\left(\tfrac{n-2}{2},\tfrac{1}{2}\right) \left(\frac{n-2}{n}\right)^{\ell+\frac{1}{2}} \left(\frac{n-2}{n}-\mathrm{i}t\right)^{-(\ell+\frac{1}{2})} \right)^{k^*}
\end{aligned}
$$

where the $p_\ell^*\left(\frac{n-2}{2},\frac{1}{2}\right)$ are given by (5),

$$
r_j = \begin{cases} 0 & j=2,3 \\ h_2+k^* & j=4 \\ r_{j-2}+h_{j-2} & j=5,\dots,p+q-1 \end{cases} \tag{8}
$$

with

$$
h_j = (\#\text{ of elements of } \{p,q-1\} \geq j) - 1, \qquad j=1,\dots,p+q-3\,, \tag{9}
$$

and

$$
k^* = \begin{cases} 1 & \text{if } p \text{ is odd and } q \text{ is even} \\ 0 & \text{all other cases} \end{cases}
$$

so that if p is even or q is odd, the exact distribution of W_2 is a GIG distribution (see Marques et al. (2010) for details), while if p is odd and q is even, the exact c.f. of W_2 may be written as

$$
\Phi_{W_2}(t) = \sum_{\ell=0}^{\infty} p_\ell^*\left(\tfrac{n-2}{2},\tfrac{1}{2}\right) \prod_{j=2}^{p+q-1} \left(\frac{n-j}{j}\right)^{r_j^*} \left(\frac{n-j}{n}-\mathrm{i}t\right)^{-r_j^*}\,,
$$

where

$$
r_j^* = \begin{cases} \frac{1}{2}+\ell & j=2 \\ r_j & j=3,\dots,p+q-1 \end{cases}
$$

for r_j given by (8)–(9) above.

We have thus the following Theorem.

Theorem 3.2 *The exact p.d.f. and c.d.f. of* $W_2 = -\log \Lambda_2$ *and* Λ_2*, where* Λ_2 *is the l.r.t. statistic to test the equality of* q *p-multivariate mean* *vectors from q p-multivariate elliptically contoured or left-orthogonal invariant distributions, based on q independent samples, the k-th of which with size n_k $(k = 1, \dots, q)$, when p is odd and q is even, under the null hypothesis of equality of the mean vectors, are, for $n = \sum_{k=1}^{q} n_k$, respectively given by*

$$f_{W_2}(w) = \sum_{\ell=0}^{\infty} p_\ell^*\left(\tfrac{n-2}{2}, \tfrac{1}{2}\right)$$
$$f^{GNIG}\left(w|r_3^*, \dots, r_{p+q-1}^*, r_2^*; \frac{n-3}{n}, \dots, \frac{n-(p+q-1)}{n}, \frac{n-2}{n}; p+q-2\right), \quad w > 0\,,$$

$$F_{W_2}(w) = \sum_{\ell=0}^{\infty} p_\ell^*\left(\tfrac{n-2}{2}, \tfrac{1}{2}\right)$$
$$F^{GNIG}\left(w|r_3^*, \dots, r_{p+q-1}^*, r_2^*; \frac{n-3}{n}, \dots, \frac{n-(p+q-1)}{n}, \frac{n-2}{n}; p+q-2\right), \quad w > 0\,,$$

$$f_{\Lambda_2}(z) = \sum_{\ell=0}^{\infty} p_\ell^*\left(\tfrac{n-2}{2}, \tfrac{1}{2}\right)$$
$$f^{GNIG}\left(-\log z|r_3^*, \dots, r_{p+q-1}^*, r_2^*; \frac{n-3}{n}, \dots, \frac{n-(p+q-1)}{n}, \frac{n-2}{n}; p+q-2\right)\frac{1}{z}\,, \quad 0 < z < 1$$

and

$$F_{\Lambda_2}(w) = \sum_{\ell=0}^{\infty} p_\ell^*\left(\tfrac{n-2}{2}, \tfrac{1}{2}\right)$$
$$\left(1 - F^{GNIG}\left(-\log z|r_3^*, \dots, r_{p+q-1}^*, r_2^*; \frac{n-3}{n}, \dots, \frac{n-(p+q-1)}{n}, \frac{n-2}{n}; p+q-2\right)\right)\,, \quad 0 < z < 1\,,$$

where the $p_\ell^*\left(\frac{n-2}{2}, \frac{1}{2}\right)$ are given by (5) and the r_j^* by (8)–(9).

4. On the exact distributions of the l.r.t. statistics to test sphericity and equality of covariance matrices

We may try to deal with the exact distributions of the l.r.t. statistics to test sphericity and equality of covariance matrices using a similar approach to the one used in the previous section.

