Sufficient Conditions for Existence of Solutions to Vectorial Differential Inclusions and Applications

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Abstract

In this paper we discuss the existence of solutions to vectorial differential inclusions, refining a result proved in Dacorogna and Marcellini [8]. We investigate sufficient conditions for existence, more flexible than those available in the literature, so that important applications can be fitted in the theory. We also study some of these applications.

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1 Introduction

In this paper we discuss the existence of $W^{1,\infty}(\Omega, \mathbb{R}^N)$ solutions to the vectorial differential inclusion problem

$$\begin{cases} Du(x) \in E, \ a.e. \ x \in \Omega, \\ u(x) = \varphi(x), \ x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$, E is a given subset of $\mathbb{R}^{N \times n}$ and $\varphi : \overline{\Omega} \to \mathbb{R}^N$. This problem has been intensively studied by Dacorogna and Marcellini [8], [9] through the Baire categories method (see also Müller and Šverák [18]). Their result provides a sufficient condition for existence of solutions related with the gradient of the boundary data. It asserts that, if this gradient belongs to a convenient set enjoying the so called relaxation property with respect to the set E (see Definition 3.1, and Theorem 3.2 due to Dacorogna and Pisante [10]) a dense set of solutions to (1.1) exists.

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In the applications, direct verification of the relaxation property is a hard task and sufficient conditions for it were also obtained by Dacorogna and Marcellini [9], namely, the approximation property, cf. Definition 3.3 and Theorem 3.4 (see also [9, Theorem 6.15] for a more general version). If such a property is satisfied, we can get as a sufficient condition for existence of solutions to (1.1)

$$D\varphi(x) \in E \cup \operatorname{int} \operatorname{Rco} E, \ \textit{a.e. } x \in \Omega, \tag{1.2}$$

where $\operatorname{int}\operatorname{Rco} E$ denotes the interior of the rank one convex hull of the set E, that is, the interior of the smallest rank one convex set containing E (see Definition 2.4).

However, there are interesting applications for which $\operatorname{int}\operatorname{Rco} E$ is empty, and thus, condition (1.2) is much too restrictive. This is the case of a well known problem solved by Kirchheim in [14] that we will discuss in Section 4. In view of this, our goal in Section 3 is to obtain sufficient conditions for the relaxation property which can be handled in the applications and which are more flexible than (1.2). More precisely, we shall be able to deal with subsets of the hull $\operatorname{Rco}_f E$ defined as

$$\operatorname{Rco}_f E = \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \le 0, \text{ for every rank one convex } f \in \mathcal{F}^E\right\},$$

where $\mathcal{F}^E = \{f : \mathbb{R}^{N\times n} \to \mathbb{R} : f|_E \le 0\}$, and which is, in general, a larger hull than $\operatorname{Rco} E$. Recovering results due to Pedregal [20] and to Müller and Šverák [18], we obtain in Theorem 2.9 the following characterization of this type of hulls for compact sets E:

$$\operatorname{Rco}_f E = \left\{ \begin{array}{l} \xi \in \mathbb{R}^{N\times n} : \ \forall\, \varepsilon > 0 \ \exists\, I \in \mathbb{N}, \ \exists\, (\lambda_i, \xi_i)_{i=1,\dots,I} \text{ with } \lambda_i > 0, \\ \displaystyle\sum_{i=1}^{I} \lambda_i = 1 \text{ satisfying } (H_I(U)), \ \xi = \sum_{i=1}^{I} \lambda_i \xi_i, \quad \sum_{\substack{i=1 \\ \xi_i \notin B_\varepsilon(E)}}^{I} \lambda_i < \varepsilon \end{array} \right\}, \tag{1.3}$$

where U is an open and bounded set containing $\operatorname{Rco}_f E$ and the property $(H_I(U))$ is introduced in Definition 2.7. Thanks to this characterization we will prove, in particular, the following result (cf. Corollary 3.7).

Theorem 1.1 *Let $E \subset \mathbb{R}^{N\times n}$ be bounded and such that E and $\operatorname{Rco}_f E$ have the approximation property with $K_\delta = \operatorname{Rco}_f E_\delta$ for some compact sets $E_\delta \subset \mathbb{R}^{N\times n}$. Then $\operatorname{int}\operatorname{Rco}_f E$ has the relaxation property with respect to E.*

From this theorem and from the Baire categories method it follows that, to ensure existence of solutions to (1.1) under the approximation property assumption, condition (1.2) can be replaced by

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{Rco}_f E, \ \textit{a.e. } x \in \Omega. \tag{1.4}$$

Based on the characterization of the elements of $\operatorname{Rco}_f E$ given in (1.3) we will prove in Theorem 3.5 a more general sufficient condition for the relaxation property which allows us to work with subsets of $\operatorname{Rco}_f E$. This is very useful in the applications because many times the entire hull $\operatorname{Rco}_f E$ is not known. Moreover, characterizing $\operatorname{Rco}_f E$ (or $\operatorname{Rco} E$) may lead to complicated formulas and thus checking condition (1.4) (or (1.2)) becomes very difficult. However, many problems can still be solved provided it is possible to work with convenient subsets of $\operatorname{Rco}_f E$. This will be the case in the several applications given in Section 4.

We start, in Section 4.1, with the problem already solved by Kirchheim [14] on the existence of a non affine map with a finite number of gradients whose values are not rank one connected and with an affine boundary condition. We will show that this example is still in the setting of the Baire categories method thanks to the sufficient conditions for the relaxation property proved in Section 3. In this case the set E is a finite set of matrices with no rank one connections. We observe that we don't need to compute $\operatorname{Rco}_f E$ and that we get existence of solutions whenever the affine boundary data φ satisfies $D\varphi \in K$, for a certain set $K \subset \operatorname{Rco}_f E$.

Then we will come back to the problem, already considered by Croce [6], of arbitrary compact isotropic subsets E of $\mathbb{R}^{2\times 2}$. Once again, our theory shows here its versatility. For this type of sets the hull $\operatorname{Rco}_f E$ was characterized by Cardaliaguet and Tahraoui [2]. Although it was proved in [6] that this hull coincides with $\operatorname{Rco} E$, we are now able to apply the Baire categories method without using this information.

Finally we consider, in Section 4.3, the case of sets E for which a constraint on the sign of the determinant is imposed on a set of isotropic matrices:

$$E = \left\{\xi \in \mathbb{R}^{n\times n} : (\lambda_1(\xi), \cdots, \lambda_n(\xi)) \in \Lambda_E,\ \det \xi > 0\right\}, \tag{1.5}$$

where Λ_E is a set contained in $\{(x_1, \cdots, x_n) \in \mathbb{R}^n : 0 < x_1 \leq \cdots \leq x_n\}$ and $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ are the singular values of the matrix ξ (cf. Section 4). Characterizing the hulls of such sets is quite complicated and the only results available were obtained by Cardaliaguet and Tahraoui [3] in dimension $n = 2$. Considering a particular class of sets E we will prove the following result (cf. Theorem 4.11).

Theorem 1.2 *Let $E = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \{(a_1, b_1), (a_2, b_2)\},\ \det \xi > 0\}$ with $0 < a_1 < b_1 < a_2 < b_2$. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let $\varphi \in C^1_{piec}(\overline{\Omega}, \mathbb{R}^2)$ be such that $D\varphi \in E \cup \operatorname{int} \operatorname{Rco}_f E$ a.e. in Ω. Then there exists a map $u \in \varphi + W^{1,\infty}_0(\Omega, \mathbb{R}^2)$ such that $Du(x) \in E$ for a.e. x in Ω.*

In addition, we are also able to establish sufficient conditions for sets E of the form (1.5) in dimension $n = 2$ and $n = 3$, working with a subset of $\operatorname{Rco}_f E$. In particular, in dimension 2, we prove the following result.

Theorem 1.3 *Let Λ_E be a subset of $\mathbb{R}^2$ containing the line segment joining two distinct points (a_1, a_2) and (b_1, b_2) such that $0 < a_1 \leq a_2$, $0 < b_1 \leq b_2$, $a_1 < b_1$, $a_2 < b_2$, and either $a_1 < a_2$ or $b_1 < b_2$. Let*

$$E = \left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_E,\ \det \xi > 0\right\},$$

$\Omega \subset \mathbb{R}^2$ *be a bounded open set and* $\varphi \in C^1_{piec}(\overline{\Omega}, \mathbb{R}^2)$ *be such that for a.e.* x *in* Ω,

$$a_1 a_2 < \det D\varphi(x) < b_1 b_2,$$

$$\lambda_2(D\varphi(x)) < \lambda_1(D\varphi(x)) \frac{b_2 - a_2}{b_1 - a_1} + \frac{a_2 b_1 - b_2 a_1}{b_1 - a_1}.$$

Then there exists a map $u \in \varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ *such that* $Du(x) \in E$ *for a.e.* x *in* Ω.

We refer to Theorems 4.12 and 4.14 for more details. We stress the fact that these results are independent of the knowledge of $\mathrm{Rco}_f\, E$, which is not known for $n > 2$, and that analogous results could be obtained in higher dimensions. In practical applications the conditions stated are easier to verify than the conditions needed to characterize $\mathrm{Rco}_f\, E$.

2 Review on the generalized notions of convexity

In this section we recall several definitions and properties of some generalized notions of convexity that will be useful throughout this paper. We refer to Dacorogna's monograph [7] and to Dacorogna and Ribeiro [11] for more details.

