The maximal subsemigroups of semigroups of transformations preserving or reversing the orientation on a finite chain

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Abstract

The study of the semigroups $\mathcal{OP}_n$ and $\mathcal{OR}_n$ respectively of all orientation-preserving transformations and of all orientation-preserving or orientation-reversing transformations on an n-element chain has began in [10] and [4]. In order to bring more insight into the subsemigroup structure of $\mathcal{OP}_n$ and $\mathcal{OR}_n$, we characterize their maximal subsemigroups.

Keywords: finite transformation semigroup, orientation-preserving and orientation-reversing transformations, maximal subsemigroups.

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Introduction and Preliminaries

For $n \in \mathbb{N}$, let $X_n = \{1 < 2 < \cdots < n\}$ be a finite chain with n elements. As usual, we denote by $\mathcal{T}_n$ the monoid (under composition) of all full transformations of X_n. For every transformation $\alpha \in \mathcal{T}_n$, we denote by $\ker\alpha$ and $\mathrm{im}\alpha$ the kernel and the image of α, respectively. The number $\mathrm{rank}\alpha = |\ker\alpha| = |\mathrm{im}\alpha|$ is called the rank of α. Given a subset U of $\mathcal{T}_n$, we denote by $E(U)$ its set of idempotents. The weight of an equivalence relation π on X_n is the number $|X_n/\pi|$. Let $A \subseteq X_n$ and let π be an equivalence relation on X_n of weight $|A|$. We say that A *is a transversal of* π (denoted by $A\#\pi$) if $|A \cap \bar{x}| = 1$ for every equivalence class $\bar{x}$ of π. A subset C of the chain X_n is said to be *convex* if $x, y \in C$ and $x \le z \le y$ together imply that $z \in C$. An equivalence relation π on X_n is convex if its classes are convex.

We say that a transformation $\alpha \in \mathcal{T}_n$ is *order-preserving* (respectively, *order-reversing*) if $x \le y$ implies that $x\alpha \le y\alpha$ (respectively, $x\alpha \ge y\alpha$), for all $x, y \in X_n$. As usual, $\mathcal{O}_n$ denotes the submonoid of $\mathcal{T}_n$ of all order-preserving transformations of X_n. This monoid has been largely studied, for instance in [1, 5, 8, 9, 12].

Let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t $(t \ge 1)$ elements from the chain X_n. We say that a is *cyclic* (respectively, *anti-cyclic*) if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ (respectively,

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$a_i < a_{i+1}$), where a_{t+1} denotes a_1. We say that a transformation $\alpha \in \mathcal{T}_n$ is *orientation-preserving* (respectively, *orientation-reversing*) if the sequence of its images is cyclic (respectively, anti-cyclic). The notion of an orientation-preserving transformation was introduced by McAlister in [10] and, independently, by Catarino and Higgins in [4]. It is easy to show that the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving, and the product of an orientation-preserving transformation by an orientation-reversing transformation is orientation-reversing (see [4]). We denote by $\mathcal{OP}_n$ (respectively, $\mathcal{OR}_n$) the monoid of all orientation-preserving (respectively, orientation-preserving or orientation-reversing) full transformations. It is clear that $\mathcal{OP}_n$ is a submonoid of $\mathcal{OR}_n$.

Since $\mathcal{O}_n$, $\mathcal{OP}_n$ and $\mathcal{OR}_n$ are regular submonoids of $\mathcal{T}_n$, the definition of the Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on $\mathcal{O}_n$, $\mathcal{OP}_n$ and $\mathcal{OR}_n$ follow immediately from well known results on regular semigroups and from their descriptions on $\mathcal{T}_n$. We have $\alpha\mathcal{L}\beta \Longleftrightarrow \mathrm{im}\alpha = \mathrm{im}\beta$ and $\alpha\mathcal{R}\beta \Longleftrightarrow \ker\alpha = \ker\beta$, for every transformations α and β. Recall also that, for the Green's relation $\mathcal{J}$, we have (on $\mathcal{O}_n$, $\mathcal{OP}_n$ and $\mathcal{OR}_n$) $\alpha\mathcal{J}\beta \Longleftrightarrow \mathrm{rank}\alpha = \mathrm{rank}\beta$, for every transformations α and β.

Regarding the monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$, presentations for them were exhibited by Catarino in [3] and by Arthur and Ruškuc in [2], the Green's relations, their sizes and ranks, among other properties, were determined by Catarino and Higgins in [4] and a description of their congruences were given in [6] by Fernandes, Gomes and Jesus. In [13] Zhao, Bo and Mei characterized the locally maximal idempotent-generated subsemigroups of $\mathcal{OP}_n$ (excluding the permutations).

In this paper, we aim to give more insight into the subsemigroup structure of the monoids $\mathcal{OP}_n$ and $\mathcal{OR}_n$ by characterizing the maximal subsemigroups of these monoids and of their ideals. In Section 1, we study the monoid $\mathcal{OP}_n$ and its ideals. First, we describe all maximal subsemigroups of $\mathcal{OP}_n$ (some of them are associated with the maximal subgroups of the additive group $\mathbb{Z}_n$). The main result of this section is the characterization of the maximal subsemigroups of the ideals of $\mathcal{OP}_n$. In Section 2, we study the monoid $\mathcal{OR}_n$ and its ideals. Again, first we describe all maximal subsemigroups of $\mathcal{OR}_n$ (some of them are associated with the maximal subgroups of the dihedral group $\mathcal{D}_n$ of order $2n$). The main result of this section is the characterization of the maximal subsemigroups of the ideals of $\mathcal{OR}_n$, which are associated with the maximal subsemigroups of the ideals of $\mathcal{OP}_n$.

1 Maximal subsemigroups of the ideals of $\mathcal{OP}_n$

Let $n \in \mathbb{N}$. The semigroup $\mathcal{OP}_n$ is the union of its $\mathcal{J}$-classes $J_1, J_2, \ldots, J_n$, where

$$J_k = \{\alpha \in \mathcal{OP}_n \mid \mathrm{rank}\alpha = k\},$$

for $k = 1, \ldots, n$. It is well known that the ideals of the semigroup $\mathcal{OP}_n$ are the unions of the $\mathcal{J}$-classes $J_1, J_2, \ldots, J_k$, i.e. the sets

$$OP(n,k) = \{\alpha \in \mathcal{OP}_n \mid \mathrm{rank}\alpha \leq k\},$$

with $k = 1, \ldots, n$. Every principal factor on $\mathcal{OP}_n$ is a Rees quotient $OP(n,k)/OP(n,k-1)$ ($2 \leq k \leq n$) of which we may think as $J_k \cup \{0\}$, where the product of two elements of J_k is taken to be zero if it falls into $OP(n,k-1)$.