4.1. On the exact distribution of the l.r.t. statistic to test sphericity

From expression (A.6) in Appendix A of Marques et al. (2010) we see that the part of the c.f. of $W_3 = -\log \Lambda_3$, where Λ_3 is the l.r.t. statistic to test sphericity in a set of p variates with a joint elliptically contoured or left-orthogonal invariant distribution, denoted by $\Phi_{2,W_3}(t)$ is, for $k^* = \lfloor p/2 \rfloor$, the c.f. of the sum of $p-k^*-1$ independent Logbeta$\left(\frac{n-1}{2}, \frac{j-1}{p}\right)$ $(j = 2, \ldots, p-k^*)$ r.v.'s, multiplied by $n/2$, with other k^* independent Logbeta$\left(\frac{n}{2}, \frac{j-1}{p} - \frac{1}{2}\right)$ $(j = p-k^*+1, \ldots, p)$ r.v.'s, multiplied by $n/2$, which, from (3) and (4) is the sum of $p-k^*-1$ independent infinite mixtures of $\Gamma\left(\frac{j-1}{p} + \ell, \frac{n-1}{n}\right)$ $(j = 2, \ldots, p-k^*; \ell = 0, 1, \ldots)$ distributions, with other k^* independent infinite mixtures of $\Gamma\left(\frac{j-1}{p} - \frac{1}{2} + \ell, \frac{n}{n}\right)$ $(j = p-k^*+1, \ldots, p; \ell = 0, 1, \ldots)$ distributions. In the case in which all the rate parameters are the same, say equal to λ, this sum of mixtures would simplify to only one infinite mixture of $\Gamma\left(\frac{p-2}{4} + \ell, \lambda\right)$ $(\ell = 0, 1, \ldots)$ distributions for even p or one infinite mixture of $\Gamma\left(\frac{p-1}{4} + \ell, \lambda\right)$ $(\ell = 0, 1, \ldots)$ distributions for odd p, since for even p we have

$$\begin{aligned}\sum_{j=1}^{p-k^*} \frac{j-1}{p} + \sum_{j=p-k^*+1}^{p} \left(\frac{j-1}{p} - \frac{1}{2}\right) &= \left\{\sum_{j=1}^{p} \frac{j-1}{p}\right\} - \frac{1}{2}\frac{p}{2} = \frac{p-1}{2} - \frac{p}{4} \\ &= \frac{p-2}{4}\end{aligned}$$

while for odd p we have

$$\begin{aligned}\sum_{j=1}^{p-k^*} \frac{j-1}{p} + \sum_{j=p-k^*+1}^{p} \left(\frac{j-1}{p} - \frac{1}{2}\right) &= \left\{\sum_{j=1}^{p} \frac{j-1}{p}\right\} - \frac{1}{2}\frac{p-1}{2} \\ &= \frac{p-1}{2} - \frac{1}{2}\frac{p-1}{2} = \frac{p-1}{4}\,.\end{aligned}$$

But then in order to approximate this sum of independent infinite mixtures of Gamma distributions by a single infinite mixture of Gamma distributions, we propose an infinite mixture of Gamma distributions with shape parameters $r + \ell$ $(\ell = 0, 1, \ldots)$, with

$$r = \begin{cases} \frac{p-2}{4} & \text{for even } p \\ \frac{p-1}{4} & \text{for odd } p \end{cases}$$

and all with rate parameter λ with the same value obtained for λ in (3.3) in Marques et al. (2010), in which we equate 4 or 6 exact moments, since in that paper analogous near-exact distributions were obtained for mixtures of Gamma distributions with the same rate parameter.

4.2. On the exact distribution of the l.r.t. statistic to test equality of covariance matrices

By using a similar procedure and similar arguments to the ones used in the previous subsection, we may approximate the exact distribution of $W_4 = -\log \Lambda_4$, where Λ_4 is the l.r.t. statistic to test the equality of q covariance matrices from p-variate elliptically contoured or left orthogonal-invariant distributions, based on q independent samples of size n, by replacing the part of the c.f. of W_4 denoted in expression (A.9) in Marques et al. (2010) by $\Phi_{2,W_4}(t)$, which, using (3) and (4), may be seen as the c.f. of the sum of $q\times\lfloor p/2\rfloor$ independent infinite mixtures of $\Gamma\left(\frac{k-2j}{q} - \left\lfloor\frac{k-2j}{q}\right\rfloor + \ell, \frac{n-1}{n} - \frac{1}{n}\left\lfloor\frac{k-2j}{q}\right\rfloor\right)$ $(k = 1,\ldots,q; j = 1,\ldots,\lfloor p/2\rfloor; \ell = 0,1,\ldots)$ distributions, plus, for odd p, the sum of q independent infinite mixtures of $\Gamma\left(\frac{2k-p-1}{2q} - \left\lfloor\frac{2k-p-1}{2q}\right\rfloor + \ell, \frac{n-1}{n} - \frac{2}{n}\left\lfloor\frac{2k-p-1}{2q}\right\rfloor\right)$ $(k = 1,\ldots,q; \ell = 0,1,\ldots)$ distributions, by the c.f. of an infinite mixture of $\Gamma\left(r+\ell,\lambda\right)$ $(\ell = 0,1,\ldots)$ distributions, where

$$\begin{aligned} r &= \left\{\sum_{j=1}^{\lfloor p/2\rfloor}\sum_{k=1}^{q}\frac{k-2j}{q} - \left\lfloor\frac{k-2j}{q}\right\rfloor\right\} + \left(\frac{2k-p-1}{2q} - \left\lfloor\frac{2k-p-1}{2q}\right\rfloor\right)^{Mod[p,2]} \\ &= \left\lfloor\frac{p+1}{2}\right\rfloor\frac{q-1}{2}, \end{aligned}$$

and λ is once again defined as in the previous subsection.

