Feb 25, 2011

Reiterated Homogenization in BV via Multiscale Convergence

Rita Ferreira
I.C.T.I. - Carnegie Mellon | Portugal, F.C.T./C.M.A. da U.N.L.
Quinta da Torre, 2829–516 Caparica, Portugal
rferreir@andrew.cmu.edu, ragf@fct.unl.pt

Irene Fonseca
Department of Mathematical Sciences
Carnegie Mellon University, Pittsburgh, PA 15213, USA
fonseca@andrew.cmu.edu

Abstract

Multiple-scale homogenization problems are treated in the space BV of functions of bounded variation, using the notion of multiple-scale convergence developed in [30]. In the case of one microscale Amar's result [3] is recovered under more general conditions; for two or more microscales new results are obtained.

AMS Classification Numbers (2010): 49J45, 35B27, 26A45

Key Words: periodic homogenization, multiscale convergence, linear growth, space BV of functions of bounded variation, BV-valued measures, Γ-convergence

1. Introduction and Main Results

Here we are concerned with the description of the macroscopic behavior of a microscopically heterogeneous system. Several approaches have been proposed to handle the minimization of oscillating functionals, such as the method of asymptotic expansions, G-convergence, H-convergence, Γ-convergence and two-scale convergence (we refer to [1] and references therein). In the case in which the microscopic properties of the system are periodic, the method of two-scale convergence has proven to be particularly successful. It was introduced by Nguetseng [37], and further developed by Allaire [1] and by Allaire and Briane [2], and it provides a mathematical rigorous justification for the formal asymptotic expansions that were commonly used in the study of homogenization problems (see [10], [34] and [40]).

In [3] Amar extended the notion of two-scale convergence to the case of bounded sequences of Radon measures with finite total variation, which was then used to study the asymptotic behavior of sequences of positively 1-homogeneous and periodically oscillating functionals with linear growth, defined in the space BV of functions of bounded variation. Precisely, the following result is given in [3].

Theorem A (cf. [3, Thm. 4.1])**.** *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with $\partial\Omega$ Lipschitz, let $Q := [0,1]^N$, and let $f : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ be a function such that*

(A1) for all $\xi \in \mathbb{R}^N$, $f(\cdot,\xi)$ is continuous and Q-periodic;

(A2) for all $y \in Q$, $f(y,\cdot)$ is convex, positively 1-homogeneous, and of class $C^1(\mathbb{R}^N\backslash\{0\})$;

(A3) there exists a constant $C > 0$ such that for all $y \in Y$, $\xi \in \mathbb{R}^N$, $\frac{1}{C}|\xi| \leqslant f(y,\xi) \leqslant C|\xi|$.

For each $\varepsilon > 0$*, let* $I_\varepsilon : BV(\Omega) \to \mathbb{R}$ *be the functional defined by*

$$I_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{\mathrm{d}Du}{\mathrm{d}\|Du\|}(x)\Big)\,\mathrm{d}\|Du\|(x) + \int_\Omega |v(x) - u(x)|^p\,\mathrm{d}x,$$

where $v \in L^{N/(N-1)}(\Omega)$*,* $p \in (1, N/(N-1)]$ *if* $N > 1$*, and* $p \in (1,\infty)$ *if* $N = 1$*, and* $\mathrm{d}Du/\mathrm{d}\|Du\|$ *represents the Radon–Nikodym derivative of* Du *with respect to its total variation* $\|Du\|$*. Then for each* $\varepsilon > 0$*, there exists a unique* $u_\varepsilon \in BV(\Omega)$ *such that*

$$I_\varepsilon(u_\varepsilon) = \min_{w\in BV(\Omega)} I_\varepsilon(w) = \inf_{w\in W^{1,1}(\Omega)} \Big\{\int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla w(x)\Big)\,\mathrm{d}x + \int_\Omega |v(x) - w(x)|^p\,\mathrm{d}x\Big\}.$$

Moreover, there exist $u \in BV(\Omega)$ *and* $\boldsymbol{\mu} \in \mathcal{M}(\Omega; BV_\#(Q))$† *such that* $\{u_\varepsilon\}_{\varepsilon>0}$ *weakly-*$\star$ *converges to* u *in* $BV(\Omega)$ *as* $\varepsilon \to 0^+$ *and, up to a subsequence,* $\{Du_\varepsilon\}_{\varepsilon>0}$ *two-scale converges to the measure on* $\Omega \times Q$*,* $\lambda_{u,\boldsymbol{\mu}} := Du dy + D_y\boldsymbol{\mu}$ *as* $\varepsilon \to 0^+$††*. Furthermore,*

$$\lim_{\varepsilon\to0^+} I_\varepsilon(u_\varepsilon) = \inf_{\substack{w\in BV(\Omega)\\ \boldsymbol{\nu}\in\mathcal{M}(\Omega;BV_\#(Y))}} I^{\mathrm{sc}}(w,\boldsymbol{\nu}) = I^{\mathrm{sc}}(u,\boldsymbol{\mu}),$$

where I^{sc} *is the two-scaled homogenized functional defined for* $w \in BV(\Omega)$ *and* $\boldsymbol{\nu} \in \mathcal{M}(\Omega; BV_\#(Q))$ *by*

$$I^{\mathrm{sc}}(w,\boldsymbol{\nu}) := \int_{\Omega\times Q} f\Big(y, \frac{\mathrm{d}\lambda_{w,\boldsymbol{\nu}}}{\mathrm{d}\|\lambda_{w,\boldsymbol{\nu}}\|}(x,y)\Big)\,\mathrm{d}\|\lambda_{w,\boldsymbol{\nu}}\|(x,y) + \int_\Omega |v(x) - w(x)|^p\,\mathrm{d}x.$$

Finally, in the minimizing pair $(u,\boldsymbol{\mu})$ *the function* $u \in BV(\Omega)$ *is uniquely determined.*

The proof of Theorem A is based on the so-called two-scale convergence method, which has the virtue of taking full advantage of the periodic microscopic properties of the media, enabling the explicit characterization of the local behavior of the system: The asymptotic behavior as $\varepsilon \to 0^+$ of the energies F_ε and of the respective minimizers u_ε is given with regard to both macroscopic and microscopic levels, through the two space variables x (the macroscopic one) and y (the microscopic one), and through the two unknowns u and $\boldsymbol{\mu}$. The next step of the two-scale convergence method is to obtain the effective or homogenized problem, that is, the limit problem only involving the macroscopic space variable x, and which has as solution the function $\bar{u}(x) := \int_Q u(x,y)\,\mathrm{d}y$. This is usually done via an average process with respect to the "fast variable" y of the two-scale homogenized problem.

For the class of functions f considered by Amar [3], Theorem A provides an alternative characterization of the homogenized problem previously obtained by Bouchitté [12], [13], and summarizes as follows:

Theorem B (cf. [12, Thm. 2.1])**.** *Let* $\Omega \subset \mathbb{R}^N$ *be an open and bounded set, let* $Y := (0,1)^N$*, and let* $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *be a function such that*

(B1) for all $\xi \in \mathbb{R}^N$*,* $f(\cdot,\xi)$ *is measurable and* Y*-periodic;*

(B2) for all $y \in Y$*,* $f(y,\cdot)$ *is convex;*

(B3) there exists a constant $C > 0$ *such that for all* $y \in Y$*,* $\xi \in \mathbb{R}^N$*,* $\frac{1}{C}|\xi| - C \leqslant f(y,\xi) \leqslant C(1 + |\xi|)$*.*

For each $\varepsilon > 0$*, let* $F_\varepsilon : L^1(\Omega) \to (-\infty,\infty]$ *be the functional defined by*

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u(x)\Big)\,\mathrm{d}x & \text{if } u \in W^{1,1}(\Omega),\\ \infty & \text{otherwise.}\end{cases}$$

† Here, and in the sequel, the subscript # stands for $Q_1 \times \cdots \times Q_n$-periodic functions (or measures), $n \in \mathbb{N}$, with respect to the variables $(y_1,\cdots,y_n)$, where each Q_i, $i \in \mathbb{N}$, is a copy of Q. We refer the reader to Section 2 for the notations used throughout this paper.

†† We will give a precise meaning for this statement further below.

Then, the sequence of functionals $\{F_\varepsilon\}_{\varepsilon>0}$ Γ*-converges as* $\varepsilon \to 0^+$ *with respect to the strong topology of* $L^1(\Omega)$ *to the functional* $F_0 : L^1(\Omega) \to (-\infty, \infty]$ *given by*

$$F_0(u) := \begin{cases} F^{\mathrm{h}}(u) & \text{if } u \in BV(\Omega), \\ \infty & \text{otherwise,} \end{cases}$$

where, for $u \in BV(\Omega)$,

$$F^{\mathrm{h}}(u) := \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x + \int_\Omega (f_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x),$$

with

$$f_{\mathrm{hom}}(\xi) := \inf\bigg\{\int_Y f(y, \xi + \nabla\psi(y))\,\mathrm{d}y\colon\ \psi \in W^{1,1}_\#(Y)\bigg\}, \quad (f_{\mathrm{hom}})^\infty(\xi) := \lim_{t\to\infty}\frac{f_{\mathrm{hom}}(t\xi)}{t},$$

and $Du = \nabla u \mathcal{L}^N{}_{\lfloor\Omega} + D^s u$ *is the Radon-Nikodym decomposition of* Du *with respect to the* N*-dimensional Lebesgue measure* $\mathcal{L}^N$*.*

We recall (see [24]) that $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converges, as $\varepsilon \to 0^+$ and with respect to the strong topology of $L^1(\Omega)$, to the functional F_0 if for all $u \in L^1(\Omega)$,

$$F_0(u) = \Gamma - \liminf_{\varepsilon\to 0^+} F_\varepsilon(u) = \Gamma - \limsup_{\varepsilon\to 0^+} F_\varepsilon(u),$$

where

$$\Gamma - \liminf_{\varepsilon\to 0^+} F_\varepsilon(u) := \inf\Big\{\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon)\colon\ u_\varepsilon \in L^1(\Omega),\ u_\varepsilon \to u \text{ in } L^1(\Omega)\Big\},$$
$$\Gamma - \limsup_{\varepsilon\to 0^+} F_\varepsilon(u) := \inf\Big\{\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon)\colon\ u_\varepsilon \in L^1(\Omega),\ u_\varepsilon \to u \text{ in } L^1(\Omega)\Big\}.$$

Moreover, under the coercivity condition in (B3), if we consider the analogous functional I_ε of [3], i.e., the functional $I_\varepsilon(u) := F_\varepsilon(u) + \int_\Omega |v-u|^p\,\mathrm{d}x$, for $u \in L^1(\Omega)$, where F_ε is as in Theorem B, and v and p are as in Theorem A, then, assuming $\partial\Omega$ Lipschitz and using the continuous injection of $BV(\Omega)$ in $L^p(\Omega)$,

$$\lim_{\varepsilon\to 0^+}\inf_{w\in L^1(\Omega)} I_\varepsilon(w) = \lim_{\varepsilon\to 0^+}\inf_{w\in W^{1,1}(\Omega)} I_\varepsilon(w) = \min_{w\in L^1(\Omega)} I_0(w) = \min_{w\in BV(\Omega)} I^{\mathrm{h}}(w),$$

where $I_0(w) := F_0(w) + \int_\Omega |v-w|^p\,\mathrm{d}x$, $I^{\mathrm{h}}(w) := F^{\mathrm{h}}(w) + \int_\Omega |v-w|^p\,\mathrm{d}x$, and F_0 and F^{h} were introduced in Theorem B. In particular, if f satisfies conditions (A1), (A2) and (A3), then $I^{\mathrm{h}}(u) = I^{\mathrm{sc}}(u, \boldsymbol{\mu})$, where I^{sc} and $(u, \boldsymbol{\mu}) \in BV(\Omega) \times \mathcal{M}(\Omega; BV_\#(Y))$ are as in the statement of Theorem A.

The proof of Theorem B relies on integral functionals of measures and their formulation by duality, while, as we mentioned before, the proof of Theorem A is based on the two-scale convergence method and is very similar to that of [1, Thm. 3.3] in which the subdifferentiability of f and the regularity and boundedness of $\nabla_\xi f$ play a crucial role. In particular, the arguments used in [3] do not apply neither under weaker regularity hypotheses than those in (A2) nor under more general linear estimates from above and from below than those in (A3).

Some questions then naturally arise: Is it possible to derive the two-scale homogenized functional under weaker hypotheses than those considered in [3]? May we establish the relation between the two-scale homogenized functional I^{sc} and the homogenized functional I^{hom} in a systematic and direct way? How to generalize this analysis to the case of multiple microscales? And to the vectorial case? The goal of this paper is precisely to give answers to these questions.

We start by recalling the notion of $(n+1)$-convergence for sequences of Radon measures introduced in [3] for $n = 1$, and generalized in [30] for any $n \in \mathbb{N}$. Let $d, m, n, N \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^N$ be an open set, and set $Y := (0,1)^N$. Let $\varrho_1, ..., \varrho_n$ be positive functions on $(0, \infty)$ such that for all $i \in \{1, \cdots, n\}$ and for all $j \in \{2, \cdots, n\}$,

$$\lim_{\varepsilon\to 0^+}\varrho_i(\varepsilon) = 0, \qquad \lim_{\varepsilon\to 0^+}\frac{\varrho_j(\varepsilon)}{\varrho_{j-1}(\varepsilon)} = 0. \tag{1.1}$$

Definition 1.1. *Let $\{\mu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^m)$ be a sequence of Radon measures with finite total variation on Ω. We say that $\{\mu_\varepsilon\}_{\varepsilon>0}$ $(n+1)$-scale converges to a Radon measure $\mu_0 \in \big(C_0(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m))\big)' \simeq \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^m)$ with finite total variation in the product space $\Omega\times Y_1\times\cdots\times Y_n$, where each Y_i is a copy of Y, if for all $\varphi \in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m))$ we have*

$$\lim_{\varepsilon\to 0^+}\int_\Omega \varphi\left(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\right)\cdot \mathrm{d}\mu_\varepsilon(x) = \int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n)\cdot \mathrm{d}\mu_0(x,y_1,\cdots,y_n),$$

in which case we write $\mu_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} \mu_0$.

This notion of convergence is justified by a compactness result, which asserts that every bounded sequence in $\mathcal{M}(\Omega;\mathbb{R}^m)$ admits a $(n+1)$-scale converging subsequence (see [30, Thm 3.2]). The (usual) weak-$\star$ limit in $\mathcal{M}(\Omega;\mathbb{R}^m)$ is the projection on Ω of the $(n+1)$-scale limit, and so the latter captures more information on the oscillatory behavior of a bounded sequence in $\mathcal{M}(\Omega;\mathbb{R}^m)$ than the former (see [30, Prop. 3.3]). This leads us to the study of the asymptotic behavior with respect to the $(n+1)$-scale convergence of first order derivatives functionals with linear growth of the form

$$F_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)},\nabla u(x)\Big)\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)},\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\mathrm{d}\|D^s u\|(x) \tag{1.2}$$

for $u \in BV(\Omega;\mathbb{R}^d)$, where

$$f^\infty(y_1,\cdots,y_n,\xi) := \limsup_{t\to\infty}\frac{f(y_1,\cdots,y_n,t\xi)}{t}$$

is the recession function of a real valued function $f:\mathbb{R}^{nN}\times\mathbb{R}^{d\times N}\to\mathbb{R}$, separately periodic in the first n variables.

We start by characterizing the $(n+1)$-scale limits of $\big\{\big(u_\varepsilon\mathcal{L}^N\lfloor_\Omega, Du_\varepsilon\lfloor_\Omega\big)\big\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^d)\times\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$, whenever $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $BV\big(\Omega;\mathbb{R}^d\big)$.

Definition 1.2. *For $i\in\mathbb{N}$, define the space $\mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$ of all $BV_\#\big(Y_i;\mathbb{R}^d\big)$-valued Radon measures $\boldsymbol{\mu}\in\mathcal{M}\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$ with finite total variation, for which there exists a $\mathbb{R}^{d\times N}$-valued Radon measure $\lambda\in\mathcal{M}_{y\#}\big(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}\big)$, with finite total variation in the product space $\Omega\times Y_1\times\cdots\times Y_i$, such that for all $B\in\mathcal{B}(\Omega\times Y_1\times\cdots\times Y_{i-1})$, $E\in\mathcal{B}(Y_i)$,*

$$\big(D_{y_i}(\boldsymbol{\mu}(B))\big)(E) = \lambda(B\times E).$$

We say that λ is the measure associated with $D_{y_i}\boldsymbol{\mu}$.

We refer the reader to [30] for more detailed considerations on the space $\mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$, $i\in\mathbb{N}$. The following result holds (see [30, Thm. 1.10]).

Theorem 1.3. *Let $\{u_\varepsilon\}_{\varepsilon>0}\subset BV(\Omega;\mathbb{R}^d)$ be such that $u_\varepsilon \overset{\star}{\rightharpoonup} u$ weakly-$\star$ in $BV(\Omega;\mathbb{R}^d)$ as $\varepsilon\to 0^+$, for some $u\in BV(\Omega;\mathbb{R}^d)$. Assume that, in addition to satisfying (1.1), the length scales $\varrho_1,\ldots,\varrho_n$ are well separated, i.e., there exists $m\in\mathbb{N}$ such that for all $i\in\{2,\cdots,n\}$,*

$$\lim_{\varepsilon\to 0^+}\left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^m\frac{1}{\varrho_i(\varepsilon)} = 0. \tag{1.3}$$

Then

a) $u_\varepsilon\mathcal{L}^N\lfloor_\Omega \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} \tau_u$, where $\tau_u\in\mathcal{M}_{y\#}\big(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^d\big)$ is the measure defined by

$$\tau_u := u\,\mathcal{L}^N\lfloor_\Omega\otimes\mathcal{L}^{nN}_{y_1,\cdots,y_n},$$

i.e., if $\varphi \in C_0\big(\Omega; C_\#\big(Y_1 \times \cdots \times Y_n; \mathbb{R}^d\big)\big)$ *then*

$$\langle \tau_u, \varphi \rangle = \int_{\Omega \times Y_1 \times \cdots \times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot u(x)\, \mathrm{d}x \mathrm{d}y_1 \cdots \mathrm{d}y_n.$$

b) there exist a subsequence $\{Du_{\varepsilon'}\}$ *of* $\{Du_\varepsilon\}$ *and, for all* $i \in \{1, \cdots, n\}$*, measures* $\boldsymbol{\mu}_i \in \mathcal{M}_\star\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i; \mathbb{R}^d)\big)$ *such that*

$$Du_{\varepsilon'} \xrightharpoonup[\varepsilon']{(n+1)\text{-}sc} \lambda_{u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n},$$

where $\lambda_{u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n} \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_n; \mathbb{R}^{d \times N}\big)$ *is the measure given by*

$$\lambda_{u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n} := Du_{\lfloor \Omega} \otimes \mathcal{L}^{nN}_{y_1, \cdots, y_n} + \sum_{i=1}^{n-1} \lambda_i \otimes \mathcal{L}^{(n-i)N}_{y_{i+1}, \cdots, y_n} + \lambda_n, \tag{1.4}$$

i.e., if $\varphi \in C_0\big(\Omega; C_\#\big(Y_1 \times \cdots \times Y_n; \mathbb{R}^{d \times N}\big)\big)$ *then*

$$\begin{aligned} \langle \lambda_{u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n}, \varphi \rangle = & \int_{\Omega \times Y_1 \times \cdots \times Y_n} \varphi(x, y_1, \cdots, y_n) : \mathrm{d}Du(x) \mathrm{d}y_1 \cdots \mathrm{d}y_n \\ & + \sum_{i=1}^{n-1} \int_{\Omega \times Y_1 \times \cdots \times Y_n} \varphi(x, y_1, \cdots, y_n) : \mathrm{d}\lambda_i(x, y_1, \cdots, y_i) \mathrm{d}y_{i+1} \cdots \mathrm{d}y_n \\ & + \int_{\Omega \times Y_1 \times \cdots \times Y_n} \varphi(x, y_1, \cdots, y_n) : \mathrm{d}\lambda_n(x, y_1, \cdots, y_n), \end{aligned}$$

and each $\lambda_i \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^{d \times N}\big)$ *is the measure associated with* $D_{y_i}\boldsymbol{\mu}_i$*,* $i \in \{1, \cdots, n\}$*.*

Using Theorem 1.3, we seek to characterize and relate the functionals

$$F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) := \inf \left\{ \liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \colon\ u_\varepsilon \in BV\big(\Omega; \mathbb{R}^d\big),\ Du_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} \lambda_{u, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n} \right\} \tag{1.5}$$

and

$$F^{\mathrm{hom}}(u) := \inf \left\{ \liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \colon\ u_\varepsilon \in BV\big(\Omega; \mathbb{R}^d\big),\ u_\varepsilon \overset{\star}{\rightharpoonup}_\varepsilon u \text{ weakly-}\star \text{ in } BV\big(\Omega; \mathbb{R}^d\big) \right\} \tag{1.6}$$

for $u \in BV\big(\Omega; \mathbb{R}^d\big)$ and $\boldsymbol{\mu}_i \in \mathcal{M}_\star\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i; \mathbb{R}^d)\big)$, $i \in \{1, \cdots, n\}$, where F_ε is given by (1.2).

Before we state our main result, we introduce some notation. Fix $k \in \mathbb{N}$ and let $g : \mathbb{R}^{kN} \times \mathbb{R}^{d \times N} \to \overline{\mathbb{R}}$ be a Borel function. We recall that the effective domain of g, $\mathrm{dom}_e g$, is the set

$$\mathrm{dom}_e g := \big\{ (y_1, \cdots, y_k, \xi) \in \mathbb{R}^{kN} \times \mathbb{R}^{d \times N} \colon\ g(y_1, \cdots, y_k, \xi) < \infty \big\},$$

while the conjugate function of g is the function $g^* : \mathbb{R}^{kN} \times \mathbb{R}^{d \times N} \to \overline{\mathbb{R}}$ defined by

$$g^*(y_1, \cdots, y_k, \xi^*) := \sup_{\xi \in \mathbb{R}^{d \times N}} \big\{ \xi : \xi^* - g(y_1, \cdots, y_k, \xi) \big\}, \quad y_1, \ldots, y_k \in \mathbb{R}^N,\ \xi^* \in \mathbb{R}^{d \times N}, \tag{1.7}$$

and the biconjugate function of g is the function $g^{**} : \mathbb{R}^{kN} \times \mathbb{R}^{d \times N} \to \overline{\mathbb{R}}$ defined by

$$g^{**}(y_1, \cdots, y_k, \xi) := \sup_{\xi^* \in \mathbb{R}^{d \times N}} \big\{ \xi^* : \xi - g^*(y_1, \cdots, y_k, \xi^*) \big\}, \quad y_1, \ldots, y_k \in \mathbb{R}^N,\ \xi^* \in \mathbb{R}^{d \times N}. \tag{1.8}$$

We define a function $g_{\hom_k} : \mathbb{R}^{(k-1)N} \times \mathbb{R}^{d\times N} \to \overline{\mathbb{R}}$ by setting

$$g_{\hom_k}(y_1, \cdots, y_{k-1}, \xi) := \inf\left\{ \int_{Y_k} g(y_1, \cdots, y_{k-1}, y_k, \xi + \nabla\psi_k(y_k))\,\mathrm{d}y_k : \ \psi_k \in W^{1,1}_{\#}\big(Y_k; \mathbb{R}^d\big) \right\} \tag{1.9}$$

for $y_1, ..., y_{k-1} \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$.