Denote by L_α, R_α and H_α the $\mathcal{L}$-class, $\mathcal{R}$-class and $\mathcal{H}$-class, respectively, of an element $\alpha \in \mathcal{OP}_n$. Since the product $\alpha\beta$, for $\alpha, \beta \in J_k$, belongs to the class J_k (if and only if $\alpha\beta \in R_\alpha \cap L_\beta$) if and only if $\mathrm{im}\alpha \# \ker\beta$, it is easy to show:

Lemma 1.1 *Let $\alpha, \beta \in J_k$, with $k = 1, 2, \ldots, n$. Then*

$$\alpha R_\beta = \begin{cases} R_{\alpha\beta} = R_\alpha & \textit{if } \ \mathrm{im}\alpha \# \ker\beta, \\ 0 & \textit{otherwise,} \end{cases} \qquad L_\alpha \beta = \begin{cases} L_{\alpha\beta} = L_\beta & \textit{if } \ \mathrm{im}\alpha \# \ker\beta, \\ 0 & \textit{otherwise,} \end{cases}$$

$$L_\alpha R_\beta = \begin{cases} J_k & \textit{if } \ \mathrm{im}\alpha \# \ker\beta, \\ 0 & \textit{otherwise} \end{cases} \qquad \textit{and} \quad \alpha H_\beta = H_\alpha \beta = \begin{cases} H_{\alpha\beta} & \textit{if } \ \mathrm{im}\alpha \# \ker\beta, \\ 0 & \textit{otherwise.} \end{cases}$$

Next, recall that Catarino and Higgins proved:

Proposition 1.2 ([4]) *Let $\alpha \in J_k$, with $k = 1, 2, \ldots, n$. Then $|H_\alpha| = k$. Moreover, if α is an idempotent, then H_α is a cyclic group of order k.*

Let G be a cyclic group of order k, with $k \in \mathbb{N}$. It is well known that there exists an one-to-one correspondence between the subgroups of G and the (positive) divisors of k. Moreover, if r is a divisor of k then there exits a (cyclic) subgroup G_r of G such that $|G_r| = r$. On the other hand, being $x \in G$, there exists a (positive) divisor r of k such that x^r is the identity of G.

Let us consider the following elements:

$$g = \left(\begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{array} \right) \in J_n \quad \text{and} \quad u_i = \left(\begin{array}{cccc|c|ccc} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i+1 & i+1 & \cdots & n \end{array} \right) \in J_{n-1},$$

for $i = 1, \ldots, n$ (with $i = n$ we take $i + 1 = 1$).

Notice that $J_n = H_g$, whence J_n is a cyclic group of order n.

We will use the following well known result (see [3, 10]).

Proposition 1.3 *$\mathcal{OP}_n = \langle u_1, g \rangle$.*

Next, we present alternative generating sets of the monoid $\mathcal{OP}_n$.

Proposition 1.4 *Let $\alpha \in J_{n-1}$ and let $\gamma \in J_n$ be a permutation of order n. Then $\mathcal{OP}_n = \langle \alpha, \gamma \rangle$.*

Proof. Since $\gamma \in J_n$ has order n, we have $\langle \gamma \rangle = J_n$ and so $g \in \langle \gamma \rangle$. From $\alpha \in J_{n-1}$, it follows that there exist $1 \le i, j \le n$ such that $\mathrm{im}\alpha = X_n \setminus \{j\}$ and $(i, i+1) \in \ker\alpha$ (by taking $i + 1 = 1$, if $i = n$). Put $s = i - j$, if $j < i$, and $s = n + i - j$, otherwise. Then, it is easy to show that $\beta = \alpha g^s \in H_{u_i}$. Now, as u_i is an idempotent of $\mathcal{OP}_n$, by Proposition 1.2, it follows that u_i is a power of β. On the other hand, it is a routine matter to show that $u_1 = g^{n+i-1} u_i g^{n-i+1}$. Thus, by Proposition 1.3, we deduce that $\mathcal{OP}_n = \langle \alpha, \gamma \rangle$. ■

For a prime divisor p of n, we put $W_p = \langle g^p \rangle = \{1, g^p, g^{2p}, \ldots, g^{n-p}\}$, which is, clearly, a cyclic group of order $\frac{n}{p}$. Furthermore, from well known results regarding finite cyclic groups, we have:

Lemma 1.5 *The groups W_p, with p a prime divisor of n, are the maximal subgroups of J_n.*

Now, we can describe the maximal subsemigroups of $\mathcal{OP}_n$.

Theorem 1.6 *A subsemigroup S of the semigroup $\mathcal{OP}_n$ is maximal if and only if $S = OP(n, n-2) \cup J_n$ or $S = OP(n, n-1) \cup W_p$, for a prime divisor p of n.*

Proof. Let S be a maximal subsemigroup of $\mathcal{OP}_n$. Then, it is clear that $OP(n, n-2) \subseteq S$ and thus $S = OP(n, n-2) \cup T$, for some subset T of $J_{n-1} \cup J_n$. By Proposition 1.4, we have $T \cap J_{n-1} = \emptyset$ or T does not contain any element of J_n of order n. In this latter case, we must have $J_{n-1} \subseteq T$, by the maximality of S. This shows that $S = OP(n, n-1) \cup T'$, for some subset T' of J_n, whence T' must be a maximal subgroup of J_n. Thus, by Lemma 1.5, we have $T' = W_p$, for some prime divisor p of n. On the other hand, if $T \cap J_{n-1} = \emptyset$ then $S \subseteq OP(n, n-2) \cup J_n$, whence $S = OP(n, n-2) \cup J_n$, by the maximality of S.

The converse part follows immediately from Proposition 1.4 and Lemma 1.5. ■

Let $n \geq 3$ and $1 \leq k \leq n-1$. In the remaining of this section, we consider the ideal $OP(n,k)$ of $\mathcal{OP}_n$.

Clearly, the maximal subsemigroups of $OP(n,1)$ are the sets of the form $OP(n,1) \setminus \{\alpha\}$, for $\alpha \in OP(n,1)$. Therefore, in what follows, we consider $k \geq 2$.