Several types of hulls in a generalized sense will be recalled here. The main result of this section is the characterization of the hull $\mathrm{Rco}_f\, E$, established in Theorem 2.9.

We start by recalling the notions of polyconvex and rank one convex functions.

Notation 2.1 *For* $\xi \in \mathbb{R}^{N \times n}$ *we let*

$$T(\xi) = (\xi, \mathrm{adj}_2 \xi, \ldots, \mathrm{adj}_{N \wedge n} \xi) \in \mathbb{R}^{\tau(N,n)},$$

where $\mathrm{adj}_s \xi$ *stands for the matrix of all* $s \times s$ *subdeterminants of the matrix* ξ, $1 \le s \le N \wedge n = \min\{N, n\}$ *and where*

$$\tau = \tau(N,n) = \sum_{s=1}^{N \wedge n} \binom{N}{s} \binom{n}{s} \quad \text{and} \quad \binom{N}{s} = \frac{N!}{s!\,(N-s)!}.$$

In particular, if $N = n = 2$, *then* $T(\xi) = (\xi, \det \xi)$.

Definition 2.2 *(i) A function* $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ *is said to be* polyconvex *if there exists a convex function* $g : \mathbb{R}^{\tau(N,n)} \to \mathbb{R} \cup \{+\infty\}$ *such that* $f(\xi) = g(T(\xi))$.

(ii) A function $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ *is said to be* rank one convex *if*

$$f(\lambda \xi + (1-\lambda)\eta) \le \lambda f(\xi) + (1-\lambda) f(\eta)$$

for every $\lambda \in [0,1]$ *and every* $\xi, \eta \in \mathbb{R}^{N \times n}$ *with* $\mathrm{rank}(\xi - \eta) = 1$.

It is well known that f polyconvex $\Rightarrow f$ rank one convex.

Next we recall the corresponding notions of convexity for sets.

Definition 2.3 *(i) We say that* $E \subset \mathbb{R}^{N\times n}$ *is* polyconvex *if there exists a convex set* $K \subset \mathbb{R}^{\tau(N,n)}$ *such that* $\left\{\xi \in \mathbb{R}^{N\times n} : T(\xi) \in K\right\} = E$.

(ii) Let $E \subset \mathbb{R}^{N\times n}$. *We say that* E *is* rank one convex *if for every* $\lambda \in [0,1]$ *and for every* $\xi, \eta \in E$ *such that* $\operatorname{rank}(\xi - \eta) = 1$*, then* $\lambda\xi + (1-\lambda)\eta \in E$.

As shown by Dacorogna and Ribeiro [11] a set E is polyconvex if and only if the following condition is satisfied, for every $I \in \mathbb{N}$

$$\left.\begin{array}{c} \displaystyle\sum_{i=1}^{I} \lambda_i T(\xi_i) = T\left(\sum_{i=1}^{I} \lambda_i \xi_i\right) \\ \xi_i \in E,\ \lambda_i \geq 0,\ \displaystyle\sum_{i=1}^{I} \lambda_i = 1 \end{array}\right\} \Rightarrow \sum_{i=1}^{I} \lambda_i \xi_i \in E.$$

Moreover, we have the following implication

$$E \text{ polyconvex} \Rightarrow E \text{ rank one convex.}$$

As in the classical convex case, for these convexity notions, related convex hulls can be considered.

Definition 2.4 *The* polyconvex *and* rank one convex hulls *of a set* $E \subset \mathbb{R}^{N\times n}$ *are, respectively, the smallest polyconvex and rank one convex sets containing* E *and are, respectively, denoted by* $\operatorname{Pco} E$ *and* $\operatorname{Rco} E$.

Obviously one has the following inclusions

$$E \subseteq \operatorname{Rco} E \subseteq \operatorname{Pco} E \subseteq \operatorname{co} E,$$

where $\operatorname{co} E$ denotes the convex hull of E.

We recall the usual characterizations for the polyconvex and rank one convex hulls. It was proved by Dacorogna and Marcellini in [9] that

$$\operatorname{Pco} E = \left\{\xi \in \mathbb{R}^{N\times n} : T(\xi) = \sum_{i=1}^{\tau+1} t_i T(\xi_i),\ \xi_i \in E,\ t_i \geq 0,\ \sum_{i=1}^{\tau+1} t_i = 1\right\} \quad (2.1)$$

and

$$\operatorname{Rco} E = \bigcup_{i\in\mathbb{N}} \mathrm{R}_i\mathrm{co}E, \quad (2.2)$$

where $\mathrm{R}_0\mathrm{co}E = E$ and

$$\mathrm{R}_{i+1}\mathrm{co}E = \left\{\xi \in \mathbb{R}^{N\times n} : \begin{array}{c} \xi = \lambda A + (1-\lambda)B,\ \lambda \in [0,1], \\ A, B \in \mathrm{R}_i\mathrm{co}E,\ \operatorname{rank}(A-B) \leq 1 \end{array}\right\},\ i \geq 0.$$

One has (see [11]) that $\operatorname{Pco} E$ and $\operatorname{Rco} E$ are open if E is open, and $\operatorname{Pco} E$ is compact if E is compact. However, in general, it isn't true that $\operatorname{Rco} E$ is compact if E is compact (see Kolář [16]).

It is well known that, for $E \subset \mathbb{R}^{N\times n}$,

$$\operatorname{co} E = \left\{ \xi \in \mathbb{R}^{N\times n} : f(\xi) \le 0, \text{ for every convex function } f \in \mathcal{F}_\infty^E \right\} \tag{2.3}$$

$$\overline{\operatorname{co} E} = \left\{ \xi \in \mathbb{R}^{N\times n} : f(\xi) \le 0, \text{ for every convex function } f \in \mathcal{F}^E \right\} \tag{2.4}$$

where $\overline{\operatorname{co} E}$ denotes the closure of the convex hull of E and

$$\begin{aligned} \mathcal{F}_\infty^E &= \left\{ f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\} : f|_E \le 0 \right\} \\ \mathcal{F}^E &= \left\{ f : \mathbb{R}^{N\times n} \to \mathbb{R} : f|_E \le 0 \right\}. \end{aligned}$$

Analogous representations to (2.3) can be obtained in the polyconvex and rank one convex cases:

$$\begin{aligned} \operatorname{Pco} E &= \left\{ \xi \in \mathbb{R}^{N\times n} : f(\xi) \le 0, \text{ for every polyconvex function } f \in \mathcal{F}_\infty^E \right\}, \\ \operatorname{Rco} E &= \left\{ \xi \in \mathbb{R}^{N\times n} : f(\xi) \le 0, \text{ for every rank one convex function } f \in \mathcal{F}_\infty^E \right\}. \end{aligned}$$

However, (2.4) can only be generalized to the polyconvex case if the sets are compact, and, in the rank one convex case, (2.4) is not true, even if compact sets are considered. In view of this, another type of hulls can be defined.

Definition 2.5 *For a set E of $\mathbb{R}^{N\times n}$, let*

$$\begin{aligned} \operatorname{co}_f E &= \left\{ \xi \in \mathbb{R}^{N\times n} : f(\xi) \le 0, \textit{ for every convex } f \in \mathcal{F}^E \right\} \\ \operatorname{Pco}_f E &= \left\{ \xi \in \mathbb{R}^{N\times n} : f(\xi) \le 0, \textit{ for every polyconvex } f \in \mathcal{F}^E \right\} \\ \operatorname{Rco}_f E &= \left\{ \xi \in \mathbb{R}^{N\times n} : f(\xi) \le 0, \textit{ for every rank one convex } f \in \mathcal{F}^E \right\}. \end{aligned}$$

Remark 2.6 *1) Notice that these hulls are closed sets. Moreover, they are, respectively, convex, polyconvex and rank one convex.*

2) For compact sets E, these are the hulls considered by Müller and Šverák [18] to establish an existence result for differential inclusions. We notice that a different definition was introduced for open sets.

Thus, as observed above, $\overline{\operatorname{co} E} = \operatorname{co}_f E$; if E is compact, then

$$\operatorname{Pco} E = \overline{\operatorname{Pco} E} = \operatorname{Pco}_f E,$$

but, in general,

$$\operatorname{Pco} E \subsetneq \overline{\operatorname{Pco} E} \subsetneq \operatorname{Pco}_f E.$$

Moreover, in general, even if E is compact,

$$\operatorname{Rco} E \subsetneq \overline{\operatorname{Rco} E} \subsetneq \operatorname{Rco}_f E.$$

Next we establish a characterization of the hull $\operatorname{Rco}_f E$ for a given compact set E. Based on the following result, we will investigate in Section 3 sufficient conditions for the relaxation property (cf. Definition 3.1) which is the key to apply the Baire categories method for vectorial differential inclusions due to Dacorogna and Marcellini [9].

Before stating the result we give a definition.