Let $f : \mathbb{R}^{nN} \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function. If $n = 1$, we set $f_{\hom} := f_{\hom_1}$, where $f_{\hom_1}$ is given by (1.9) for $k = 1$ and with g replaced by f, that is,

$$f_{\hom}(\xi) := \inf\left\{ \int_{Y_1} f(y_1, \xi + \nabla\psi_1(y_1))\,\mathrm{d}y_1 : \ \psi_1 \in W^{1,1}_{\#}\big(Y_1; \mathbb{R}^d\big) \right\}.$$

If $n = 2$, we define $f_{\hom} := (f_{\hom_2})_{\hom_1}$, which is the function given by (1.9) for $k = 1$ and with g replaced by $f_{\hom_2}$, where the latter is the function given by (1.9) for $k = 2$ and with g replaced by f. Precisely,

$$f_{\hom}(\xi) := \inf\left\{ \int_{Y_1} f_{\hom_2}(y_1, \xi + \nabla\psi_1(y_1))\,\mathrm{d}y_1 : \ \psi_1 \in W^{1,1}_{\#}\big(Y_1; \mathbb{R}^d\big) \right\},$$

where

$$f_{\hom_2}(y_1, \xi) := \inf\left\{ \int_{Y_2} f(y_1, y_2, \xi + \nabla\psi_2(y_2))\,\mathrm{d}y_2 : \ \psi_2 \in W^{1,1}_{\#}\big(Y_2; \mathbb{R}^d\big) \right\}.$$

Similarly, if $n = 3$ we define $f_{\hom} := \big((f_{\hom_3})_{\hom_2}\big)_{\hom_1}$, i.e.,

$$f_{\hom}(\xi) := \inf\left\{ \int_{Y_1} (f_{\hom_3})_{\hom_2}(y_1, \xi + \nabla\psi_1(y_1))\,\mathrm{d}y_1 : \ \psi_1 \in W^{1,1}_{\#}\big(Y_1; \mathbb{R}^d\big) \right\},$$

where

$$(f_{\hom_3})_{\hom_2}(y_1, \xi) := \inf\left\{ \int_{Y_2} f_{\hom_3}(y_1, y_2, \xi + \nabla\psi_2(y_2))\,\mathrm{d}y_2 : \ \psi_2 \in W^{1,1}_{\#}\big(Y_2; \mathbb{R}^d\big) \right\},$$

with

$$f_{\hom_3}(y_1, y_2, \xi) := \inf\left\{ \int_{Y_3} f(y_1, y_2, y_3, \xi + \nabla\psi_3(y_3))\,\mathrm{d}y_3 : \ \psi_3 \in W^{1,1}_{\#}\big(Y_3; \mathbb{R}^d\big) \right\}.$$

Recursively, for $n \in \mathbb{N}$ we set

$$f_{\hom} := \Big((f_{\hom_n})_{\hom_{n-1}} \cdots\Big)_{\hom_1}. \tag{1.10}$$

Consider the following conditions:

($\mathcal{F}1$) for all $\xi \in \mathbb{R}^{d\times N}$, $f(\cdot, \xi)$ is $Y_1 \times \cdots \times Y_n$-periodic;

($\mathcal{F}2$) for all $y_1, ..., y_n \in \mathbb{R}^N$, $f(y_1, \cdots, y_n, \cdot)$ is convex;

($\mathcal{F}3$) there exists $C > 0$ such that for all $y_1, ..., y_n \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$,

$$f(y_1, \cdots, y_n, \xi) \leqslant C(1 + |\xi|);$$

($\mathcal{F}4$) for all $\delta > 0$ there exist $c_\delta \in \mathbb{R}^N$, $b_\delta \in \mathbb{R}$, such that $|c_\delta| \to 0$ as $\delta \to 0^+$, and for all $y_1, ..., y_n \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$,

$$f(y_1, \cdots, y_n, \xi) + c_\delta \cdot \xi + b_\delta \geqslant 0;$$

($\mathcal{F}4$)' there exists $C > 0$ such that for all $y_1, ..., y_n \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$,

$$f(y_1, \cdots, y_n, \xi) \geqslant \frac{1}{C}|\xi| - C;$$

($\mathcal{F}5$) for every $y_1', ..., y_n' \in \mathbb{R}^N$, $\delta > 0$, there exists $\tau = \tau(y_1', \cdots, y_n', \delta)$ such that for all $y_1, ..., y_n \in \mathbb{R}^N$ with $|(y_1', \cdots, y_n') - (y_1, \cdots, y_n)| \leqslant \tau$, and for all $\xi \in \mathbb{R}^{d\times N}$,

$$|f(y_1', \cdots, y_n', \xi) - f(y_1, \cdots, y_n, \xi)| \leqslant \delta(1 + |\xi|);$$

($\mathcal{F}6$) for all $\delta > 0$ there exists $\tilde{a}_\delta \in L^1_\#(Y_1 \times \cdots \times Y_n)$ such that $\delta \|\tilde{a}_\delta\|_{L^1_\#(Y_1\times\cdots\times Y_n)} \to 0$ as $\delta \to 0^+$, and there exits $\tau_\delta > 0$ such that for all $y_1, ..., y_{n-1}, y_1', ..., y_{n-1}' \in \mathbb{R}^N$ with $|(y_1, \cdots, y_{n-1}) - (y_1', \cdots, y_{n-1}')| \leqslant \tau_\delta$, and for all y_n, $\xi \in \mathbb{R}^{d\times N}$,

$$f(y_1, \cdots, y_{n-1}, y_n, \xi) \geqslant \delta\, \tilde{a}_\delta(y_1', \cdots, y_{n-1}', y_n) + (1 + o(1)) f(y_1', \cdots, y_{n-1}', y_n, \xi)$$

(as $\delta \to 0^+$). If $n \geqslant 3$, then we assume in addition that for a.e. $y_{n-1}, y_n \in \mathbb{R}^N$ we have $\tilde{a}_\delta(\cdot, y_{n-1}, y_n) \in C_\#(Y_1 \times \cdots \times Y_{n-2})$ with $\|\tilde{a}_\delta(\cdot, y_{n-1}, y_n)\|_{C_\#(Y_1\times\cdots\times Y_{n-2})} \in L^1(Y_{n-1} \times Y_n)$;

($\mathcal{F}7$) there exist $\alpha \in (0, 1)$ and $L, C > 0$, such that for all $y_1, ..., y_n \in \mathbb{R}^N$, for all $\xi \in \mathbb{R}^{d\times N}$ with $|\xi| = 1$, and for all $t \geqslant L$,

$$\left| f^\infty(y_1, \cdots, y_n, \xi) - \frac{f(y_1, \cdots, y_n, t\xi)}{t} \right| \leqslant \frac{C}{t^\alpha};$$

($\mathcal{F}8$) the conjugate function f^* of f is a bounded function on its effective domain, $\mathrm{dom}_e f^*$.

The next proposition will be used to establish integral representations for the multiple-scale functional F^{sc} in (1.5) and for the homogenized functional F^{hom} in (1.6).

Proposition 1.4. *Let $f : \mathbb{R}^{nN} \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function satisfying hypotheses ($\mathcal{F}1$), ($\mathcal{F}3$) and ($\mathcal{F}4$). For $\eta > 0$, let f_η be the function defined by $f_\eta(y_1, \cdots, y_n, \xi) := f(y_1, \cdots, y_n, \xi) + \eta|\xi|$. Then,*

(i) For all $y_1, ..., y_n \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$, the limit

$$\lim_{\eta\to 0^+} ((f_\eta)^{**})^\infty(y_1, ..., y_n, \xi) =: ((f_{0^+})^{**})^\infty(y_1, ..., y_n, \xi) \tag{1.11}$$

exists, $((f_{0^+})^{**})^\infty : \mathbb{R}^{nN} \times \mathbb{R}^N \to \mathbb{R}$ is positively 1-homogeneous and convex in the last variable, and $(f^{**})^\infty \leqslant ((f_{0^+})^{**})^\infty \leqslant (f^\infty)^{**}$.

Furthermore, if in addition

a) f also satisfies ($\mathcal{F}2$), then $((f_{0^+})^{**})^\infty \equiv f^\infty$;

b) $d = 1$ and f also satisfies ($\mathcal{F}7$), then $((f_{0^+})^{**})^\infty \equiv (f^\infty)^{**}$.

(ii) For all $\xi \in \mathbb{R}^N$, the limit

$$\lim_{\eta\to 0^+} \big(((f_\eta)^{**})_{\mathrm{hom}}\big)^\infty(\xi) =: \big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty(\xi) \tag{1.12}$$

exists, with $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty : \mathbb{R}^N \to \mathbb{R}$ positively 1-homogeneous, convex, and such that

$$\big((f^{**})_{\mathrm{hom}}\big)^\infty \leqslant \big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty \leqslant \big(((f_{0^+})^{**})^\infty\big)_{\mathrm{hom}} \leqslant \big((f^\infty)^{**}\big)_{\mathrm{hom}}.$$

Furthermore, if in addition

a) f also satisfies ($\mathcal{F}2$) and ($\mathcal{F}8$), then $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty \equiv (f_{\mathrm{hom}})^\infty = (f^\infty)_{\mathrm{hom}}$;

b) f also satisfies ($\mathcal{F}2$) and ($\mathcal{F}7$), then $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty \equiv (f^\infty)_{\mathrm{hom}}$;

c) $d = 1$ and f also satisfies ($\mathcal{F}7$), then $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty \equiv \big((f^\infty)^{**}\big)_{\mathrm{hom}}$.

Remark 1.5. *Hypothesis ($\mathcal{F}7$) is common within variational problems with linear growth conditions (see, for example, [14, Sect. 4], [9]). We will prove (see Lemma 3.12 below) that under hypotheses ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$)' and ($\mathcal{F}7$), we have $(f_{\rm hom})^\infty = (f^\infty)_{\rm hom}$; in the scalar case, these conditions also ensure the equality $(f^{**})^\infty = (f^\infty)^{**}$. Other sufficient conditions to guarantee that $(f_{\rm hom})^\infty = (f^\infty)_{\rm hom}$ are ($\mathcal{F}1$)–($\mathcal{F}4$) and ($\mathcal{F}8$) (see Lemma 3.11 below), which is an hypothesis on f^* that is often considered when dealing with duality problems (see, for example, [42, Ch. II.4]).*

Unless stated otherwise, we will always assume that the length scales $\varrho_1, ..., \varrho_n$ satisfy (1.1) and (1.3). A simple example of such functions is the case in which for all $i \in \{1, \cdots, n\}$, $\varrho_i = \varepsilon^i$. Our main result is the following.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with $\partial\Omega$ Lipschitz, let $Y_i := (0,1)^N$, $i \in \{1, \cdots, n\}$, and let $f : \mathbb{R}^{nN} \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function satisfying ($\mathcal{F}1$)–($\mathcal{F}4$), ($\mathcal{F}5$) and ($\mathcal{F}6$). Then, for all $(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) \in BV\big(\Omega; \mathbb{R}^d\big) \times \mathcal{M}_\star\big(\Omega; BV_\#\big(Y_1; \mathbb{R}^d\big)\big) \times \cdots \times \mathcal{M}_\star\big(\Omega \times Y_1 \times \cdots \times Y_{n-1}; BV_\#\big(Y_n; \mathbb{R}^d\big)\big)$,*

$$\begin{aligned} F^{\rm sc}(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) = &\int_{\Omega\times Y_1\times\cdots\times Y_n} f\Big(y_1, \cdots, y_n, \frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}}{\mathrm{d}\mathcal{L}^{(n+1)N}}(x, y_1, \cdots, y_n)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &+ \int_{\Omega\times Y_1\times\cdots\times Y_n} f^\infty\Big(y_1, \cdots, y_n, \frac{\mathrm{d}\lambda^{s}_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}}{\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}\|}(x, y_1, \cdots, y_n)\Big)\,\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}\|(x, y_1, \cdots, y_n). \end{aligned} \tag{1.13}$$

Moreover, for all $u \in BV\big(\Omega; \mathbb{R}^d\big)$,

$$\begin{aligned} F^{\rm hom}(u) &= \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d)),\dots,\\ \boldsymbol{\mu}_n\in\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{n-1};BV_\#(Y_n;\mathbb{R}^d))}} F^{\rm sc}(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) \\ &= \int_\Omega f_{\rm hom}(\nabla u(x))\,\mathrm{d}x + \int_\Omega (f_{0^+,\rm hom})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x), \end{aligned} \tag{1.14}$$

where $(f_{0^+,\rm hom})^\infty := \big(((f_{0^+})^{**})_{\rm hom}\big)^\infty$ is the function defined by (1.12) (note that in view of ($\mathcal{F}2$), $(f_\eta)^{**} \equiv f_\eta$).

Furthermore, if in addition

(i) f satisfies one of the two conditions ($\mathcal{F}4$)' or ($\mathcal{F}8$), then $(f_{0^+,\rm hom})^\infty \equiv (f_{\rm hom})^\infty$;

(ii) f satisfies ($\mathcal{F}7$), then $(f_{0^+,\rm hom})^\infty \equiv (f^\infty)_{\rm hom}$.

We remark that in Theorem 1.6 we do not assume coercivity nor boundedness from below of f. The main ingredients of the proof are the unfolding operator (see [19], [21]; see also [31]) and Reshetnyak's continuity- and lower semicontinuity-type results. The approach via the unfolding operator, in connection with the notion of two-scale convergence and in the framework of homogenization problems, sometimes referred as periodic unfolding method, has already been adopted by other authors in the Sobolev setting (see, for example, [19], [20], [31]).

We use the convexity hypothesis ($\mathcal{F}2$) when establishing the lower bound for the infimum defining $F^{\rm sc}$, which is based on a sequential lower semicontinuity argument. We start by proving that the $(n+1)$-scale convergence of a sequence of measures absolutely continuous with respect to the Lebesgue measure is equivalent to the weak-$\star$ convergence on the product space $\Omega \times Y_1 \times \cdots \times Y_n$ in the sense of measures of the unfolded sequence, i.e., the image through the unfolding operator of the original sequence (see Lemma 3.4). Then we prove that the energy F_ε does not increase by means of the unfolding operator (see Lemma 3.2). In order to conclude we need sequential lower semicontinuity of the functional

$$\begin{aligned} F(\lambda) := &\int_{\Omega\times Y_1\times\cdots\times Y_n} f\Big(y_1, \cdots, y_n, \frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mathcal{L}^{(n+1)N}}(x, y_1, ..., y_n)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &+ \int_{\Omega\times Y_1\times\cdots\times Y_n} f^\infty\Big(y_1, \cdots, y_n, \frac{\mathrm{d}\lambda^s}{\mathrm{d}\|\lambda^s\|}(x, y_1, ..., y_n)\Big)\,\mathrm{d}\|\lambda^s\|(x, y_1, ..., y_n) \end{aligned}$$

for $\lambda \in \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N})$, with respect to weak-$\star$ convergence in the sense of measures, which requires convexity of f in the last variable (see, for example, [4]). In the scalar case $d=1$ we can overcome this difficulty by a relaxation argument with respect to the weak topology of $W^{1,1}(\Omega)$, which cannot be applied in the vectorial case since quasiconvexity is a weaker condition than convexity (see, for example, [22]). As a corollary of Theorem 1.6, we obtain the following result concerning the scalar case $d=1$.

Corollary 1.7. *Let $\Omega\subset\mathbb{R}^N$ be an open and bounded set with $\partial\Omega$ Lipschitz, let $Y_i:=(0,1)^N$, $i\in\{1,\cdots,n\}$, and let $f:\mathbb{R}^{nN}\times\mathbb{R}^N\to\mathbb{R}$ be a Borel function satisfying conditions ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$), ($\mathcal{F}5$) and ($\mathcal{F}6$) with $d=1$ and with $o(1)$ replaced by $-|o(1)|$ in ($\mathcal{F}6$). Then, for all $(u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n)\in BV(\Omega)\times\mathcal{M}_\star(\Omega;BV_\#(Y_1))\times\cdots\times\mathcal{M}_\star(\Omega\times Y_1\times\cdots Y_{n-1};BV_\#(Y_n))$,*

$$\begin{aligned}F^{\rm sc}(u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n)=&\int_{\Omega\times Y_1\times\cdots\times Y_n} f^{**}\Big(y_1,\cdots,y_n,\frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}}{\mathrm{d}\mathcal{L}^{(n+1)N}}(x,y_1,\cdots,y_n)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\ &+\int_{\Omega\times Y_1\times\cdots\times Y_2}((f_{0^+})^{**})^\infty\Big(y_1,\cdots,y_n,\frac{\mathrm{d}\lambda^s_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}}{\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|}(x,y_1,\cdots,y_n)\Big)\,\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(x,y_1,\cdots,y_n),\end{aligned}\tag{1.15}$$

where $((f_{0^+})^{**})^\infty$ is the function defined by (1.11). Moreover, for all $u\in BV(\Omega)$,

$$\begin{aligned}F^{\rm hom}(u)=&\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1)),\,\ldots,\\ \boldsymbol{\mu}_n\in\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{n-1};BV_\#(Y_n))}} F^{\rm sc}(u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n)\\ =&\int_\Omega (f^{**})_{\rm hom}(\nabla u(x))\,\mathrm{d}x+\int_\Omega\big(((f_{0^+})^{**})_{\rm hom}\big)^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x),\end{aligned}\tag{1.16}$$

where $\big(((f_{0^+})^{**})_{\rm hom}\big)^\infty$ is the function defined by (1.12).

Furthermore, if in addition

(i) f satisfies the coercivity condition ($\mathcal{F}4$)', then $((f_{0^+})^{**})^\infty\equiv(f^{**})^\infty$ and $\big(((f_{0^+})^{**})_{\rm hom}\big)^\infty\equiv\big((f^{**})_{\rm hom}\big)^\infty$;

(ii) f satisfies ($\mathcal{F}7$), then $((f_{0^+})^{**})^\infty\equiv(f^\infty)^{**}$ and $\big(((f_{0^+})^{**})_{\rm hom}\big)^\infty\equiv\big((f^\infty)^{**}\big)_{\rm hom}$.

Remark 1.8. (Comments on the hypotheses) *(i) If f is bounded from below, then ($\mathcal{F}4$) is satisfied: it suffices to take $c_\delta\equiv 0$ and $b_\delta\equiv -b$, where $b:=\inf f\in\mathbb{R}$. Hypothesis ($\mathcal{F}4$) may be regarded as a stronger version of the condition*

$(\mathcal{F}4)^\star$ for all $\delta>0$ there exists $b_\delta\in\mathbb{R}$ such that for all $y_1,...,y_n$, $\xi\in\mathbb{R}^N$,

$$f(y_1,\cdots,y_n,\xi)+\delta|\xi|+b_\delta\geqslant 0,$$

so f cannot decrease as $-|\xi|$ but it can decrease as $-|\xi|^\alpha$ with $\alpha\in(0,1)$: If $\tilde f:\mathbb{R}^{nN}\times\mathbb{R}^{d\times N}\to[0,\infty)$ is a nonnegative function, and $b\in\mathbb{R}$, $c>0$, then for all $\alpha\in(0,1)$,

$$f(y_1,\cdots,y_n,\xi):=\tilde f(y_1,\cdots,y_n,\xi)-c|\xi|^\alpha+b$$

is a function satisfying $(\mathcal{F}4)^\star$. We do not assume $(\mathcal{F}4)^\star$ in place of ($\mathcal{F}4$) in Theorem 1.6 and Corollary 1.7 because in general the former is not inherit neither by $f_{\rm hom}$ nor by f^{**} from f, whereas the latter is.

We observe that if f is lower semicontinuous and independent of $(y_1,\cdots,y_n)$, then f satisfies $(\mathcal{F}4)^\star$ if, and only if, it satisfies

$$\liminf_{|\xi|\to\infty}\frac{f(\xi)}{|\xi|}\geqslant 0.\tag{1.17}$$

Moreover, if f is in addition convex, then (1.17) is a necessary and sufficient condition for the sequentially lower semicontinuity with respect to weak-$\star$ convergence in the sense of measures of the functional

$$u\in L^1\big(\Omega;\mathbb{R}^{d\times N}\big)\mapsto\int_\Omega f(u(x))\,\mathrm{d}x.$$

Furthermore, (1.17) yields

$$\liminf_{\varepsilon\to 0^+}\int_\Omega f(u_\varepsilon(x))\,\mathrm{d}x \geqslant \int_\Omega f\Big(\frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mathcal{L}^N}(x)\Big)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\lambda^s}{\mathrm{d}\|\lambda^s\|}(x)\Big)\,\mathrm{d}\|\lambda^s\|(x)$$

whenever $u_\varepsilon \mathcal{L}^N\lfloor_\Omega \overset{\star}{\rightharpoonup} \lambda$ *weakly-*$\star$ *in* $\mathcal{M}\big(\Omega;\mathbb{R}^{d\times N}\big)$ *(see [32, Thm. 5.21]). This fact will be used when establishing (1.14) and (1.16).*

(ii) If f *satisfies a growth condition of the form* $|f(y_1,\cdots,y_n,\xi)| \leqslant C(1+|\xi|)$ *and is convex in the last variable, then (see [11]) (*$\mathcal{F}5$*) holds if, and only if, the function* $\bar f : \mathbb{R}^{nN}\times\mathbb{R}^{d\times N}\times[0,\infty)\to\mathbb{R}$ *defined by*

$$\bar f(y_1,\cdots,y_n,\xi,t) := \begin{cases} t f\big(y_1,\cdots,y_n,\frac{\xi}{t}\big) & \text{if } t>0,\\ f^\infty(y_1,\cdots,y_n,\xi) & \text{if } t=0,\end{cases}$$

is continuous. In particular, if f *is continuous, positively 1-homogeneous in the last variable, and satisfies (*$\mathcal{F}2$*), (*$\mathcal{F}3$*), and (*$\mathcal{F}4$*)***, then it also satisfies (*$\mathcal{F}5$*) since in this setting* $\bar f$ *is continuous.*

The continuity of $\bar f$ *will be crucial in our analysis in order to apply Reshetnyak's continuity- and lower semicontinuity-type results (see Lemmas 3.5 and 3.6 below).*

(iii) Hypothesis ($\mathcal{F}6$*) is a weaker version of the hypothesis*

($\mathcal{F}6$)' *there exist a continuous, positive function* ω *satisfying* $\omega(0)=0$*, and a function* $a\in L^1_\#(Y_n)$ *such that for all* $y_1,\dots,y_{n-1}$*,* $y'_1,\dots,y'_{n-1}$*,* y_n*,* $\xi\in\mathbb{R}^{d\times N}$*, we*

$$\begin{aligned}&|f(y_1,\cdots,y_{n-1},y_n,\xi) - f(y'_1,\cdots,y'_{n-1},y_n,\xi)|\\ &\leqslant \omega(|(y_1,\cdots,y_{n-1})-(y'_1\cdots,y'_{n-1})|)\big(a(y_n)+f(y_1,\cdots,y_n,\xi)\big),\end{aligned}$$

which often appears in the literature (see, for example, [16], [41]).

If f *is of the form* $f(y_1,\cdots,y_n,\xi) := g(y_1,\cdots,y_{n-1})h(y_n,\xi)$*, where* g *is a continuous and* $Y_1\times\cdots\times Y_{n-1}$*-periodic function, and* h *is a function satisfying (*$\mathcal{F}1$*)–(*$\mathcal{F}5$*), then* f *satisfies (*$\mathcal{F}1$*)–(*$\mathcal{F}6$*); in particular, we may consider* $g\equiv 1$*, which corresponds to the case of one microscale (i.e.,* $n=1$*) and so, in this situation, (*$\mathcal{F}6$*) is trivially satisfied. Other simple examples of functions satisfying (*$\mathcal{F}1$*)–(*$\mathcal{F}6$*) are functions of the form* $f(y_1,\cdots,y_n,\xi) := g(y_1,\cdots,y_n)h(\xi)$*, where* g *is continuous and* $Y_1\times\cdots\times Y_n$*-periodic, and* h *satisfies (*$\mathcal{F}2$*)–(*$\mathcal{F}4$*).*

Remark 1.9. *(i) Equalities (1.13) and the first one in (1.14) are valid under the more general growth condition from below (*$\mathcal{F}4$*)** *(introduced in Remark 1.8 (i)). The reason why this condition is not enough in order to conclude the second equality in (1.14) is that in general it is not inherited by* f_{hom}*, while (*$\mathcal{F}4$*) is and this ensures that* f_{hom} *satisfies (1.17), which, as we will see, will play a crucial role in the proof.*

(ii) In Theorem 1.6 and Corollary 1.7, we need the length scales to satisfy condition (1.3) only to establish the equalities (1.14) and (1.16) involving F^{hom}*.*

In the case in which $n=1$ and $d=1$, we recover Amar's integral representation [3] of the two-scale homogenized functional F^{sc} under more general conditions (see Remark 1.8 (ii) and (iii)). Furthermore, if we assume *a priori* compactness of a diagonal infimizing sequence for the sequence of functionals $\{F_\varepsilon\}_{\varepsilon>0}$, we recover Amar's result [3] under more general conditions. We observe that even if *a priori* compactness of a diagonal infimizing sequence is assumed in Theorem A, the coercivity condition is still needed to validate the arguments in [3]. We also recover Bouchitté's integral representation [12] of the effective energy F^{hom} without assuming coercivity of f and without assuming convexity of f in the second variable, but assuming continuity in the first one in order to apply Reshetnyak Continuity Theorem, while in [12] f is assumed to be convex in the second variable and coercive, but only measurable and Y-periodic in the first variable.

If $n=1$ and $d>1$ in Theorem 1.6, then we recover De Arcangelis and Gargiulo's integral representation [26] of the effective energy F^{hom} without assuming f to be bounded from below, but assuming f to be continuous in the first variable and convex in the second one, while in [26] f is only required to be nonnegative, measurable and Y-periodic in the first variable and continuous in the second one. As we mentioned before, our hypotheses are related to the periodic unfolding method and Reshetnyak Continuity Theorem's hypotheses.

In the case in which $n \geqslant 2$, Theorem 1.6 and Corollary 1.7 provide new results in the literature in that, to the best of our knowledge, the homogenization of nonlinear periodically oscillating functionals with linear growth and characterized by $n \geqslant 2$ microscales has not yet been carried out.

Finally, in the framework of homogenization by Γ-convergence in the BV setting and for $n = 1$ we also mention the works by Braides and Chiatò Piat [15] and Carbone, Cioranescu, De Arcangelis and Gaudiello [17] concerning the convex case; and Bouchitté, Fonseca and Mascarenhas [14, Sect. 4.3], Attouch, Buttazzo and Michaille [7, Sect. 12.3] and Babadjian and Millot [8] regarding the nonconvex case.

This paper is organized as follows. In Section 2, we collect the necessary notation and we recall some basic properties of ($\mathbb{R}^m$-valued) Radon measures and of functions of bounded variation. We also recall some results established in [30] that will be used in the subsequent sections. In Section 3 we prove Proposition 1.4 and Theorem 1.6, and in Section 4 we prove Corollary 1.7.