5. Near-exact distributions

Although the approximations proposed in the previous section for the l.r.t. statistics to test sphericity and equality of covariance matrices provide crude approximations to the exact distributions of those statistics and their logarithms, such approximations are in fact not as accurate as one would wish. Anyway, even if they were judged to be sufficiently accurate, for practical applications we would still have to face the problem of the truncation of the series corresponding to their p.d.f.'s and c.d.f.'s as also happens with the exact distributions obtained for the l.r.t. statistics to test the independence of several sets of variables or the equality of several mean vectors in section 3.

The development of near-exact distributions, based on such exact distributions or approximations, which will take the form of finite mixtures of

GNIG distributions for the negative logarithm of such l.r.t. statistics, will solve such a problem and provide us with very accurate approximations to the exact distributions.

The near-exact distributions proposed for the four basic l.r.t. statistics addressed in this paper, which are: (i) the l.r.t. statistic to test the independence of several sets of variables; (ii) the l.r.t. statistic to test the equality of several mean vectors; (iii) the l.r.t. statistic to test for sphericity; and (iv) the l.r.t. statistic to test the equality of several covariance matrices, have c.f.'s which may be written, for the negative logarithm of those statistics (generally denoted here by W), as

$$\Phi^*_W(t) = \sum_{k=0}^{m^*} p_k \left\{ \lambda^{b+j}(\lambda - \mathrm{i}t)^{-(b+j)} \prod_{j=2}^{p} \left(\frac{n-j}{n}\right)^{r_j} \left(\frac{n-1}{n} - \mathrm{i}t\right)^{-r_j} \right\} \tag{10}$$

which yields as near-exact distributions for W, the generic negative logarithm of the l.r.t. statistic and for Λ, the l.r.t. statistic itself, distributions with p.d.f.'s and c.d.f.'s respectively given by

$$f_W(w) = \sum_{k=0}^{m^*} p_k\, f^{GNIG}\left(w \,|\, r_2, \ldots, r_p, r_k^+; \frac{n-2}{n}, \ldots, \frac{n-p}{n}, \lambda; p\right)$$

$$F_W(w) = \sum_{k=0}^{m^*} p_k\, F^{GNIG}\left(w \,|\, r_2, \ldots, r_p, r_k^+; \frac{n-2}{n}, \ldots, \frac{n-p}{n}, \lambda; p\right)$$

$$f_\Lambda(z) = \sum_{k=0}^{m^*} p_k\, f^{GNIG}\left(-\log z \,|\, r_2, \ldots, r_p, r_k^+; \frac{n-2}{n}, \ldots, \frac{n-p}{n}, \lambda; p\right) \frac{1}{z}\,, \qquad 0 < z < 1$$

$$F_\Lambda(z) = \sum_{k=0}^{m^*} p_k \left(1 - F^{GNIG}\left(-\log\, z \,|\, r_2, \ldots, r_p, r_k^+; \frac{n-2}{n}, \ldots, \frac{n-p}{n}, \lambda; p\right)\right), \qquad 0 < z < 1\,,$$

where p is to be taken as $p+q-1$ for the test of equality of q p-variate mean vectors, the r_j are given by:

(i) expressions (A.3) and (A.4) in Marques et al. (2010) for the l.r.t. statistic to test the independence of m sets of elliptically contoured or left-orthogonal invariant distributed r.v.'s,

(ii) expressions (8) and (9) for the l.r.t. statistic to test the equality of mean vectors of q populations with elliptically or left-orthogonal invariant distributions,

(iii) expression (A.8) in Marques et al. (2010) for the l.r.t. statistic to test sphericity in a set of elliptically contoured or left-orthogonal invariant r.v.'s,

(iv) expressions (A.14) – (A.18) in Marques et al. (2010) for the l.r.t. statistic to test the equality of covariance matrices of q elliptically contoured or left-orthogonal invariant sets of r.v.'s,

r_k^+ is given by

(i) $\frac{k^*}{2}+k$ for the l.r.t. statistic to test independence of m sets of variables,

(ii) $\frac{1}{2}+k$ for the l.r.t. statistic to test the equality of q p-variate mean vectors,

(iii) $\frac{p-2}{4}$ for even p or $\frac{p-1}{4}$ for odd p, for the l.r.t. statistic to test sphericity,