Notice that, as every element $\alpha \in \mathcal{O}_n$ of rank $r-1$, for $2 \leq r \leq n-1$, is expressible as a product of elements of $\mathcal{O}_n$ of rank r (see [7]) and every element $\beta \in \mathcal{OP}_n$ admits a decomposition $\beta = g^t\alpha$, for some $1 \leq t \leq n$ and $\alpha \in \mathcal{O}_n$ (see [4]), we deduce that every element of J_{r-1} is a product of elements of J_r, for $2 \leq r \leq n-1$. Thus, we have:

Lemma 1.7 *$OP(n,k) = \langle J_k \rangle$.*

Let us denote by Λ_k the collection of all subsets of X_n of cardinality k. Since two elements of J_k are $\mathcal{L}$-related if and only if they have the same image, an $\mathcal{L}$-class of J_k (which coincides with an $\mathcal{L}$-class of $\mathcal{OP}_n$, as J_k is regular) is completely determined by the image set of its transformations. Therefore a typical $\mathcal{L}$-class of J_k has the form

$$L_A = \{\alpha \in J_k \mid \mathrm{im}\alpha = A\},$$

with $A \in \Lambda_k$.

Let Ω_k be the collection of all equivalence relations π on X_n of weight k such that, for all $\overline{x} \in X_n/\pi$, either $\overline{x}$ or $X_n \setminus \overline{x}$ is a convex subset of X_n. Since two transformations of J_k are $\mathcal{R}$-related if and only if they have the same kernel, an $\mathcal{R}$-class of J_k (which coincides with an $\mathcal{R}$-class of $\mathcal{OP}_n$, as J_k is regular) is completely determined by the kernel of any of its elements. A typical $\mathcal{R}$-class of J_k has then the form

$$R_\pi = \{\alpha \in J_k \mid \ker\alpha = \pi\},$$

with $\pi \in \Omega_k$.

Finally, it follows that a typical $\mathcal{H}$-class of J_k has the form

$$H_{(\pi,A)} = R_\pi \cap L_A,$$

with $\pi \in \Omega_k$ and $A \in \Lambda_k$.

Notice that, for any $\pi \in \Omega_k$ and for any $\alpha \in R_\pi$, it is easy to show that $H_\alpha \cap \mathcal{O}_n = \emptyset$ if and only if $(1,n) \in \pi$ (i.e. π contains a non-convex class). Observe also that, being $O(n,k) = OP(n,k) \cap \mathcal{O}_n$ (the ideal of $\mathcal{O}_n$ of all elements of rank less than or equal to k) and $J'_k = J_k \cap \mathcal{O}_n$ (the $\mathcal{J}$-class of $\mathcal{O}_n$ of all elements of rank equal to k), a typical $\mathcal{L}$-class of J'_k has the form $L_A \cap \mathcal{O}_n$, with $A \in \Lambda_k$, and a typical $\mathcal{R}$-class of J'_k has the form $R_\pi \cap \mathcal{O}_n$, with $\pi \in \Omega'_k = \{\pi \in \Omega_k \mid (1,n) \notin \pi\}$.

Proposition 1.8 *Let C be any subset of J_k containing $J_k \cap \mathcal{O}_n$ and at least one element from each $\mathcal{R}$-class of J_k. Then $OP(n,k) = \langle C \rangle$.*

Proof. First, let α be an element of C with kernel $\{\{1,k{+}1,\ldots,n\},\{2\},\ldots,\{k\}\}$. Let β be any order-preserving transformation with image $\{1,\ldots,k\}$ such that $\mathrm{im}\alpha \# \ker\beta$. Then, $\ker(\alpha\beta) = \ker\alpha$ and $\mathrm{im}(\alpha\beta) = \mathrm{im}\beta$, from which it follows that the idempotent power of $\alpha\beta$ is the transformation $\left(\begin{array}{ccc|ccc} 1 & \cdots & k & k+1 & \cdots & n \\ 1 & \cdots & k & 1 & \cdots & 1 \end{array}\right)$. Therefore

$$\begin{aligned} \gamma &= \left(\begin{array}{ccc|ccc} 1 & \cdots & k & k+1 & \cdots & n \\ 2 & \cdots & k+1 & k+1 & \cdots & k+1 \end{array}\right)\left(\begin{array}{ccc|ccc} 1 & \cdots & k & k+1 & \cdots & n \\ 1 & \cdots & k & 1 & \cdots & 1 \end{array}\right) \\ &= \left(\begin{array}{cccc|ccc} 1 & \cdots & k-1 & k & k+1 & \cdots & n \\ 2 & \cdots & k & 1 & 1 & \cdots & 1 \end{array}\right) \in \langle C \rangle. \end{aligned}$$

Furthermore, as γ generates a cycle group of order k, we have $H_\gamma \subseteq \langle C \rangle$.

Now, let $\varepsilon = \gamma^k = \left(\begin{array}{ccc|ccc} 1 & \cdots & k & k+1 & \cdots & n \\ 1 & \cdots & k & k & \cdots & k \end{array}\right)$ be the idempotent of H_γ and let H be any $\mathcal{H}$-class contained in $R_\varepsilon = R_\gamma$. Since the elements of H have the same kernel that $\varepsilon \in \mathcal{O}_n$, then H has an order-preserving element τ. From $\varepsilon \mathcal{R} \tau$ it follows that $\varepsilon\tau = \tau$, whence $\mathrm{im}\varepsilon \# \ker\tau$ and so, by Lemma 1.1, we have $H_\varepsilon \tau = H_\tau$. As $\tau \in C$ and $H_\varepsilon \subseteq \langle C \rangle$, we also have $H = H_\tau \subseteq \langle C \rangle$. Hence $R_\varepsilon \subseteq \langle C \rangle$.

Next, let $\pi \in \Omega_k$ be such that $(1,n) \notin \pi$. Then, there exists an order-preserving transformation $\tau \in L_\varepsilon \cap R_\pi$. Since $\varepsilon \in L_\varepsilon \cap R_\varepsilon = L_\tau \cap R_\varepsilon$, we have $\tau\varepsilon = \tau$, whence $\mathrm{im}\tau \# \ker\varepsilon$ and so, by Lemma 1.1, we obtain $\tau R_\varepsilon = R_\tau = R_\pi$. As $\tau \in C$ and $R_\varepsilon \subseteq \langle C \rangle$, it follows that $R_\pi \subseteq \langle C \rangle$.

Finally, let $\pi \in \Omega_k$ be such that $(1,n) \in \pi$ and let $\tau \in C \cap R_\pi$. Take an order-preserving idempotent ε' such that $\mathrm{im}\varepsilon' = \mathrm{im}\tau$. Then, $\varepsilon' \in L_{\varepsilon'} \cap R_{\varepsilon'} = L_\tau \cap R_{\varepsilon'}$, whence $\tau\varepsilon' = \tau$ and so $\mathrm{im}\tau \# \ker\varepsilon'$. Thus, by Lemma 1.1, we have $\tau R_{\varepsilon'} = R_\tau = R_\pi$. As $\tau \in C$ and $R_{\varepsilon'} \subseteq \langle C \rangle$ (by the previous case), it follows that $R_\pi \subseteq \langle C \rangle$.