2. Notation and preliminaries

2.1. Notation

In the sequel Z is a σ-compact separable metric space, Ω is an open subset of $\mathbb{R}^N$, $N \in \mathbb{N}$, and $Y := (0,1)^N$ is the reference cell. For each $i \in \mathbb{N}$, Y_i stands for a copy of Y. Given $x \in \mathbb{R}^N$, we write $[x]$ and $\langle x \rangle$ to denote the integer and the fractional part of x componentwise, respectively, so that $x = [x] + \langle x \rangle$ and $[x] \in \mathbb{Z}^N$, $\langle x \rangle \in Y$.

Let $n, m \in \mathbb{N}$. If $x, y \in \mathbb{R}^m$, then $x \cdot y$ stands for the Euclidean inner product of x and y, and $|x| := \sqrt{x \cdot x}$ for the Euclidean norm of x. The space of $(m \times n)$-dimensional matrices will be identified with $\mathbb{R}^{mn}$, and we write $\mathbb{R}^{m\times n}$. If $\xi = (\xi_{ij})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$, $\zeta = (\zeta_{ij})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n} \in \mathbb{R}^{m\times n}$, then

$$\xi : \zeta := \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij}\zeta_{ij}$$

represents the inner product of ξ and ζ, while $|\xi| := \sqrt{\xi : \xi}$ denotes the norm of ξ. If $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, then $a \otimes b$ stands for the $(m \times n)$-dimensional rank-one matrix defined by $a \otimes b := (a_i b_j)_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$.

Let $g : \mathbb{R}^{nN} \to \mathbb{R}^m$ be a function. We denote the Lipschitz constant of g on a set $D \subset \mathbb{R}^{nN}$ by $\mathrm{Lip}(g; D)$; if D coincides with the domain of g we omit its dependence. We say that g is $Y_1 \times \cdots \times Y_n$-periodic if for all $i \in \{1, \cdots, n\}$, $\kappa \in \mathbb{Z}^N$, $y_1, ..., y_n \in \mathbb{R}^N$, one has $g(y_1, \cdots, y_i + \kappa, \cdots, y_n) = g(y_1, \cdots, y_i, \cdots, y_n)$.

We will consider the Banach spaces

$$C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m) := \big\{g \in C(\mathbb{R}^{nN}; \mathbb{R}^m) \colon\ g \text{ is } Y_1 \times \cdots \times Y_n\text{-periodic}\big\}$$

endowed with the supremum norm $\|\cdot\|_\infty$, and $C_0(Z; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$, which is the closure with respect to the supremum norm $\|\cdot\|_\infty$ of $C_c(Z; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$. The latter is the space of all functions $g : Z \times \mathbb{R}^{nN} \to \mathbb{R}^m$ such that for all $z \in Z$, $g(z, \cdot) \in C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m)$ and for all $y_1, ..., y_n \in \mathbb{R}^N$, $g(\cdot, y_1, \ldots, y_n) \in C_c(Z; \mathbb{R}^m)$. The spaces $C^k_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m)$, $C^\infty_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m)$, $C^k_c(Z; C^k_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$, $C^\infty_c(Z; C^\infty_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$, $C^k_0(Z; C^k_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$ and $C^\infty_0(Z; C^\infty_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$ are now defined in an obvious way.

If $m = 1$ the co-domain will often be omitted (e.g., we write $C_0(Z)$ instead of $C_0(Z; \mathbb{R})$).

$\mathcal{C}$ represents a generic positive constant, whose value may change from expression to expression, and ε stands for a positive small parameter, often considered as taking its values on a positive sequence converging to zero; in this case, ε' represents a subsequence of ε, and we write $\varepsilon' \prec \varepsilon$.

2.2. Measure theory

For $m \in \mathbb{N}$, the m-dimensional Lebesgue measure is denoted by $\mathcal{L}^m$.

The Borel σ-algebra on Z is denoted by $\mathcal{B}(Z)$, and $\mathcal{M}(Z;\mathbb{R}^m)$ is the Banach space of all Radon measures $\lambda : \mathcal{B}(Z) \to \mathbb{R}^m$ endowed with the total variation norm $\|\cdot\|$.

If $\varphi \in C_0(Z)$ and $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathcal{M}(Z;\mathbb{R}^m)$, then we set

$$\int_Z \varphi(z)\,\mathrm{d}\lambda(z) := \left(\int_Z \varphi(z)\,\mathrm{d}\lambda_1(z), \cdots, \int_Z \varphi(z)\,\mathrm{d}\lambda_m(z)\right).$$

If $\varphi = (\varphi_1, \cdots, \varphi_m) \in C_0(Z;\mathbb{R}^m)$ and $\lambda \in \mathcal{M}(Z;\mathbb{R})$, then we define

$$\int_Z \varphi(z)\,\mathrm{d}\lambda(z) := \left(\int_Z \varphi_1(z)\,\mathrm{d}\lambda(z), \cdots, \int_Z \varphi_m(z)\,\mathrm{d}\lambda(z)\right).$$

We write $\mathcal{M}_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m)$ and $\mathcal{M}_{y\#}(Z\times Y_1\times\cdots\times Y_n;\mathbb{R}^m)$ to denote the duals of $C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m)$ and $C_0(Z;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m))$, respectively.

Let $E \subset \mathbb{R}^n$ be a Borel set and let $\mu : \mathcal{B}(E) \to [0,\infty]$ be a positive Radon measure. If $\lambda \in \mathcal{M}(E;\mathbb{R}^m)$, then (see for example, [32]) by Lebesgue Decomposition Theorem we can decompose λ as $\lambda = \lambda^{ac} + \lambda^s = \frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mu}\mu_{\lfloor E} + \lambda^s$, where λ^{ac} is absolutely continuous with respect to μ, λ^s and μ are mutually singular.

2.3. The space of functions of bounded variation

A function $u : \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, is said to be a function of bounded variation, and we write $u \in BV\big(\Omega;\mathbb{R}^d\big)$, if $u \in L^1\big(\Omega;\mathbb{R}^d\big)$ and its distributional derivative Du belongs to $\mathcal{M}\big(\Omega;\mathbb{R}^{d\times N}\big)$, that is, if there exists a measure $Du \in \mathcal{M}\big(\Omega;\mathbb{R}^{d\times N}\big)$ such that for all $\phi \in C_c(\Omega)$, $j \in \{1,\cdots,d\}$ and $i \in \{1,\cdots,N\}$ one has

$$\int_\Omega u_j(x)\frac{\partial\phi}{\partial x_i}(x)\,\mathrm{d}x = -\int_\Omega \phi(x)\,\mathrm{d}D_i u_j(x),$$

where $u = (u_1,\cdots,u_d)$ and $Du_j = (D_1u_j,\cdots,D_Nu_j)$. The space $BV\big(\Omega;\mathbb{R}^d\big)$ is a Banach space when endowed with the norm $\|u\|_{BV(\Omega;\mathbb{R}^d)} := \|u\|_{L^1(\Omega;\mathbb{R}^d)} + \|Du\|(\Omega)$.

We will also consider the space $BV_\#\big(Y;\mathbb{R}^d\big) := \big\{u \in BV_{\mathrm{loc}}\big(\mathbb{R}^N;\mathbb{R}^d\big) :\ u \text{ is } Y\text{-periodic}\big\}$, endowed with the norm of $BV\big(Y;\mathbb{R}^d\big)$. Notice that if $u \in BV_\#\big(Y;\mathbb{R}^d\big)$, then $Du \in \mathcal{M}_\#\big(Y;\mathbb{R}^{d\times N}\big)$.

We will consider the weak-$\star$ convergence in $BV\big(\Omega;\mathbb{R}^d\big)$. We recall that $\{u_j\}_{j\in\mathbb{N}} \subset BV\big(\Omega;\mathbb{R}^d\big)$ is said to weakly-$\star$ converge in $BV\big(\Omega;\mathbb{R}^d\big)$ to some $u \in BV\big(\Omega;\mathbb{R}^d\big)$ if $u_j \to u$ (strongly) in $L^1\big(\Omega;\mathbb{R}^d\big)$ and $Du_j \overset{\star}{\rightharpoonup} Du$ weakly-$\star$ in $\mathcal{M}\big(\Omega;\mathbb{R}^{d\times N}\big)$.

If $u \in BV\big(\Omega;\mathbb{R}^d\big)$, then $Du = \nabla u\mathcal{L}^N{}_{\lfloor\Omega} + D^s u$ is the Radon-Nikodym decomposition of Du with respect to $\mathcal{L}^N{}_{\lfloor\Omega}$.

2.4. Some preliminary results

We start this subsection by providing a simple example of a measure in the space $\mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$, $i \in \mathbb{N}$, introduced in Section 1. For simplicity, assume $i = 1$, and let $\tau \in \mathcal{M}(\Omega;\mathbb{R})$ be a real-valued Radon measure with finite total variation, let $v \in BV_\#\big(Y;\mathbb{R}^d\big)$, and consider the mapping $\boldsymbol{\mu} : B \in \mathcal{B}(\Omega) \mapsto \tau(B)v \in BV_\#\big(Y;\mathbb{R}^d\big)$. Then $\boldsymbol{\mu} \in \mathcal{M}_*\big(\Omega;BV_\#\big(Y;\mathbb{R}^d\big)\big)$, and $\lambda := \tau\otimes Dv \in \mathcal{M}_{y\#}\big(\Omega\times Y;\mathbb{R}^{d\times N}\big)$ is the measure associated with $D_y\boldsymbol{\mu} : B \in \mathcal{B}(\Omega) \mapsto D_y(\boldsymbol{\mu}(B)) = \tau(B)Dv$ in the sense of Definition 1.2, that is, $\langle\lambda,\varphi\rangle := \int_{\Omega\times Y}\varphi(x,y)\,\mathrm{d}\tau(x)\mathrm{d}Dv(y)$, $\varphi \in C_0\big(\Omega;C_\#\big(Y;\mathbb{R}^{d\times N}\big)\big)$.

We refer the reader to [30] for more detailed considerations on the space $\mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$, $i \in \mathbb{N}$.

The next result shows that Theorem 1.3 fully characterizes the $(n+1)$-scale limit of bounded sequences in $BV\big(\Omega;\mathbb{R}^d\big)$ (see [30, Prop. 1.11]).

Proposition 2.1. *Let $u \in BV(\Omega;\mathbb{R}^d)$ and for $i \in \{1,\cdots,n\}$, let $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1}; BV_\#(Y_i;\mathbb{R}^d))$. Then there exists a bounded sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset BV(\Omega;\mathbb{R}^d)$ for which a) and b) of Theorem 1.3 hold (with ε' replaced by ε).*

Remark 2.2. *Since every $(n+1)$-scale convergent sequence in $\mathcal{M}(\Omega;\mathbb{R}^m)$ is also a weakly-$\star$ convergent sequence in the sense of measures (see [30, Prop. 3.3]), it follows that any such sequence is bounded in $\mathcal{M}(\Omega;\mathbb{R}^m)$.*

We now recall a density type result proved in [30, Prop. 3.14], which will play an important role in the proof of our main results.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set such that $\partial\Omega$ is Lipschitz. Let $u \in BV(\Omega;\mathbb{R}^d)$, and for each $i \in \{1,\cdots,n\}$, let $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega \times Y_1 \times\cdots\times Y_{i-1}; BV_\#(Y_i;\mathbb{R}^d))$. Then there exist sequences $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega};\mathbb{R}^d)$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega; C_\#^\infty(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$ satisfying*

$$\begin{aligned} &u_j \overset{\star}{\rightharpoonup}_j u \ \text{ weakly-}\star \text{ in } BV(\Omega;\mathbb{R}^d), \quad \lim_{j\to\infty}\int_\Omega |\nabla u_j(x)|\,\mathrm{d}x = \|Du\|(\Omega),\\ &\Big(\nabla u_j + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}\Big)\mathcal{L}^{(n+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_n} \overset{\star}{\rightharpoonup}_j\\ &\qquad\qquad \overset{\star}{\rightharpoonup}_j \lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n} \ \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}),\\ &\lim_{j\to\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n}\Big|\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\ &\qquad\qquad = \|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n), \end{aligned} \tag{2.1}$$

where $\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}$ is the measure defined in (1.4), and

$$\begin{aligned} &\tilde\lambda_j \overset{\star}{\rightharpoonup} \tilde\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n} \ \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\times\mathbb{R}),\\ &\lim_{j\to\infty}\|\tilde\lambda_j\|(\Omega\times Y_1\times\cdots\times Y_n) = \|\tilde\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n), \end{aligned} \tag{2.2}$$

where, for any $B \in \mathcal{B}(\Omega\times Y_1\times\cdots\times Y_n)$,

$$\begin{aligned} &\tilde\lambda_j(B) := \bigg(\int_B\Big(\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n, \mathcal{L}^{(n+1)N}(B)\bigg),\\ &\tilde\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}(B) := \Big(\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}(B), \mathcal{L}^{(n+1)N}(B)\Big). \end{aligned}$$

Finally, we recall that in view of Riemann-Lebesgue's Lemma, if $\varphi \in C(\Omega; C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m))$ then

$$\varphi\Big(\,\cdot\,, \frac{\cdot}{\varrho_1(\varepsilon)}, \cdots, \frac{\cdot}{\varrho_n(\varepsilon)}\Big) \overset{\star}{\rightharpoonup} \int_{Y_1\times\cdots\times Y_n}\varphi(\,\cdot\,,y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n \tag{2.3}$$

weakly-$\star$ in $L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$. In particular, if $\varphi \in C_0(\Omega; C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^d))$ then (2.3) holds weakly-$\star$ in $L^\infty(\Omega;\mathbb{R}^m)$.

Also, if $a : \mathbb{R}^{nN} \to \mathbb{R}$ is a $Y_1\times\cdots\times Y_n$-periodic function such that for some $1 \leqslant p \leqslant \infty$ and for a.e. $y_n \in Y_n$ we have $a(\cdot,y_n) \in C_\#(Y_1\times\cdots\times Y_{n-1})$ and $\|a(\cdot,y_n)\|_{C_\#(Y_1\times\cdots\times Y_{n-1})} \in L^p(Y_n)$, then (see [27])

$$\begin{cases} a\big(\frac{\cdot}{\varrho_1(\varepsilon)},\cdots,\frac{\cdot}{\varrho_n(\varepsilon)}\big) \rightharpoonup \bar a \ \text{ weakly in } L^p_{\mathrm{loc}}(\mathbb{R}^N) & \text{if } 1 \leqslant p < \infty,\\ a\big(\frac{\cdot}{\varrho_1(\varepsilon)},\cdots,\frac{\cdot}{\varrho_n(\varepsilon)}\big) \overset{\star}{\rightharpoonup} \bar a \ \text{ weakly-}\star \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}^N) & \text{if } p = \infty, \end{cases} \tag{2.4}$$

where $\bar a := \int_{Y_1\times\cdots\times Y_n} a(y_1,\cdots,y_n)\,\mathrm{d}y_1\ldots\mathrm{d}y_n$.

3. Proof of Theorem 1.6

Throughout this section we will assume that $n = 2$. The cases in which $n = 1$ or $n \geqslant 3$ do not bring any additional technical difficulties.

For $n = 2$ the energies F_ε in (1.2) take the form

$$F_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon)}, \frac{x}{\varrho_2(\varepsilon)}, \nabla u(x)\Big)\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{x}{\varrho_1(\varepsilon)}, \frac{x}{\varrho_2(\varepsilon)}, \frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\mathrm{d}\|D^s u\|(x) \tag{3.1}$$

for $u \in BV\big(\Omega; \mathbb{R}^d\big)$, where, we recall, $\varrho_1, \varrho_2 : (0, \infty) \to (0, \infty)$ are functions satisfying (1.1) (with $n = 2$) and f^∞ is the recession function associated with f. Due to the convexity hypothesis $(\mathcal{F}2)$, the limit superior defining f^∞ is actually a limit (see, for example, [32]), so that $f^\infty : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is given by by

$$f^\infty(y_1, y_2, \xi) := \lim_{t\to\infty} \frac{f(y_1, y_2, t\xi)}{t}.$$

Moreover, under hypotheses $(\mathcal{F}1)$–$(\mathcal{F}3)$ and $(\mathcal{F}4)^\star$ on f, we have that f^∞ is a Borel function satisfying $(\mathcal{F}1)$, $(\mathcal{F}2)$, and the growth condition

$$0 \leqslant f^\infty(y_1, y_2, \xi) \leqslant C|\xi|. \tag{3.2}$$

Notice that in view of $(\mathcal{F}3)$, $(\mathcal{F}4)^\star$ and (3.2), the functional F_ε is well defined (in $\mathbb{R}$) for every $u \in BV\big(\Omega; \mathbb{R}^d\big)$.

In Theorem 3.1 below we will establish (1.13). We will use the unfolding operator (see [19], [21]; see also [31]): For $\varrho > 0$, $\mathcal{T}_\varrho : L^1(\Omega; \mathbb{R}^m) \to L^1(\mathbb{R}^N; L^1_\#(Y_2; \mathbb{R}^m))$ is defined by

$$\mathcal{T}_\varrho(g)(x, y_2) := \tilde{g}\Big(\varrho\Big[\frac{x}{\varrho}\Big] + \varrho(y_2 - [y_2])\Big) \quad \text{for } x, y_2 \in \mathbb{R}^N,\ g \in L^1(\Omega; \mathbb{R}^m),$$

where $\tilde{g}$ is the extension by zero of g to $\mathbb{R}^N$. Clearly $\mathcal{T}_\varrho$ is linear, and for every $g \in L^1(\Omega; \mathbb{R}^m)$

$$\|\mathcal{T}_\varrho(g)\|_{L^1(\Omega\times Y_2;\mathbb{R}^m)} \leqslant \|\mathcal{T}_\varrho(g)\|_{L^1(\mathbb{R}^N\times Y_2;\mathbb{R}^m)} = \|\tilde{g}\|_{L^1(\mathbb{R}^N;\mathbb{R}^m)} = \|g\|_{L^1(\Omega;\mathbb{R}^m)}, \tag{3.3}$$

and

$$\lim_{\varrho\to 0^+} \int_{\mathbb{R}^N\times Y_2} |\tilde{g}(x) - \mathcal{T}_\varrho(g)(x, y_2)|\,\mathrm{d}x\mathrm{d}y_2 = 0 \tag{3.4}$$

(see [31, Prop. A.1]).

Similarly, we define the operator $\mathcal{A}_\varrho : L^1(\Omega \times Y_2; \mathbb{R}^m) \to L^1(\mathbb{R}^N; L^1_\#(Y_1; L^1(Y_2; \mathbb{R}^m)))$ by

$$\begin{aligned}&\mathcal{A}_\varrho(h)(x, y_1, y_2)\\ &\quad := \tilde{h}\Big(\varrho\Big[\frac{x}{\varrho}\Big] + \varrho(y_1 - [y_1]), y_2\Big) = \mathcal{T}_\varrho(h(\cdot, y_2))(x, y_1) \quad \text{for } x, y_1 \in \mathbb{R}^N,\ y_2 \in Y_2,\ h \in L^1(\Omega \times Y_2; \mathbb{R}^m),\end{aligned}$$

where $\tilde{h}$ is the extension by zero of h to $\mathbb{R}^N \times Y_2$. $\mathcal{A}_\varrho$ is linear, and for all $h \in L^1(\Omega \times Y_2; \mathbb{R}^m)$,

$$\|\mathcal{A}_\varrho(h)\|_{L^1(\Omega\times Y_1\times Y_2;\mathbb{R}^m)} \leqslant \|\mathcal{A}_\varrho(h)\|_{L^1(\mathbb{R}^N\times Y_1\times Y_2;\mathbb{R}^m)} = \|\tilde{h}\|_{L^1(\mathbb{R}^N\times Y_2;\mathbb{R}^m)} = \|h\|_{L^1(\Omega\times Y_2;\mathbb{R}^m)} \tag{3.5}$$

by (3.3) and Fubini's Theorem. Moreover, we notice that for a.e. $y_2 \in Y_2$, we have

$$\lim_{\varrho\to 0^+} \int_{\mathbb{R}^N\times Y_1} \big|\tilde{h}(x, y_2) - \mathcal{T}_\varrho(h(\cdot, y_2))(x, y_1)\big|\,\mathrm{d}x\mathrm{d}y_1 = 0$$

by (3.4), and

$$\int_{\mathbb{R}^N\times Y_1} \big|\tilde{h}(x, y_2) - \mathcal{T}_\varrho(h(\cdot, y_2))(x, y_1)\big|\,\mathrm{d}x\mathrm{d}y_1 \leqslant 2\int_{\mathbb{R}^N} \big|\tilde{h}(x, y_2)\big|\,\mathrm{d}x \in L^1(Y_2),$$

where we used (3.3) to obtain

$$\int_{\mathbb{R}^N\times Y_1} \left|\mathcal{T}_\varrho(h(\cdot,y_2))(x,y_1)\right| \mathrm{d}x\mathrm{d}y_1 = \int_{\mathbb{R}^N} \left|\tilde{h}(x,y_2)\right| \mathrm{d}x.$$

Thus, Lebesgue Dominated Convergence Theorem yields

$$\begin{aligned}
&\lim_{\varrho\to 0^+} \int_{\mathbb{R}^N\times Y_1\times Y_2} \left|\tilde{h}(x,y_2) - \mathcal{A}_\varrho(h)(x,y_1,y_2)\right| \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad = \lim_{\varrho\to 0^+} \int_{\mathbb{R}^N\times Y_1\times Y_2} \left|\tilde{h}(x,y_2) - \mathcal{T}_\varrho(h(\cdot,y_2))(x,y_1)\right| \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 = 0.
\end{aligned}$$

Theorem 3.1. *Let $\Omega\subset\mathbb{R}^N$ be an open, bounded set with $\partial\Omega$ Lipschitz, let $Y_1 = Y_2 := (0,1)^N$, and let $f:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}$ be a Borel function satisfying conditions ($\mathcal{F}$1)–($\mathcal{F}$3), ($\mathcal{F}$4)*, ($\mathcal{F}$5), ($\mathcal{F}$6) for $n=2$. Then (1.13) holds (with $n=2$).*

The proof of Theorem 3.1 is hinged on some lemmas. The first lemma "unfolds" the rapidly oscillating sequence.

Lemma 3.2. *Under the same hypotheses of Theorem 3.1, if $\{v_\varepsilon\}_{\varepsilon>0}\subset L^1(\Omega;\mathbb{R}^{d\times N})$ is a bounded sequence then, for all $\eta>0$,*

$$\liminf_{\varepsilon\to 0^+} \int_\Omega f_\eta\Big(\frac{x}{\varrho_1(\varepsilon)}, \frac{x}{\varrho_2(\varepsilon)}, v_\varepsilon(x)\Big)\,\mathrm{d}x \geqslant \liminf_{\varepsilon\to 0^+} \int_{\Omega\times Y_1\times Y_2} f_\eta\big(y_1,y_2,\mathcal{A}_{\varrho_1(\varepsilon)}\big(\mathcal{T}_{\varrho_2(\varepsilon)}(v_\varepsilon)\big)(x,y_1,y_2)\big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2, \tag{3.6}$$

where $f_\eta(y_1,y_2,\xi) := f(y_1,y_2,\xi) + \eta|\xi|$.