(iv) $\left\lfloor\frac{p+1}{2}\right\rfloor\frac{q-1}{2}$ for the l.r.t. statistic to test the equality of q covariance matrices,

and λ assumes the value

(i),(ii) $\frac{n-2}{n}$ for the l.r.t. statistics to test independence of several sets of variables and the equality of several mean vectors,

(iii),(iv) of λ in (3.3) of Marques et al. (2010) when we match 4 or 6 exact moments, for the l.r.t. statistics to test sphericity and to test the equality of several covariance matrices,

and where the depth of the GNIG distributions is indeed $p-1$ for the l.r.t. of independence, where $r_2=0$ and $p+q-3$ for the l.r.t. of equality of mean vectors, where $r_2=r_3=0$.

In the above expressions, m^* represents the number of exact moments equated to obtain the near-exact distribution, and the weights p_k, for $k=0,\ldots,m^*-1$, being determined by equating the first m^* moments from (10) and the first exact moments of W, that is, in such a way that

$$\left.\frac{\partial^h}{\partial t^h}\Phi_W(t)\right|_{t=0} = \left.\frac{\partial^h}{\partial t^h}\Phi_W^*(t)\right|_{t=0}\,, \qquad h=1,\ldots,m^*\,,$$

where $\Phi_W(t)$ represents the exact c.f. of W and $\Phi_W^*(t)$ the near-exact c.f. of W in (10). The weight p_{m^*} is obtained through the relation

$$p_{m^*} = 1-\sum_{k=0}^{m^*-1} p_k\,.$$

These near-exact distributions are thus, if m^* exact moments are matched, mixtures of m^*+1 GNIG distributions with shape parameters $r_2, \ldots, r_p, r_k^+$ and rate parameters $\frac{n-2}{n}, \ldots, \frac{n-p}{n}, \lambda$.

The number of exact moments matched by these near-exact distributions may be virtually as many as one wishes.

If the number of exact moments matched is equal to two, four or six, the near-exact distributions developed in this paper will indeed have a slightly more complicated structure than the near-exact distributions in Marques et al. (2010) that match the same number of exact moments, since while the previously developed near-exact distributions would correspond respectively to a single GNIG distribution or to a mixture of two or three GNIG distributions, we now have a mixture of respectively three, five or seven GNIG distributions. However, the near-exact distributions proposed in the present paper enjoy several advantages, among which are the following:

- they may equate virtually as many of the exact moments as one wishes,
- they show even better asymptotic characteristics than the near-exact distributions previously proposed, with even better performances for very small sample sizes,
- for a given number of exact moments matched, they always show a better performance than the near-exact distributions previously proposed,
- the system of equations to be solved in order to determine the weights p_k $(k = 0, \ldots, m^* - 1)$ is linear and as such very simple to be solved,
- it is possible to obtain an *a priori* upper-bound on the probability error, that is, an upper-bound on the difference between the near-exact and the exact c.d.f..

In Tables $1-4$ we may analyze the values for the measure

$$\Delta = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{\Phi_W(t) - \Phi_W^*(t)}{t} \right| dt$$

where $\Phi_W(t)$ represents the exact c.f. of W and $\Phi_W^*(t)$ the near-exact c.f. of W in (10), with

$$\max_{w \in I\!R^+} |F_W(w) - F_W^*(w)| \leq \Delta$$

and

$$\max_{z \in]0,1[} |F_\Lambda(z) - F_\Lambda^*(z)| \leq \Delta\,,$$

where $F_W(\,\cdot\,)$ and $F_W^*(\,\cdot\,)$ represent respectively the exact c.d.f. of W and the near-exact c.d.f. corresponding to $\Phi_W^*(t)$ in (10) and $F_\Lambda(\,\cdot\,)$ and $F_\Lambda^*(\,\cdot\,)$

represent respectively the exact c.d.f. of Λ and the near-exact c.d.f. corresponding to $\Phi_W^*(t)$ in (10). See Marques et al. (2010) and Coelho & Mexia (2010) for details on the use of the measure Δ as a measure of proximity between distributions.

Tables $1-4$ display values of the measure Δ for the near-exact distributions proposed in this paper for the cases in which 4, 6, 10 and 15 exact moments are matched, as well as for the near-exact distributions proposed in Marques et al. (2010) in which 4 and 6 exact moments were matched. These are designated by M2GNIG and M3GNIG respectively (since they are mixtures of respectively 2 and 3 GNIG distributions).

All the computations relative to the l.r.t. statistics to test sphericity and the equality of several covariance matrices, for the newly developed near-exact distributions use a value of λ which is equal to the value obtained for λ in (3.3) of Marques et al. (2010) when equating 4 exact moments.

From Table 1 we may see that the new near-exact distributions besides showing even better performance for very small sample sizes than the already very good performance of the former near-exact distributions, also will show an an improved asymptotic behavior, in terms of sample size and number of variables involved, compared to that of the former near-exact distributions since they display a more accentuated decrease in the values of the measure Δ even for small increases in the sample size or in the number of variables involved than the former near-exact distributions.