Hence, we proved that $J_k \subseteq \langle C \rangle$ and so, by Lemma 1.7, we obtain $OP(n,k) = \langle C \rangle$, as required. ∎

Since $O(n,k) = \langle E(J_k \cap \mathcal{O}_n) \rangle$ (see [7]) and each $\mathcal{R}$-class of J_k contains at least one idempotent, we have:

Corollary 1.9 *$OP(n,k) = \langle E(J_k) \rangle$.*

Notice that, it is easy to show that, in fact, each $\mathcal{R}$-class of J_k contains at least two idempotents. Moreover, as $2 \leq k \leq n-1$, it also is easy to show that each $\mathcal{L}$-class of J_k contains at least two idempotents.

Let Λ be a non-empty proper subset of Λ_k and let Ω be a non-empty proper subset of Ω_k (respectively, of Ω'_k). The pair (Λ, Ω) is called a *coupler* of (Λ_k, Ω_k) (respectively, of (Λ_k, Ω'_k)) if the following three conditions are satisfied (see [11]):

1. For every $A \in \Lambda$ and $\pi \in \Omega$, A is not a transversal of π;
2. For every $B \in \Lambda_k \setminus \Lambda$, there exists $\pi \in \Omega$ such that $B \# \pi$;
3. For every $\rho \in \Omega_k \setminus \Omega$ (respectively, $\rho \in \Omega'_k \setminus \Omega$), there exists $A \in \Lambda$ such that $A \# \rho$.

Next, we consider the following subsets of $OP(n,k)$:

1. $S_A = OP(n,k-1) \cup (J_k \setminus L_A)$, for each $A \in \Lambda_k$;
2. $S_\pi = OP(n,k-1) \cup (J_k \setminus R_\pi)$, for each $\pi \in \Omega_k$;
3. $S_{(\Lambda,\Omega)} = OP(n,k-1) \cup (\bigcup\{L_A \mid A \in \Lambda\}) \cup (\bigcup\{R_\pi \mid \pi \in \Omega\})$, for each coupler (Λ, Ω) of (Λ_k, Ω_k).

It is routine matter to prove that each of these subsets is a (proper) subsemigroup of $OP(n,k)$.

Before we give the description of the maximal subsemigroups of the ideals of the semigroup $\mathcal{OP}_n$, we recall the following result presented by the first and third author in [5] (see also [12]).

Theorem 1.10 ([5]) *Let $n \geq 3$ and $2 \leq k \leq n-1$. Then a subsemigroup of $O(n,k)$ is maximal if and only if it belongs to one of the following types:*

1. *$S_A \cap \mathcal{O}_n$, with $A \in \Lambda_k$;*
2. *$S_\pi \cap \mathcal{O}_n$, with $\pi \in \Omega'_k$ such that π does not admit an interval of Λ as a transversal;*
3. *$S'_{(\Lambda,\Omega)} = O(n,k-1) \cup (\bigcup\{L_A \cap \mathcal{O}_n \mid A \in \Lambda\}) \cup (\bigcup\{R_\pi \cap \mathcal{O}_n \mid \pi \in \Omega\})$, with (Λ, Ω) a coupler of (Λ_k, Ω'_k).*

Lemma 1.11 *Let S be a maximal subsemigroup of $OP(n,k)$. Then $S = \bigcup\{H_\alpha \mid \alpha \in S\}$.*

Proof. Let $T = \bigcup\{H_\alpha \mid \alpha \in S\}$. First, notice that, by Corollary 1.9, there exists $\varepsilon \in E(J_k)$ such that $\varepsilon \notin S$. Hence, $H_\varepsilon \cap S = \emptyset$ and so $S \subseteq T \subsetneq OP(n,k)$. The result follows by proving that T is a subsemigroup of $OP(n,k)$. Clearly, by the maximality of S (and Lemma 1.7), we have $OP(n,k-1) \subsetneq S$. So, if suffices to show that, for all $\alpha, \beta \in T \cap J_k$ such that $\alpha\beta \in J_k$, we get $\alpha\beta \in T$. Therefore, let $\alpha, \beta \in T \cap J_k$ be such that $\alpha\beta \in J_k$. Take $\alpha', \beta' \in S$ such that $\alpha \in H_{\alpha'}$ and $\beta \in H_{\beta'}$. Then $\mathrm{im}\alpha' = \mathrm{im}\alpha \# \ker\beta = \ker\beta'$ and $\alpha\beta \in R_\alpha \cap L_\beta$, whence $\alpha'\beta' \in R_{\alpha'} \cap L_{\beta'} = R_\alpha \cap L_\beta = H_{\alpha\beta}$ and so, as $\alpha'\beta' \in S$, we obtain $\alpha\beta \in H_{\alpha'\beta'} \subseteq T$, as required. ■

Now, we have:

Theorem 1.12 *Let $n \geq 3$ and $2 \leq k \leq n-1$. Then a subsemigroup of $OP(n,k)$ is maximal if and only if it belongs to one of the following types:*

1. *S_A, with $A \in \Lambda_k$;*
2. *S_π, with $\pi \in \Omega_k$;*
3. *$S_{(\Lambda,\Omega)}$, with (Λ, Ω) a coupler of (Λ_k, Ω_k).*

Proof. We begin by showing that each of these subsemigroups of $OP(n,k)$ is maximal.

First, let $A \in \Lambda_k$ and let $\alpha \in L_A$. Take an idempotent $\varepsilon \in (J_k \setminus L_A) \cap R_\alpha$. As $L_\varepsilon \subseteq S_A$ and, by Lemma 1.1, $L_\varepsilon \alpha = L_A$, we have $\langle S_A, \alpha \rangle = OP(n,k)$. Thus, S_A is maximal.

Similarly, being $\pi \in \Omega_k$ and $\alpha \in R_\pi$, the $\mathcal{L}$-class L_α contains at least an idempotent $\varepsilon \in J_k \setminus R_\pi$ and so $R_\varepsilon \subseteq S_\pi$ and, by Lemma 1.1, $\alpha R_\varepsilon = R_\pi$, whence $\langle S_\pi, \alpha \rangle = OP(n,k)$. Thus, S_π is maximal.