Proof. Fix $\eta>0$ and $\delta>0$. Let $b_\eta\in\mathbb{R}$ be given by ($\mathcal{F}$4)* (see Remark 1.8), and let $\tilde{a}_\delta\in L^1_\#(Y_1\times Y_2)$ and $\tau_\delta>0$ be given by ($\mathcal{F}$6). Then

$$f_\eta(\cdot,\cdot,\cdot) \geqslant -b_\eta, \tag{3.7}$$

and, for all $y_1, y_1', y_2\in\mathbb{R}^N$, $\xi\in\mathbb{R}^{d\times N}$ such that $|y_1-y_1'|\leqslant\tau_\delta$,

$$f_\eta(y_1,y_2,\xi) \geqslant \delta\tilde{a}_\delta(y_1',y_2) + (1+o(1))f_\eta(y_1',y_2,\xi) - o(1)\eta|\xi| \;\; (\text{as } \delta\to 0^+). \tag{3.8}$$

Set $c := \sup_\varepsilon \|v_\varepsilon\|_{L^1(\Omega;\mathbb{R}^{d\times N})}$, $\varepsilon_1 := \varrho_1(\varepsilon)$ and $\varepsilon_2 := \varrho_2(\varepsilon)$. Define

$$Z_{\varepsilon_2} := \big\{\kappa\in\mathbb{Z}^N:\; \varepsilon_2(\kappa+Y_2)\cap\overline{\Omega}\neq\emptyset\big\}, \quad \Omega_{\varepsilon_2} := \mathrm{int}\Big(\bigcup_{\kappa\in Z_{\varepsilon_2}} \varepsilon_2(\kappa+\overline{Y}_2)\Big). \tag{3.9}$$

Notice that $\Omega\subset\Omega_{\varepsilon_2}$ and, by (3.3),

$$\sup_{\varepsilon>0} \|\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\|_{L^1(\mathbb{R}^N\times Y_2;\mathbb{R}^{d\times N})} \leqslant c. \tag{3.10}$$

Recalling that $\tilde{v}_\varepsilon$ stands for the extension by zero to the whole $\mathbb{R}^N$ of v_ε, using ($\mathcal{F}$3), a change of variables and ($\mathcal{F}$1), in this order, we obtain

$$\begin{aligned}
\int_\Omega f_\eta\Big(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},v_\varepsilon(x)\Big)\,\mathrm{d}x &= \int_{\Omega_{\varepsilon_2}} f_\eta\Big(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},\tilde{v}_\varepsilon(x)\Big)\,\mathrm{d}x - \int_{\Omega_{\varepsilon_2}\backslash\Omega} f_\eta\Big(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},0\Big)\,\mathrm{d}x \\
&\geqslant \sum_{\kappa\in Z_{\varepsilon_2}} \int_{\varepsilon_2(\kappa+Y_2)} f_\eta\Big(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},\tilde{v}_\varepsilon(x)\Big)\,\mathrm{d}x - C\mathcal{L}^N\big(\Omega_{\varepsilon_2}\backslash\Omega\big) \\
&= \sum_{\kappa\in Z_{\varepsilon_2}} \int_{Y_2} f_\eta\Big(\frac{\varepsilon_2}{\varepsilon_1}\kappa + \frac{\varepsilon_2}{\varepsilon_1}y_2, y_2, \tilde{v}_\varepsilon(\varepsilon_2\kappa+\varepsilon_2 y_2)\Big)\,\varepsilon_2^N\mathrm{d}y_2 - C\mathcal{L}^N\big(\Omega_{\varepsilon_2}\backslash\Omega\big).
\end{aligned} \tag{3.11}$$

Since $\left[\frac{x}{\varepsilon_2}\right] = \kappa$ whenever $x \in \varepsilon_2(\kappa + Y_2)$, $\mathcal{L}^N(\varepsilon_2(\kappa + Y_2)) = \varepsilon_2^N$ and $[y_2] = 0$ for all $y_2 \in Y_2$, in view of the definition of $\mathcal{T}_{\varepsilon_2}(v_\varepsilon)$, by Fubini's Theorem, and from (3.11) we get

$$\begin{aligned}
&\int_\Omega f_\eta\Big(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, v_\varepsilon(x)\Big)\,\mathrm{d}x \\
&\quad\geqslant \sum_{\kappa\in Z_{\varepsilon_2}} \int_{\varepsilon_2(\kappa+Y_2)} \bigg(\int_{Y_2} f_\eta\Big(\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2, y_2, \tilde{v}_\varepsilon\Big(\varepsilon_2\Big[\frac{x}{\varepsilon_2}\Big] + \varepsilon_2 y_2\Big)\Big)\,\mathrm{d}y_2\bigg)\,\mathrm{d}x - C\mathcal{L}^N\big(\Omega_{\varepsilon_2}\backslash\Omega\big) \\
&\quad= \int_{\Omega_{\varepsilon_2}\times Y_2} f_\eta\Big(\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2, y_2, \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_2 - C\mathcal{L}^N\big(\Omega_{\varepsilon_2}\backslash\Omega\big) \\
&\quad\geqslant \int_{\Omega\times Y_2} f_\eta\Big(\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2, y_2, \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_2 - (b_\eta + C)\mathcal{L}^N\big(\Omega_{\varepsilon_2}\backslash\Omega\big),
\end{aligned} \tag{3.12}$$

where in the last inequality we used (3.7).

By (1.1) there exists $\varepsilon_\delta > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_\delta$ one has $0 < \varepsilon_2/\varepsilon_1 < \tau_\delta/2\sqrt{N}$. For any such ε,

$$\sup_{x\in\Omega, y_2\in Y_2} \Big|\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2 - \frac{x}{\varepsilon_1}\Big| = \sup_{x\in\Omega, y_2\in Y_2} \Big| - \frac{\varepsilon_2}{\varepsilon_1}\Big\langle\frac{x}{\varepsilon_2}\Big\rangle + \frac{\varepsilon_2}{\varepsilon_1}y_2\Big| < \tau_\delta,$$

thus (3.8) and (3.10) yield

$$\begin{aligned}
&\int_{\Omega\times Y_2} f_\eta\Big(\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2, y_2, \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_2 \\
&\quad\geqslant \delta\int_{\Omega\times Y_2} \tilde{a}_\delta\Big(\frac{x}{\varepsilon_1}, y_2\Big)\,\mathrm{d}x\mathrm{d}y_2 + \big(1 + o(1)\big)\int_{\Omega\times Y_2} f_\eta\Big(\frac{x}{\varepsilon_1}, y_2, \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_2 - |o(1)|\eta c.
\end{aligned} \tag{3.13}$$

Defining Z_{ε_1} and Ω_{ε_1} as in (3.9) (with ε_2 and Y_2 replaced by ε_1 and Y_1, respectively), and reasoning as in (3.11)–(3.12), we conclude that

$$\begin{aligned}
&\int_{\Omega\times Y_2} f_\eta\Big(\frac{x}{\varepsilon_1}, y_2, \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_2 \\
&\quad\geqslant \int_{\Omega\times Y_1\times Y_2} f_\eta\big(y_1, y_2, \mathcal{A}_{\varepsilon_1}\big(\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\big)(x,y_1,y_2)\big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 - (b_\eta + C)\mathcal{L}^N\big(\Omega_{\varepsilon_1}\backslash\Omega\big).
\end{aligned} \tag{3.14}$$

By the Riemann–Lebesgue Lemma we have that for a.e. $y_2 \in Y_2$, $\tilde{a}_\delta(\cdot/\varepsilon_1, y_2) \rightharpoonup \int_{Y_1} \tilde{a}_\delta(y_1, y_2)\,\mathrm{d}y_1$ weakly in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Hence,

$$\liminf_{\varepsilon\to0^+} \int_{\Omega\times Y_2} \tilde{a}_\delta\Big(\frac{x}{\varepsilon_1}, y_2\Big)\,\mathrm{d}x\mathrm{d}y_2 \geqslant \mathcal{L}^N(\Omega)\int_{Y_1\times Y_2} \tilde{a}_\delta(y_1, y_2)\,\mathrm{d}y_1\mathrm{d}y_2, \tag{3.15}$$

where we have also used Fatou's Lemma and Fubini's Theorem.

In view of (3.12)–(3.15), we obtain

$$\begin{aligned}
&\liminf_{\varepsilon\to0^+} \int_\Omega f_\eta\Big(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, v_\varepsilon(x)\Big)\,\mathrm{d}x \\
&\quad\geqslant \big(1 + o(1)\big)\liminf_{\varepsilon\to0^+} \int_{\Omega\times Y_1\times Y_2} f_\eta\big(y_1, y_2, \mathcal{A}_{\varepsilon_1}\big(\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\big)(x,y_1,y_2)\big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\
&\qquad + \delta\mathcal{L}^N(\Omega)\int_{Y_1\times Y_2} \tilde{a}_\delta(y_1, y_2)\,\mathrm{d}y_1\mathrm{d}y_2 - |o(1)|\eta c,
\end{aligned} \tag{3.16}$$

where we also used the convergences $\mathcal{L}^N\big(\Omega_{\varepsilon_1}\backslash\Omega\big), \mathcal{L}^N\big(\Omega_{\varepsilon_2}\backslash\Omega\big) \to 0$ as $\varepsilon \to 0^+$, since $\partial\Omega$ is Lipschitz and so $\mathcal{L}^N(\partial\Omega) = 0$. Finally, recalling that $\delta\|\tilde{a}_\delta\|_{L^1_\#(Y_1\times Y_2)} \to 0$ as $\delta \to 0^+$, passing (3.16) to the limit as $\delta \to 0^+$ we get (3.6). □

Remark 3.3. *The previous proof can be easily generalized to the case in which $n \geqslant 3$ by using (2.4) in place of Riemann–Lebesgue Lemma (see (3.15)).*

We now show that, similarly to what happens in the L^p-case with $p \in (1,\infty)$ (see [21, Prop. 2.14]), 3-scale convergence of a sequence of measures absolutely continuous with respect to the Lebesgue measure is equivalent to a weak-$\star$ convergence in the sense of measures in a product space of the unfolded sequence.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, let $\{v_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega;\mathbb{R}^{d\times N})$ be a bounded sequence and let $\lambda \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times Y_2;\mathbb{R}^{d\times N})$. Then $v_\varepsilon \mathcal{L}^N\lfloor_\Omega \xrightarrow[\varepsilon]{3\text{-}sc} \lambda$ if, and only if, $\mathcal{A}_{\varrho_1(\varepsilon)}\big(\mathcal{T}_{\varrho_2(\varepsilon)}(v_\varepsilon)\big)\mathcal{L}^{3N}\lfloor_{\Omega\times Y_1\times Y_2} \overset{\star}{\rightharpoonup} \lambda$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega \times Y_1 \times Y_2;\mathbb{R}^{d\times N})$ as $\varepsilon \to 0^+$.*

Proof. For $\delta > 0$, define the sets

$$W_\delta := \big\{\kappa \in \mathbb{Z}^N:\ \delta(\kappa + Y) \subset \Omega\big\}, \qquad \Omega_\delta := \operatorname{int}\Big(\bigcup_{\kappa\in W_\delta} \delta(\kappa+\overline{Y})\Big).$$

Take $\phi \in C^1_c(\Omega)$, $\psi_1 \in C^1_\#(Y_1)$ and $\psi_2 \in C^1_\#(Y_2;\mathbb{R}^{d\times N})$, and let $\varphi := \phi\psi_1\psi_2$. Set $\varepsilon_1 := \varrho_1(\varepsilon)$ and $\varepsilon_2 := \varrho_2(\varepsilon)$. By (1.1) we can find $\bar\varepsilon > 0$ such that for all $0 < \varepsilon \leqslant \bar\varepsilon$ one has

$$\operatorname{dist}(\operatorname{supp}\phi, \Omega\backslash\Omega_{\varepsilon_1}) > 2\varepsilon_1\sqrt{N}, \quad \operatorname{dist}(\operatorname{supp}\phi, \Omega\backslash\Omega_{\varepsilon_2}) > 2\varepsilon_1\sqrt{N}. \tag{3.17}$$

Fix any such ε. Using (3.17), the definition of $\mathcal{A}_{\varepsilon_1}$, Fubini's Theorem, and the equalities $\left[\frac{x}{\varepsilon_1}\right] = \kappa$ if $x \in \varepsilon_1(\kappa + Y_1)$ and $[y_1] = 0$ if $y_1 \in Y_1$, in this order, we get

$$\begin{aligned}
&\int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2) : \mathcal{A}_{\varepsilon_1}\big(\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\big)(x,y_1,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad= \int_{\Omega_{\varepsilon_1}\times Y_1\times Y_2} \varphi(x,y_1,y_2) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1(y_1 - [y_1]), y_2\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad= \int_{Y_1\times Y_2}\Big(\sum_{\kappa\in W_{\varepsilon_1}} \int_{\varepsilon_1(\kappa+Y_1)} \varphi(x,y_1,y_2) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(\varepsilon_1\kappa + \varepsilon_1 y_1, y_2)\,\mathrm{d}x\Big)\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned} \tag{3.18}$$

Performing the change of variables $x = \varepsilon_1\kappa + \varepsilon_1\zeta$, by Fubini's Theorem the last integral in (3.18) becomes

$$\int_{Y_1\times Y_2}\Big(\sum_{\kappa\in W_{\varepsilon_1}} \int_{Y_1} \varphi(\varepsilon_1\kappa + \varepsilon_1\zeta, y_1, y_2) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(\varepsilon_1\kappa + \varepsilon_1 y_1, y_2)\,\varepsilon_1^N\,\mathrm{d}y_1\Big)\mathrm{d}\zeta\mathrm{d}y_2. \tag{3.19}$$

Considering now the change of variables $y_1 = \frac{x}{\varepsilon_1} - \kappa$, and using again Fubini's Theorem, (3.19) reduces to

$$\begin{aligned}
&\int_{Y_1\times Y_2}\Big(\sum_{\kappa\in W_{\varepsilon_1}} \int_{\varepsilon_1(\kappa+Y_1)} \varphi\Big(\varepsilon_1\kappa + \varepsilon_1\zeta, \frac{x}{\varepsilon_1} - \kappa, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\,\mathrm{d}x\Big)\mathrm{d}\zeta\mathrm{d}y_2 \\
&\quad= \int_{Y_1\times Y_2}\Big(\int_{\Omega_{\varepsilon_1}} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1\zeta, \frac{x}{\varepsilon_1}, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\,\mathrm{d}x\Big)\mathrm{d}\zeta\mathrm{d}y_2 \\
&\quad= \int_{\Omega_{\varepsilon_1}\times Y_1\times Y_2} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 y_1, \frac{x}{\varepsilon_1}, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2,
\end{aligned} \tag{3.20}$$

where in the first equality we used the Y_1-periodicity of ψ_1.

We claim that if $x \in \Omega\backslash\Omega_{\varepsilon_1} \cup \Omega\backslash\Omega_{\varepsilon_2}$ then

$$\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 Y_1\Big) \cap \operatorname{supp}\phi = \emptyset. \tag{3.21}$$

In fact, if there was $z \in (\varepsilon_1[\frac{x}{\varepsilon_1}] + \varepsilon_1 Y_1) \cap \operatorname{supp}\phi$, then $z = \varepsilon_1[\frac{x}{\varepsilon_1}] + \varepsilon_1 y_1$ for some $y_1 \in Y_1$ and, by (3.17),

$$2\varepsilon_1\sqrt{N} < \operatorname{dist}(\operatorname{supp}\phi, x) \leqslant |z - x| = \Big|\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 y_1 - x\Big| = \Big| - \varepsilon_1\Big\langle\frac{x}{\varepsilon_1}\Big\rangle + \varepsilon_1 y_1\Big| \leqslant 2\varepsilon_1\sqrt{N},$$

which is a contradiction. Hence, (3.21) holds. Consequently,

$$\begin{aligned}\int_{\Omega_{\varepsilon_1}\times Y_1\times Y_2} &\varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 y_1, \frac{x}{\varepsilon_1}, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x, y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &= \int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 y_1, \frac{x}{\varepsilon_1}, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x, y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2.\end{aligned} \tag{3.22}$$

Arguing as in (3.18)–(3.20), we have

$$\begin{aligned}&\int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 y_1, \frac{x}{\varepsilon_1}, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x, y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &= \int_{Y_1\times Y_2}\Big(\sum_{\kappa\in W_{\varepsilon_2}}\int_{\varepsilon_2(\kappa+Y_2)} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 y_1, \frac{x}{\varepsilon_1}, y_2\Big) : v_\varepsilon(\varepsilon_2\kappa + \varepsilon_2 y_2)\,\mathrm{d}x\Big)\mathrm{d}y_1\mathrm{d}y_2 \\ &= \int_{Y_1\times Y_2}\Big(\sum_{\kappa\in W_{\varepsilon_2}}\int_{Y_2} \varphi\Big(\varepsilon_1\Big[\frac{\varepsilon_2}{\varepsilon_1}\kappa + \frac{\varepsilon_2}{\varepsilon_1}\zeta\Big] + \varepsilon_1 y_1, \frac{\varepsilon_2}{\varepsilon_1}\kappa + \frac{\varepsilon_2}{\varepsilon_1}\zeta, y_2\Big) : v_\varepsilon(\varepsilon_2\kappa + \varepsilon_2 y_2)\,\varepsilon_2^N\mathrm{d}y_2\Big)\mathrm{d}y_1\mathrm{d}\zeta \\ &= \int_{Y_1\times Y_2}\Big(\sum_{\kappa\in W_{\varepsilon_2}}\int_{\varepsilon_2(\kappa+Y_2)} \varphi\Big(\varepsilon_1\Big[\frac{\varepsilon_2}{\varepsilon_1}\kappa + \frac{\varepsilon_2}{\varepsilon_1}\zeta\Big] + \varepsilon_1 y_1, \frac{\varepsilon_2}{\varepsilon_1}\kappa + \frac{\varepsilon_2}{\varepsilon_1}\zeta, \frac{x}{\varepsilon_2} - \kappa\Big) : v_\varepsilon(x)\,\mathrm{d}x\Big)\mathrm{d}y_1\mathrm{d}\zeta \\ &= \int_{Y_1\times Y_2}\Big(\int_{\Omega_{\varepsilon_2}} \varphi\Big(\varepsilon_1\Big[\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}\zeta\Big] + \varepsilon_1 y_1, \frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}\zeta, \frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\Big)\mathrm{d}y_1\mathrm{d}\zeta \\ &= \int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \varphi\Big(\varepsilon_1\Big[\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2\Big] + \varepsilon_1 y_1, \frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2, \frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2,\end{aligned} \tag{3.23}$$

where in the fourth equality we used the Y_2-periodicity of ψ_2.
In view of (3.18)–(3.20) and (3.22)–(3.23), we conclude that

$$\begin{aligned}\int_{\Omega\times Y_1\times Y_2} &\varphi(x, y_1, y_2) : \mathcal{A}_{\varepsilon_1}\big(\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\big)(x, y_1, y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &= \int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \phi(a_\varepsilon(x, y_1, y_2))\psi_1(b_\varepsilon(x, y_2))\psi_2\Big(\frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2,\end{aligned} \tag{3.24}$$

where

$$a_\varepsilon(x, y_1, y_2) := \varepsilon_1\Big[\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2\Big] + \varepsilon_1 y_1,\ \ b_\varepsilon(x, y_2) := \frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2,\ \ x, y_1, y_2 \in \mathbb{R}^N.$$

Notice that for all $x \in \Omega$, $y_1 \in Y_1$ and $y_2 \in Y_2$,

$$|a_\varepsilon(x, y_1, y_2) - x| \leqslant 2\sqrt{N}(\varepsilon_1 + \varepsilon_2), \qquad \Big|b_\varepsilon(x, y_2) - \frac{x}{\varepsilon_1}\Big| \leqslant 2\sqrt{N}\,\frac{\varepsilon_2}{\varepsilon_1}. \tag{3.25}$$

Using (3.24) and (3.17), we obtain

$$
\begin{aligned}
&\Big| \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2) : \mathcal{A}_{\varepsilon_1}\big(\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\big)(x,y_1,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 - \int_\Omega \varphi\Big(x,\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\Big| \\
&\quad = \Big| \int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \phi(a_\varepsilon(x,y_1,y_2))\psi_1(b_\varepsilon(x,y_2))\psi_2\Big(\frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\
&\qquad\qquad - \int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \phi(x)\psi_1\Big(\frac{x}{\varepsilon_1}\Big)\psi_2\Big(\frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\Big| \\
&\quad \leqslant \|\psi_2\|_{L^\infty_\#(Y_2;\mathbb{R}^{d\times N})} \int_{\Omega\times Y_1\times Y_2} \Big|\phi(a_\varepsilon(x,y_1,y_2))\psi_1(b_\varepsilon(x,y_2)) - \phi(x)\psi_1\Big(\frac{x}{\varepsilon_1}\Big)\Big| |v_\varepsilon(x)|\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad \leqslant \|\psi_2\|_{L^\infty_\#(Y_2;\mathbb{R}^{d\times N})} \int_{\Omega\times Y_1\times Y_2} \Big(\|\phi\|_{L^\infty(\Omega)}\mathrm{Lip}(\psi)\Big|b_\varepsilon(x,y_2) - \frac{x}{\varepsilon_1}\Big| \\
&\qquad\qquad + \|\psi_1\|_{L^\infty_\#(Y_1)}\mathrm{Lip}(\phi)\big|a_\varepsilon(x,y_1,y_2) - x\big|\Big)|v_\varepsilon(x)|\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad \leqslant \mathcal{C}\Big(\varepsilon_1 + \varepsilon_2 + \frac{\varepsilon_2}{\varepsilon_1}\Big),
\end{aligned}
\tag{3.26}
$$

where in the last inequality we used (3.25) and the fact that $\sup_\varepsilon \|v_\varepsilon\|_{L^1(\Omega;\mathbb{R}^{d\times N})} < \infty$.

Since functions of the form $\varphi = \phi\psi_1\psi_2$ are dense in $C_0\big(\Omega; C_\#\big(Y_1{\times}Y_2;\mathbb{R}^{d\times N}\big)\big)$, and since $\big\{\mathcal{A}_{\varrho_1(\varepsilon)}\big(\mathcal{T}_{\varrho_2(\varepsilon)}(v_\varepsilon)\big)\big\} \subset L^1\big(\Omega\times Y_1\times Y_2;\mathbb{R}^{d\times N}\big)$, $\{v_\varepsilon\}\subset L^1\big(\Omega;\mathbb{R}^{d\times N}\big)$ are bounded sequences (see (3.3) and (3.5)), using a density argument, (1.1), and passing (3.26) to the limit as $\varepsilon\to 0^+$, we conclude that $v_\varepsilon \mathcal{L}^N\lfloor_\Omega \xrightarrow{3\text{-}sc}_{\varepsilon} \lambda$ if, and only if, $\mathcal{A}_{\varrho_1(\varepsilon)}\big(\mathcal{T}_{\varrho_2(\varepsilon)}(v_\varepsilon)\big)\mathcal{L}^{3N}\lfloor_{\Omega\times Y_1\times Y_2} \overset{\star}{\rightharpoonup} \lambda$ weakly-$\star$ in $\mathcal{M}_{y\#}\big(\Omega\times Y_1\times Y_2;\mathbb{R}^{d\times N}\big)$ as $\varepsilon\to 0^+$. □

The next lemma is a Reshetnyak continuity type result for functions not necessarily positively 1-homogeneous, and similar to [35, Thm. 5] (see [25] for related results).

Lemma 3.5. *Let $U\subset\mathbb{R}^l$ be an open set such that $\mathcal{L}^l(U)<\infty$. Let $g: U\times\mathbb{R}^m\to\mathbb{R}$ be a function such that $\bar g: U\times\mathbb{R}^m\times[0,\infty)\to\mathbb{R}$ defined by*

$$
\bar g(z,\xi,t) := \begin{cases} t g\big(z,\frac{\xi}{t}\big) & \text{if } t>0,\\ g^\infty(z,\xi) & \text{if } t=0,\end{cases}
\tag{3.27}
$$

is continuous and bounded on $U\times\mathbb{S}^m$, where $g^\infty(z,\xi) := \limsup_{t\to\infty} g(z,t\xi)/t$ is the recession function of g and $\mathbb{S}^m$ is the unit sphere in $R^m\times\mathbb{R}$. If $\lambda\in\mathcal{M}(U;\mathbb{R}^m)$, let $\tilde\lambda\in\mathcal{M}(U;\mathbb{R}^m\times\mathbb{R})$ denote the measure defined by $\tilde\lambda(\cdot) := \big(\lambda(\cdot),\mathcal{L}^l(\cdot)\big)$. Assume that $\lambda_j,\lambda\in\mathcal{M}(U;\mathbb{R}^m)$ are such that

$$
\tilde\lambda_j \overset{\star}{\rightharpoonup}_j \tilde\lambda \quad \text{weakly-}\star \text{ in } \mathcal{M}(U;\mathbb{R}^m\times\mathbb{R}), \qquad \lim_{j\to\infty}\|\tilde\lambda_j\|(U) = \|\tilde\lambda\|(U).
\tag{3.28}
$$

Then

$$
\begin{aligned}
&\lim_{j\to\infty}\Big\{ \int_U g\Big(z,\frac{\mathrm{d}\lambda_j^{ac}}{\mathrm{d}\mathcal{L}^l}(z)\Big)\,\mathrm{d}z + \int_U g^\infty\Big(z,\frac{\mathrm{d}\lambda_j^s}{\mathrm{d}\|\lambda_j^s\|}(z)\Big)\,\mathrm{d}\|\lambda_j^s\|(z)\Big\} \\
&\quad = \int_U g\Big(z,\frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mathcal{L}^l}(z)\Big)\,\mathrm{d}z + \int_U g^\infty\Big(z,\frac{\mathrm{d}\lambda^s}{\mathrm{d}\|\lambda^s\|}(z)\Big)\,\mathrm{d}\|\lambda^s\|(z).
\end{aligned}
\tag{3.29}
$$

Proof. Since $\bar g$ is a continuous and bounded function on $U\times\mathbb{S}^m$, in view of (3.28) Reshetnyak Continuity Theorem (see [38], and also [5, Thm. 2.39]) yields

$$
\lim_{j\to\infty}\int_U \bar g\Big(z,\frac{\mathrm{d}\tilde\lambda_j}{\mathrm{d}\|\tilde\lambda_j\|}(z)\Big)\mathrm{d}\|\tilde\lambda_j\|(z) = \int_U \bar g\Big(z,\frac{\mathrm{d}\tilde\lambda}{\mathrm{d}\|\tilde\lambda\|}(z)\Big)\mathrm{d}\|\tilde\lambda\|(z).
\tag{3.30}
$$

We claim that (3.30) reduces to (3.29). In fact, writing the Lebesgue decomposition of an arbitrary $\mu \in \mathcal{M}(U;\mathbb{R}^m)$ with respect to $\mathcal{L}^l$ as

$$\mu = \frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}\mathcal{L}^l\lfloor_U + \mu^s,$$

then

$$\tilde{\mu} = \Big(\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}, 1\Big)\mathcal{L}^l\lfloor_U + (\mu^s, 0), \quad \|\tilde{\mu}\| = \Big|\Big(\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}, 1\Big)\Big|\mathcal{L}^l\lfloor_U + \|\mu^s\|, \tag{3.31}$$

are the Lebesgue decomposition of $\tilde{\mu}$ and $\|\tilde{\mu}\|$ with respect to $\mathcal{L}^l$, respectively.