By analyzing Table 2 we may see how also for the l.r.t. statistic used to test the equality of q p-multivariate mean vectors, assuming equality of the covariance matrices, the near-exact distributions proposed in this paper exhibit improved behavior for increasing sample size, increasing dimension (p) and also for increasing values of q, the number of mean vectors being tested. Although this also happens for the former near-exact distributions, namely for the M2GNIG and M3GNIG near-exact distributions, this behavior is now more accentuated for the newly proposed near-exact distributions. We may also notice that, as expected, the larger the number of exact moments matched, the more accentuated is the asymptotic behavior of the new near-exact distributions, since the decrease in the values of the measure Δ is even more substantial for increasing values of n, p and q when we consider the near-exact distributions that match a larger number of exact moments.

From Table 3 we may see that, as with the other l.r.t. statistics, for the l.r.t. statistic used to test sphericity, the near-exact distributions proposed in this paper provide improved approximations both for increasing sample sizes as well as for increasing dimension. Once again use of the near-exact distributions that match more exact moments results in even better approximations and more marked improvement as sample size and/or dimension increases.

Table 1 – Values of the measure Δ for several near-exact distributions for the l.r.t. statistic to test the independence of m groups of variables, the k-th of group of which contains p_k variables ($k = 1, \ldots, m$).

			former near-exact dists.		new near-exact dists. (# of exact moments matched)			
m	p_k	n	M2GNIG	M3GNIG	4	6	10	15
3	{3,5,6}	16	2.16×10^{-11}	3.81×10^{-15}	3.84×10^{-13}	1.24×10^{-16}	1.09×10^{-21}	5.88×10^{-27}
		50	1.33×10^{-11}	5.53×10^{-15}	2.04×10^{-14}	2.93×10^{-18}	5.81×10^{-25}	9.96×10^{-32}
		100	2.21×10^{-12}	1.21×10^{-15}	7.95×10^{-16}	3.17×10^{-20}	5.76×10^{-28}	2.04×10^{-36}
3	{5,7,10}	24	3.09×10^{-13}	9.10×10^{-18}	2.41×10^{-15}	1.48×10^{-19}	2.74×10^{-27}	2.65×10^{-34}
		50	8.16×10^{-13}	1.11×10^{-16}	1.33×10^{-15}	6.73×10^{-20}	1.80×10^{-27}	2.97×10^{-35}
		100	1.86×10^{-13}	2.04×10^{-17}	7.16×10^{-17}	1.15×10^{-21}	3.69×10^{-30}	1.75×10^{-39}
3	{7,11,15}	35	7.50×10^{-15}	1.18×10^{-19}	2.65×10^{-17}	2.98×10^{-22}	2.92×10^{-31}	2.08×10^{-40}
		50	4.14×10^{-14}	1.71×10^{-18}	6.90×10^{-17}	1.09×10^{-21}	2.90×10^{-30}	2.87×10^{-39}
		100	1.78×10^{-14}	7.68×10^{-19}	7.04×10^{-18}	4.56×10^{-23}	2.46×10^{-32}	1.34×10^{-42}
4	{3,5,7,7}	24	5.83×10^{-13}	1.60×10^{-17}	4.25×10^{-15}	2.68×10^{-19}	2.13×10^{-27}	2.90×10^{-34}
		50	1.32×10^{-12}	1.56×10^{-16}	1.97×10^{-15}	9.54×10^{-20}	2.21×10^{-27}	2.67×10^{-35}
		100	2.89×10^{-13}	5.37×10^{-17}	1.01×10^{-16}	1.52×10^{-21}	4.07×10^{-30}	1.37×10^{-39}
4	{7,7,9,10}	35	6.74×10^{-15}	1.01×10^{-19}	2.39×10^{-17}	2.55×10^{-22}	2.24×10^{-31}	1.36×10^{-40}
		50	3.25×10^{-14}	1.22×10^{-18}	5.42×10^{-17}	7.76×10^{-22}	1.70×10^{-30}	1.30×10^{-39}
		100	1.28×10^{-14}	4.85×10^{-19}	5.08×10^{-18}	2.89×10^{-23}	1.21×10^{-32}	4.76×10^{-43}
5	{3,5,5,5,4}	24	5.45×10^{-13}	1.45×10^{-17}	3.98×10^{-15}	2.43×10^{-19}	1.79×10^{-27}	2.19×10^{-34}
		50	1.12×10^{-12}	1.23×10^{-16}	1.67×10^{-15}	7.57×10^{-20}	1.54×10^{-27}	1.57×10^{-35}
		100	2.38×10^{-13}	4.09×10^{-17}	8.32×10^{-17}	1.16×10^{-21}	2.69×10^{-30}	7.56×10^{-40}
5	{5,5,7,7,9}	35	1.36×10^{-14}	1.92×10^{-19}	4.43×10^{-17}	4.95×10^{-22}	4.76×10^{-31}	2.49×10^{-40}
		50	6.23×10^{-14}	2.16×10^{-18}	9.47×10^{-17}	1.38×10^{-21}	2.98×10^{-30}	1.96×10^{-39}
		100	2.36×10^{-14}	1.62×10^{-18}	8.43×10^{-18}	4.80×10^{-23}	1.88×10^{-32}	6.11×10^{-43}
5	{9,9,11,11,8}	50	4.70×10^{-16}	2.46×10^{-21}	7.19×10^{-19}	1.60×10^{-24}	8.83×10^{-35}	6.89×10^{-46}
		100	2.25×10^{-15}	6.03×10^{-20}	8.10×10^{-19}	1.82×10^{-24}	1.12×10^{-34}	3.68×10^{-46}
		200	5.39×10^{-16}	5.20×10^{-21}	4.69×10^{-20}	3.39×10^{-26}	2.25×10^{-37}	1.93×10^{-50}
6	{11,11,15,15,19,21}	100	9.06×10^{-18}	2.25×10^{-23}	2.77×10^{-21}	6.28×10^{-28}	3.57×10^{-40}	3.01×10^{-54}
		200	1.66×10^{-17}	5.30×10^{-24}	1.21×10^{-21}	1.99×10^{-28}	6.07×10^{-41}	9.08×10^{-56}