Finally, regarding the subsemigroups of type 3, let (Λ, Ω) be a coupler of (Λ_k, Ω_k). As $\Lambda_k \setminus \Lambda \neq \emptyset$ and $\Omega_k \setminus \Omega \neq \emptyset$, we may take $\alpha \in R_\rho \cap L_B$, for some $\rho \in \Omega_k \setminus \Omega$ and $B \in \Lambda_k \setminus \Lambda$. Then, there exist $\pi \in \Omega$ and $A \in \Lambda$ such that $B \# \pi$ and $A \# \rho$. Now, by Lemma 1.1, we have $\alpha R_\pi = R_\alpha = R_\rho$. As $R_\pi \subseteq S_{(\Lambda,\Omega)}$, we obtain $R_\rho \subseteq \langle S_{(\Lambda,\Omega)}, \alpha \rangle$. On the other hand, by Lemma 1.1, we also have $L_A R_\rho = J_k$. Since $L_A \in S_{(\Lambda,\Omega)}$, we deduce that $\langle S_{(\Lambda,\Omega)}, \alpha \rangle = OP(n,k)$. Thus, $S_{(\Lambda,\Omega)}$ is maximal.

For the converse part, let S be a maximal subsemigroup of the ideal $OP(n,k)$.

If $S \cap R_\pi = \emptyset$, for some $\pi \in \Omega_k$, then $S = S_\pi$, by the maximality of S. Similarly, if $S \cap L_A = \emptyset$, for some $A \in \Lambda_k$, then $S = S_A$. Thus, admit that S contains at least one element from each $\mathcal{R}$-class and each $\mathcal{L}$-class of J_k. If $S \cap \mathcal{O}_n = O(n,k)$ then $S = OP(n,k)$, by Proposition 1.8. Therefore $S \cap \mathcal{O}_n \subsetneq O(n,k)$. Let $\bar{S}$ be any maximal subsemigroup of $O(n,k)$ such that $S \cap \mathcal{O}_n \subseteq \bar{S}$. Now, by Theorem 1.10, we have three possible cases for $\bar{S}$.

First, suppose that $\bar{S} = S_\pi \cap \mathcal{O}_n$, for some $\pi \in \Omega'_k$. Then, as $S \cap R_\pi \neq \emptyset$, we may take $\alpha \in S \cap R_\pi$. Since $\pi \in \Omega'_k$, we have $H_\alpha \cap \mathcal{O}_n \neq \emptyset$. Now, as $H_\alpha \subseteq S$ (by Lemma 1.11), we have $(S \cap \mathcal{O}_n) \cap R_\pi \neq \emptyset$, whence $\bar{S} \cap R_\pi \neq \emptyset$, which is a contradiction. Thus, $\bar{S}$ cannot be of this type.

Secondly, we suppose that $\bar{S} = S_{A_1} \cap \mathcal{O}_n$, for some $A_1 \in \Lambda_k$. Let $A_1, \ldots, A_r$ be the $r \geq 1$ distinct elements of Λ_k such that, for all $A \in \Lambda_k$, $L_A \cap \mathcal{O}_n \cap S = \emptyset$ if and only if $A \in \{A_1, \ldots, A_r\}$. Notice that, for $i \in \{1, \ldots, r\}$, we have $L_{A_i} \cap S \neq \emptyset$ and, as a consequence of Lemma 1.11, if $\alpha \in L_{A_i} \cap S$ then $(1,n) \in \ker\alpha$. Now, let

$$\Omega = \{\pi \in \Omega_k \mid R_\pi \cap L_{A_i} \cap S \neq \emptyset, \text{ for some } i \in \{1, \ldots, r\}\}.$$

Notice that, clearly, $\Omega \neq \emptyset$. Also, let

$$\Lambda = \{A \in \Lambda_k \mid A \text{ is not a transversal of } \pi, \text{ for all } \pi \in \Omega\}.$$

Observe that, as $(1,n) \in \pi$, for all $\pi \in \Omega$, then $\{A \in \Lambda_k \mid 1, n \in A\} \subseteq \Lambda$ and so, in particular, $\Lambda \neq \emptyset$. Furthermore, it is a routine matter to check that the pair (Λ, Ω) is a coupler of (Λ_k, Ω_k). Next, we show that $S \cap J_k \subseteq S_{(\Lambda,\Omega)}$. Take $\alpha \in S \cap J_k$. If $\mathrm{im}\alpha \in \Lambda$, then $\alpha \in \bigcup\{L_A \mid A \in \Lambda\}$. Thus, let us suppose that $\mathrm{im}\alpha \notin \Lambda$. Then we have to consider two cases. If $\alpha \in L_{A_i}$, for some $i \in \{1, \ldots, r\}$, then $\ker\alpha \in \Omega$, whence $\alpha \in \bigcup\{R_\pi \mid \pi \in \Omega\}$. Now, let $\alpha \notin L_{A_i}$, for all $i \in \{1, \ldots, r\}$. Then, there exists $\pi \in \Omega$ such that $\mathrm{im}\alpha \# \pi$. As

$\pi \in \Omega$, there exists $i \in \{1, \ldots, r\}$ such that $R_\pi \cap L_{A_i} \cap S \neq \emptyset$. Take $\beta \in R_\pi \cap L_{A_i} \cap S \neq \emptyset$. Hence $\mathrm{im}\alpha \# \pi = \ker\beta$ and so $\alpha\beta \in R_\alpha \cap L_\beta = R_\alpha \cap L_{A_i}$. Moreover, $\alpha\beta \in S$, whence $\alpha\beta \in R_{\alpha\beta} \cap L_{A_i} \cap S$. Then $\ker\alpha = \ker(\alpha\beta) \in \Omega$, from which it follows that $\alpha \in \bigcup\{R_\pi \mid \pi \in \Omega\}$. So, we have proved that $\alpha \in S_{(\Lambda,\Omega)}$. Therefore $S \subseteq S_{(\Lambda,\Omega)}$ and thus $S = S_{(\Lambda,\Omega)}$, by the maximality of S.

Finally, suppose that $\bar{S} = S'_{(\Lambda',\Omega')}$, for some coupler (Λ', Ω') of (Λ_k, Ω'_k). Let

$$\Lambda = \{A \in \Lambda' \mid L_A \cap S \cap \left(\bigcup\{R_\pi \mid \pi \in \Omega'_k \setminus \Omega'\}\right) \neq \emptyset\},$$

which is a nonempty subset of Λ_k (as $S \cap R_\pi \neq \emptyset$, for all $\pi \in \Omega_k$). Also, let

$$\Omega = \{\pi \in \Omega_k \mid A \text{ is not a transversal of } \pi, \text{ for all } A \in \Lambda\}.$$