In view of the Besicovitch Derivation Theorem, for $\mathcal{L}^l$-a.e. $z \in U$, we have

$$\frac{\mathrm{d}\tilde{\mu}}{\mathrm{d}\|\tilde{\mu}\|}(z) = \frac{\big(\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}(z), 1\big)}{\big|\big(\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}(z), 1\big)\big|}, \tag{3.32}$$

and for $\|\mu^s\|$-a.e. $z \in U$, we have

$$\frac{\mathrm{d}\tilde{\mu}}{\mathrm{d}\|\tilde{\mu}\|}(z) = \Big(\frac{\mathrm{d}\mu^s}{\mathrm{d}\|\mu^s\|}(z), 0\Big). \tag{3.33}$$

From (3.31)–(3.33), and taking into account the positive 1-homogeneity of $(\xi, t) \in \mathbb{R}^m \times [0,\infty) \mapsto \bar{g}(z,\xi,t)$, we deduce that

$$\begin{aligned}\int_U \bar{g}\Big(z, \frac{\mathrm{d}\tilde{\mu}}{\mathrm{d}\|\tilde{\mu}\|}(z)\Big)\mathrm{d}\|\tilde{\mu}\|(z) &= \int_U \bar{g}\Big(z, \frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}(z), 1\Big)\,\mathrm{d}z + \int_U \bar{g}\Big(z, \frac{\mathrm{d}\mu^s}{\mathrm{d}\|\mu^s\|}(z), 0\Big)\,\mathrm{d}\|\mu^s\|(z) \\ &= \int_U g\Big(z, \frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}(z)\Big)\,\mathrm{d}z + \int_U g^\infty\Big(z, \frac{\mathrm{d}\mu^s}{\mathrm{d}\|\mu^s\|}(z)\Big)\,\mathrm{d}\|\mu^s\|(z),\end{aligned} \tag{3.34}$$

where in the last equality we used the definition of $\bar{g}$. By (3.34) we conclude that (3.30) reduces to (3.29). □

Next we prove a Reshetnyak lower semicontinuity type result for functions not necessarily positively 1-homogeneous (see also [23], [33]).

Lemma 3.6. *Let $U \subset \mathbb{R}^l$ be an open set such that $\mathcal{L}^l(U) < \infty$. Let $g : U \times \mathbb{R}^m \to \mathbb{R}$ be a function satisfying $|g(z,\xi)| \leqslant C(1+|\xi|)$, for some $C > 0$ and for every $(z,\xi) \in U \times \mathbb{R}^m$, and such that for all $z \in U$, $g(z,\cdot)$ is convex. Assume further that for all $\bar{z} \in U$ and $\delta > 0$, there exists $\tau = \tau(\bar{z},\delta) > 0$ such that for all $z \in U$ with $|z - \bar{z}| < \tau$, and $\xi \in \mathbb{R}^m$, we have $|g(\bar{z},\xi) - g(z,\xi)| \leqslant \delta(1+|\xi|)$. If $\lambda_j, \lambda \in \mathcal{M}(U;\mathbb{R}^m)$ are such that $\lambda_j \overset{\star}{\rightharpoonup}_j \lambda$ weakly-$\star$ in $\mathcal{M}(U;\mathbb{R}^m)$ as $j \to \infty$, then*

$$\begin{aligned}\liminf_{j\to\infty}&\Big\{\int_U g\Big(z, \frac{\mathrm{d}\lambda_j^{ac}}{\mathrm{d}\mathcal{L}^l}(z)\Big)\,\mathrm{d}z + \int_U g^\infty\Big(z, \frac{\mathrm{d}\lambda_j^s}{\mathrm{d}\|\lambda_j^s\|}(z)\Big)\,\mathrm{d}\|\lambda_j^s\|(z)\Big\} \\ &\geqslant \int_U g\Big(z, \frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mathcal{L}^l}(z)\Big)\,\mathrm{d}z + \int_U g^\infty\Big(z, \frac{\mathrm{d}\lambda^s}{\mathrm{d}\|\lambda^s\|}(z)\Big)\,\mathrm{d}\|\lambda^s\|(z).\end{aligned} \tag{3.35}$$

Proof. Let $\lambda_j, \lambda \in \mathcal{M}(U;\mathbb{R}^m)$ be such that $\lambda_j \overset{\star}{\rightharpoonup}_j \lambda$ weakly-$\star$ in $\mathcal{M}(U;\mathbb{R}^m \times \mathbb{R})$. Defining $\tilde{\lambda}_j, \tilde{\lambda} \in \mathcal{M}(U;\mathbb{R}^m \times \mathbb{R})$ as in Lemma 3.5, we see that $\tilde{\lambda}_j \overset{\star}{\rightharpoonup}_j \tilde{\lambda}$ weakly-$\star$ in $\mathcal{M}(U;\mathbb{R}^m \times \mathbb{R})$.

Let $\bar{g} : U \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be the function introduced in (3.27). Then (see Remark 1.8 (ii)) $\bar{g}$ is a continuous function, and $|\bar{g}(z,\xi,t)| \leqslant 2C|(\xi,t)|$ for all $(z,\xi,t) \in U \times \mathbb{R}^m \times [0,\infty)$. Moreover, since for each $i \in \mathbb{N}$ there exist functions $a_i : U \to \mathbb{R}$ and $b_i : U \to \mathbb{R}^m$ such that

$$g(z,\xi) = \sup_{i\in\mathbb{N}}\big\{a_i(z) + b_i(z)\cdot\xi\big\}, \quad g^\infty(z,\xi) = \sup_{i\in\mathbb{N}}\big\{b_i(z)\cdot\xi\big\},$$

(see [32, Prop. 2.77]), we have that for all $(z,\xi,t) \in U \times \mathbb{R}^m \times [0,\infty)$,

$$\bar{g}(z,\xi,t) = \sup_{i\in\mathbb{N}}\big\{a_i(z)t + b_i(z)\cdot\xi\big\}.$$

Thus for all $z \in U$, $(\xi, t) \in \mathbb{R}^m \times [0, \infty) \mapsto \bar{g}(z, \xi, t)$ is convex and positively 1-homogeneous. So, Reshetnyak Lower Semicontinuity Theorem (see [38], and also [5, Thm. 2.38]) yields

$$\liminf_{j\to\infty} \int_U \bar{g}\Big(z, \frac{\mathrm{d}\tilde{\lambda}_j}{\mathrm{d}\|\tilde{\lambda}_j\|}(z)\Big)\mathrm{d}\|\tilde{\lambda}_j\|(z) \geqslant \int_U \bar{g}\Big(z, \frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\|\tilde{\lambda}\|}(z)\Big)\mathrm{d}\|\tilde{\lambda}\|(z). \tag{3.36}$$

Finally, we observe that by (3.34), (3.36) reduces to (3.35). □

Proof of Theorem 3.1. Fix $(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in BV\big(\Omega; \mathbb{R}^d\big) \times \mathcal{M}_\star\big(\Omega; BV_\#\big(Y_1; \mathbb{R}^d\big)\big) \times \mathcal{M}_\star\big(\Omega \times Y_1; BV_\#\big(Y_2; \mathbb{R}^d\big)\big)$, and set

$$\begin{aligned} G(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) := &\int_{\Omega\times Y_1\times Y_2} f\Big(y_1, y_2, \frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x, y_1, y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &+ \int_{\Omega\times Y_1\times Y_2} f^\infty\Big(y_1, y_2, \frac{\mathrm{d}\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x, y_1, y_2)\Big)\,\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x, y_1, y_2). \end{aligned}$$

We will proceed in two steps.

Step 1. We start by proving that

$$F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \geqslant G(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2).$$

Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to zero as $h \to \infty$, and by Proposition 2.1 let $\{u_h\}_{h\in\mathbb{N}} \subset BV\big(\Omega; \mathbb{R}^d\big)$ be a bounded sequence such that $Du_h \xrightarrow[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$. We claim that

$$\liminf_{h\to\infty} F_{\varepsilon_h}(u_h) \geqslant G(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2). \tag{3.37}$$

Since $\{Du_h\}_{h\in\mathbb{N}}$ is bounded in $\mathcal{M}\big(\Omega; \mathbb{R}^{d\times N}\big)$ (see Remark 2.2), in view of $(\mathcal{F}3)$, $(\mathcal{F}4)^\star$ and (3.2), we have that $\{F_{\varepsilon_h}(u_h)\}_{h\in\mathbb{N}}$ is bounded. Therefore, we may assume without loss of generality that the limit inferior in (3.37) is actually a limit and that this limit is finite (which is true up to a subsequence).

By Proposition 2.3 (with $\boldsymbol{\mu}_i = 0$), for each $h \in \mathbb{N}$ we can find a sequence $\big\{u^{(h)}_j\big\}_{j\in\mathbb{N}} \subset W^{1,1}\big(\Omega; \mathbb{R}^d\big)$ such that

$$\begin{aligned} &u^{(h)}_j \xrightarrow{\star}_j u_h \quad \text{weakly-}\star \text{ in } BV\big(\Omega; \mathbb{R}^d\big), \\ &\tilde{\lambda}^{(h)}_j \xrightarrow{\star}_j \tilde{\lambda}_h \quad \text{weakly-}\star \text{ in } \mathcal{M}(\Omega; \mathbb{R}^{d\times N} \times \mathbb{R}), \quad \lim_{j\to\infty} \|\tilde{\lambda}^{(h)}_j\|(\Omega) = \|\tilde{\lambda}_h\|(\Omega), \end{aligned} \tag{3.38}$$

where, for $B \in \mathcal{B}(\Omega)$,

$$\tilde{\lambda}^{(h)}_j(B) := \bigg(\int_B \nabla u^{(h)}_j(x)\,\mathrm{d}x, \mathcal{L}^N(B)\bigg), \quad \tilde{\lambda}_h(B) := \big(Du_h(B), \mathcal{L}^N(B)\big).$$

Under hypotheses $(\mathcal{F}1)$–$(\mathcal{F}3)$, $(\mathcal{F}4)^\star$, $(\mathcal{F}5)$ (see also Remark 1.8 (ii)), it can be shown that for fixed $h \in \mathbb{N}$, Lemma 3.5 applies to $U := \Omega$ and $g(x, \xi) := f\big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \xi\big)$, which ensures the continuity of the functional F_{ε_h} with respect to the convergence (3.38), that is, $\lim_{j\to\infty} F_{\varepsilon_h}\Big(u^{(h)}_j\Big) = F_{\varepsilon_h}(u_h)$. Consequently,

$$\lim_{h\to\infty}\lim_{j\to\infty} F_{\varepsilon_h}\Big(u^{(h)}_j\Big) = \lim_{h\to\infty} F_{\varepsilon_h}(u_h). \tag{3.39}$$

Moreover, given $\varphi \in C_0\big(\Omega; C_\#\big(Y_1 \times Y_2; \mathbb{R}^{d\times N}\big)\big)$ we have

$$\begin{aligned} \lim_{h\to\infty}\lim_{j\to\infty} \int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big) : \nabla u^{(h)}_j(x)\,\mathrm{d}x &= \lim_{h\to\infty} \int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big) : \mathrm{d}Du_h(x) \\ &= \int_{\Omega\times Y_1\times Y_2} \varphi(x, y_1, y_2) : \mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(x, y_1, y_2), \end{aligned} \tag{3.40}$$

where we have used the weak-$\star$ convergence $\nabla u_j^{(h)}\mathcal{L}^N\lfloor\Omega \overset{\star}{\rightharpoonup}_j Du_h$ in $\mathcal{M}\big(\Omega;\mathbb{R}^{d\times N}\big)$, and the 3-scale convergence $Du_h \xrightharpoonup[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$. In addition, in view of (3.38),

$$\sup_{h\in\mathbb{N}}\sup_{j\in\mathbb{N}}\int_\Omega \big|\nabla u_j^{(h)}(x)\big|\,\mathrm{d}x < \infty. \tag{3.41}$$

Using the separability of $C_0\big(\Omega;C_\#\big(Y_1\times Y_2;\mathbb{R}^{d\times N}\big)\big)$ and a diagonalization argument, from (3.39), (3.40) and (3.41), we can find a sequence $\{j_h\}$ such that $j_h\to\infty$ as $h\to\infty$, and such that $w_h := u_{j_h}^{(h)}$ satisfies

$$w_h \in W^{1,1}\big(\Omega;\mathbb{R}^d\big),\quad \nabla w_h\mathcal{L}^N\lfloor\Omega \xrightharpoonup[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2},\quad \lim_{h\to\infty} F_{\varepsilon_h}(w_h) = \lim_{h\to\infty} F_{\varepsilon_h}(u_h). \tag{3.42}$$

Set $c := \sup_h \|\nabla w_h\|_{L^1(\Omega;\mathbb{R}^{d\times N})} < \infty$ and fix $\eta>0$. Then by Lemmas 3.2 and 3.4, and by Lemma 3.6 applied to $U := \Omega\times Y_1\times Y_2$ and $g(x,y_1,y_2,\xi) := f_\eta(y_1,y_2,\xi)$, where $f_\eta(y_1,y_2,\xi) := f(y_1,y_2,\xi)+\eta|\xi|$, we conclude that

$$\begin{aligned}\lim_{h\to\infty} F_{\varepsilon_h}(u_h)+\eta c &= \lim_{h\to\infty} F_{\varepsilon_h}(w_h)+\eta c \geqslant \liminf_{h\to\infty}\int_\Omega f_\eta\Big(\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)},\nabla w_h(x)\Big)\,\mathrm{d}x\\ &\geqslant \liminf_{h\to\infty}\int_{\Omega\times Y_1\times Y_2} f_\eta\big(y_1,y_2,\mathcal{A}_{\varrho_1(\varepsilon_h)}\big(\mathcal{T}_{\varrho_2(\varepsilon_h)}\big(\nabla w_h\big)\big)(x,y_1,y_2)\big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \geqslant F^{\mathrm{sc}}_\eta(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2),\end{aligned} \tag{3.43}$$

where

$$\begin{aligned}F^{\mathrm{sc}}_\eta(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) := &\int_{\Omega\times Y_1\times Y_2} f_\eta\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x,y_1,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\ &+\int_{\Omega\times Y_1\times Y_2} f_\eta^\infty\Big(y_1,y_2,\frac{\mathrm{d}\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x,y_1,y_2)\Big)\,\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x,y_1,y_2).\end{aligned} \tag{3.44}$$

Since $f_\eta^\infty(y_1,y_2,\xi) = f^\infty(y_1,y_2,\xi)+\eta|\xi|$, from (3.43) we deduce that

$$\lim_{h\to\infty} F_{\varepsilon_h}(u_h)+\eta c \geqslant G(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)+\eta\|\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(\Omega\times Y_1\times Y_2).$$

Finally, letting $\eta\to 0^+$ we obtain (3.37).

Step 2. We prove that

$$F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \leqslant G(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2). \tag{3.45}$$

Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ be a sequence of positive numbers converging to zero as $h\to\infty$, and let $\{u_j\}_{j\in\mathbb{N}}\subset C^\infty\big(\overline{\Omega};\mathbb{R}^d\big)$, $\big\{\psi_j^{(1)}\big\}_{j\in\mathbb{N}}\subset C_c^\infty\big(\Omega;C_\#^\infty\big(Y_1;\mathbb{R}^d\big)\big)$ and $\big\{\psi_j^{(2)}\big\}_{j\in\mathbb{N}}\subset C_c^\infty\big(\Omega;C_\#^\infty\big(Y_1\times Y_2;\mathbb{R}^d\big)\big)$ be the sequences given by Proposition 2.3. For each $h,j\in\mathbb{N}$ define $u_{h,j}\in C^\infty\big(\overline{\Omega};\mathbb{R}^d\big)$ by

$$u_{h,j}(x) := u_j(x)+\varrho_1(\varepsilon_h)\psi_j^{(1)}\Big(x,\frac{x}{\varrho_1(\varepsilon_h)}\Big)+\varrho_2(\varepsilon_h)\psi_j^{(2)}\Big(x,\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)}\Big). \tag{3.46}$$

Using (1.1), (2.3), and (2.1), in this order, we have that for all $\varphi\in C_0\big(\Omega;C_\#\big(Y_1\times Y_2;\mathbb{R}^{d\times N}\big)\big)$

$$\lim_{j\to\infty}\lim_{h\to\infty}\int_\Omega \varphi\Big(x,\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)}\Big):\nabla u_{h,j}(x)\,\mathrm{d}x = \int_{\Omega\times Y_1\times Y_2}\varphi(x,y_1,y_2):\mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(x,y_1,y_2). \tag{3.47}$$

Moreover,

$$\begin{aligned}F_{\varepsilon_h}(u_{h,j}) &= \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)},\nabla u_{h,j}(x)\Big)\,\mathrm{d}x\\ &= \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)},\nabla u_j(x)+\big(\nabla_{y_1}\psi_j^{(1)}\big)\Big(x,\frac{x}{\varrho_1(\varepsilon_h)}\Big)+\big(\nabla_{y_2}\psi_j^{(2)}\big)\Big(x,\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)}\Big)+\vartheta_{h,j}(x)\Big)\,\mathrm{d}x,\end{aligned}$$

where

$$\vartheta_{h,j}(x) := \varrho_1(\varepsilon_h)\big(\nabla_x\psi_j^{(1)}\big)\Big(x,\frac{x}{\varrho_1(\varepsilon_h)}\Big) + \varrho_2(\varepsilon_h)\big(\nabla_x\psi_j^{(2)}\big)\Big(x,\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)}\Big) + \frac{\varrho_2(\varepsilon_h)}{\varrho_1(\varepsilon_h)}\big(\nabla_{y_1}\psi_j^{(2)}\big)\Big(x,\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)}\Big).$$

We claim that if $K \subset \mathbb{R}^{d\times N}$ is a compact set then there exists a positive constant $C(K)$, depending only on K, such that for all $y_1, y_2 \in \mathbb{R}^N$, $\xi, \xi' \in K$,

$$|f(y_1,y_2,\xi) - f(y_1,y_2,\xi')| \leqslant C(K)|\xi - \xi'|. \tag{3.48}$$

In fact, the continuity of f (see Remark 1.8 (ii)) and $(\mathcal{F}1)$ ensure that there exists a positive constant $c(K)$ only depending on K such that for all $y_1, y_2 \in \mathbb{R}^N$, $\xi \in K$,

$$|f(y_1,y_2,\xi)| \leqslant c(K). \tag{3.49}$$

On the other hand, by $(\mathcal{F}2)$ (see, for example, [32, Thm. 4.36]) $f(y_1,y_2,\cdot)$ is locally Lipschitz with

$$\mathrm{Lip}(f(y_1,y_2,\cdot);B(0;r)) \leqslant \frac{\sqrt{d\times N}}{r}\sup\big\{|f(y_1,y_2,\xi) - f(y_1,y_2,\xi')| : \xi,\xi' \in B(0,2r)\big\}. \tag{3.50}$$

From (3.49) and (3.50), we deduce that (3.48) holds.

Taking into account (1.1), in view of (3.48) for each $j \in \mathbb{N}$ we can find a positive constant C_j independent of ε such that

$$F_{\varepsilon_h}(u_{h,j}) \leqslant \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)},\nabla u_j(x) + \big(\nabla_{y_1}\psi_j^{(1)}\big)\Big(x,\frac{x}{\varrho_1(\varepsilon_h)}\Big) + \big(\nabla_{y_2}\psi_j^{(2)}\big)\Big(x,\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)}\Big)\Big)\,\mathrm{d}x + C_j\int_\Omega|\vartheta_{h,j}(x)|\,\mathrm{d}x, \tag{3.51}$$

with, for all $j \in \mathbb{N}$,

$$\lim_{h\to\infty}\int_\Omega|\vartheta_{h,j}(x)|\,\mathrm{d}x = 0. \tag{3.52}$$

Furthermore, the function

$$g_j(x,y_1,y_2) := f\big(y_1,y_2,\nabla u_j(x) + \big(\nabla_{y_1}\psi_j^{(1)}\big)(x,y_1) + \big(\nabla_{y_2}\psi_j^{(2)}\big)(x,y_1,y_2)\big)$$

belongs to $C(\overline{\Omega};C_\#(Y_1\times Y_2))$, hence by (2.3)

$$\lim_{h\to\infty}\int_\Omega g_j\Big(x,\frac{x}{\varrho_1(\varepsilon_h)},\frac{x}{\varrho_2(\varepsilon_h)}\Big)\,\mathrm{d}x = \int_{\Omega\times Y_1\times Y_2} g_j(x,y_1,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2. \tag{3.53}$$

From (3.51)–(3.53) we conclude that

$$\begin{aligned}&\limsup_{j\to\infty}\limsup_{h\to\infty}F_{\varepsilon_h}(u_{h,j})\\ &\quad\leqslant \limsup_{j\to\infty}\int_{\Omega\times Y_1\times Y_2} f\big(y_1,y_2,\nabla u_j(x) + \big(\nabla_{y_1}\psi_j^{(1)}\big)(x,y_1) + \big(\nabla_{y_2}\psi_j^{(2)}\big)(x,y_1,y_2)\big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\ &\quad= G(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2),\end{aligned} \tag{3.54}$$

where in the last equality we invoked Lemma 3.5 applied to $U := \Omega\times Y_1\times Y_2$ and $g(x,y_1,y_2,\xi) := f(y_1,y_2,\xi)$, and also (2.2).

Using the separability of $C_0\big(\Omega; C_\#\big(Y_1\times Y_2;\mathbb{R}^{d\times N}\big)\big)$ and a diagonalization argument, from (3.47) and (3.54), and noticing that $\{u_{h,j}\}_{h,j\in\mathbb{N}}$ is a bounded sequence in $W^{1,1}\big(\Omega;\mathbb{R}^d\big)$, we can find subsequences $h_k \prec h$ and $j_k \prec j$ such that $u_{h_k,j_k}\in C^\infty\big(\overline{\Omega};\mathbb{R}^d\big)$ satisfies

$$\nabla u_{h_k,j_k}\mathcal{L}^N\lfloor_\Omega \xrightharpoonup[\varepsilon_{h_k}]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}, \quad \limsup_{k\to\infty} F_{\varepsilon_{h_k}}(u_{h_k,j_k}) \leqslant G(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2). \tag{3.55}$$

Finally, consider the sequence $\{w_h\}_{h\in\mathbb{N}}\subset BV\big(\Omega;\mathbb{R}^d\big)$ defined by

$$w_h := \begin{cases} u_{h_k,j_k} & \text{if } h = h_k \text{ for some } k\in\mathbb{N},\\ v_h & \text{if } h\neq h_k \text{ for all } k\in\mathbb{N},\end{cases}$$

where $\{v_h\}_{h\in\mathbb{N}}\subset BV\big(\Omega;\mathbb{R}^d\big)$ is a sequence such that $Dv_h \xrightharpoonup[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ (which exists by Proposition 2.1). Then $Dw_h \xrightharpoonup[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$, and so by (3.55)

$$F^{\text{sc}}(u,\boldsymbol{\mu}_1\boldsymbol{\mu}_2) \leqslant \liminf_{h\to\infty} F_{\varepsilon_h}(w_h) \leqslant \limsup_{k\to\infty} F_{\varepsilon_{h_k}}(u_{h_k,j_k}) \leqslant G(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2).$$

This concludes the proof of Theorem 3.1. □

The next theorem concerns the first equality in (1.14) relating the three-scale homogenized functional, F^{sc}, and the effective energy, F^{hom}.

Theorem 3.7. *Under the hypotheses of Theorem 3.1, assume further that the length scales ϱ_1, ϱ_2 satisfy the condition (1.3). Then, for all $u\in BV\big(\Omega;\mathbb{R}^d\big)$,*

$$F^{\text{hom}}(u) = \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\text{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2).$$

Proof. Let $u\in BV\big(\Omega;\mathbb{R}^d\big)$ be given. We will proceed in two steps.

Step 1. We prove that

$$F^{\text{hom}}(u) \geqslant \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\text{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2). \tag{3.56}$$

Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to zero as $h\to\infty$, and let $\{u_h\}_{h\in\mathbb{N}}\subset BV\big(\Omega;\mathbb{R}^d\big)$ be a sequence weakly-$\star$ converging to u in $BV\big(\Omega;\mathbb{R}^d\big)$ as $h\to\infty$. By $(\mathcal{F}3)$, $(\mathcal{F}4)^\star$ and (3.2), $\liminf_{h\to\infty} F_{\varepsilon_h}(u_h)\in\mathbb{R}$. Using Theorem 1.3, we can find a subsequence $h_k\prec h$ and measures $\bar{\boldsymbol{\mu}}_1\in\mathcal{M}_\star\big(\Omega;BV_\#\big(Y_1;\mathbb{R}^d\big)\big)$, $\bar{\boldsymbol{\mu}}_2\in\mathcal{M}_\star\big(\Omega\times Y_1;BV_\#\big(Y_2;\mathbb{R}^d\big)\big)$, such that

$$\lim_{k\to\infty} F_{\varepsilon_{h_k}}(u_{h_k}) = \liminf_{h\to\infty} F_{\varepsilon_h}(u_h), \quad u_{h_k} \xrightharpoonup[\varepsilon_{h_k}]{3\text{-}sc} u\mathcal{L}^N\lfloor_\Omega\otimes\mathcal{L}^{2N}_{y_1,y_2}, \quad Du_{h_k} \xrightharpoonup[\varepsilon_{h_k}]{3\text{-}sc} \lambda_{u,\bar{\boldsymbol{\mu}}_1,\bar{\boldsymbol{\mu}}_2}.$$

Hence, taking into account Theorem 3.1 (see (3.37)),

$$\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\text{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \leqslant F^{\text{sc}}(u,\bar{\boldsymbol{\mu}}_1,\bar{\boldsymbol{\mu}}_2) \leqslant \liminf_{h\to\infty} F_{\varepsilon_h}(u_h).$$

Taking the infimum over all sequences $\{u_h\}_{h\in\mathbb{N}}$ as above, we deduce that (3.56) holds.