By analyzing the values of the measure Δ in Table 4 we see that, for the l.r.t. statistic to test the equality of q p-variate covariance matrices, the newly proposed near-exact distributions once again display improved behavior and provide excellent approximations to the exact distributions, as also happened for all the other l.r.t. statistics addressed in this paper. Once again the near-exact distributions proposed in this paper show better behavior than the previously proposed approximations, with improved performance both for increasing sample sizes as well as for increasing dimension (p) and also for increasing number of matrices involved (q).

Also, given the oscillating sign nature of the weights p_k in (10), for all the new near-exact distributions, we may take the absolute value of the last of these weights, that is p_{m^*-1}, as a rough upper-bound on the difference between the near-exact and exact c.d.f.'s. This means that before we determine the value for Δ in each single case, which may indeed take some time, we may obtain from the absolute value of p_{m^*-1} a rough upper-bound on this value, which may come in quite handy.

Table 2 – Values of the measure Δ for several near-exact distributions for the l.r.t. statistic to test the equality of q p-variate mean vectors, for odd p and even q

			former near-exact dists.		new near-exact dists. (# of exact moments matched)			
p	q	n	M2GNIG	M3GNIG	4	6	10	15
5	4	30	2.82×10^{-9}	5.61×10^{-12}	9.16×10^{-12}	1.15×10^{-14}	5.88×10^{-20}	1.16×10^{-24}
		50	7.06×10^{-10}	1.36×10^{-12}	7.69×10^{-13}	3.75×10^{-16}	6.51×10^{-22}	1.06×10^{-27}
		100	9.62×10^{-11}	1.17×10^{-13}	2.46×10^{-14}	3.06×10^{-18}	4.12×10^{-25}	1.17×10^{-32}
5	10	70	1.29×10^{-11}	6.51×10^{-15}	9.36×10^{-15}	9.72×10^{-19}	1.10×10^{-25}	5.46×10^{-33}
		100	5.02×10^{-12}	2.02×10^{-15}	1.73×10^{-15}	9.16×10^{-20}	2.82×10^{-27}	1.79×10^{-35}
7	4	35	6.64×10^{-10}	9.36×10^{-13}	1.76×10^{-12}	1.28×10^{-15}	3.71×10^{-21}	2.56×10^{-26}
		50	2.67×10^{-10}	3.59×10^{-13}	3.30×10^{-13}	1.25×10^{-16}	1.43×10^{-22}	1.57×10^{-28}
		100	3.88×10^{-11}	3.41×10^{-14}	1.12×10^{-14}	1.11×10^{-18}	1.02×10^{-25}	2.01×10^{-33}
7	10	100	2.02×10^{-12}	5.72×10^{-16}	7.31×10^{-16}	2.83×10^{-20}	4.91×10^{-28}	1.65×10^{-36}
		200	2.98×10^{-13}	4.91×10^{-17}	2.61×10^{-17}	2.65×10^{-22}	3.25×10^{-31}	1.78×10^{-41}
15	4	65	1.30×10^{-11}	6.40×10^{-15}	1.11×10^{-14}	1.25×10^{-18}	1.68×10^{-25}	1.17×10^{-32}
		100	4.52×10^{-12}	1.76×10^{-15}	1.57×10^{-15}	8.04×10^{-20}	2.35×10^{-27}	1.42×10^{-35}
		200	6.81×10^{-13}	1.55×10^{-16}	5.70×10^{-17}	7.70×10^{-22}	1.59×10^{-30}	1.55×10^{-40}
15	10	200	3.94×10^{-14}	2.93×10^{-18}	3.67×10^{-18}	1.79×10^{-23}	5.45×10^{-33}	6.01×10^{-44}
		300	1.28×10^{-14}	6.76×10^{-19}	5.23×10^{-19}	1.17×10^{-24}	7.50×10^{-35}	7.23×10^{-47}
25	4	120	6.00×10^{-13}	1.13×10^{-16}	1.53×10^{-16}	3.26×10^{-21}	1.80×10^{-29}	1.26×10^{-38}
		200	1.64×10^{-13}	3.60×10^{-17}	1.47×10^{-17}	1.22×10^{-22}	1.02×10^{-31}	3.68×10^{-42}
25	10	260	4.78×10^{-15}	1.65×10^{-19}	2.67×10^{-19}	4.65×10^{-25}	1.86×10^{-35}	1.26×10^{-47}
		300	3.25×10^{-15}	9.98×10^{-20}	1.36×10^{-19}	1.81×10^{-25}	2.21×10^{-36}	1.22×10^{-48}
		500	7.82×10^{-16}	1.54×10^{-20}	1.17×10^{-20}	5.81×10^{-27}	1.90×10^{-38}	2.38×10^{-52}