Clearly, $\Omega' \subseteq \Omega$, whence $\Omega \neq \emptyset$. Furthermore, it is a routine matter to check that the pair (Λ, Ω) is a coupler of (Λ_k, Ω_k). Next, we aim to prove that $S = S_{(\Lambda,\Omega)}$. First, observe that, from the definition of Λ and from $S \cap \mathcal{O}_n \subseteq S'_{(\Lambda',\Omega')}$ in view of Lemma 1.11, we deduce that $R_\pi \cap L_A \cap S = \emptyset$, for all $\pi \in \Omega'_k \setminus \Omega'$ and $A \in \Lambda_k \setminus \Lambda$. Now, take $\alpha \in J_k \cap S$ and suppose that $\alpha \notin S_{(\Lambda,\Omega)}$. Then, $\mathrm{im}\alpha \in \Lambda_k \setminus \Lambda$ and $\ker\alpha \in \Omega_k \setminus \Omega$. Hence, there exists $A \in \Lambda$ such that $A \# \ker\alpha$. Thus, by the definition of Λ, we may take $\beta \in L_A \cap S \cap (\bigcup\{R_\pi \mid \pi \in \Omega'_k \setminus \Omega'\})$ and so, as $\mathrm{im}\beta = A \# \ker\alpha$, we have $\beta\alpha \in R_\beta \cap L_\alpha \cap S$, i.e. $\beta\alpha \in S$, $\mathrm{im}(\beta\alpha) = \mathrm{im}\alpha \in \Lambda_k \setminus \Lambda$ and $\ker(\beta\alpha) = \ker(\beta) \in \Omega'_k \setminus \Omega'$, which contradicts the above deduction. Therefore $\alpha \in S_{(\Lambda,\Omega)}$. It follows that $S \subseteq S_{(\Lambda,\Omega)}$ and then $S = S_{(\Lambda,\Omega)}$, by the maximality of S, as required. ■

2 Maximal subsemigroups of the ideals of $\mathcal{OR}_n$

Let $n \in \mathbb{N}$. As for $\mathcal{OP}_n$, the semigroup $\mathcal{OR}_n$ is the union of its $\mathcal{J}$-classes $\bar{J}_1, \bar{J}_2, \ldots, \bar{J}_n$, where

$$\bar{J}_k = \{\alpha \in \mathcal{OR}_n \mid \mathrm{rank}\alpha = k\}$$

for $k = 1, \ldots, n$. Notice that $\bar{J}_k \cap \mathcal{OP}_n$ is the $\mathcal{J}$-class J_k of $\mathcal{OP}_n$, for $k = 1, \ldots, n$, and $\bar{J}_1 = J_1$ and $\bar{J}_2 = J_2$. Analogously to $\mathcal{OP}_n$, the ideals of the semigroup $\mathcal{OR}_n$ are the unions of the $\mathcal{J}$-classes $\bar{J}_1, \bar{J}_2, \ldots, \bar{J}_k$, i.e. the sets

$$OR(n,k) = \{\alpha \in \mathcal{OR}_n \mid \mathrm{rank}\alpha \leq k\},$$

with $k = 1, \ldots, n$. To avoid ambiguity, we denote by $\bar{L}_\alpha$, $\bar{R}_\alpha$ and $\bar{H}_\alpha$ the $\mathcal{L}$-class, $\mathcal{R}$-class and $\mathcal{H}$-class, respectively, of an element $\alpha \in \mathcal{OR}_n$. Observe that, for $\alpha \in \mathcal{OP}_n$, the sets $\bar{L}_\alpha \cap \mathcal{OP}_n$, $\bar{R}_\alpha \cap \mathcal{OP}_n$ and $\bar{H}_\alpha \cap \mathcal{OP}_n$ are respectively the $\mathcal{L}$-class L_α, the $\mathcal{R}$-class R_α and the $\mathcal{H}$-class H_α of $\mathcal{OP}_n$.

Taking the product of two elements of $\bar{J}_k$, for $k = 2, \ldots, n$, as being zero if it falls into $OR(n, k-1)$, a result similar to Lemma 1.1 holds for elements of $\mathcal{OR}_n$.

Lemma 2.1 *Let $\alpha, \beta \in \bar{J}_k$, with $k = 1, 2, \ldots, n$. Then*

$$\alpha\bar{R}_\beta = \begin{cases} \bar{R}_{\alpha\beta} = \bar{R}_\alpha & \text{if } \mathrm{im}\alpha \# \ker\beta, \\ 0 & \text{otherwise,} \end{cases} \qquad \bar{L}_\alpha\beta = \begin{cases} \bar{L}_{\alpha\beta} = \bar{L}_\beta & \text{if } \mathrm{im}\alpha \# \ker\beta, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{L}_\alpha\bar{R}_\beta = \begin{cases} \bar{J}_k & \text{if } \mathrm{im}\alpha \# \ker\beta, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \alpha\bar{H}_\beta = \bar{H}_\alpha\beta = \begin{cases} \bar{H}_{\alpha\beta} & \text{if } \mathrm{im}\alpha \# \ker\beta, \\ 0 & \text{otherwise.} \end{cases}$$

As $\mathcal{OR}_1 = \mathcal{OP}_1$ and $\mathcal{OR}_2 = \mathcal{OP}_2$, in what follows, we consider $n \geq 3$.

Next, recall that a dihedral group $\mathcal{D}_n$ of order $2n$ can abstractly be defined by the group presentation

$$\langle x, y \mid x^n = y^2 = 1, xy = yx^{-1}\rangle.$$

Let

$$h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix} \in \bar{J}_n.$$

Hence, we have $\bar{J}_n = \langle g, h \rangle$ and, as $g^n = h^2 = (gh)^2 = 1$, it is easy to see that $\bar{J}_n$ is a dihedral group of order $2n$. Furthermore, Catarino and Higgins proved:

Proposition 2.2 ([4]) *Let $\alpha \in \bar{J}_k$, with $k = 3, \ldots, n$. Then $|\bar{H}_\alpha| = 2k$. Moreover, if α is an idempotent, then $\bar{H}_\alpha$ is a dihedral group of order $2k$.*

Thus, each $\mathcal{H}$-class of rank k of $\mathcal{OR}_n$ has k orientation-preserving transformations and k orientation-reversing transformations, for $k \in \{3, \ldots, n\}$.

Notice that, since $\bar{J}_1 = J_1$ and $\bar{J}_2 = J_2$, for $\alpha \in \bar{J}_k$ with $k = 1, 2$, we have $|\bar{H}_\alpha| = k$.

Let us consider again the dihedral group $\mathcal{D}_n = \langle x, y \mid x^n = y^2 = 1, xy = yx^{-1} \rangle$ of order $2n$. Observe that $\mathcal{D}_n = \{1 = x^0, x, x^2, \ldots, x^{n-1}\} \cup \{y, xy, x^2y, \ldots, x^{n-1}y\}$. It is easy to show that the subgroups of $\mathcal{D}_n$ are of the form $\langle x^d \rangle$ (a cyclic group of order n/d) and of the form $\langle x^d, x^i y \rangle$ (a dihedral group of order $2n/d$), for each positive divisor d of n and each $0 \le i < d$. It follows that $\langle x \rangle$ and $\langle x^p, x^i y \rangle$, with p a prime divisor of n and $0 \le i < p$, are the maximal subgroups of $\mathcal{D}_n$.