Step 2. We show that

$$F^{\text{hom}}(u) \leqslant \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\text{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2). \tag{3.57}$$

Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to zero as $h\to\infty$, and take $\boldsymbol{\mu}_1\in\mathcal{M}_\star\big(\Omega;BV_\#\big(Y_1;\mathbb{R}^d\big)\big)$, $\boldsymbol{\mu}_2\in\mathcal{M}_\star\big(\Omega\times Y_1;BV_\#\big(Y_2;\mathbb{R}^d\big)\big)$. Reasoning as in the proof of (3.45), we can find a subsequence $h_k\prec h$ and a sequence $\{v_k\}_{k\in\mathbb{N}}\subset C^\infty\big(\overline{\Omega};\mathbb{R}^d\big)$ such that (see (3.46) and (3.55))

$$\lim_{k\to\infty}\int_\Omega |v_k-u|\,\mathrm{d}x = 0, \quad \nabla v_k\mathcal{L}^N\lfloor_\Omega \xrightharpoonup[\varepsilon_{h_k}]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}, \quad \limsup_{k\to\infty} F_{\varepsilon_{h_k}}(v_k) \leqslant F^{\text{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2).$$

Consequently, we also have that $Dv_k \overset{\star}{\rightharpoonup} Du$ weakly-$\star$ in $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ as $k\to\infty$. Finally, define

$$u_h := \begin{cases} v_k & \text{if } h = h_k \text{ for some } k\in\mathbb{N},\\ u & \text{otherwise.}\end{cases}$$

Then $u_h \overset{\star}{\rightharpoonup} u$ weakly-$\star$ in $BV(\Omega;\mathbb{R}^d)$ as $h\to\infty$, so that

$$F^{\mathrm{hom}}(u) \leqslant \liminf_{h\to\infty} F_{\varepsilon_h}(u_h) \leqslant \limsup_{k\to\infty} F_{\varepsilon_{h_k}}(v_k) \leqslant F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2),$$

from which we get (3.57) by taking the infimum over all $\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega; BV_\#(Y_1;\mathbb{R}^d))$ and $\boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1; BV_\#(Y_2;\mathbb{R}^d))$. □

Remark 3.8. *We observe that Theorems 3.1 and 3.7 hold if $(\mathcal{F}4)^\star$ is replaced by $(\mathcal{F}4)$ (see also Remark 1.8 (i)).*

In order to establish the integral representation for the effective energy F^{hom} stated in Theorem 1.6 we will need some auxiliary results. The first one is a measurable selection criterion (see [31, Lemma 3.10]; see also [18]).

Lemma 3.9. *Let Z be a separable metric space, let T be a measurable space and let $\Gamma: T\to 2^Z$ be a multifunction such that for every $t\in T$, $\Gamma(t)\subset Z$ is nonempty and open, and for every $z\in Z$, $\{t\in T:\ z\in\Gamma(t)\}$ is measurable. Then Γ admits a measurable selection, i.e., there exists a measurable function $\gamma: T\to Z$ such that for all $t\in T$, $\gamma(t)\in\Gamma(t)$.*

Next, we observe that the following result is a simple consequence of [35, Thm. 6] (see also [23] in the case where $d=1$ and g is coercive).

Lemma 3.10. *Assume that $\Omega\subset\mathbb{R}^N$ is an open and bounded set with $\partial\Omega$ Lipschitz, and let $g:\mathbb{R}^{d\times N}\to\mathbb{R}$ be a convex function such that for all $\xi\in\mathbb{R}^{d\times N}$ and for some constant $M>0$, $|g(\xi)|\leqslant M(1+|\xi|)$. Then, for all $\delta>0$ and for all $u\in BV(\Omega;\mathbb{R}^{d\times N})$, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}\subset W^{1,1}(\Omega;\mathbb{R}^{d\times N})$ such that $u_j \overset{\star}{\rightharpoonup} u$ weakly-$\star$ in $BV(\Omega;\mathbb{R}^{d\times N})$ as $j\to\infty$, and*

$$\int_\Omega g(\nabla u(x))\,\mathrm{d}x + \int_\Omega g^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x) + \delta \geqslant \lim_{j\to\infty}\int_\Omega g(\nabla u_j(x))\,\mathrm{d}x.$$

The next two lemmas provide sufficient conditions under which equality $(g_{\mathrm{hom}})^\infty = (g^\infty)_{\mathrm{hom}}$ holds.

Lemma 3.11. *Let $g:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}$ be a Borel function satisfying conditions $(\mathcal{F}1)$–$(\mathcal{F}4)$ and $(\mathcal{F}8)$. Then,*

$$(g_{\mathrm{hom}})^\infty = (g^\infty)_{\mathrm{hom}}. \tag{3.58}$$

Proof. We start by observing that, arguing as in [6, Thm. 4], we can prove a similar result to [13, Lemme 3.5]: If $h:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}$ is a Borel function satisfying hypotheses $(\mathcal{F}1)$–$(\mathcal{F}4)$, then for all $y_1\in\mathbb{R}^N$, $\xi^*\in\mathbb{R}^{d\times N}$ (see (1.7) and (1.9)),

$$(h_{\mathrm{hom}_2})^*(y_1,\xi^*) = \inf_{\Psi_2\in E_\#(Y_2;\mathbb{R}^{d\times N})}\int_{Y_2} h^*(y_1,y_2,\xi^*+\Psi_2(y_2))\,\mathrm{d}y_2 \tag{3.59}$$

where, for $k\in\mathbb{N}$,

$$E_\#(Y_k;\mathbb{R}^{d\times N}) := \Big\{\Psi=(\Psi_{ij})\in L^\infty_\#(Y_k;\mathbb{R}^{d\times N}):\ \int_{Y_k}\Psi(y_k)\,\mathrm{d}y_k = 0,\ \mathrm{div}\,\Psi_{i\cdot}=0 \text{ for all } i\in\{1,\cdots,d\}\Big\}.$$

Similarly, since $h_{\hom_2} : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is also a Borel function satisfying conditions $(\mathcal{F}1)$–$(\mathcal{F}4)$, we have that for all $\xi^* \in \mathbb{R}^{d\times N}$,

$$(h_{\hom})^*(\xi^*) = \inf_{\Psi_1 \in E_{\#}(Y_1;\mathbb{R}^{d\times N})} \int_{Y_1} (h_{\hom_2})^*(y_1, \xi^* + \Psi_1(y_1))\,\mathrm{d}y_1. \tag{3.60}$$

Moreover, for all $y_1, y_2 \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$ (see, for example, [39, Thm. 13.3, Lemma 7.42]),

$$h^\infty(y_1,y_2,\xi) = \sup_{(y_1,y_2,\xi^*)\in \operatorname{dom}_e h^*} \xi : \xi^*, \quad (h_{\hom_2})^\infty(y_1,\xi) = \sup_{(y_1,\xi^*)\in \operatorname{dom}_e (h_{\hom_2})^*} \xi : \xi^*. \tag{3.61}$$

If, in addition, h^* is bounded from above in $\operatorname{dom}_e h^*$, then we claim that for all $y_1, y_2 \in \mathbb{R}^N$, $\xi^* \in \mathbb{R}^{d\times N}$,

$$(h^\infty)^*(y_1,y_2,\xi^*) = \begin{cases} 0 & \text{if } (y_1,y_2,\xi^*) \in \operatorname{dom}_e h^*, \\ \infty & \text{otherwise.} \end{cases} \tag{3.62}$$

Indeed, under this additional hypothesis, we have that for each $y_1, y_2 \in \mathbb{R}^N$ the set $\{\xi^* \in \mathbb{R}^{d\times N} : (y_1,y_2,\xi^*) \in \operatorname{dom}_e h^*\}$ is convex and closed. Hence (see, for example, [28], [39]), the indicator function $\chi_{\operatorname{dom}_e h^*}$, that is, the function defined by

$$\chi_{\operatorname{dom}_e h^*}(y_1,y_2,\xi^*) := \begin{cases} 0 & \text{if } (y_1,y_2,\xi^*) \in \operatorname{dom}_e h^*, \\ \infty & \text{otherwise,} \end{cases}$$

coincides with its biconjugate function $(\chi_{\operatorname{dom}_e h^*})^{**}$. On the other hand, defining for each $t > 0$,

$$h_t(y_1,y_2,\xi) := \frac{h(y_1,y_2,t\xi) - h(y_1,y_2,0)}{t}, \quad y_1,y_2 \in \mathbb{R}^N, \xi \in \mathbb{R}^{d\times N},$$

due to the convexity hypothesis we have that for all $y_1, y_2 \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$, $t \in \mathbb{R}^+ \mapsto h_t(y_1,y_2,\xi)$ is nondecreasing and

$$\sup_{t>0} h_t(y_1,y_2,\xi) = \lim_{t\to\infty} h_t(y_1,y_2,\xi) = h^\infty(y_1,y_2,\xi).$$

Furthermore, it can be shown that for all $y_1, y_2 \in \mathbb{R}^N$, $\xi, \xi^* \in \mathbb{R}^{d\times N}$,

$$\begin{aligned} &\inf_{t>0} h_t^*(y_1,y_2,\xi^*) = \lim_{t\to\infty} h_t^*(y_1,y_2,\xi^*) = \chi_{\operatorname{dom}_e h^*}(y_1,y_2,\xi^*), \\ &\Big(\inf_{t>0} h_t^*\Big)^{**}(y_1,y_2,\xi) = \Big(\sup_{t>0} h_t\Big)^{*}(y_1,y_2,\xi) = (h^\infty)^*(y_1,y_2,\xi), \end{aligned}$$

so that (3.62) follows from the equality $(\chi_{\operatorname{dom}_e h^*})^{**} = \chi_{\operatorname{dom}_e h^*}$.

We now establish equality (3.58) in two steps. Notice that both $g_{\hom_2}$ and $g_{\hom}$, as well as their respective recession functions, are real-valued Borel functions satisfying similar conditions to $(\mathcal{F}1)$–$(\mathcal{F}4)$.

Step 1. We prove that $(g_{\hom_2})^\infty = (g^\infty)_{\hom_2}$.

Inequality $(g_{\hom_2})^\infty \leqslant (g^\infty)_{\hom_2}$ follows from the definitions of both functions and using Lebesgue Dominated Convergence Theorem taking into account $(\mathcal{F}3)$ and $(\mathcal{F}4)$.

We claim that to prove that $(g_{\hom_2})^\infty \geqslant (g^\infty)_{\hom_2}$, it suffices to show that

$$\operatorname{dom}_e (g_{\hom_2})^* \supset \operatorname{dom}_e \big((g^\infty)_{\hom_2}\big)^*. \tag{3.63}$$

In fact, if (3.63) holds then by (3.61) we have that

$$(g_{\hom_2})^\infty \geqslant \big((g^\infty)_{\hom_2}\big)^\infty. \tag{3.64}$$

Since $(g^\infty)_{\hom_2}$ is positively 1-homogeneous in the last variable, we have that $\big((g^\infty)_{\hom_2}\big)^\infty = (g^\infty)_{\hom_2}$, which together with (3.64) yields $(g_{\hom_2})^\infty \geqslant (g^\infty)_{\hom_2}$.

We now prove (3.63). Let $(y_1, \xi^*) \in \mathrm{dom}_e\big((g^\infty)_{\hom_2}\big)^*$. Then, by (3.59) (with g replaced by g^∞), there exists $\Psi_2 \in E_\#\big(Y_2; \mathbb{R}^{d\times N}\big)$ such that

$$\int_{Y_2} (g^\infty)^*(y_1, y_2, \xi^* + \Psi_2(y_2))\,\mathrm{d}y_2 < \infty,$$

and so (3.62) ensures that for a.e. $y_2 \in Y_2$ we have $(y_1, y_2, \xi^* + \Psi_2(y_2)) \in \mathrm{dom}_e g^*$. From (3.59) and ($\mathcal{F}8$) we conclude that

$$(g_{\hom_2})^*(y_1, \xi^*) \leqslant \int_{Y_2} g^*(y_1, y_2, \xi^* + \Psi_2(y_2))\,\mathrm{d}y_2 \leqslant \mathcal{C} < \infty.$$

Thus, $(y_1, \xi^*) \in \mathrm{dom}_e(g_{\hom_2})^*$, which proves (3.63). So, $(g_{\hom_2})^\infty = (g^\infty)_{\hom_2}$ and, consequently,

$$\Big((g_{\hom_2})^\infty\Big)_{\hom_1} = \Big((g^\infty)_{\hom_2}\Big)_{\hom_1} = (g^\infty)_{\hom}, \tag{3.65}$$

where in the last equality we used definition (1.10).

Step 2. We prove that $(g_{\hom})^\infty = (g^\infty)_{\hom}$.

It suffices to observe that ($\mathcal{F}3$), ($\mathcal{F}8$) and (3.59) imply that $(g_{\hom_2})^*$ is also bounded on its effective domain. Hence, reasoning as before and in view of (3.60),

$$\Big((g_{\hom_2})^\infty\Big)_{\hom_1} = \Big((g_{\hom_2})_{\hom_1}\Big)^\infty = (g_{\hom})^\infty. \tag{3.66}$$

Thus, from (3.65)–(3.66) we conclude that $(g_{\hom})^\infty = (g^\infty)_{\hom}$. □

Lemma 3.12. *Let $g : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function satisfying conditions ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$)' and ($\mathcal{F}7$). Then $(g_{\hom})^\infty = (g^\infty)_{\hom}$.*

Proof. Note that ($\mathcal{F}7$) is equivalent to requiring that there exist constants $C, L > 0$ and $\alpha \in (0,1)$ such that given y_1, $y_2 \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{d\times N}$ arbitrarily, then for all $t \in \mathbb{R}$ such that $t|\xi| > L$,

$$\left| g^\infty(y_1, y_2, \xi) - \frac{g(y_1, y_2, t\xi)}{t} \right| \leqslant C\frac{|\xi|^{1-\alpha}}{t^\alpha}. \tag{3.67}$$

We now prove that

$$(g_{\hom_2})^\infty = (g^\infty)_{\hom_2}. \tag{3.68}$$

Inequality $(g_{\hom_2})^\infty \leqslant (g^\infty)_{\hom_2}$ follows from the definitions of both functions and Fatou's Lemma taking into account ($\mathcal{F}3$) and ($\mathcal{F}4$)'.

Conversely, fix $y_1 \in \mathbb{R}$, $\xi \in \mathbb{R}^{d\times N}$. By definition of infimum, for each $t \geqslant 1$ we can find $\psi_t \in W^{1,1}_\#\big(Y_2; \mathbb{R}^d\big)$ such that

$$\int_{Y_2} \frac{g(y_1, y_2, t\xi + t\nabla\psi_t(y_2))}{t}\,\mathrm{d}y_2 \leqslant \frac{g_{\hom_2}(y_1, t\xi)}{t} + \frac{1}{t}. \tag{3.69}$$

In particular, (3.69), together with ($\mathcal{F}3$) and ($\mathcal{F}4$)', yields

$$\int_{Y_2} |\xi + \nabla\psi_t(y_2)|\,\mathrm{d}y_2 \leqslant \bar{C}(1 + |\xi|), \tag{3.70}$$

for some positive constant $\bar{C}$ independent of t.

By definition of $(g^\infty)_{\hom_2}$,

$$
\begin{aligned}
(g^\infty)_{\hom_2}(y_1,\xi) &\leqslant \int_{Y_2} g^\infty(y_1,y_2,\xi+\nabla\psi_t(y_2))\,\mathrm{d}y_2\\
&\leqslant \frac{CL}{t} + \int_{Y_2\cap\{y_2\colon t|\xi+\nabla\psi_t(y_2)|>L\}} g^\infty(y_1,y_2,\xi+\nabla\psi_t(y_2))\,\mathrm{d}y_2,
\end{aligned}
$$

where we used the fact that in view of $(\mathcal{F}3)$, $g^\infty(y_1,y_2,\xi)\leqslant C|\xi|$. Invoking, in addition, (3.67), $(\mathcal{F}4)$' and (3.69), in this order, we have

$$
\begin{aligned}
(g^\infty)_{\hom_2}(y_1,\xi) &\leqslant \frac{CL}{t} + \int_{Y_2\cap\{y_2\colon t|\xi+\nabla\psi_t(y_2)|>L\}} \frac{g(y_1,y_2,t\xi+t\nabla\psi_t(y_2))}{t} + C\frac{|\xi+\nabla\psi_t(y_2)|^{1-\alpha}}{t^\alpha}\,\mathrm{d}y_2\\
&\leqslant \frac{C(L+1)}{t} + \int_{Y_2} \frac{g(y_1,y_2,t\xi+t\nabla\psi_t(y_2))}{t}\,\mathrm{d}y_2 + \frac{C}{t^\alpha}\int_{Y_2}|\xi+\nabla\psi_t(y_2)|^{1-\alpha}\,\mathrm{d}y_2\\
&\leqslant \frac{C(L+1)+1}{t} + \frac{g_{\hom_2}(y_1,t\xi)}{t} + \frac{C}{t^\alpha}\big(\bar{C}(1+|\xi|)\big)^{1-\alpha},
\end{aligned}
\tag{3.71}
$$

where in the last estimate we also used Hölder's Inequality together with (3.70). Letting $t\to\infty$, we conclude that $(g^\infty)_{\hom_2}\leqslant(g_{\hom_2})^\infty$. Thus, (3.68) holds. Consequently,

$$
\Big((g_{\hom_2})^\infty\Big)_{\hom_1} = \Big((g^\infty)_{\hom_2}\Big)_{\hom_1} = (g^\infty)_{\hom}.
$$

Next we show that

$$
\Big((g_{\hom_2})^\infty\Big)_{\hom_1} = \big((g_{\hom_2})_{\hom_1}\big)^\infty,
\tag{3.72}
$$

which will finish the proof since, by definition, $\big((g_{\hom_2})_{\hom_1}\big)^\infty = (g_{\hom})^\infty$.

In view of the hypotheses on g and using definition (1.9), it can be shown that $g_{\hom_2}:\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}$ is a Borel function satisfying conditions $(\mathcal{F}1)$, $(\mathcal{F}3)$ and $(\mathcal{F}4)$'. If we prove that $g_{\hom_2}$ also satisfies $(\mathcal{F}7)$ then, reasoning as in the proof of (3.68), we deduce that (3.72) holds.

Let $C,L>0$ and $\alpha\in(0,1)$ be given by $(\mathcal{F}7)$ for g. Fix $y_1\in\mathbb{R}^N$ and $\xi\in\mathbb{R}^{d\times N}$ such that $|\xi|=1$. Let $t\geqslant\tilde{L}:=\max\{1,L\}$. Using (3.68) and (3.71), we have

$$
(g_{\hom_2})^\infty(y_1,\xi) - \frac{g_{\hom_2}(y_1,t\xi)}{t} = (g^\infty)_{\hom_2}(y_1,\xi) - \frac{g_{\hom_2}(y_1,t\xi)}{t} \leqslant \frac{C(L+1)+1}{t} + \frac{C}{t^\alpha}\big(2\bar{C}\big)^{1-\alpha} \leqslant \frac{C_1}{t^\alpha},
\tag{3.73}
$$

where C_1 is a positive constant independent of t.

Conversely, for each $0<\delta<1$ we can find $\psi_\delta\in W^{1,1}_\#\big(Y_2,\mathbb{R}^d\big)$ such that

$$
\int_{Y_2} g^\infty(y_1,y_2,\xi+\nabla\psi_\delta(y_2))\,\mathrm{d}y_2 \leqslant (g^\infty)_{\hom_2}(y_1,\xi)+\delta,
\tag{3.74}
$$

so that, in view of $(\mathcal{F}3)$ and $(\mathcal{F}4)$',

$$
\frac{1}{C}\int_{Y_2}|\xi+\nabla\psi_\delta(y_2)|\,\mathrm{d}y_2 \leqslant C|\xi|+\delta < C+1.
\tag{3.75}
$$

From (3.68), (3.74) and (3.67), and taking into account that $g^\infty\geqslant 0$, we conclude that

$$
\begin{aligned}
&\frac{g_{\hom_2}(y_1,t\xi)}{t} - (g_{\hom_2})^\infty(y_1,\xi)\\
&\leqslant \int_{Y_2}\frac{g(y_1,y_2,t\xi+t\nabla\psi_\delta(y_2))}{t} - g^\infty(y_1,y_2,\xi+\nabla\psi_\delta(y_2))\,\mathrm{d}y_2+\delta\\
&\leqslant C\int_{Y_2}\frac{|\xi+\nabla\psi_\delta(y_2)|^{1-\alpha}}{t^\alpha}\,\mathrm{d}y_2 + \int_{Y_2\cap\{y_2\colon t|\xi+\nabla\psi_\delta(y_2)|\leqslant L\}}\frac{g(y_1,y_2,t\xi+t\nabla\psi_\delta(y_2))}{t}\,\mathrm{d}y_2+\delta\\
&\leqslant \frac{C(C^2+C)^{1-\alpha}}{t^\alpha} + \frac{C(1+L)}{t}+\delta,
\end{aligned}
\tag{3.76}
$$

where in the last inequality we also used Hölder's Inequality together with (3.75), and ($\mathcal{F}3$). Letting $\delta \to 0^+$ in (3.76), using the fact that $t \geqslant t^\alpha$ whenever $t \geqslant 1$ together with (3.73), we deduce that g_{hom_2} satisfies ($\mathcal{F}7$). □

We now prove Proposition 1.4.

Proof of Proposition 1.4. Without loss of generality we may assume that the parameter $\eta > 0$ takes values on a sequence of positive numbers converging to zero.

(i) We start by observing that for fixed $y_1, y_2 \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$, the sequences $\{f_\eta(y_1,y_2,\xi)\}_{\eta>0}$, $\{(f_\eta)^{**}(y_1,y_2,\xi)\}_{\eta>0}$ and $\{((f_\eta)^{**})^\infty(y_1,y_2,\xi)\}_{\eta>0}$ are decreasing (as $\eta\to 0^+$), so that the respective limits as $\eta \to 0^+$ exist and are given by the infimum in $\eta > 0$.

Recalling definition (1.8) and in view of ($\mathcal{F}3$) and ($\mathcal{F}4$), we have that the biconjugate function f^{**} of f is such that for all $y_1, y_2 \in \mathbb{R}^N$, $f^{**}(y_1,y_2,\cdot)$ is a convex function which coincides with the convex envelop $\mathcal{C}f(y_1,y_2,\cdot)$ of $f(y_1,y_2,\cdot)$ (see, for example, [32, Thm. 4.92]). Precisely, for all $(y_1,y_2,\xi) \in \mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}$,

$$f^{**}(y_1,y_2,\xi) = \mathcal{C}f(y_1,y_2,\xi) := \sup\big\{g(\xi)\colon\ g:\mathbb{R}^{d\times N}\to\mathbb{R} \text{ convex},\ g(\cdot)\leqslant f(y_1,y_2,\cdot)\big\}. \tag{3.77}$$

Note that the same holds true for $(f_\eta)^{**}$. Consequently, $((f_\eta)^{**})^\infty$ is a convex function, since the recession function of a convex function is a convex function. Moreover, for all $\eta > 0$, we have that

$$f^{**} \leqslant (f_\eta)^{**} \leqslant f_\eta, \tag{3.78}$$

and so, using the fact that the pointwise limit of a sequence of convex functions is a convex function, passing (3.78) to the limit as $\eta \to 0^+$ we get

$$\lim_{\eta\to 0^+} (f_\eta)^{**}(y_1,y_2,\xi) = f^{**}(y_1,y_2,\xi). \tag{3.79}$$

In view of (3.78), $(f^{**})^\infty \leqslant ((f_\eta)^{**})^\infty \leqslant (f_\eta)^\infty$; thus, letting $\eta\to 0^+$ and observing that $(f_\eta)^\infty(y_1,y_2,\xi) = f^\infty(y_1,y_2,\xi) + \eta|\xi|$, we have

$$(f^{**})^\infty(y_1,y_2,\xi) \leqslant ((f_{0^+})^{**})^\infty(y_1,y_2,\xi) \leqslant (f^\infty)^{**}(y_1,y_2,\xi), \tag{3.80}$$

where we also used the fact that both functions $(f^{**})^\infty$ and $((f_{0^+})^{**})^\infty$ are convex in the last variable, since the recession function of a convex function is also a convex function. We further observe that $((f_{0^+})^{**})^\infty$ is positively 1-homogeneous in the last variable because it is the pointwise limit of a sequence of positively 1-homogeneous functions in the last variable.

(i)–a) If, in addition, f also satisfies ($\mathcal{F}2$), then $(f^{**})^\infty = f^\infty = (f^\infty)^{**}$, which, together with (3.80), implies that $((f_{0^+})^{**})^\infty \equiv f^\infty$.

(i)–b) Assume that $d = 1$ and that, in addition, f also satisfies ($\mathcal{F}7$).

In the scalar case $d = 1$ the notions of convexity and quasiconvexity agree (see, for example, [22, Thms. 5.3, 6.9]), therefore f^{**} is alternatively given by

$$f^{**}(y_1,y_2,\xi) = \inf\left\{\int_Y f(y_1,y_2,\xi+\nabla\varphi(y))\,\mathrm{d}y\colon\ \varphi\in W_0^{1,\infty}(Y)\right\} \tag{3.81}$$

for $(y_1,y_2,\xi)\in\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N$.