Definition 2.7 *Let U be a subset of $\mathbb{R}^{N\times n}$ and, for some integer $I \geq 1$, let $\xi_i \in \mathbb{R}^{N\times n}$ and $\lambda_i > 0$, $i = 1, ..., I$ be such that $\sum_{i=1}^{I} \lambda_i = 1$. We say that $(\lambda_i, \xi_i)_{1\leq i\leq I}$ satisfy $(H_I(U))$ if*

(i) in the case $I = 1$, $\xi_1 \in U$;

(ii) in the case $I = 2$, $\xi_1, \xi_2 \in U$ and $\operatorname{rank}(\xi_1 - \xi_2) = 1$;

(iii) in the case $I > 2$, up to a permutation, $\xi_1, \xi_2 \in U$, $\operatorname{rank}(\xi_1 - \xi_2) = 1$ and defining

$$\begin{cases} \mu_1 = \lambda_1 + \lambda_2, & \eta_1 = \frac{\lambda_1\xi_1+\lambda_2\xi_2}{\lambda_1+\lambda_2} \\ \mu_i = \lambda_{i+1}, & \eta_i = \xi_{i+1},\ 2 \leq i \leq I-1 \end{cases}$$

then $(\mu_i, \eta_i)_{1\leq i\leq I-1}$ satisfy $(H_{I-1}(U))$.

Remark 2.8 *The property in the above definition was introduced in [7, page 174], but here we have the additional condition that the vertices of the "chain" must be elements of a given set. Moreover, we notice that in the above definition, in particular, all $\xi_i \in U$.*

Theorem 2.9 *Let $E \subset \mathbb{R}^{N\times n}$ be a compact set and let U be an open and bounded subset of $\mathbb{R}^{N\times n}$ containing $\operatorname{Rco}_f E$. Then*

$$\operatorname{Rco}_f E = \left\{ \begin{array}{l} \xi \in \mathbb{R}^{N\times n} :\ \forall\, \varepsilon > 0\ \exists\, I \in \mathbb{N},\ \exists\, (\lambda_i, \xi_i)_{i=1,...,I} \text{ with } \lambda_i > 0, \\ \displaystyle\sum_{i=1}^{I} \lambda_i = 1 \text{ satisfying } (H_I(U)),\ \xi = \sum_{i=1}^{I} \lambda_i \xi_i,\quad \sum_{\substack{i=1 \\ \xi_i \notin B_\varepsilon(E)}}^{I} \lambda_i < \varepsilon \end{array} \right\}, \tag{2.5}$$

where $B_\varepsilon(E) = \{\xi \in \mathbb{R}^{N\times n} :\ \operatorname{dist}(\xi; E) < \varepsilon\}$.

Remark 2.10 *1) The fact that $\operatorname{Rco}_f E$ is included in the set on the right hand side of (2.5) was obtained by means of Young measures by Müller and Šverák [18, Theorem 2.1] as a refinement of a result due to Pedregal [20]. Below we recall the proof without mentioning Young measures.*

2) Since, for compact sets E, $\operatorname{Rco}_f E \subseteq \operatorname{co} E$, the set U can be chosen to be any convex open set containing E.

3) This result should be compared with the following characterization of $\operatorname{Rco} E$ which follows trivially from (2.2): for any set $E \subset \mathbb{R}^{N\times n}$,

$$\operatorname{Rco} E = \left\{ \begin{array}{l} \xi \in \mathbb{R}^{N\times n} :\ \exists\, I \in \mathbb{N},\ \exists\, (\lambda_i, \xi_i)_{i=1,...,I} \text{ with } \lambda_i > 0,\ \displaystyle\sum_{i=1}^{I} \lambda_i = 1 \\ \text{satisfying } (H_I(\operatorname{Rco} E)),\ \xi = \displaystyle\sum_{i=1}^{I} \lambda_i \xi_i,\quad \xi_i \in E,\ \forall\, i = 1, ..., I \end{array} \right\}.$$

Proof. Let us call X the set on the right hand side of the identity to be proved.

First we show that $X \subseteq \operatorname{Rco}_f E$. Let $\xi \in X$ and let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be any rank one convex function such that $f_{|E} \leq 0$. We will show that $f(\xi) \leq 0$ by verifying that $f(\xi) \leq \delta$ for any $\delta > 0$.

We start by noticing that, since f is continuous, it is uniformly continuous in $\overline{U}$. Thus, fix $\delta > 0$ and let $\gamma > 0$ be such that

$$\forall\ \eta_1, \eta_2 \in \overline{U},\ |\eta_1 - \eta_2| \leq \gamma \ \Rightarrow\ |f(\eta_1) - f(\eta_2)| \leq \delta.$$

Now, let $\varepsilon > 0$ be such that $\varepsilon \leq \min\{\delta, \gamma\}$ and $B_\varepsilon(E) \subset \overline{U}$. By definition of X, let $I_\varepsilon \in \mathbb{N}$, $(\lambda_i^\varepsilon, \xi_i^\varepsilon)_{i=1,\dots,I_\varepsilon}$ with $\lambda_i^\varepsilon > 0$, $\displaystyle\sum_{i=1}^{I_\varepsilon} \lambda_i^\varepsilon = 1$ satisfying $(H_{I_\varepsilon}(U))$, be such that $\xi = \displaystyle\sum_{i=1}^{I_\varepsilon} \lambda_i^\varepsilon \xi_i^\varepsilon$, $\displaystyle\sum_{\substack{i=1 \\ \xi_i^\varepsilon \notin B_\varepsilon(E)}}^{I_\varepsilon} \lambda_i^\varepsilon < \varepsilon$. Then, using the rank one convexity of f, we have

$$f(\xi) = f\left(\sum_{i=1}^{I_\varepsilon} \lambda_i^\varepsilon \xi_i^\varepsilon\right) \leq \sum_{i=1}^{I_\varepsilon} \lambda_i^\varepsilon f(\xi_i^\varepsilon) = \sum_{\substack{i=1 \\ \xi_i^\varepsilon \notin B_\varepsilon(E)}}^{I_\varepsilon} \lambda_i^\varepsilon f(\xi_i^\varepsilon) + \sum_{\substack{i=1 \\ \xi_i^\varepsilon \in B_\varepsilon(E)}}^{I_\varepsilon} \lambda_i^\varepsilon f(\xi_i^\varepsilon).$$

We are going to estimate the last two sums. For the first one we have

$$\sum_{\substack{i=1 \\ \xi_i^\varepsilon \notin B_\varepsilon(E)}}^{I_\varepsilon} \lambda_i^\varepsilon f(\xi_i^\varepsilon) \leq C \sum_{\substack{i=1 \\ \xi_i^\varepsilon \notin B_\varepsilon(E)}}^{I_\varepsilon} \lambda_i^\varepsilon \leq C\delta,$$

where $C := \max\limits_{\overline{U}} f$. For the second sum we use the uniform continuity of f in $\overline{U}$. Since $\xi_i^\varepsilon \in B_\varepsilon(E)$, we can consider $\eta_i^\varepsilon \in E$ such that $|\eta_i^\varepsilon - \xi_i^\varepsilon| < \varepsilon$. Then,

$$f(\xi_i^\varepsilon) \leq f(\xi_i^\varepsilon) - f(\eta_i^\varepsilon) \leq |f(\xi_i^\varepsilon) - f(\eta_i^\varepsilon)| \leq \delta.$$

We then conclude that $f(\xi) \leq (1 + C)\delta$. Since δ is arbitrarily small, we obtain $f(\xi) \leq 0$, as we wanted.

We will now prove the other inclusion, $\operatorname{Rco}_f E \subseteq X$. We suppose by contradiction that $\xi \in \operatorname{Rco}_f E$ and $\xi \notin X$. Then, there exists $\varepsilon > 0$ such that, for every $I \in \mathbb{N}$ and for every (λ_i, ξ_i) satisfying $(H_I(U))$ with $\xi = \sum_{i=1}^{I} \lambda_i \xi_i$ we have

$$\sum_{\substack{i=1 \\ \xi_i \notin B_\varepsilon(E)}}^{I} \lambda_i \geq \varepsilon.$$

Defining, for $\eta \in U$, $f(\eta) := \operatorname{dist}(\eta; E)$ and

$$g(\eta) := \inf\left\{ \begin{array}{c} \displaystyle\sum_{i=1}^{I} \mu_i f(\eta_i) : \quad I \in \mathbb{N},\ (\mu_i, \eta_i)_{i=1,\dots,I} \text{ with } \mu_i > 0,\ \sum_{i=1}^{I} \mu_i = 1 \\ \text{satisfying } (H_I(U)) \end{array} \right\},$$

it is trivial to see that $0 \leq g < +\infty$ and $g_{|E} = 0$. Moreover, for any $(\mu_i, \eta_i)_{i=1,\dots,I}$ as in the definition of $g(\xi)$, the contradiction assumption gives

$$\sum_{i=1}^{I} \mu_i f(\eta_i) \geq \sum_{\substack{i=1\\ \eta_i \notin B_\varepsilon(E)}}^{I} \mu_i f(\eta_i) \geq \sum_{\substack{i=1\\ \eta_i \notin B_\varepsilon(E)}}^{I} \mu_i \varepsilon \geq \varepsilon^2 > 0$$

and therefore $g(\xi) > 0$. Finally, we show that g is rank one convex on U according to the definition in [18], that is, if $A, B \in U$ with $\mathrm{rank}(A-B) = 1$ and $\lambda A + (1-\lambda)B \in U$, $\forall\, \lambda \in (0,1)$ then $g(\lambda A + (1-\lambda)B) \leq \lambda g(A) + (1-\lambda) g(B)$. To achieve this, it is enough to observe that if $(\mu_i, A_i)_{i=1,\dots,I}$ and $(\gamma_i, B_i)_{i=1,\dots,J}$ are as in the definition of $g(A)$ and $g(B)$, respectively, then $((\lambda\mu_i, A_i), ((1-\lambda)\gamma_i, B_i))$ satisfy the conditions in the definition of $g(\lambda A + (1-\lambda)B)$.