Now, for a prime divisor p of n and $0 \le i < p$, consider the dihedral group $V_{p,i} = \langle g^p, g^i h \rangle$ of order $2n/p$. Then, the above observation can be rewrote as:

Lemma 2.3 *The group $J_n = \langle g \rangle$ and the groups $V_{p,i}$, with p a prime divisor of n and $0 \le i < p$, are the maximal subgroups of $\bar{J}_n$.*

Next, we recall the following well known result (see [3, 10]).

Proposition 2.4 *$\mathcal{OR}_n = \langle u_1, g, h \rangle$.*

In fact, more generally, we have:

Proposition 2.5 *Let $\alpha \in \bar{J}_{n-1}$, γ an element of J_n of order n and $\beta \in \bar{J}_n \setminus J_n$. Then $\mathcal{OR}_n = \langle \alpha, \gamma, \beta \rangle$.*

Proof. If $\alpha \in \bar{J}_{n-1} \cap \mathcal{OP}_n$ then, by Proposition 1.4, we have $\mathcal{OP}_n = \langle \alpha, \gamma \rangle$. If $\alpha \in \bar{J}_{n-1} \setminus \mathcal{OP}_n$ then $\alpha\beta \in \bar{J}_{n-1} \cap \mathcal{OP}_n$ and, again by Proposition 1.4, we obtain $\mathcal{OP}_n = \langle \alpha\beta, \gamma \rangle$. Therefore, $u_1, g \in \langle \alpha, \gamma, \beta \rangle$. As $\beta \in \bar{J}_n \setminus \mathcal{OP}_n$, there exists $i \in \{1, \ldots, n\}$ such that $\beta = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & \cdots & n-1 & n \\ i-1 & i-2 & \cdots & 1 & n & \cdots & i+1 & i \end{pmatrix}$. On the other hand, the transformation $\delta = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & \cdots & n-1 & n \\ n-i+2 & n-i+3 & \cdots & n & 1 & \cdots & n-i & n-i+1 \end{pmatrix}$ is an element of $\mathcal{OP}_n$ and $h = \beta\delta \in \langle \alpha, \gamma, \beta \rangle$. Therefore, by Proposition 2.4, we deduce that $\mathcal{OR}_n = \langle \alpha, \gamma, \beta \rangle$. ∎

We have now all the ingredients to describe the maximal subsemigroups of $\mathcal{OR}_n$.

Theorem 2.6 *A subsemigroup S of the semigroup $\mathcal{OR}_n$ is maximal if and only if $S = OR(n, n-2) \cup \bar{J}_n$ or $S = OR(n, n-1) \cup J_n$ or $S = OR(n, n-1) \cup V_{p,i}$, for some prime divisor p of n and $0 \le i < p$.*

Proof. Let S be a maximal subsemigroup of $\mathcal{OR}_n$. Then, by Proposition 2.5, we have $S = OR(n, n-2) \cup T$, for some $T \subset (\bar{J}_{n-1} \cup \bar{J}_n)$ such that $T \cap \bar{J}_{n-1} = \emptyset$ or T does not contain any element of J_n of order n or $T \cap (\bar{J}_n \setminus J_n) = \emptyset$. In the latter two cases, we must have $\bar{J}_{n-1} \subseteq T$, by the maximality of S. Thus, $S = OR(n, n-1) \cup T'$, for some $T' \subset \bar{J}_n$. Clearly, T' must be a maximal subgroup of $\bar{J}_n$, whence $S = OR(n, n-1) \cup J_n$ or $S = OR(n, n-1) \cup V_{p,i}$, for some prime divisor p of n and $0 \le i < p$, accordingly with Lemma 2.3. On the other hand, if $T \cap \bar{J}_{n-1} = \emptyset$ then $S \subseteq OR(n, n-2) \cup \bar{J}_n$ and so $S = OR(n, n-2) \cup \bar{J}_n$, by the the maximality of S.

The converse part follows immediately from Proposition 2.5 and Lemma 2.3. ∎

From now on we consider the ideals $OR(n,k)$ of $\mathcal{OR}_n$, for $k \in \{1,\ldots,n-1\}$. Since $OR(n,1) = OP(n,1)$ and $OR(n,2) = OP(n,2)$, in what follows, we take $k \geq 3$.

Notice that, as $\alpha h \in OP(n,k)$, for all $\alpha \in OR(n,k) \setminus OP(n,k)$, by using Lemma 1.7, it is easy to conclude:

Lemma 2.7 *$OR(n,k) = \langle \bar{J}_k \rangle$.*

In fact, moreover, we have:

Proposition 2.8 *$OR(n,k) = \langle J_k, \alpha \rangle$, for all $\alpha \in \bar{J}_k \setminus J_k$.*

Proof. Let $\alpha \in \bar{J}_k \setminus J_k$ and take an idempotent $\varepsilon \in \bar{L}_\alpha$. Since $\mathrm{im}\alpha = \mathrm{im}\varepsilon \# \ker\varepsilon$, we have $\alpha \bar{R}_\varepsilon = \bar{R}_\alpha$, by Lemma 2.1. Hence, $\alpha(\bar{R}_\varepsilon \cap J_k) = \bar{R}_\alpha \setminus J_k$ and so $\bar{R}_\alpha = (\bar{R}_\alpha \cap J_k) \cup (\bar{R}_\alpha \setminus J_k) = (\bar{R}_\alpha \cap J_k) \cup \alpha(\bar{R}_\varepsilon \cap J_k) \subseteq \langle J_k, \alpha \rangle$.

Now, let ε' be an idempotent of $\bar{R}_\alpha$ and take $\alpha' \in \bar{H}_{\varepsilon'} \setminus J_k$. Notice that $\alpha' \in \langle J_k, \alpha \rangle$. As $\mathrm{im}\varepsilon' \# \ker\varepsilon' = \ker\alpha'$, we have $\bar{L}_{\varepsilon'}\alpha' = \bar{L}_{\alpha'} = \bar{L}_{\varepsilon'}$, by Lemma 2.1. Thus $(\bar{L}_{\varepsilon'} \cap J_k)\alpha' = \bar{L}_{\varepsilon'} \setminus J_k$, whence $\bar{L}_{\varepsilon'} = (\bar{L}_{\varepsilon'} \cap J_k) \cup (\bar{L}_{\varepsilon'} \setminus J_k) = (\bar{L}_{\varepsilon'} \cap J_k) \cup (\bar{L}_{\varepsilon'} \cap J_k)\alpha' \subseteq \langle J_k, \alpha \rangle$.