Since f_η is a Borel function satisfying conditions ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$)' and ($\mathcal{F}7$), using (3.81) and arguing as in the proof of Lemma 3.12, it can be shown that $(f_\eta)^{**}$ also satisfies ($\mathcal{F}7$) and that $((f_\eta)^{**})^\infty = ((f_\eta)^\infty)^{**}$. Consequently,

$$((f_{0^+})^{**})^\infty(y_1,y_2,\xi) = \lim_{\eta\to 0^+}((f_\eta)^{**})^\infty(y_1,y_2,\xi) = \lim_{\eta\to 0^+}((f_\eta)^\infty)^{**}(y_1,y_2,\xi) = (f^\infty)^{**}(y_1,y_2,\xi), \tag{3.82}$$

where the last equality may be proved in a similar way as (3.79) (with f replaced by f^∞).

(ii) Just as (i) above, we can be shown that the limit (1.12) exists and defines a positively 1-homogeneous convex function $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty : \mathbb{R}^N \to \mathbb{R}$.

By (1.9), ($\mathcal{F}3$) and ($\mathcal{F}4$), there exists a constant $M > 0$ such that for all y_1, $y_2 \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$,

$$|f(y_1,y_2,\xi)| \leqslant M(1+|\xi|), \quad |f_{\mathrm{hom}_2}(y_1,\xi)| \leqslant M(1+|\xi|), \quad |f_{\mathrm{hom}}(\xi)| \leqslant M(1+|\xi|). \tag{3.83}$$

Using in addition (3.79), Lebesgue Dominated Convergence Theorem yields

$$\lim_{\eta\to 0^+} ((f_\eta)^{**})_{\mathrm{hom}_2}(y_1,\xi) \leqslant (f^{**})_{\mathrm{hom}_2},$$

which, together with inequality $((f_\eta)^{**})_{\mathrm{hom}_2} \geqslant (f^{**})_{\mathrm{hom}_2}$, implies that

$$\lim_{\eta\to 0^+} ((f_\eta)^{**})_{\mathrm{hom}_2}(y_1,\xi) = (f^{**})_{\mathrm{hom}_2}(y_1,\xi).$$

Similar arguments ensure that

$$\lim_{\eta\to 0^+} ((f_\eta)^{**})_{\mathrm{hom}}(\xi) = \lim_{\eta\to 0^+} (((f_\eta)^{**})_{\mathrm{hom}_2})_{\mathrm{hom}_1}(\xi) = ((f^{**})_{\mathrm{hom}_2})_{\mathrm{hom}_1}(\xi) = (f^{**})_{\mathrm{hom}}(\xi), \tag{3.84}$$

and that

$$\lim_{\eta\to 0^+} \big(((f_\eta)^{**})^\infty\big)_{\mathrm{hom}}(\xi) = \big(((f_{0^+})^{**})^\infty\big)_{\mathrm{hom}}(\xi) \leqslant \big((f^\infty)^{**}\big)_{\mathrm{hom}}(\xi), \tag{3.85}$$

with $((f_{0^+})^{**})^\infty$ the function defined by (1.11), where in the last inequality we used (3.80).

Using the fact that if g is a function satisfying ($\mathcal{F}3$) and ($\mathcal{F}4$) then $(g_{\mathrm{hom}})^\infty \leqslant (g^\infty)_{\mathrm{hom}}$, passing to the limit as $\eta \to 0^+$ the chain of inequalities

$$\big((f^{**})_{\mathrm{hom}}\big)^\infty \leqslant \big(((f_\eta)^{**})_{\mathrm{hom}}\big)^\infty \leqslant \big(((f_\eta)^{**})^\infty\big)_{\mathrm{hom}},$$

from (3.85) we obtain

$$\big((f^{**})_{\mathrm{hom}}\big)^\infty(\xi) \leqslant \big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty(\xi) \leqslant \big(((f_{0^+})^{**})^\infty\big)_{\mathrm{hom}}(\xi) \leqslant \big((f^\infty)^{**}\big)_{\mathrm{hom}}(\xi). \tag{3.86}$$

(ii)–a) Assume that, in addition, f also satisfies ($\mathcal{F}2$) and ($\mathcal{F}8$).

In this case, from (3.86) we get

$$(f_{\mathrm{hom}})^\infty \leqslant (f_{0^+,\mathrm{hom}})^\infty \leqslant (f^\infty)_{\mathrm{hom}}, \tag{3.87}$$

where $(f_{0^+,\mathrm{hom}})^\infty := \big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty = \lim_{\eta\to 0^+} \big((f_\eta)_{\mathrm{hom}}\big)^\infty(\xi)$, since $(f_\eta)^{**} = f_\eta$. To conclude that $(f_{0^+,\mathrm{hom}})^\infty \equiv (f_{\mathrm{hom}})^\infty = (f^\infty)_{\mathrm{hom}}$ it suffices to apply Lemma 3.11 to f, taking into account (3.87).

(ii)–b) Assume that, in addition, f also satisfies ($\mathcal{F}2$) and ($\mathcal{F}7$).

As before, using (1.9), equality $(f_\eta)^\infty(y_1,y_2,\xi) = f^\infty(y_1,y_2,\xi)+\eta|\xi|$, and Lebesgue Dominated Convergence Theorem together with (3.83), we obtain

$$\lim_{\eta\to 0^+} ((f_\eta)^\infty)_{\mathrm{hom}}(\xi) = (f^\infty)_{\mathrm{hom}}(\xi). \tag{3.88}$$

By Lemma 3.12 applied to f_η, we conclude that for all $\eta > 0$, $((f_\eta)_{\mathrm{hom}})^\infty = ((f_\eta)^\infty)_{\mathrm{hom}}$, which, together with (3.88), yields $(f_{0^+,\mathrm{hom}})^\infty \equiv (f^\infty)_{\mathrm{hom}}$.

(ii)–c) Assume that $d = 1$ and that, in addition, f also satisfies ($\mathcal{F}7$) (with $d = 1$).

As we observed in (i)–b), $(f_\eta)^{**}$ is a Borel function satisfying conditions ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$)' and ($\mathcal{F}7$). Applying Lemma 3.12 to $(f_\eta)^{**}$, using the first equality in (3.85) and by (3.82),

$$\begin{aligned}\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty(\xi) &= \lim_{\eta\to0^+}\big(((f_\eta)^{**})_{\mathrm{hom}}\big)^\infty(\xi) = \lim_{\eta\to0^+}\big(((f_\eta)^{**})^\infty\big)_{\mathrm{hom}}(\xi)\\ &= \big(((f_{0^+})^{**})^\infty\big)_{\mathrm{hom}}(\xi) = \big((f^\infty)^{**}\big)_{\mathrm{hom}}(\xi).\end{aligned}$$

This concludes the proof of Proposition 1.4. □

We finally prove Theorem 1.6.

Proof of Theorem 1.6. By Theorem 3.1 and Remark 3.8, we have that (1.13) holds.

We observe that in view of ($\mathcal{F}1$)–($\mathcal{F}4$), we have that both f_{hom_2} and f_{hom} are real-valued Borel functions, satisfying ($\mathcal{F}1$)–($\mathcal{F}4$), and we can find a constant $M>0$ such that for all y_1, $y_2\in\mathbb{R}^N$, $\xi\in\mathbb{R}^{d\times N}$,

$$|f(y_1,y_2,\xi)|\leqslant M(1+|\xi|),\quad |f_{\mathrm{hom}_2}(y_1,\xi)|\leqslant M(1+|\xi|),\quad |f_{\mathrm{hom}}(\xi)|\leqslant M(1+|\xi|). \tag{3.89}$$

Moreover, since ($\mathcal{F}4$) holds for f_{hom},

$$\liminf_{|\xi|\to\infty}\frac{f_{\mathrm{hom}}(\xi)}{|\xi|}\geqslant 0. \tag{3.90}$$

The first equality in (1.14) is given by Theorem 3.7 (see also Remark 3.8). To prove the second equality in (1.14) we will proceed in several steps.

Step 1. We show that for all $u\in BV\big(\Omega;\mathbb{R}^d\big)$,

$$\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\geqslant\int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x+\int_\Omega(f_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^su}{\mathrm{d}\|D^su\|}(x)\Big)\,\mathrm{d}\|D^su\|(x). \tag{3.91}$$

Fix $(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\in BV(\Omega;\mathbb{R}^d)\times\mathcal{M}_\star\big(\Omega;BV_\#\big(Y_1;\mathbb{R}^d\big)\big)\times\mathcal{M}_\star\big(\Omega\times Y_1;BV_\#\big(Y_2;\mathbb{R}^d\big)\big)$, and let $\{u_j\}_{j\in\mathbb{N}}\subset C^\infty\big(\overline{\Omega};\mathbb{R}^d\big)$, $\big\{\psi_j^{(1)}\big\}_{j\in\mathbb{N}}\subset C_c^\infty\big(\Omega;C_\#^\infty\big(Y_1;\mathbb{R}^d\big)\big)$ and $\big\{\psi_j^{(2)}\big\}_{j\in\mathbb{N}}\subset C_c^\infty\big(\Omega;C_\#^\infty\big(Y_1\times Y_2;\mathbb{R}^d\big)\big)$ be sequences given by Proposition 2.3.

By (1.13), applying Lemma 3.5 to $U:=\Omega\times Y_1\times Y_2$ and $g(x,y_1,y_2,\xi):=f(y_1,y_2,\xi)$ (see also Remark 1.8 (ii)), and using the definitions of f_{hom_2} and f_{hom} together with Fubini's Theorem, we conclude that

$$\begin{aligned}F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) &= \int_{\Omega\times Y_1\times Y_2} f\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x,y_1,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\ &\quad+\int_{\Omega\times Y_1\times Y_2} f^\infty\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x,y_1,y_2)\Big)\,\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x,y_1,y_2)\\ &=\lim_{j\to\infty}\int_{\Omega\times Y_1\times Y_2} f\big(y_1,y_2,\nabla u_j(x)+\big(\nabla_{y_1}\psi_j^{(1)}\big)(x,y_1)+\big(\nabla_{y_2}\psi_j^{(2)}\big)(x,y_1,y_2)\big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\ &\geqslant\liminf_{j\to\infty}\int_{\Omega\times Y_1} f_{\mathrm{hom}_2}\big(y_1,\nabla u_j(x)+\big(\nabla_{y_1}\psi_j^{(1)}\big)(x,y_1)\big)\,\mathrm{d}x\mathrm{d}y_1\\ &\geqslant\liminf_{j\to\infty}\int_\Omega f_{\mathrm{hom}}(\nabla u_j(x))\,\mathrm{d}x\\ &\geqslant\int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x+\int_\Omega(f_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^su}{\mathrm{d}\|D^su\|}(x)\Big)\,\mathrm{d}\|D^su\|(x),\end{aligned}$$

where in the last inequality we have used [32, Thm. 5.21] (see also Remark 1.8 (i)) taking into account that $\nabla u_j\mathcal{L}^N\lfloor_\Omega \overset{\star}{\rightharpoonup} Du$ weakly-$\star$ in $\mathcal{M}\big(\Omega;\mathbb{R}^{d\times N}\big)$ as $j\to\infty$, and that f_{hom} is a real-valued convex function satisfying (3.90). Taking the infimum over all $\boldsymbol{\mu}_1\in\mathcal{M}_\star\big(\Omega;BV_\#\big(Y_1;\mathbb{R}^d\big)\big)$ and $\boldsymbol{\mu}_2\in\mathcal{M}_\star\big(\Omega\times Y_1;BV_\#\big(Y_2;\mathbb{R}^d\big)\big)$, we obtain (3.91).

Step 2. We prove that for all $u \in W^{1,1}(\Omega;\mathbb{R}^d)$,

$$\inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d)) \\ \boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \leqslant \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x. \tag{3.92}$$

Fix $\eta > 0$, and let $0 < \tau < \eta$ be such that for all measurable sets $D \subset \Omega$ with $\mathcal{L}^N(D) \leqslant \tau$,

$$\int_D (1 + |\nabla u(x)|)\,\mathrm{d}x \leqslant \eta. \tag{3.93}$$

In view of (3.83), without loss of generality we may assume that for all $x \in \Omega$,

$$f_{\mathrm{hom}}(\nabla u(x)) \in \mathbb{R}. \tag{3.94}$$

Fix $0 < \delta < \tau$, and consider the multifunction $\Gamma_1^\delta : \Omega \to 2^{W^{1,1}_\#(Y_1;\mathbb{R}^d)}$ defined, for each $x \in \Omega$, by

$$\Gamma_1^\delta(x) := \left\{ \psi_1 \in W^{1,1}_\#(Y_1;\mathbb{R}^d) \colon \int_{Y_1} f_{\mathrm{hom}_2}(y_1, \nabla u(x) + \nabla\psi_1(y_1))\,\mathrm{d}y_1 < f_{\mathrm{hom}}(\nabla u(x)) + \delta \right\}.$$

By (3.94), for all $x \in \Omega$ one has $\Gamma_1^\delta(x) \neq \emptyset$. Moreover, if $\{\psi_j\}_{j\in\mathbb{N}} \subset W^{1,1}_\#(Y_1;\mathbb{R}^d)\backslash\Gamma_1^\delta(x)$ is a sequence converging in $W^{1,1}_\#(Y_1;\mathbb{R}^d)$ to some ψ, then, taking into account (3.83) and the continuity of $f_{\mathrm{hom}_2}(y_1,\cdot)$, by Lebesgue Dominated Convergence Theorem we deduce that $\psi \in W^{1,1}_\#(Y_1;\mathbb{R}^d)\backslash\Gamma_1^\delta(x)$. Thus, $\Gamma_1^\delta(x)$ is an open subset of $W^{1,1}_\#(Y_1;\mathbb{R}^d)$. Furthermore, given $\psi_1 \in W^{1,1}_\#(Y_1;\mathbb{R}^d)$, the measurability of the function

$$x \mapsto \int_{Y_1} f_{\mathrm{hom}_2}(y_1, \nabla u(x) + \nabla\psi_1(y_1))\,\mathrm{d}y_1 - f_{\mathrm{hom}}(\nabla u(x)) - \delta$$

ensures the measurability of the set $\{x \in \Omega \colon \ \psi_1 \in \Gamma_1^\delta(x)\}$. Thus, by Lemma 3.9 we can find a measurable selection $\bar\psi_1 : \Omega \to W^{1,1}_\#(Y_1;\mathbb{R}^d)$ of Γ_1^δ. Moreover, by Lusin's Theorem, $\bar\psi_1 \in L^1(\Omega_\delta;W^{1,1}_\#(Y_1;\mathbb{R}^d))$ for a suitable measurable set $\Omega_\delta \subset \Omega$ such that $\mathcal{L}^N(\Omega\backslash\Omega_\delta) \leqslant \delta$. Since for a.e. $x \in \Omega_\delta$ one has $\bar\psi_1(x) \in \Gamma_1^\delta(x)$, in view of (3.83) and (3.93) we obtain

$$\int_{\Omega_\delta\times Y_1} f_{\mathrm{hom}_2}(y_1, \nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1))\,\mathrm{d}x\mathrm{d}y_1 \leqslant \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x + M\eta + \eta\mathcal{L}^N(\Omega), \tag{3.95}$$

where we also used the fact that $0 < \delta < \tau < \eta$.

Similarly, let $0 < \bar\tau < \delta$ be such that for all measurable sets $E \subset \Omega_\delta \times Y$ with $\mathcal{L}^{2N}(E) \leqslant \bar\tau$,

$$\int_E (1 + |\nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1)|)\,\mathrm{d}x\mathrm{d}y_1 \leqslant \eta. \tag{3.96}$$

As before, we may assume without loss of generality that for all $(x,y_1) \in \Omega_\delta \times Y_1$ we have $f_{\mathrm{hom}_2}(y_1, \nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1)) \in \mathbb{R}$. Moreover, fixed $0 < \gamma < \bar\tau$, the multifunction $\Gamma_2^\gamma : \Omega_\delta \times Y_1 \to 2^{W^{1,1}_\#(Y_2;\mathbb{R}^d)}$ defined, for each $(x,y_1) \in \Omega_\delta \times Y_1$, by

$$\begin{aligned}\Gamma_2^\gamma(x,y_1) := \Big\{ \psi_2 \in W^{1,1}_\#(Y_2;\mathbb{R}^d) \colon \int_{Y_2} & f(y_1,y_2,\nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1) + \nabla\psi_2(y_2))\,\mathrm{d}y_2 \\ & < f_{\mathrm{hom}_2}(y_1, \nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1)) + \gamma \Big\},\end{aligned}$$

is such that for all $(x,y_1) \in \Omega_\delta \times Y_1$, $\Gamma_2^\gamma(x,y_1)$ is a nonempty and open subset of $W^{1,1}_\#(Y_2;\mathbb{R}^d)$, and for all $\psi_2 \in W^{1,1}_\#(Y_2;\mathbb{R}^d)$, the set $\{(x,y_1) \in \Omega_\delta \times Y_1 \colon \ \psi_2 \in \Gamma_2^\gamma(x,y_1)\}$ is measurable. Hence, by Lemma 3.9

we can find a measurable selection $\bar{\psi}_2 : \Omega_\delta \times Y_1 \to W^{1,1}_{\#}\big(Y_2;\mathbb{R}^d\big)$ of Γ_2^γ. Moreover, by Lusin's Theorem, $\bar{\psi}_2 \in L^1\big(E_\gamma; W^{1,1}_{\#}\big(Y_2;\mathbb{R}^d\big)\big)$ for a suitable measurable set $E_\gamma \subset \Omega_\delta \times Y_1$ such that $\mathcal{L}^N(\Omega_\delta \times Y_1 \backslash E_\gamma) \leqslant \gamma$. Since for a.e. $(x,y_1) \in E_\gamma$ one has $\bar{\psi}_2(x,y_1) \in \Gamma_2^\gamma(x,y_1)$, in view of (3.83) and (3.96) we get

$$\begin{aligned}&\int_{E_\gamma \times Y_2} f(y_1,y_2,\nabla u(x) + \nabla_{y_1}\bar{\psi}_1(x,y_1) + \nabla_{y_2}\bar{\psi}_2(x,y_1y_2))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad\leqslant \int_{\Omega_\delta\times Y_1} f_{\mathrm{hom}_2}(y_1,\nabla u(x) + \nabla_{y_1}\bar{\psi}_1(x,y_1))\,\mathrm{d}x\mathrm{d}y_1 + M\eta + \eta\mathcal{L}^N(\Omega).\end{aligned} \tag{3.97}$$

Finally, define $\psi_1 \in L^1\big(\Omega; W^{1,1}_{\#}\big(Y_1;\mathbb{R}^d\big)\big)$, $\psi_2 \in L^1\big(\Omega\times Y_1; W^{1,1}_{\#}\big(Y_2;\mathbb{R}^d\big)\big)$ by setting $\psi_1(x) := \bar{\psi}_1(x)$ if $x \in \Omega_\delta$, $\psi_1(x) := 0$ if $x \in \Omega\backslash\Omega_\delta$, $\psi_2(x,y_1) := \bar{\psi}_2(x,y_1)$ if $(x,y_1) \in E_\gamma$, and $\psi_2(x,y_1) := 0$ if $(x,y_1) \in (\Omega\times Y_1)\backslash E_\gamma$. Using the usual identification of an integrable function with a measure, elements of $L^1\big(\Omega; W^{1,1}_{\#}\big(Y_1;\mathbb{R}^d\big)\big)$ and $L^1\big(\Omega\times Y_1; W^{1,1}_{\#}\big(Y_2;\mathbb{R}^d\big)\big)$ can be seen as elements of $\mathcal{M}_\star\big(\Omega; BV_{\#}\big(Y_1;\mathbb{R}^d\big)\big)$ and $\mathcal{M}_\star\big(\Omega\times Y_1; BV_{\#}\big(Y_2;\mathbb{R}^d\big)\big)$, respectively. Considering this identification (see also (1.4)), we have

$$\lambda_{u,\psi_1,\psi_2\lfloor\Omega\times Y_1\times Y_2} = \nabla u\mathcal{L}^{3N}_{\lfloor\Omega\times Y_1\times Y_2} + \nabla_{y_1}\psi_1\mathcal{L}^{3N}_{\lfloor\Omega\times Y_1\times Y_2} + \nabla_{y_2}\psi_2\mathcal{L}^{3N}_{\lfloor\Omega\times Y_1\times Y_2}. \tag{3.98}$$

From (1.13), (3.98), (3.83), (3.93), (3.96), (3.97) and (3.95), in this order, we deduce that

$$\begin{aligned}&\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_{\#}(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_{\#}(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \\ &= \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_{\#}(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_{\#}(Y_2;\mathbb{R}^d))}} \bigg\{\int_{\Omega\times Y_1\times Y_2} f\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x,y_1,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &\qquad + \int_{\Omega\times Y_1\times Y_2} f^\infty\Big(y_1,y_2,\frac{\mathrm{d}\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x,y_1,y_2)\Big)\,\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x,y_1,y_2)\bigg\} \\ &\leqslant \int_{\Omega\times Y_1\times Y_2} f(y_1,y_2,\nabla u(x) + \nabla_{y_1}\psi_1(x,y_1) + \nabla_{y_2}\psi_2(x,y_1,y_2))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &= \int_{(\Omega\backslash\Omega_\delta)\times Y_1\times Y_2} f(y_1,y_2,\nabla u(x))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad + \int_{((\Omega_\delta\times Y_1)\backslash E_\gamma)\times Y_2} f(y_1,y_2,\nabla u(x) + \nabla_{y_1}\bar{\psi}_1(x,y_1))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad + \int_{E_\gamma\times Y_2} f(y_1,y_2,\nabla u(x) + \nabla_{y_1}\bar{\psi}_1(x,y_1) + \nabla_{y_2}\bar{\psi}_2(x,y_1y_2))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &\leqslant 2M\eta + \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x + 2\big(M\eta + \eta\mathcal{L}^N(\Omega)\big).\end{aligned}$$

Letting $\eta \to 0^+$, we obtain (3.92).

Step 3. We prove that if $(\mathcal{F}4)$' is satisfied, then the converse of (3.91) holds for all $u \in BV\big(\Omega;\mathbb{R}^d\big)$.

Indeed, let $u \in BV\big(\Omega;\mathbb{R}^d\big)$. Since $f_{\mathrm{hom}} : \mathbb{R}^{d\times N} \to \mathbb{R}$ is a convex function satisfying (3.83), in view of Lemma 3.10 for all $\eta > 0$ we can find a sequence $\{u_j\}_{j\in\mathbb{N}} \subset W^{1,1}\big(\Omega;\mathbb{R}^d\big)$ weakly-$\star$ converging to u in $BV\big(\Omega;\mathbb{R}^d\big)$ and such that

$$\lim_{j\to\infty}\int_\Omega f_{\mathrm{hom}}(\nabla u_j(x))\,\mathrm{d}x \leqslant \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x + \int_\Omega (f_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x) + \eta.$$

Under the present hypotheses on f, it can be checked that F^{hom} is sequentially lower semicontinuous with respect to the weak-$\star$ convergence in $BV\big(\Omega;\mathbb{R}^d\big)$. Hence, using Theorem 3.7 and (3.92),

$$\begin{aligned}&\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_{\#}(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_{\#}(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \leqslant \liminf_{j\to\infty} \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_{\#}(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_{\#}(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u_j,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \\ &\quad\leqslant \lim_{j\to\infty}\int_\Omega f_{\mathrm{hom}}(\nabla u_j(x))\,\mathrm{d}x \leqslant \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x + \int_\Omega (f_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x) + \eta,\end{aligned}$$

from which we conclude Step 3 by letting $\eta \to 0^+$.

Step 4. We establish the second equality in (1.14).

Let $u \in BV\big(\Omega;\mathbb{R}^d\big)$, and fix $\eta > 0$ (which, without loss of generality, we assume will take values on a sequence of positive numbers converging to zero). Then f_η (we recall, $f_\eta(y_1,y_2,\xi) := f(y_1,y_2,\xi) + \eta|\xi|$) satisfies conditions $(\mathcal{F}1)$–$(\mathcal{F}3)$, $(\mathcal{F}4)$', $(\mathcal{F}5)$; condition $(\mathcal{F}6)$, which was only used in Lemma 3.2, reads slightly different for f_η than for f (see (3.8)), but it can be checked that this difference is innocuous. So, in view of Steps 1, 2 and 3 applied to f_η,

$$\inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F_\eta^{\rm sc}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) = \int_\Omega f_{\eta,\rm hom}(\nabla u(x))\,\mathrm{d}x + \int_\Omega (f_{\eta,\rm hom})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x), \tag{3.99}$$

where $F_\eta^{\rm sc}$ is the functional given by (3.44), and where $f_{\eta,\rm hom} := (f_\eta)_{\rm hom}$.