These properties of g allow us to apply the extension result [18, Lemma 2.3] which ensures that there exists a rank one convex function $G : \mathbb{R}^{N\times n} \to \mathbb{R}$ coinciding with g in a neighborhood of $\mathrm{Rco}_f\, E$. This yields the desired contradiction, since we are assuming that $\xi \in \mathrm{Rco}_f\, E$. □

3 Sufficient conditions for the relaxation property

The Baire categories method, developed by Dacorogna and Marcellini [9] for solving vectorial differential inclusions, relies on a fundamental property, called relaxation property, cf. Definition 3.1 below. Due to the difficulty in dealing with this property in the applications, sufficient conditions for it were also obtained in [9]. They ensure existence of solutions to the differential inclusion boundary value problem when the gradient of the boundary data is in the interior of the rank one convex hull of the set where the differential inclusion is to be solved. However, in some examples this hull turns out to be too restrictive. Therefore, our goal in this section is to find more flexible sufficient conditions for the relaxation property. This will allow us to handle the problems considered in Section 4.

We start by recalling the relaxation property introduced by Dacorogna and Marcellini [9] and their related existence theorem for differential inclusions, here in a more general version due to Dacorogna and Pisante [10].

Definition 3.1 (Relaxation Property) *Let E, $K \subset \mathbb{R}^{N\times n}$. We say that K has the relaxation property with respect to E if, for every bounded open set $\Omega \subset \mathbb{R}^n$ and for every affine function u_ξ, such that $Du_\xi(x) = \xi$ and $Du_\xi(x) \in K$, there exists a sequence $u_\nu \in Aff_{piec}(\overline{\Omega}; \mathbb{R}^N)$ such that*

$$u_\nu \in u_\xi + W_0^{1,\infty}(\Omega; \mathbb{R}^N), \quad Du_\nu(x) \in E \cup K, \text{ a.e. } x \text{ in } \Omega,$$

$$u_\nu \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{1,\infty}(\Omega; \mathbb{R}^N), \quad \lim_{\nu\to+\infty} \int_\Omega \mathrm{dist}(Du_\nu(x); E)\, dx = 0.$$

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $E \subset \mathbb{R}^{N\times n}$ and $K \subset \mathbb{R}^{N\times n}$ be compact and bounded, respectively. Assume that K has the relaxation property with respect to E. Let $\varphi \in Aff_{piec}\left(\overline{\Omega};\mathbb{R}^N\right)$ be such that*

$$D\varphi(x) \in E \cup K, \text{ a.e. } x \text{ in } \Omega.$$

Then there exists (a dense set of) $u \in \varphi + W_0^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$ such that

$$Du(x) \in E, \text{ a.e. } x \text{ in } \Omega.$$

Moreover, if K is open, φ can be taken in $C^1_{piec}(\overline{\Omega};\mathbb{R}^N)$.

A sufficient condition for the relaxation property is the approximation property [9, Definition 6.12 and Theorem 6.14] (see also [7, Definition 10.6 and Theorem 10.9]) that we recall next.

Definition 3.3 (Approximation property) *Let $E \subset K \subset \mathbb{R}^{N\times n}$. The sets E and K are said to have the approximation property if there exists a family of closed sets E_δ and K_δ, $\delta > 0$, such that*

(i) $E_\delta \subset K_\delta \subset \operatorname{int} K$ *for every* $\delta > 0$;

(ii) $\forall\, \varepsilon > 0\ \exists\, \delta_0 > 0 :\ \operatorname{dist}(\eta; E) \leq \varepsilon,\ \forall\, \eta \in E_\delta,\ \delta \in (0, \delta_0]$;

(iii) $\eta \in \operatorname{int} K \ \Rightarrow\ \exists\, \delta_0 > 0 :\ \eta \in K_\delta,\ \forall\, \delta \in (0, \delta_0]$.

We therefore have the following theorem.

Theorem 3.4 *Let $E \subset \mathbb{R}^{N\times n}$ be compact and assume $\operatorname{Rco} E$ has the approximation property with $K_\delta = \operatorname{Rco} E_\delta$. Then $\operatorname{int}\operatorname{Rco} E$ has the relaxation property with respect to E.*

In the spirit of the approximation property, we establish a sufficient condition for the relaxation property such that larger sets than the rank one convex hull of E can be considered as the set K in Theorem 3.2. More precisely, we will show that hulls like $\operatorname{Rco}_f E$ are likely to enjoy the relaxation property.

We can now state our main theorem.

Theorem 3.5 *Let E, K be two bounded subsets of $\mathbb{R}^{N\times n}$ and let, for $\delta > 0$, E_δ, K_δ be sets verifying the following conditions:*

(i) $\forall\, \varepsilon > 0\ \exists\, \delta_0 > 0 :\ \operatorname{dist}(\eta; E) \leq \varepsilon,\ \forall\, \eta \in E_\delta,\ \delta \in (0, \delta_0]$;

(ii) $\eta \in \operatorname{int} K \ \Rightarrow\ \exists\, \delta_0 > 0 :\ \eta \in K_\delta,\ \forall\, \delta \in (0, \delta_0]$;

(iii) $\forall\, \delta > 0\ \forall\, \xi \in K_\delta\ \exists\, I \in \mathbb{N},\ \exists\, (\lambda_i, \xi_i)_{1\leq i\leq I}$ *with* $\lambda_i > 0,\ \sum_{i=1}^{I} \lambda_i = 1,$

$$\xi_i \in \mathbb{R}^{N\times n}, \text{ satisfying } (H_I(\operatorname{int} K)) \text{ and } \xi = \sum_{i=1}^{I} \lambda_i \xi_i,\ \sum_{\substack{i=1\\ \xi_i \notin E_\delta}}^{I} \lambda_i < \delta.$$

Then $\operatorname{int} K$ has the relaxation property with respect to E.

Remark 3.6 *1) In the sense of Proposition 3.13, Theorem 3.5 is a generalization of the usual approximation property for the hulls* Rco_f*.*

2) This result is analogous to Theorem 6.15 in [9] and allows us to work with subsets of $\mathrm{Rco}_f\, E$*.*

Before proving the result we establish two corollaries.

Corollary 3.7 *Let* $E \subset \mathbb{R}^{N\times n}$ *be bounded. Assume that there exist compact sets* $E_\delta \subset \mathbb{R}^{N\times n}$ *such that, defining* $K_\delta = \mathrm{Rco}_f\, E_\delta$ *and* $K = \mathrm{Rco}_f\, E$*,* $K_\delta \subset \mathrm{int}\, K$ *and conditions* (i) *and* (ii) *of Theorem 3.5 are satisfied. Then* $\mathrm{int}\,\mathrm{Rco}_f\, E$ *has the relaxation property with respect to* E*.*

Remark 3.8 *One can easily see that Theorem 3.5 is also true if in condition* (iii) *we replace* E_δ *by* $B_\delta(E)$*. In this case, Corollary 3.7 follows directly from Theorem 3.5 and Proposition 3.13 below. In the proof that we present here we have to consider artificial approximating sets* E_δ*.*

Proof of Corollary 3.7. Let $\widetilde{E}_\delta = B_\delta(E_\delta)$ and let $\widetilde{K}_\delta = \mathrm{Rco}_f\, E_\delta$. We will show the result as an application of Theorem 3.5 for this choice of approximating sets.

By the hypotheses on E_δ and K_δ one can easily see that conditions (i) and (ii) of Theorem 3.5 are still satisfied by $\widetilde{E}_\delta$ and $\widetilde{K}_\delta$. Condition (iii) follows from Theorem 2.9 applied to $\mathrm{Rco}_f\, E_\delta$, noticing that we are assuming that E_δ is compact, and taking $U = \mathrm{int}\,\mathrm{Rco}_f\, E$ which contains $\mathrm{Rco}_f\, E_\delta$ by hypothesis. □

The following result was already proved by Ribeiro [21].

Corollary 3.9 *Let* $E, K \subset \mathbb{R}^{N\times n}$ *be such that* E *is compact and* K *is bounded. Assume that the following condition holds:*

(H) *given* $\delta > 0$*, there exists* $L = L(\delta, E, K) \in \mathbb{N}$ *such that*

$$\begin{aligned}
&\forall\, \xi \in \mathrm{int}\, K \setminus B_\delta(E)\\
&\exists\, \eta_1, ..., \eta_J \in \mathbb{R}^{N\times n},\ J \in \mathbb{N},\ J \le L,\ \mathrm{rank}(\eta_j) = 1,\ j = 1, ..., J\\
&[\xi + \eta_1 + ... + \eta_{j-1} - \eta_j, \xi + \eta_1 + ... + \eta_{j-1} + \eta_j] \subset \mathrm{int}\, K,\ j = 1, ..., J,\\
&\xi + \eta_1 + ... + \eta_J \in B_\delta(E),
\end{aligned}$$

where $[A, B]$ *represents the segment joining the matrices* A *and* B*. Then* $\mathrm{int}\, K$ *has the relaxation property with respect to* E*.*

Remark 3.10 *Using the same ideas of the following proof, it turns out that, under the conditions of Corollary 3.9, the set* K *is contained in* $\mathrm{Rco}_f\, E$*.*

Proof of Corollary 3.9. We will prove that

$$E_\delta = \mathrm{int}\, K \cap B_\delta(E) \qquad \text{and} \qquad K_\delta = (\mathrm{int}\, K \cap E) \cup (\mathrm{int}\, K \setminus B_\delta(E))$$

satisfy conditions (i), (ii) and (iii) of Theorem 3.5.