Finally, as $\mathrm{im}\varepsilon' \# \ker\varepsilon' = \ker\alpha$, we have $\bar{L}_{\varepsilon'} \bar{R}_\alpha = \bar{J}_k$, again by Lemma 2.1. Therefore, $\bar{J}_k \subseteq \langle J_k, \alpha \rangle$ and so, by Lemma 2.7, $OR(n,k) = \langle J_k, \alpha \rangle$, as required. ∎

As an immediate consequence of Proposition 2.8, we have:

Corollary 2.9 *$OR(n,k-1) \cup J_k$ is a maximal subsemigroup of $OR(n,k)$.*

Also, combining Proposition 2.8 with Corollary 1.9, we have:

Corollary 2.10 *$OR(n,k) = \langle E(J_k), \alpha \rangle$, for all $\alpha \in \bar{J}_k \setminus J_k$.*

Before we present our description of the maximal subsemigroups of the ideals of $\mathcal{OR}_n$, we prove the following result:

Proposition 2.11 *Let S be a maximal subsemigroup of $OR(n,k)$ containing at least one orientation-reversing transformation of rank k. Then $S = \bigcup\{\bar{H}_\alpha \mid \alpha \in S \cap \mathcal{OP}_n\}$.*

Proof. Let $\alpha \in S$. As clearly $OR(n,k-1) \subseteq S$, it suffices to consider $\alpha \in \bar{J}_k$. Take $\beta \in \bar{H}_\alpha$ and suppose that $\beta \notin S$. Hence, by the maximality of S, we have $OR(n,k) = \langle S, \beta \rangle$. Let $\tau \in \bar{J}_k \setminus S$. Then, there exist $t \geq 0$, $r_0, r_1, \ldots, r_t \geq 0$ and $\alpha_1, \ldots, \alpha_t \in S$ such that $\tau = \beta^{r_0}\alpha_1\beta^{r_1}\alpha_2 \cdots \beta^{r_{t-1}}\alpha_t\beta^{r_t}$. As $\alpha\mathcal{H}\beta$, it follow that $\tau' = \alpha^{r_0}\alpha_1\alpha^{r_1}\alpha_2 \cdots \alpha^{r_{t-1}}\alpha_t\alpha^{r_t}\mathcal{H}\tau$. Furthermore, $\tau' \in S$. Thus, for all $\tau \in \bar{J}_k$, $\bar{H}_\tau \cap S \neq \emptyset$, from which it follows that $E(J_k) \subseteq S$. Since S also contains an orientation-reversing transformation of rank k, by Corollary 2.10, we have $S = OR(n,k)$, a contradiction. Therefore $\bar{H}_\alpha \subseteq S$. This shows that $\bar{H}_\alpha \subseteq S$ for all $\alpha \in S$, i.e. $\bigcup\{\bar{H}_\alpha \mid \alpha \in S\} \subseteq S$ and thus $\bigcup\{\bar{H}_\alpha \mid \alpha \in S\} = S$. Since each $\mathcal{H}$-class of $\mathcal{OR}_n$ contains an orientation-preserving transformation, we obtain $S = \bigcup\{\bar{H}_\alpha \mid \alpha \in S \cap \mathcal{OP}_n\}$, as required. ∎

In general, if S' is a subsemigroup of $OP(n,k)$ containing $OP(n,k-1)$, then (using an argument similar to that considered in the proof of Lemma 1.11) it is easy to show that $S = \bigcup\{\bar{H}_\alpha \mid \alpha \in S'\}$ is a subsemigroup of $OR(n,k)$. Furthermore, if $S' \subsetneq OP(n,k)$ then S is also a proper subsemigroup of $OR(n,k)$. In fact, in this case, by Corollary 1.9, there exists $\varepsilon \in E(J_k)$ such that $\varepsilon \notin S'$. It follows that $\bar{H}_\varepsilon \cap S' = \emptyset$ and so also $\varepsilon \notin S$.

Finally, we have:

Theorem 2.12 *Let $n \geq 4$ and $3 \leq k \leq n-1$. Then, a subsemigroup S of $OR(n,k)$ is maximal if and only if $S = OR(n,k-1) \cup J_k$ or $S = \bigcup\{\bar{H}_\alpha \mid \alpha \in S'\}$, for some maximal subsemigroup S' of $OP(n,k)$.*

Proof. First, let S be a maximal subsemigroup of $OR(n,k)$ and admit that $S \neq OR(n,k-1) \cup J_k$. Then S must contain an orientation-reversing transformation of rank k and so $S = \bigcup\{\bar{H}_\alpha \mid \alpha \in S \cap \mathcal{OP}_n\}$, by Proposition 2.11. Clearly, $S \cap \mathcal{OP}_n$ is a proper subsemigroup of $OP(n,k)$, whence there exists a maximal subsemigroup S' of $OP(n,k)$ such that $S \cap \mathcal{OP}_n \subseteq S'$. Then, by the above observation, $\bigcup\{\bar{H}_\alpha \mid \alpha \in S'\}$ is a proper subsemigroup of $OR(n,k)$ and, as it contains S, it follows that $S = \bigcup\{\bar{H}_\alpha \mid \alpha \in S'\}$, by the maximality of S.

Conversely, if $S = OR(n,k-1) \cup J_k$, then S is a maximal subsemigroup of $OR(n,k)$, by Corollary 2.9. Hence, let us admit that $S = \bigcup\{\bar{H}_\alpha \mid \alpha \in S'\}$, for some maximal subsemigroup S' of $OP(n,k)$. Then, by the above observation, S is a proper subsemigroup of $OR(n,k)$. Moreover, S must contain an orientation-reversing transformation of rank k. Let $\hat{S}$ be a maximal subsemigroup of $OR(n,k)$ such that $S \subseteq \hat{S}$. Then $\hat{S}$ also contains an orientation-reversing transformation of rank k and so, by Proposition 2.11, $\hat{S} = \bigcup\{\bar{H}_\alpha \mid \alpha \in \hat{S} \cap \mathcal{OP}_n\}$. On the other hand, $S' \subseteq S \cap \mathcal{OP}_n \subseteq \hat{S} \cap \mathcal{OP}_n \subsetneq OP(n,k)$, whence $S' = S \cap \mathcal{OP}_n$, by the maximality of S'. It follows that $S = \hat{S}$ and thus S is a maximal subsemigroup of $OR(n,k)$, as required. ■

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