In order to pass (3.99) to the limit as $\eta \to 0^+$, we start by observing that for fixed $(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \in BV\big(\Omega;\mathbb{R}^d\big) \times \mathcal{M}_\star\big(\Omega;BV_\#\big(Y_1;\mathbb{R}^d\big)\big) \times \mathcal{M}_\star\big(\Omega\times Y_1;BV_\#\big(Y_2;\mathbb{R}^d\big)\big)$, $\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ has finite total variation and $\{F_\eta^{\rm sc}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\}_{\eta>0}$ is a bounded decreasing sequence, and so

$$\begin{aligned} \lim_{\eta\to 0^+} \inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F_\eta^{\rm sc}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) &= \inf_{\eta>0} \inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F_\eta^{\rm sc}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\\ = \inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} \inf_{\eta>0} F_\eta^{\rm sc}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) &= \inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\rm sc}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2). \end{aligned} \tag{3.100}$$

Furthermore, using Lebesgue Dominated Convergence Theorem together with (3.89), in view of (3.84) (observing that thanks to $(\mathcal{F}2)$, $f^{**} = f$ and $(f_\eta)^{**} = f_\eta$) and of (1.12) we get

$$\lim_{\eta\to 0^+} \int_\Omega f_{\eta,\rm hom}(\nabla u(x))\,\mathrm{d}x = \int_\Omega f_{\rm hom}(\nabla u(x))\,\mathrm{d}x, \tag{3.101}$$

and

$$\lim_{\eta\to 0^+} \int_\Omega (f_{\eta,\rm hom})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x) = \int_\Omega (f_{0^+,\rm hom})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x). \tag{3.102}$$

From (3.99), (3.100), (3.101) and (3.102), we conclude Step 4.

Finally, we observe that

a) if, in addition, f satisfies $(\mathcal{F}4)$', then by Step 1–Step 4, we have that $(f_{0^+,\rm hom})^\infty \equiv (f_{\rm hom})^\infty$;

b) if, in addition, f satisfies $(\mathcal{F}8)$, then by Proposition 1.4 (ii)–a), $(f_{0^+,\rm hom})^\infty \equiv (f_{\rm hom})^\infty = (f^\infty)_{\rm hom}$;

c) if, in addition, f satisfies $(\mathcal{F}7)$, then Proposition 1.4 (ii)–b) yields $(f_{0^+,\rm hom})^\infty \equiv (f^\infty)_{\rm hom}$. □

4. Proof of Corollary 1.7

As in the previous section, below we will assume, without loss of generality ,that $n = 2$, since the generalization to an arbitrary $n \in \mathbb{N}$ does not bring any additional technical difficulties.

The proof of Corollary 1.7 relies on Theorems 1.6 and on the next lemma concerning properties inherited by f^{**} from f.

Lemma 4.1. *Assume that $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a function satisfying conditions $(\mathcal{F}1)$, $(\mathcal{F}3)$, $(\mathcal{F}4)$', $(\mathcal{F}5)$ and $(\mathcal{F}6)$ with $d = 1$. Then the biconjugate function f^{**} of f is a real-valued Borel function in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, and verifies conditions $(\mathcal{F}1)$, $(\mathcal{F}3)$, $(\mathcal{F}4)$', $(\mathcal{F}5)$ and $(\mathcal{F}6)$ with $d = 1$.*

Proof. By (3.77) and since $f_1 \leqslant f_2$ implies that $\mathcal{C}f_1 \leqslant \mathcal{C}f_2$, the only nontrivial condition to verify is ($\mathcal{F}5$). Fix $(y_1', y_2') \in \mathbb{R}^N \times \mathbb{R}^N$ and $\delta > 0$ arbitrarily. Set $\bar\delta := \delta/(1+2C^2)$, where C is given by ($\mathcal{F}3$) and ($\mathcal{F}4$)', and let $\bar\tau = \bar\tau(y_1', y_2', \bar\delta)$ be given by ($\mathcal{F}5$) for f and for such $\bar\delta$.

Fix $\xi \in \mathbb{R}^N$ and $(y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N$ such that $|(y_1', y_2') - (y_1, y_2)| \leqslant \bar\tau$. By (3.81), for each $\epsilon > 0$ we can find $\varphi_\epsilon \in W_0^{1,\infty}(Y)$ such that

$$\int_Y f(y_1, y_2, \xi + \nabla\varphi_\epsilon(y))\,\mathrm{d}y \leqslant f^{**}(y_1, y_2, \xi) + \epsilon, \tag{4.1}$$

and so,

$$\begin{aligned} f^{**}(y_1', y_2', \xi) - f^{**}(y_1, y_2, \xi) &\leqslant \int_Y \Big(f(y_1', y_2', \xi + \nabla\varphi_\epsilon(y)) - f(y_1, y_2, \xi + \nabla\varphi_\epsilon(y))\Big)\,\mathrm{d}y + \epsilon \\ &\leqslant \int_Y \bar\delta(1 + |\xi + \nabla\varphi_\epsilon(y)|)\,\mathrm{d}y + \epsilon, \end{aligned} \tag{4.2}$$

where in the last inequality we used ($\mathcal{F}5$) for f.

In view of (4.1), ($\mathcal{F}3$) and ($\mathcal{F}4$)', we have that $\frac{1}{C}\|\xi + \nabla\varphi_\epsilon\|_{L^1(Y;\mathbb{R}^N)} - C \leqslant C(1+|\xi|) + \epsilon$. Thus, from (4.2) we deduce that

$$f^{**}(y_1', y_2', \xi) - f^{**}(y_1, y_2, \xi) \leqslant \bar\delta(1 + C^2(2+|\xi|)) + (\bar\delta C + 1)\epsilon \leqslant \delta(1+|\xi|) + (\delta C + 1)\epsilon.$$

Letting $\epsilon \to 0^+$, we conclude that

$$f^{**}(y_1', y_2', \xi) - f^{**}(y_1, y_2, \xi) \leqslant \delta(1+|\xi|).$$

Interchanging the roles between (y_1', y_2', ξ) and (y_1, y_2, ξ), we prove that $f^{**}(y_1, y_2, \xi) - f^{**}(y_1', y_2', \xi) \leqslant \delta(1 + |\xi|)$ also holds. Thus f^{**} satisfies ($\mathcal{F}5$). □

Proof of Corollary 1.7. We proceed in two steps.

Step 1. We prove that if in addition f satisfies ($\mathcal{F}4$)', then (1.15) holds with $(f_{0^+}^{**})^\infty$ replaced by $(f^{**})^\infty$, and (1.16) holds with $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty$ replaced by $\big((f^{**})_{\mathrm{hom}}\big)^\infty$.

Substep 1.1. We show that the infima (1.5) and (1.6) remain unchanged if we substitute f by its biconjugate function f^{**}.

Fix $(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in BV(\Omega) \times \mathcal{M}_\star\big(\Omega; BV_\#(Y_1)\big) \times \mathcal{M}_\star(\Omega \times Y_1; BV_\#(Y_2))$, and define

$$F^{**,\mathrm{sc}}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) := \inf\Big\{ \liminf_{\varepsilon\to 0^+} F_\varepsilon^{**}(u_\varepsilon) :\, u_\varepsilon \in BV(\Omega),\, Du_\varepsilon \xrightarrow[\varepsilon]{\text{3-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2} \Big\}$$

and

$$F^{**,\mathrm{hom}}(u) := \inf\Big\{ \liminf_{\varepsilon\to 0^+} F_\varepsilon^{**}(u_\varepsilon) \colon\ u_\varepsilon \in BV(\Omega),\ u_\varepsilon \overset{\star}{\rightharpoonup}_\varepsilon u \text{ weakly-}\star \text{ in } BV(\Omega) \Big\},$$

where F_ε^{**} is the functional given by (3.1) for $d = 1$ and with f replaced by f^{**}.

Notice that by Lemma 4.1 and Remark 1.8 (ii), f^{**} is a real-valued continuous function in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ satisfying conditions ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$)', ($\mathcal{F}5$) and ($\mathcal{F}6$) with $d = 1$.

Since $f^{**} \leqslant f$, we have that $F^{**,\mathrm{sc}}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \leqslant F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$ and $F^{**,\mathrm{hom}}(u) \leqslant F^{\mathrm{hom}}(u)$. To prove the opposite inequalities, we start by observing that in view of (3.38)–(3.42) the following equalities hold:

$$\begin{aligned} F^{**,\mathrm{sc}}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= \inf\Big\{ \liminf_{\varepsilon\to 0^+} F_\varepsilon^{**}(u_\varepsilon) :\, u_\varepsilon \in W^{1,1}(\Omega),\, \nabla u_\varepsilon \mathcal{L}^N\lfloor_\Omega \xrightarrow[\varepsilon]{\text{3-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2} \Big\} \\ &= \inf\Big\{ \liminf_{\varepsilon\to 0^+} \int_\Omega f^{**}\Big(\frac{x}{\varrho_1(\varepsilon)}, \frac{x}{\varrho_2(\varepsilon)}, \nabla u_\varepsilon(x)\Big)\mathrm{d}x :\, u_\varepsilon \in W^{1,1}(\Omega),\, \nabla u_\varepsilon \mathcal{L}^N\lfloor_\Omega \xrightarrow[\varepsilon]{\text{3-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2} \Big\}. \end{aligned}$$

Moreover, a similar argument to (3.38)–(3.42) ensures that also

$$\begin{aligned} F^{**,\mathrm{hom}}(u) &= \inf\Big\{ \liminf_{\varepsilon\to 0^+} F^{**}_\varepsilon(u_\varepsilon)\colon\ u_\varepsilon \in W^{1,1}(\Omega),\ u_\varepsilon \xrightharpoonup{\star}_\varepsilon u \text{ weakly-}\star \text{ in } BV(\Omega)\Big\} \\ &= \inf\Big\{ \liminf_{\varepsilon\to 0^+} \int_\Omega f^{**}\Big(\frac{x}{\varrho_1(\varepsilon)}, \frac{x}{\varrho_2(\varepsilon)}, \nabla u_\varepsilon(x)\Big)\mathrm{d}x\colon\ u_\varepsilon \in W^{1,1}(\Omega),\ u_\varepsilon \xrightharpoonup{\star}_\varepsilon u \text{ weakly-}\star \text{ in } BV(\Omega)\Big\}. \end{aligned}$$

Fix $\delta > 0$. We can find a sequence $\{\varepsilon_h\}_{h\in\mathbb{N}}$ of positive numbers converging to zero as $h\to\infty$, and a sequence $\{u_h\}_{h\in\mathbb{N}} \subset W^{1,1}(\Omega)$ such that $\nabla u_h \mathcal{L}^N\lfloor\Omega \xrightharpoonup{3\text{-}sc}_{\varepsilon_h} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ and

$$F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) + \delta \geqslant \lim_{h\to\infty} \int_\Omega f^{**}\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_h(x)\Big)\mathrm{d}x.$$

On the other hand (see, for example, [36, Cor. 3.13]; see also [28, Chapter X]), since f is a continuous function satisfying ($\mathcal{F}3$) and ($\mathcal{F}4$)', for each $h\in\mathbb{N}$ there exist a sequence $\{u_j^{(h)}\}_{j\in\mathbb{N}} \subset W^{1,1}(\Omega)$ weakly converging to u_h in $W^{1,1}(\Omega)$ and such that

$$\int_\Omega f^{**}\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_h(x)\Big)\mathrm{d}x = \lim_{j\to\infty} \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_j^{(h)}(x)\Big)\mathrm{d}x.$$

Hence,

$$F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) + \delta \geqslant \lim_{h\to\infty}\lim_{j\to\infty} \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_j^{(h)}(x)\Big)\mathrm{d}x, \tag{4.3}$$

and for all $\varphi \in C_0(\Omega; C_\#(Y_1\times Y_2;\mathbb{R}^N))$,

$$\begin{aligned} \lim_{h\to\infty}\lim_{j\to\infty} \int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big)\cdot \nabla u_j^{(h)}(x)\,\mathrm{d}x &= \lim_{h\to\infty} \int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big)\cdot \nabla u_h(x)\,\mathrm{d}x \\ &= \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(x,y_1,y_2). \end{aligned} \tag{4.4}$$

Using a diagonalization argument and the separability of $C_0(\Omega; C_\#(Y_1\times Y_2;\mathbb{R}^N))$, from (4.3), (4.4) and ($\mathcal{F}4$)' we can find a sequence $\{j_h\}_{h\in\mathbb{N}}$ such that $j_h\to\infty$ as $h\to\infty$, $v_h := u_{j_h}^{(h)} \in W^{1,1}(\Omega)$, $\nabla v_h \mathcal{L}^N\lfloor\Omega \xrightharpoonup{3\text{-}sc}_{\varepsilon_h} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ and

$$F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) + \delta \geqslant \lim_{h\to\infty} \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla v_h(x)\Big)\mathrm{d}x \geqslant F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2),$$

where in the last inequality we used the definition of $F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)$. Letting $\delta\to 0^+$, we conclude that $F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \geqslant F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)$.

The proof of inequality $F^{**,\mathrm{hom}}(u) \geqslant F^{\mathrm{hom}}(u)$ is similar. Thus, we conclude that $F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) = F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)$ and $F^{**,\mathrm{hom}}(u) = F^{\mathrm{hom}}(u)$.

Substep 1.2. Finally, we observe that in view of Theorem 1.6 (i) and Lemma 4.1, we have that for all $(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \in BV(\Omega)\times\mathcal{M}_\star(\Omega; BV_\#(Y_1))\times\mathcal{M}_\star(\Omega\times Y_1; BV_\#(Y_2))$,

$$\begin{aligned} F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) :=& \int_{\Omega\times Y_1\times Y_2} f^{**}\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x,y_1,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &+ \int_{\Omega\times Y_1\times Y_2} (f^{**})^\infty\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x,y_1,y_2)\Big)\,\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x,y_1,y_2) \end{aligned}$$

and

$$\begin{aligned} F^{**,\mathrm{hom}}(u) &= \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))}} F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \\ &= \int_\Omega (f^{**})_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x + \int_\Omega ((f^{**})_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x), \end{aligned}$$

and this, together with Substep 1.1, completes the proof of Step 1.

Step 2. We establish Corollary 1.7.

Fix $\eta > 0$ (which, without loss of generality, we assume will take values on a sequence of positive numbers converging to zero), and let $F^{\rm sc}_\eta$ and $F^{\rm hom}_\eta$ be the functionals given by (1.5) and (1.6) for $d = 1$, respectively, with f replaced by f_η.

Assuming ($\mathcal{F}6$) with $o(1)$ replaced by $-|o(1)|$ in ($\mathcal{F}6$), it can be shown that we may use Step 1 for f_η. Thus, for every $(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in BV(\Omega) \times \mathcal{M}_\star(\Omega; BV_\#(Y_1)) \times \mathcal{M}_\star(\Omega \times Y_1; BV_\#(Y_2))$,

$$\begin{aligned} F^{\rm sc}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = &\int_{\Omega\times Y_1\times Y_2} (f_\eta)^{**}\Big(y_1, y_2, \frac{{\rm d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{{\rm d}\mathcal{L}^{3N}}(x, y_1, y_2)\Big)\, {\rm d}x{\rm d}y_1{\rm d}y_2 \\ &+ \int_{\Omega\times Y_1\times Y_2} ((f_\eta)^{**})^\infty\Big(y_1, y_2, \frac{{\rm d}\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{{\rm d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x, y_1, y_2)\Big)\, {\rm d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x, y_1, y_2) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} F^{\rm hom}_\eta(u) &= \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))}} F^{\rm sc}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \\ &= \int_\Omega ((f_\eta)^{**})_{\rm hom}(\nabla u(x))\, {\rm d}x + \int_\Omega \big(((f_\eta)^{**})_{\rm hom}\big)^\infty\Big(\frac{{\rm d}D^s u}{{\rm d}\|D^s u\|}(x)\Big)\, {\rm d}\|D^s u\|(x). \end{aligned} \tag{4.6}$$

In order to pass (4.5) and (4.6) to the limit as $\eta \to 0^+$, we start by observing that for fixed $(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in BV(\Omega) \times \mathcal{M}_\star\big(\Omega; BV_\#(Y_1)) \times \mathcal{M}_\star(\Omega \times Y_1; BV_\#(Y_2))$, the sequences $\{F^{\rm sc}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)\}_{\eta>0}$ and $\{F^{\rm hom}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)\}_{\eta>0}$ are decreasing (as $\eta \to 0^+$), so that the respective limits as $\eta \to 0^+$ exist and are given by the infimum in $\eta > 0$.

Let $\{u_\varepsilon\}_{\varepsilon>0} \subset BV(\Omega)$ be such that $Du_\varepsilon \xrightarrow[\varepsilon]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$. Then $\{Du_\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathcal{M}\big(\Omega; \mathbb{R}^N\big)$ (see Remark 2.2), and so since $(f_\eta)^\infty(y_1, y_2, \xi) = f^\infty(y_1, y_2, \xi) + \eta|\xi|$, we have

$$F^{\rm sc}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \leqslant \liminf_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon) + \eta C,$$

where C is a constant independent of ε. Letting $\eta \to 0^+$ and then taking the infimum over all such sequences $\{u_\varepsilon\}_{\varepsilon>0}$, we conclude that $\lim_{\eta\to0^+} F^{\rm sc}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \leqslant F^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$. Conversely, since for all $\eta > 0$, $f_\eta \geqslant f$, we have that $F^{\rm sc}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \geqslant F^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$. Hence,

$$\lim_{\eta\to0^+} F^{\rm sc}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = F^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2). \tag{4.7}$$

Similar arguments ensure that

$$\lim_{\eta\to0^+} F^{\rm hom}_\eta(u) = F^{\rm hom}(u). \tag{4.8}$$

Moreover, as in (3.100),

$$\lim_{\eta\to0^+} \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\rm sc}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2). \tag{4.9}$$

So, letting $\eta \to 0^+$ in (4.5) and (4.6), thanks to (4.7), (4.8), (4.9), (3.79), (3.80), (3.84), (3.86) and Lebesgue Dominated Convergence Theorem together with ($\mathcal{F}3$) and ($\mathcal{F}4$), we obtain (1.15) and (1.16).

Finally, we observe that in view of Step 1, if f satisfies in addition ($\mathcal{F}4$)', then

$$((f_{0^+})^{**})^\infty \equiv (f^{**})^\infty \text{ and } \big(((f_{0^+})^{**})_{\rm hom}\big)^\infty \equiv \big((f^{**})_{\rm hom}\big)^\infty.$$

Moreover, if, in addition to ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$), ($\mathcal{F}5$) and ($\mathcal{F}6$), with $o(1)$ replaced by $-|o(1)|$ in ($\mathcal{F}6$), f satisfies the condition ($\mathcal{F}7$), then by Proposition 1.4 (i)–b) and (ii)–c),

$$((f_{0^+})^{**})^\infty \equiv (f^\infty)^{**} \text{ and } \big(((f_{0^+})^{**})_{\rm hom}\big)^\infty \equiv \big((f^\infty)^{**}\big)_{\rm hom}. \qquad \square$$

Acknowledgements

The authors thank the Center for Nonlinear Analysis (NSF Grants No. DMS-0405343 and DMS-0635983) for its support during the preparation of this paper. The research of R. Ferreira was supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the Carnegie Mellon | Portugal Program under Grant SFRH/BD/35695/2007, the Financiamento Base 20010 ISFL–1–297, PTDC/MAT/109973/2009 and UTA−CMU/MAT/0005/2009. The research of I. Fonseca was partially supported by the National Science Foundation under Grants No. DMS-0401763 and DMS-0905778.

References

[1] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal. **23** (1992), 1482–1518

[2] G. Allaire, M. Briane, *Multiscale convergence and reiterated homogenisation*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), 297–342

[3] M. Amar, *Two-scale convergence and homogenization on $BV(\Omega)$*, Asymptotic Analysis **16** (1998), 65–84

[4] L. Ambrosio, G. Buttazzo, *Weak lower semicontinuous envelope of functionals defined on a space of measures*, Ann. Mat. Pura Appl. (4) **150** (1988), 311–339

[5] L. Ambrosio, N. Fusco, D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, New York, 2000

[6] H. Attouch, *Variational properties of epi-convergence. Applications to limit analysis problems in mechanics and duality theory*, Multifunctions and integrands (Proceeding Catania), Lecture Notes in Math., Springer, Berlin, 1984

[7] H. Attouch, G. Buttazzo, G. Michaille, *Variational analysis in Sobolev and BV spaces*, Variational analysis in Sobolev and BV spaces, MPS/SIAM Series on Optimization, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2006

[8] J.F. Babadjian, V. Millot, *Homogenization of variational problems in manifold valued BV-spaces*, Calc. Var. Partial Differential Equations **36** (2009), 7–47

[9] J.F. Babadjian, E. Zappale, H. Zorgati, *Dimensional reduction for energies with linear growth involving the bending moment*, J. Math. Pures Appl. (9) **90** (2008), 520–549

[10] A. Bensoussan, J.-L. Lions, G. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland Publishing Co., Amsterdam, 1978

[11] M. Boni, *Quasi additività e quasi subadditività nell'integrale ordinario del calcolo delle variazioni alla Weierstrass*, Rend. Ist. Mat. Univ. Trieste **6** (1974), 51–70

[12] G. Bouchitté, *Homogénéisation sur* BV(Ω) *de fonctionnelles intégrales à croissance linéaire. Application à un problème d'analyse limite en plasticité*, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), 785–788

[13] G. Bouchitté, *Convergence et relaxation de fonctionnelles du calcul des variations à croissance linéaire. Application à l'homogénéisation en plasticité*, Ann. Fac. Sci. Toulouse Math. Ser. 5 **8** (1986/87), 7–36

[14] G. Bouchitté, I. Fonseca, M.L. Mascarenhas, *A global method for relaxation*, Arch. Rational Mech. Anal. **145** (1998), 51–98

[15] A. Braides, V. Chiadò Piat, *A derivation formula for convex integral functionals defined on $BV(\Omega)$*, J. Convex Anal. **2** (1995), 69–85

[16] A. Braides, A. Defranceschi, *Homogenization of multiple integrals*, Oxford Lecture Series in Mathematics and its Applications **12**, The Clarendon Press Oxford University Press, New York, 1998

[17] L. Carbone, D. Cioranescu, R. De Arcangelis, A. Gaudiello, *Homogenization of unbounded functionals and nonlinear elastomers. The case of the fixed constraints set*, ESAIM Control Optim. Calc. Var. **10** (2004), 53–83

[18] C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Mathematics **580**, Springer-Verlag, Berlin, 1977

[19] D. Cioranescu, A. Damlamian, R. De Arcangelis, *Homogenization of nonlinear integrals via the periodic unfolding method*, C. R. Math. Acad. Sci. Paris **339** (2004), 77–82

[20] D. Cioranescu, A. Damlamian, R. De Arcangelis, *Homogenization of quasiconvex integrals via the periodic unfolding method*, SIAM J. Math. Anal. **37** (2006), 1435–1453

[21] D. Cioranescu, A. Damlamian, G. Griso, *The periodic unfolding method in homogenization*, SIAM J. Math. Anal. **40** (2008), 1585–1620

[22] B. Dacorogna, *Direct methods in the calculus of variations*, Applied Mathematical Sciences **78**, Springer, New York, 2008

[23] G. Dal Maso, *Integral representation on* $BV(\Omega)$ *of* Γ*-limits of variational integrals*, Manuscripta Math. **30** (1980), 387–416

[24] G. Dal Maso, *An introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and their Applications **8**, Birkhäuser Boston Inc., Boston, 1993

[25] S. Delladio, *Lower semicontinuity and continuity of functions of measures with respect to the strict convergence*, Proc. Roy. Soc. Edinburgh Sect. A **119** (1991), 265–278

[26] R. De Arcangelis, G. Gargiulo, *Homogenization of integral functionals with linear growth defined on vector-valued functions*, Nonlinear Differential Equations Appl. **2** (1995), 371–416

[27] P. Donato, *Alcune osservazioni sulla convergenza debole di funzioni non uniformemente oscillanti*, Ricerche Mat. **32** (1983), 203–219

[28] I. Ekeland, R. Témam, *Convex analysis and variational problems*, Stud. Math. Appl. 1, NorthHolland, Amsterdam, American Elsevier, New York, 1976

[29] L. Evans, *Weak convergence methods for nonlinear partial differential equations*, CBMS, vol. 74, Amer. Math. Soc., Washington, DC, 1990

[30] R. Ferreira, I. Fonseca, *Characterization of the Multiscale Limit Associated with Bounded Sequences in* BV*, submitted*

[31] I. Fonseca, S. Krömer, *Multiple Integrals under Differential Constraints: Two-Scale Convergence and Homogenization*, Indiana Univ. Math. J. **59** (2010), 427–457

[32] I. Fonseca, G. Leoni, *Modern methods in the calculus of variations:* L^p *spaces*, Springer Monographs in Mathematics, Springer, New York, 2007

[33] M. Giaquinta, G. Modica, J. Souček, *Functionals with linear growth in the calculus of variations. I, II*, Comment. Math. Univ. Carolin. **20** (1979), 143–156, 157–172

[34] V. Jikov, S. Kozlov, O. Oleĭnik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994

[35] J. Kristensen, F. Rindler, *Relaxation of signed integral functionals in BV*, Calc. Var. Partial Differ. Equ. **37** (2010), 29–62

[36] P. Marcellini, C. Sbordone, *Semicontinuity problems in the calculus of variations*, Nonlinear Anal. **4** (1989), 241–257

[37] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. **20** (1989), 608–623

[38] Y.G. Reshetnyak, *The weak convergence of completely additive vector-valued set functions*, Sibirsk. Mat. Ž. **9** (1968), 1386–1394

[39] R.T. Rockafellar, *Convex Analysis*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997

[40] E. Sanchez-Palencia, *Nonhomogeneous media and vibration theory*, Springer-Verlag, Berlin, 1980

[41] J. Serrin, *On the definition and properties of certain variational integrals*, Trans. Amer. Math. Soc. **101** (1961), 139–167

[42] R. Témam, *Mathematical problems in plasticity*, North-Holland, Amsterdam, 1980