Condition (i) is trivial. To get condition (ii), we observe that if $\eta \in \operatorname{int} K$ then either $\eta \in E$, and thus $\eta \in K_\delta$ for every $\delta > 0$, or $\eta \notin E$. In this last case, since E is compact, $\operatorname{dist}(\eta; E) > 0$ which entails (ii).

It remains to show condition (iii). Let $\delta > 0$, $\xi \in K_\delta$ and consider $L = L(\delta, E, K) \in \mathbb{N}$ as in the hypothesis. If $\xi \in \operatorname{int} K \cap E$, then condition (iii) is satisfied with $I = 1$ and $(\lambda_i, \xi_i)_{i=1} = (1, \xi)$ and we are left with the case $\xi \in \operatorname{int} K \setminus B_\delta(E)$. Applying the hypothesis, we have

$$\begin{aligned}
&\exists\ \eta_1, ..., \eta_J \in \mathbb{R}^{N\times n},\ J \in \mathbb{N},\ J \le L,\ \operatorname{rank}(\eta_j) = 1,\ j = 1, ..., J \\
&[\xi + \eta_1 + ... + \eta_{j-1} - \eta_j, \xi + \eta_1 + ... + \eta_{j-1} + \eta_j] \subset \operatorname{int} K,\ j = 1, ..., J, \\
&\xi + \eta_1 + ... + \eta_J \in B_\delta(E).
\end{aligned}$$

Thus, by iteratively writing convex combinations using the matrices η_i, $i = 1, \cdots, J$, we obtain

$$\xi = \frac{1}{2}(\xi - \eta_1) + \frac{1}{2^2}(\xi + \eta_1 - \eta_2) + ... + \frac{1}{2^J}(\xi + \eta_1 + ... + \eta_{J-1} - \eta_J) + \frac{1}{2^J}(\xi + \eta_1 + ... + \eta_J). \tag{3.1}$$

We notice that if we take

$$\begin{cases} \lambda_j = \frac{1}{2^j},\ \text{if } 1 \le j \le J \\ \lambda_{J+1} = \frac{1}{2^J} \end{cases} \quad \text{and} \quad \begin{cases} \xi_j = \xi + \eta_1 + ... + \eta_{j-1} - \eta_j,\ \text{if } 1 \le j \le J \\ \xi_{J+1} = \xi + \eta_1 + ... + \eta_J, \end{cases}$$

then $(\lambda_j, \xi_j)_{1 \le j \le J+1}$ satisfy $(H_{J+1}(\operatorname{int} K))$ and (3.1) can be rewritten in the form

$$\xi = \sum_{j=1}^{J+1} \lambda_j \xi_j, \quad \text{with} \quad \sum_{\substack{j=1 \\ \xi_j \notin B_\delta(E)}}^{J+1} \lambda_j \le 1 - \frac{1}{2^J} \le 1 - \frac{1}{2^L}.$$

If all $\xi_j \in B_\delta(E)$, then $\displaystyle\sum_{\substack{j=1 \\ \xi_j \notin B_\delta(E)}}^{J+1} \lambda_j = 0$ and we have achieved condition (iii). Otherwise, for each $\xi_j \in \operatorname{int} K \setminus B_\delta(E)$ we apply again the hypothesis and get, for some $I_j \le L$,

$$\xi_j = \sum_{l=1}^{I_j+1} \widetilde{\lambda}_l^j \widetilde{\xi}_l^j \ \text{ such that } \ (\widetilde{\lambda}_l^j, \widetilde{\xi}_l^j)_{1 \le l \le I_j+1} \text{ satisfy } (H_{I_j+1}(\operatorname{int} K))$$

and

$$\sum_{\substack{l=1 \\ \widetilde{\xi}_l^j \notin B_\delta(E)}}^{I_j+1} \widetilde{\lambda}_l^j \le 1 - \frac{1}{2^L}.$$

Therefore

$$\xi = \sum_{\substack{j=1 \\ \xi_j \in B_\delta(E)}}^{J+1} \lambda_j \xi_j + \sum_{\substack{j=1 \\ \xi_j \notin B_\delta(E)}}^{J+1} \sum_{l=1}^{I_j+1} \lambda_j \widetilde{\lambda}_l^j \widetilde{\xi}_l^j\ ,$$

where the scalars and matrices in the above expression satisfy $(H_{\tilde{I}}(\operatorname{int} K))$ for a certain $\tilde{I} \in \mathbb{N}$, and

$$\sum_{\substack{j=1 \\ \xi_j \notin B_\delta(E)}}^{J+1} \sum_{\substack{l=1 \\ \tilde{\xi}_l^j \notin B_\delta(E)}}^{I_j+1} \lambda_j \widetilde{\lambda}_l^j \leq \sum_{\substack{j=1 \\ \xi_j \notin B_\delta(E)}}^{J+1} \lambda_j \left(1 - \frac{1}{2^L}\right) \leq \left(1 - \frac{1}{2^L}\right)^2.$$

Of course, after a finite number of iterations of this procedure one gets condition (iii). $\square$

In order to prove Theorem 3.5 we will show the following lemma.

Lemma 3.11 *Let $I > 1$ be an integer and let $U \subset \mathbb{R}^{N\times n}$ be an open set. For $1 \leq i \leq I$, let $\xi_i \in \mathbb{R}^{N\times n}$ and $\lambda_i > 0$ be such that $\sum_{i=1}^{I} \lambda_i = 1$ and $(\lambda_i, \xi_i)_{1\leq i\leq I}$ satisfy $(H_I(U))$. Denote by ξ the sum $\sum_{i=1}^{I} \lambda_i \xi_i$ and let u_ξ be an affine map such that $Du_\xi = \xi$. Then, for any given $\varepsilon > 0$ and any bounded open set $\Omega \subset \mathbb{R}^n$, there exist $u_\varepsilon \in Aff_{piec}(\overline{\Omega}; \mathbb{R}^N)$ and disjoint open sets $\Omega_\varepsilon^i \subset \Omega$ such that*

$$u_\varepsilon \in u_\xi + W_0^{1,\infty}(\Omega; \mathbb{R}^N),$$
$$Du_\varepsilon(x) \in U \cup B_\varepsilon(\xi), \text{ a.e. } x \in \Omega, \quad Du_\varepsilon(x) = \xi_i, \text{ a.e. } x \in \Omega_\varepsilon^i, \ i = 1, ..., I,$$
$$||u_\varepsilon - u_\xi||_{L^\infty} \leq \varepsilon, \quad |\operatorname{meas}(\Omega_\varepsilon^i) - \lambda_i \operatorname{meas}(\Omega)| \leq \varepsilon, \ i = 1, ..., I.$$

The proof of the lemma relies on the following approximation result, due to Müller and Sychev [19, Lemma 3.1], which is a refinement of a classical result. For $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ we will denote by $a \otimes b$ the $N \times n$ matrix whose (i, j) entry is $a_i b_j$.

Lemma 3.12 (Approximation lemma) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $A, B \in \mathbb{R}^{N\times n}$ be such that $A - B = a \otimes b$, with $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$. Let $b_3, \ldots, b_k \in \mathbb{R}^n$, $k \geq n$, be such that $0 \in \operatorname{int} \operatorname{co}\{b, -b, b_3, \ldots, b_k\}$ and, for $t \in [0, 1]$, let φ be an affine map such that*

$$D\varphi(x) = \xi = tA + (1 - t)B, \ x \in \overline{\Omega}$$

(i.e. $A = \xi + (1 - t)\, a \otimes b$ and $B = \xi - ta \otimes b$). Then, for every $\varepsilon > 0$, there exists $u \in Aff_{piec}(\overline{\Omega}; \mathbb{R}^N)$ and there exist disjoint open sets $\Omega_A, \Omega_B \subset \Omega$, such that

$$\begin{cases} |\operatorname{meas}(\Omega_A) - t \ \operatorname{meas}(\Omega)| \leq \varepsilon, \ |\operatorname{meas}(\Omega_B) - (1 - t) \ \operatorname{meas}(\Omega)| \leq \varepsilon \\ u(x) = \varphi(x), \ x \in \partial\Omega \\ |u(x) - \varphi(x)| \leq \varepsilon, \ x \in \Omega \\ Du(x) = \begin{cases} A & \text{in } \Omega_A \\ B & \text{in } \Omega_B \end{cases} \\ Du(x) \in \xi + \{(1 - t)\, a \otimes b, -ta \otimes b, a \otimes b_3, \ldots, a \otimes b_k\}, \ \text{a.e. } x \text{ in } \Omega. \end{cases}$$

Proof of Lemma 3.11. We prove the result by induction on I.

If $I = 2$ it suffices to apply Lemma 3.12 choosing $|b_3|, ..., |b_k|$ sufficiently small so that $|a \otimes b_i| \le \varepsilon$ for $i = 3, ..., k$.