Table 3 – Values of the measure Δ for several near-exact distributions for the l.r.t. statistic to test sphericity

		former near-exact dists.		new near-exact dists. (# of exact moments matched)			
p	n	M2GNIG	M3GNIG	4	6	10	15
3	5	2.27×10^{-6}	5.78×10^{-7}	6.47×10^{-8}	1.20×10^{-8}	1.54×10^{-10}	2.06×10^{-12}
	15	2.76×10^{-7}	1.24×10^{-9}	1.46×10^{-9}	7.40×10^{-12}	5.96×10^{-15}	3.59×10^{-18}
6	8	3.57×10^{-9}	3.74×10^{-12}	4.79×10^{-11}	6.76×10^{-14}	1.64×10^{-17}	2.01×10^{-21}
	18	1.14×10^{-9}	5.05×10^{-12}	2.62×10^{-12}	1.67×10^{-15}	2.42×10^{-21}	2.53×10^{-26}
	30	3.03×10^{-10}	8.19×10^{-13}	2.37×10^{-13}	5.99×10^{-17}	3.12×10^{-23}	1.84×10^{-29}
10	12	6.59×10^{-11}	1.23×10^{-14}	4.26×10^{-13}	1.15×10^{-16}	2.24×10^{-22}	7.89×10^{-28}
	22	6.13×10^{-11}	2.56×10^{-14}	1.09×10^{-13}	1.80×10^{-17}	3.13×10^{-24}	2.12×10^{-30}
	35	2.22×10^{-11}	8.30×10^{-15}	1.51×10^{-14}	1.16×10^{-18}	7.09×10^{-26}	3.93×10^{-33}
	50	8.90×10^{-12}	2.69×10^{-15}	2.91×10^{-15}	1.16×10^{-19}	2.15×10^{-27}	1.69×10^{-35}
15	17	5.34×10^{-12}	9.62×10^{-17}	3.97×10^{-14}	4.52×10^{-18}	8.84×10^{-24}	7.78×10^{-30}
	27	9.72×10^{-12}	1.17×10^{-15}	2.92×10^{-14}	3.95×10^{-18}	5.44×10^{-25}	2.26×10^{-31}
	50	3.03×10^{-12}	4.34×10^{-16}	2.64×10^{-15}	1.57×10^{-19}	6.79×10^{-27}	3.90×10^{-34}
	100	5.04×10^{-13}	8.02×10^{-18}	1.09×10^{-16}	1.88×10^{-21}	9.26×10^{-30}	1.30×10^{-38}
20	22	3.14×10^{-13}	1.80×10^{-17}	5.96×10^{-16}	9.86×10^{-21}	1.83×10^{-29}	1.01×10^{-37}
	35	7.39×10^{-13}	1.30×10^{-17}	5.38×10^{-16}	8.09×10^{-21}	2.14×10^{-29}	3.62×10^{-38}
	50	4.19×10^{-13}	4.12×10^{-18}	1.48×10^{-16}	1.34×10^{-21}	1.46×10^{-30}	4.95×10^{-40}
	100	8.07×10^{-14}	1.87×10^{-19}	6.98×10^{-18}	1.88×10^{-23}	1.99×10^{-33}	2.05×10^{-44}
25	27	8.79×10^{-14}	2.48×10^{-18}	2.23×10^{-16}	3.92×10^{-21}	1.30×10^{-29}	7.19×10^{-38}
	50	2.12×10^{-13}	1.16×10^{-17}	1.56×10^{-16}	2.54×10^{-21}	1.03×10^{-29}	3.64×10^{-38}
	100	4.89×10^{-14}	1.95×10^{-18}	8.97×10^{-18}	4.85×10^{-23}	2.99×10^{-32}	3.66×10^{-42}