Now, let $I > 2$ and consider $(\lambda_i, \xi_i)_{1 \le i \le I}$ as in the hypothesis. Up to a permutation, and defining

$$\begin{cases} \mu_1 = \lambda_1 + \lambda_2, & \eta_1 = \frac{\lambda_1 \xi_1 + \lambda_2 \xi_2}{\lambda_1 + \lambda_2} \\ \mu_i = \lambda_{i+1}, & \eta_i = \xi_{i+1},\ 2 \le i \le I-1, \end{cases}$$

we have $\xi_1, \xi_2 \in U$, $\text{rank}(\xi_1 - \xi_2) = 1$ and $(\mu_i, \eta_i)_{1 \le i \le I-1}$ satisfy $(H_{I-1}(U))$. Then the induction hypothesis ensures the existence of $v \in Aff_{piec}(\overline{\Omega}; \mathbb{R}^N)$ and disjoint open sets $\widetilde{\Omega}^i_\varepsilon \subset \Omega,\ i = 1, ..., I-1$ such that

$$\begin{aligned} & v \in u_\xi + W^{1,\infty}_0(\Omega; \mathbb{R}^N), \\ & Dv(x) \in U \cup B_\varepsilon(\xi),\ a.e.\ x \in \Omega, \quad Dv(x) = \eta_i,\ a.e.\ x \in \widetilde{\Omega}^i_\varepsilon,\ i = 1, ..., I-1 \\ & ||v - u_\xi||_{L^\infty} \le \tfrac{\varepsilon}{2}, \quad |\text{meas}(\widetilde{\Omega}^i_\varepsilon) - \mu_i \text{meas}(\Omega)| \le \tfrac{\varepsilon}{2},\ i = 1, ..., I-1. \end{aligned}$$

Since $(\mu_i, \eta_i)_{1 \le i \le I-1}$ satisfy $(H_{I-1}(U))$, then $\eta_1 \in U$. Now let $0 < \delta < \varepsilon$ be such that the neighborhood of η_1, $\overline{B_\delta(\eta_1)}$, is contained in U and apply again Lemma 3.12 in $\widetilde{\Omega}^1_\varepsilon$ to obtain $w \in Aff_{piec}(\overline{\widetilde{\Omega}^1_\varepsilon}; \mathbb{R}^N)$ and disjoint open sets $\Omega^i_\varepsilon \subset \widetilde{\Omega}^1_\varepsilon,\ i = 1, 2$ such that

$$\begin{aligned} & w \in v + W^{1,\infty}_0(\widetilde{\Omega}^1_\varepsilon; \mathbb{R}^N), \\ & Dw(x) \in \{\xi_1, \xi_2\} \cup B_\delta(\eta_1),\ a.e.\ x \in \widetilde{\Omega}^1_\varepsilon, \quad Dw(x) = \xi_i,\ a.e.\ x \in \Omega^i_\varepsilon,\ i = 1, 2 \\ & ||w - v||_{L^\infty} \le \frac{\varepsilon}{2}, \quad \left| \text{meas}(\Omega^i_\varepsilon) - \frac{\lambda_i}{\lambda_1 + \lambda_2} \text{meas}(\widetilde{\Omega}^1_\varepsilon) \right| \le \frac{\varepsilon}{2},\ i = 1, 2. \end{aligned} \tag{3.2}$$

We then obtain the desired result taking Ω^1_ε and Ω^2_ε as above, $\Omega^i_\varepsilon = \widetilde{\Omega}^{i-1}_\varepsilon$ for $i = 3, ..., I$ and

$$u_\varepsilon = \begin{cases} v \text{ in } \Omega \setminus \widetilde{\Omega}^1_\varepsilon, \\ w \text{ in } \widetilde{\Omega}^1_\varepsilon. \end{cases}$$

In fact we only need to verify that $|\text{meas}(\Omega^i_\varepsilon) - \lambda_i \text{meas}(\Omega)| \le \varepsilon$ for $i = 1, 2$:

$$\begin{aligned} |\text{meas}(\Omega^i_\varepsilon) - \lambda_i \text{meas}(\Omega)| & \le \left| \text{meas}(\Omega^i_\varepsilon) - \tfrac{\lambda_i}{\lambda_1 + \lambda_2} \text{meas}(\widetilde{\Omega}^1_\varepsilon) \right| + \\ & \quad + \left| \tfrac{\lambda_i}{\lambda_1 + \lambda_2} \text{meas}(\widetilde{\Omega}^1_\varepsilon) - \lambda_i \text{meas}(\Omega) \right| \\ & \le \tfrac{\varepsilon}{2} + \tfrac{\lambda_i}{\lambda_1 + \lambda_2} \left| \text{meas}(\widetilde{\Omega}^1_\varepsilon) - (\lambda_1 + \lambda_2) \text{meas}(\Omega) \right| \\ & = \tfrac{\varepsilon}{2} + \tfrac{\lambda_i}{\lambda_1 + \lambda_2} \left| \text{meas}(\widetilde{\Omega}^1_\varepsilon) - \mu_1 \text{meas}(\Omega) \right| \\ & \le \varepsilon, \end{aligned}$$

where we have used (3.2). □

We can now prove Theorem 3.5.

Proof of Theorem 3.5. Let Ω be an open bounded subset of $\mathbb{R}^n$, $\xi \in \operatorname{int} K$ and let us denote by u_ξ an affine map such that $Du_\xi = \xi$. We want to construct a sequence $u_\varepsilon \in Aff_{piec}(\overline{\Omega};\mathbb{R}^N)$ such that

$$u_\varepsilon \in u_\xi + W_0^{1,\infty}(\Omega;\mathbb{R}^N), \quad Du_\varepsilon(x) \in E \cup \operatorname{int} K, \ a.e.\ x \in \Omega,$$
$$u_\varepsilon \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{1,\infty}(\Omega;\mathbb{R}^N), \quad \lim_{\varepsilon \to 0^+} \int_\Omega \operatorname{dist}(Du_\varepsilon(x);E)\,dx = 0.$$

Fix $\varepsilon > 0$. From condition (ii), and since $\xi \in \operatorname{int} K$, we have $\xi \in K_\delta$ for $\delta \le \delta_0$. Choose $0 < \delta \le \delta_0$ such that $\delta < \varepsilon$ and

$$\operatorname{dist}(\eta;E) \le \varepsilon, \ \forall\, \eta \in E_\delta. \tag{3.3}$$

This is possible from condition (i). We then apply condition (iii) to obtain $I = I(\delta) \in \mathbb{N}$, $(\lambda_i,\xi_i)_{1\le i\le I}$ with $\lambda_i > 0$, $\sum_{i=1}^I \lambda_i = 1$, $\xi_i \in \mathbb{R}^{N\times n}$ satisfying $(H_I(\operatorname{int} K))$ and such that

$$\xi = \sum_{i=1}^{I} \lambda_i \xi_i \quad \text{and} \quad \sum_{\substack{i=1\\ \xi_i \notin E_\delta}}^{I} \lambda_i < \delta. \tag{3.4}$$

By Lemma 3.11 we now get $u_\varepsilon \in Aff_{piec}(\overline{\Omega};\mathbb{R}^N)$ and disjoint open sets $\Omega_\varepsilon^i \subset \Omega$ such that, for ε sufficiently small,

$$\begin{aligned} &u_\varepsilon \in u_\xi + W_0^{1,\infty}(\Omega;\mathbb{R}^N),\\ &Du_\varepsilon(x) \in \operatorname{int} K,\ a.e.\ x \in \Omega, \quad Du_\varepsilon(x) = \xi_i,\ a.e.\ x \in \Omega_\varepsilon^i,\ i = 1,...,I,\\ &||u_\varepsilon - u_\xi||_{L^\infty} \le \varepsilon, \quad |\operatorname{meas}(\Omega_\varepsilon^i) - \lambda_i \operatorname{meas}(\Omega)| \le \frac{\varepsilon}{I},\ i = 1,...,I. \end{aligned} \tag{3.5}$$

Since K is bounded, up to a subsequence, we have $u_\varepsilon \overset{*}{\rightharpoonup} u_\xi$ in $W^{1,\infty}(\Omega;\mathbb{R}^N)$. We will finish the proof by verifying that

$$\lim_{\varepsilon \to 0^+} \int_\Omega \operatorname{dist}(Du_\varepsilon(x);E)\,dx = 0.$$

Since E and K are bounded there exists a positive constant c such that

$$\operatorname{dist}(\eta;E) \le c, \ \forall\, \eta \in \operatorname{int} K. \tag{3.6}$$

Then, using (3.3), (3.6), (3.5) and (3.4), we obtain the following estimates,

$$\begin{aligned}
&\int_\Omega \operatorname{dist}(Du_\varepsilon(x);E)\,dx = \\
&= \sum_{\substack{i=1\\ \xi_i\in E_\delta}}^{I} \int_{\Omega_\varepsilon^i} \operatorname{dist}(\xi_i;E)\,dx + \sum_{\substack{i=1\\ \xi_i\notin E_\delta}}^{I} \int_{\Omega_\varepsilon^i} \operatorname{dist}(\xi_i;E)\,dx + \\
&\qquad + \int_{\Omega\setminus\left(\cup_{i=1}^I \Omega_\varepsilon^i\right)} \operatorname{dist}(Du_\varepsilon(x);E)\,dx \\
&\le \varepsilon\operatorname{meas}(\Omega) + c \sum_{\substack{i=1\\ \xi_i\notin E_\delta}}^{I} \operatorname{meas}(\Omega_\varepsilon^i) + c\operatorname{meas}\left(\Omega\setminus\left(\cup_{i=1}^I \Omega_\varepsilon^i\right)\right) \\
&\le \varepsilon\operatorname{meas}(\Omega) + c \sum_{\substack{i=1\\ \xi_i\notin E_\delta}}^{I} \left(\frac{\varepsilon}{I} + \lambda_i \operatorname{meas}(\Omega)\right) + c\,\varepsilon \\
&\le \varepsilon\operatorname{meas}(\Omega) + c\,\varepsilon + c\,\varepsilon\operatorname{meas}(\Omega) + c\,\varepsilon.
\end{aligned}$$

This completes the proof. □

As already mentioned, the characterization of $\operatorname{Rco}_f E$ obtained in Section 2 entails a similar condition to condition (*iii*) of Theorem 3.5 under the approximation property assumption. This is stated in the next proposition.