Table 4 – Values of the measure Δ for several near-exact distributions for the l.r.t. statistic to test the equality of q covariance matrices of dimension p

			former near-exact dists.		new near-exact dists. (# of exact moments matched)			
p	q	n	M2GNIG	M3GNIG	4	6	10	15
5	3	7	2.24×10^{-7}	7.42×10^{-10}	5.23×10^{-9}	1.98×10^{-11}	2.39×10^{-14}	1.15×10^{-16}
		50	3.85×10^{-11}	2.11×10^{-12}	3.64×10^{-13}	7.62×10^{-17}	8.60×10^{-23}	1.50×10^{-29}
		100	1.03×10^{-11}	1.29×10^{-13}	1.23×10^{-14}	4.41×10^{-19}	2.05×10^{-26}	4.93×10^{-34}
5	7	7	9.86×10^{-8}	5.42×10^{-10}	2.13×10^{-8}	9.79×10^{-11}	4.71×10^{-15}	6.07×10^{-20}
		50	1.25×10^{-10}	1.04×10^{-13}	5.51×10^{-13}	4.43×10^{-17}	1.11×10^{-24}	9.17×10^{-32}
		100	1.25×10^{-11}	2.32×10^{-14}	1.49×10^{-14}	2.97×10^{-19}	5.79×10^{-28}	8.55×10^{-37}
7	5	9	5.17×10^{-9}	6.54×10^{-12}	3.33×10^{-10}	2.51×10^{-13}	1.61×10^{-18}	5.80×10^{-24}
		50	2.39×10^{-11}	8.90×10^{-15}	3.22×10^{-14}	1.55×10^{-18}	4.00×10^{-26}	6.34×10^{-33}
		100	1.83×10^{-12}	7.32×10^{-15}	5.43×10^{-16}	1.00×10^{-20}	2.94×10^{-29}	1.99×10^{-37}
7	10	9	2.35×10^{-9}	2.53×10^{-12}	3.82×10^{-10}	3.18×10^{-13}	4.23×10^{-19}	2.84×10^{-26}
		50	1.88×10^{-11}	2.72×10^{-14}	1.24×10^{-13}	4.70×10^{-18}	1.39×10^{-26}	4.68×10^{-35}
		100	1.83×10^{-12}	1.63×10^{-14}	3.44×10^{-15}	3.28×10^{-20}	6.58×10^{-30}	4.00×10^{-40}
10	5	12	1.64×10^{-10}	3.25×10^{-14}	3.88×10^{-12}	5.14×10^{-16}	8.45×10^{-24}	2.41×10^{-31}
		50	3.59×10^{-12}	1.34×10^{-14}	3.68×10^{-15}	1.96×10^{-19}	3.01×10^{-28}	3.68×10^{-39}
		100	1.67×10^{-13}	9.61×10^{-16}	2.46×10^{-16}	2.31×10^{-21}	1.97×10^{-31}	1.08×10^{-43}
10	15	12	7.13×10^{-12}	—× —	1.84×10^{-13}	5.58×10^{-18}	5.73×10^{-27}	1.56×10^{-36}
		50	2.81×10^{-13}	—× —	3.51×10^{-16}	1.24×10^{-21}	3.48×10^{-33}	5.51×10^{-43}
		100	1.97×10^{-14}	1.03×10^{-20}	5.23×10^{-18}	3.29×10^{-24}	8.09×10^{-36}	5.82×10^{-46}

6. Conclusions

In all cases, for a given number of exact moments matched, the newly proposed near-exact distributions show smaller values of the discrepancy measure Δ than the former near-exact distributions,indicating in this way that an excellent approximation to the exact distribution has been obtained.

Although it is clear that, for a given number of exact moments matched, the near-exact distributions proposed in this paper have a somewhat more complicated structure than the ones proposed in Marques et al. (2010), since they will correspond to a mixture of a larger number of GNIG distributions, the near-exact distributions proposed here are still quite manageable when using adequate software on most computers available nowadays. Moreover, the computation of the parameters involved in these near-exact distributions, which have to be computed by matching some of the exact moments, cause absolutely no problems since the systems of equations to be solved are always linear and as such very easy to solve. This would not be the case with the formerly proposed near-exact distributions when the number of exact moments matched is larger than 4.

With the proposed technique we may essentially get near-exact distributions that match as many of the exact moments as we wish and which will be as close to the exact distribution as we wish. As we may see from the tables in the previous section, in many situations such near-exact distributions, even when a quite small number of exact moments is matched, the discrepancy measure Δ is remarkably small. In practice, for example when computing p-values and quantiles, they may be used with confidence, even for applications requiring very good approximations, instead of the exact distributions, which are considerably less manageable.

Moreover, the technique used to build the near-exact distributions proposed in this paper may be readily extended to any l.r.t. statistic which may be built as the product of any number of independent l.r.t. statistics of the type studied in this paper, just by applying the methods and ideas in Coelho & Marques (2009).

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