Proposition 3.13 *Let $E \subset \mathbb{R}^{N\times n}$ be a bounded set and for $\delta > 0$ let E_δ be compact sets such that $\operatorname{Rco}_f E_\delta \subset \operatorname{int}\operatorname{Rco}_f E$ and*

(i) $\forall\ \varepsilon > 0\ \exists\ \delta_0 > 0 :\ \operatorname{dist}(\eta;E) \le \varepsilon, \forall\ \eta \in E_\delta,\ \delta \in (0,\delta_0]$;

(ii) $\eta \in \operatorname{int}\operatorname{Rco}_f E \ \Rightarrow\ \exists\ \delta_0 > 0 :\ \eta \in \operatorname{Rco}_f E_\delta,\ \forall\ \delta \in (0,\delta_0]$.

Then the following condition is satisfied:

$$\forall\ \delta > 0\ \forall\ \xi \in \operatorname{Rco}_f E_\delta\ \exists\ I \in \mathbb{N},\ \exists\ (\lambda_i,\xi_i)_{1\le i\le I} \text{ with } \lambda_i > 0,\ \sum_{i=1}^I \lambda_i = 1,$$

$$\xi_i \in \mathbb{R}^{N\times n},\ \text{ satisfying } (H_I(\operatorname{int}\operatorname{Rco}_f E)) \text{ and } \xi = \sum_{i=1}^I \lambda_i\xi_i,\quad \sum_{\substack{i=1\\ \xi_i\notin B_\delta(E)}}^{I} \lambda_i < \delta.$$

Proof. Let $\delta > 0$ and $\xi \in \operatorname{Rco}_f E_\delta$. Since $\operatorname{Rco}_f E_\delta \subset \operatorname{int}\operatorname{Rco}_f E$, $\xi \in \operatorname{int}\operatorname{Rco}_f E$ so, using (*ii*), we conclude that $\xi \in \operatorname{Rco}_f E_\mu$ for all $\mu \le \mu_1$. Thus, by the characterization of $\operatorname{Rco}_f E_\mu$ stated in Theorem 2.9 with $U = \operatorname{int}\operatorname{Rco}_f E$, for every $\mu \le \mu_1$ and for every $\varepsilon > 0$, there exist $I \in \mathbb{N}$ and $(\lambda_i,\eta_i)_{i=1,\dots,I}$ with $\lambda_i > 0$, $\sum_{i=1}^I \lambda_i = 1$ satisfying $(H_I(U))$ and such that

$$\xi = \sum_{i=1}^I \lambda_i\eta_i, \qquad \sum_{\substack{i=1\\ \eta_i\notin B_\varepsilon(E_\mu)}}^{I} \lambda_i < \varepsilon. \tag{3.7}$$

On the other hand, by (i), for every $\gamma > 0$ there exists $\mu_2 > 0$ such that

$$\operatorname{dist}(\eta; E) \leq \gamma, \ \forall\, \eta \in E_\mu \text{ with } \mu \leq \mu_2. \tag{3.8}$$

Choosing in the previous conditions $\gamma = \frac{\delta}{3}$, $\mu_0 \leq \min\{\mu_1, \mu_2\}$, $\varepsilon = \frac{\delta}{3}$ we conclude, by (3.7), that there exist $I \in \mathbb{N}$ and $(\lambda_i, \eta_i)_{i=1,\dots,I}$ with $\lambda_i > 0$, $\sum_{i=1}^{I} \lambda_i = 1$ satisfying $(H_I(U))$ and such that

$$\xi = \sum_{i=1}^{I} \lambda_i \eta_i, \qquad \sum_{\substack{i=1 \\ \eta_i \notin B_{\delta/3}(E_{\mu_0})}}^{I} \lambda_i < \frac{\delta}{3}.$$

To obtain the desired condition, we observe that

$$\sum_{\substack{i=1 \\ \eta_i \notin B_\delta(E)}}^{I} \lambda_i \leq \sum_{\substack{i=1 \\ \eta_i \notin B_{2\delta/3}(E_{\mu_0})}}^{I} \lambda_i < \frac{2\delta}{3} < \delta.$$

Indeed, if $\eta_i \notin B_\delta(E)$ then $\operatorname{dist}(\eta_i; E) \geq \delta$. From (3.8), $\operatorname{dist}(\eta; E) \leq \frac{\delta}{3}$, $\forall\, \eta \in E_{\mu_0}$. This means that $E_{\mu_0} \subset B_{\delta/3}(E)$ and thus $\operatorname{dist}(\eta_i; E_{\mu_0}) \geq \frac{2\delta}{3}$. □

4 Applications

We will now recall some properties of isotropic sets and investigate similar properties when a restriction on the sign of the determinant is considered. These results will be useful in the study of some differential inclusions related with this type of sets which we present in subsections 4.2 and 4.3.

We start by giving the precise definition of isotropic set.

Definition 4.1 *Let E be a subset of $\mathbb{R}^{n\times n}$. We say E is isotropic if $RES \subseteq E$ for every R, S in the orthogonal group $\mathcal{O}(n)$.*

Isotropic sets can be easily described by means of the singular values of its matrices. Indeed, let $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ denote the singular values of the matrix ξ, that is, the eigenvalues of the matrix $\sqrt{\xi\xi^t}$, then the isotropic sets E of $\mathbb{R}^{n\times n}$ are those which can be written in the form

$$E = \{\xi \in \mathbb{R}^{n\times n} : (\lambda_1(\xi), \cdots, \lambda_n(\xi)) \in \Lambda_E\}, \tag{4.1}$$

where Λ_E is a set contained in $\{(x_1, \cdots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq \cdots \leq x_n\}$. This is a consequence of some properties of the singular values that we recall next.

The following decomposition holds (see [13]): for every matrix $\xi \in \mathbb{R}^{n\times n}$ there exist $R, S \in \mathcal{O}(n)$ such that

$$\xi = R\operatorname{diag}(\lambda_1(\xi), \cdots, \lambda_n(\xi))S = R \begin{pmatrix} \lambda_1(\xi) & & \\ & \ddots & \\ & & \lambda_n(\xi) \end{pmatrix} S \tag{4.2}$$

and, for every $\xi \in \mathbb{R}^{n\times n}$, $R, S \in \mathcal{O}(n)$

$$\lambda_i(\xi) = \lambda_i(R\xi S).$$

Moreover, one has

$$\prod_{i=1}^{n} \lambda_i(\xi) = |\det \xi| \qquad \text{and} \qquad \sum_{i=1}^{n} (\lambda_i(\xi))^2 = |\xi|^2.$$

In particular, in the 2×2 case, λ_1 and λ_2 are given by

$$\lambda_1(\xi) = \frac{1}{2}\left[\sqrt{|\xi|^2 + 2|\det \xi|} - \sqrt{|\xi|^2 - 2|\det \xi|}\right]$$
$$\lambda_2(\xi) = \frac{1}{2}\left[\sqrt{|\xi|^2 + 2|\det \xi|} + \sqrt{|\xi|^2 - 2|\det \xi|}\right].$$

The functions λ_i are continuous, $\xi \to \prod_{i=k}^{n} \lambda_i(\xi)$ is polyconvex for any $1 \leq k \leq n$ and λ_n is a norm. From this, clearly if the set Λ_E in (4.1) is compact (respectively, open) then E is also compact (respectively, open). On the other hand, if E is compact the set Λ_E can be taken to be compact and if E is open (4.1) holds for an open set $\Lambda_E \subset \mathbb{R}^n$.

In this section we will also be interested in sets of the form

$$E = \left\{\xi \in \mathbb{R}^{n\times n} : (\lambda_1(\xi), \cdots, \lambda_n(\xi)) \in \Lambda_E,\ \det \xi \geq 0\right\}, \tag{4.3}$$

where, as before, Λ_E is a set contained in $\{(x_1, \cdots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq \cdots \leq x_n\}$. We observe that these are not isotropic sets, but just a class of $\mathcal{SO}(n)$ invariant sets, where $\mathcal{SO}(n)$ denotes the special orthogonal group.

Theorem 4.2 *If $E \subseteq \mathbb{R}^{n\times n}$ has the form (4.3) for some compact set Λ_E, then $\mathrm{Rco}_f\, E$ has the same form, with $\Lambda_{\mathrm{Rco}_f\, E}$ also compact.*

Proof. Let $E \subseteq \mathbb{R}^{n\times n}$ be a set of the form (4.3) for some compact set Λ_E. Then it is trivial to conclude that $\mathrm{Rco}_f\, E$ is compact and $\mathrm{Rco}_f\, E \subseteq \{\xi \in \mathbb{R}^{n\times n} :\ \det \xi \geq 0\}$, this follows from the fact that $\xi \to -\det \xi$ is rank one convex. We will show that

$$\mathrm{Rco}_f E = \left\{\xi \in \mathbb{R}^{n\times n} :\ (\lambda_1(\xi), \cdots, \lambda_n(\xi)) \in \Lambda_{\mathrm{Rco}_f\, E}, \det \xi \geq 0\right\}$$

where

$$\Lambda_{\mathrm{Rco}_f\, E} = \{x \in \mathbb{R}^n : x = (\lambda_1(\xi), \cdots, \lambda_n(\xi)) \text{ for some } \xi \in \mathrm{Rco}_f E\}.$$

Notice that, in particular, $\Lambda_{\mathrm{Rco}_f\, E}$ is compact. To achieve the desired representation of $\mathrm{Rco}_f\, E$ we only need to show that if $\xi \notin \mathrm{Rco}_f\, E$ then for every $R, S \in \mathcal{SO}(n)$ one has $R\xi S \notin \mathrm{Rco}_f\, E$. Let $\xi \notin \mathrm{Rco}_f\, E$, then there exists a rank one convex function $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ such that $f|_E \leq 0$ and $f(\xi) > 0$. Let $R, S \in \mathcal{SO}(n)$ and define $f_1(\eta) := f(R^{-1}\eta S^{-1})$. Then f_1 is rank one convex and for all $\eta \in E$, $f_1(\eta) = f(R^{-1}\eta S^{-1}) \leq 0$, as $R^{-1}\eta S^{-1} \in E$. However $f_1(R\xi S) = f(\xi) > 0$ and so $R\xi S$ doesn't belong to $\mathrm{Rco}_f\, E$. □

4.1 Non-affine map with a finite number of gradients without rank one connections

In [14], Kirchheim proved the existence of non-affine maps with a finite number of values for the gradient but without rank one connections between them (see also the result obtained by Kirchheim and Preiss, cf. [15, Corollary 4.40], where a non-affine map whose gradient takes five possible values not rank one connected was constructed). Kirchheim's result is the following.

Theorem 4.3 *Let $N, n \geq 2$, $m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there is a set $E = \{\xi_1, ..., \xi_m\} \subset \mathbb{R}^{N\times n}$ such that*

$$\operatorname{rank}(\xi_i - \xi_j) = \min\{N, n\}, \text{ if } i \neq j$$

and there are $\xi \notin E$ and $u \in u_\xi + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ such that

$$Du(x) \in E, \text{ a.e. } x \in \Omega,$$

where u_ξ represents a map such that $Du_\xi = \xi$.

This theorem was obtained thanks to an abstract result also due to Kirchheim (cf. [14, Theorem 5]). What we want to show in this section is that the same result can also be achieved by the Baire categories method, cf. Theorem 3.2. Evidently, since, as described in the statement of the theorem, the elements of the set E are not rank one connected, the gradient of the affine boundary data ξ does not belong to $\operatorname{Rco} E = E$. Therefore, to prove the relaxation property required to apply Theorem 3.2, one cannot use the usual approximation property (cf. Definition 3.3). However, as we will see, this difficulty can be overcome by means of Corollary 3.9. Indeed, the set E constructed by Kirchheim is such that the gradient of the boundary data ξ belongs to the interior of $\operatorname{Rco}_f E$ (cf. Remark 3.10).

We recall in the following lemma the properties of the set E constructed by Kirchheim. For the construction of the set we refer again to [14].

Lemma 4.4 *Let $N, n \geq 2$ and denote by $B_{\frac{1}{2}}(0)$ the open ball of $\mathbb{R}^{N\times n}$ centered at 0 and with radius $\frac{1}{2}$. Then there exists a set $E = \{\xi_1, ..., \xi_m\} \subset \mathbb{R}^{N\times n}$, $m \in \mathbb{N}$, such that*

$$\operatorname{rank}(\xi_i - \xi_j) = \min\{N, n\}, \text{ if } i \neq j$$

and $\operatorname{dist}(\xi; B_{\frac{1}{2}}(0)) > 0$, *for every $\xi \in E$. Moreover, for every $\xi \in E$ there exists $\mathcal{M}_\xi \subset \mathbb{R}^{N\times n}$ such that*

i) $\mathcal{M}_\xi \subset \xi + \{\mu \in \mathbb{R}^{N\times n} : \operatorname{rank} \mu = 1\}$

ii) $\mathcal{M}_\xi \subset B_{\frac{1}{2}}(0)$, $\#\mathcal{M}_\xi < 4Nn$

iii) $\partial B_{\frac{1}{2}}(0) \subset \bigcup_{\xi \in E} \operatorname{int}(\operatorname{co}(\{\xi\} \cup \mathcal{M}_\xi))$.

We now prove Theorem 4.3, for the set E considered in the previous lemma, using the Baire categories method. This proof was already obtained by Ribeiro in [21] but we include it here for the sake of completeness.

Proof of Theorem 4.3. We consider a set E with the properties of the above lemma and, with the same notations of the lemma, we define

$$K = B_{\frac{1}{2}}(0) \cup \bigcup_{\xi \in E} \operatorname{int}(\operatorname{co}(\{\xi\} \cup \mathcal{M}_\xi)).$$

Notice that K is an open and bounded set. Since $K \setminus E$ is non empty (for instance, it contains 0) let $\overline{\xi} \in K \setminus E$. We will show that K has the relaxation property with respect to E by applying Corollary 3.9. Once this is proved, we conclude by Theorem 3.2 that there exists $u \in u_{\overline{\xi}} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ such that $Du \in E$ with $\overline{\xi} \notin E$.

We only need to ensure condition (H) of Corollary 3.9. Let $\delta > 0$ and $\eta \in K \setminus B_\delta(E)$. If $\eta \in B_{\frac{1}{2}}(0)$, by definition of K and condition $iii)$ of Lemma 4.4 one can easily reduce to the case $\eta \in \operatorname{int}(\operatorname{co}(\{\xi\} \cup \mathcal{M}_\xi)) \setminus B_{\frac{1}{2}}(0)$, with $\xi \in E$, moving along any rank one direction.

Consider now the case where $\eta \in \operatorname{int}(\operatorname{co}(\{\xi\} \cup \mathcal{M}_\xi)) \setminus B_{\frac{1}{2}}(0)$, for some $\xi \in E$. In this case we can write

$$\eta = \sum_{j=1}^{k} \lambda_j \mu_j + \left(1 - \sum_{j=1}^{k} \lambda_j\right) \xi,$$

for some $\lambda_j \in (0,1)$ such that $\sum_{j=1}^{k} \lambda_j < 1$ and for some $\mu_j \in \mathcal{M}_\xi$. By condition $ii)$ of Lemma 4.4 we have $k < 4Nn$.

Let $j^* \in \{1, ..., k\}$ be such that $\lambda_{j^*}|\xi - \mu_{j^*}| = \max_{1 \le j \le k} \lambda_j |\xi - \mu_j|$ and consider the rank one direction $\xi - \mu_{j^*}$. We will show that it is possible to find $c > 0$, independent of η, such that $\eta_1 = c(\xi - \mu_{j^*})$ satisfies $[\eta - \eta_1, \eta + \eta_1] \subset \operatorname{int} K$. In particular, since $\operatorname{dist}(\xi; B_{\frac{1}{2}}(0)) > 0$, for every $\xi \in E$, and $\mu_j \in B_{\frac{1}{2}}(0)$ by condition $ii)$ of Lemma 4.4, it follows that, for some $C > 0$ independent of η and ξ,

$$|\eta_1| \geq C. \tag{4.4}$$

To find the constant c we proceed in the following way. We notice that, for

$$|t| < \min\left\{\lambda_{j^*}, 1 - \sum_{j=1}^{k} \lambda_j\right\},$$

$$\eta + t(\xi - \mu_{j^*}) = \sum_{\substack{j=1 \\ j \neq j^*}}^{k} \lambda_j \mu_j + (\lambda_{j^*} - t)\mu_{j^*} + \left(1 - \sum_{j=1}^{k} \lambda_j + t\right) \xi \in \operatorname{int}(\operatorname{co}(\{\xi\} \cup \mathcal{M}_\xi)).$$

Thus we only need to show that

$$\min \left\{ \lambda_{j^*}, 1 - \sum_{j=1}^{k} \lambda_j \right\} \geq c > 0.$$

The estimate for λ_{j^*} follows from

$$\delta < |\eta - \xi| \leq \sum_{j=1}^{k} \lambda_j |\mu_j - \xi| \leq 4Nn\lambda_{j^*} |\mu_{j^*} - \xi| \leq 4Nn\lambda_{j^*} \max_{\substack{\xi \in E \\ \mu \in \mathcal{M}_\xi}} |\mu - \xi|.$$

On the other hand, since

$$\begin{aligned} \frac{1}{2} &\leq |\eta| \leq \sum_{j=1}^{k} \lambda_j |\mu_j| + \left(1 - \sum_{j=1}^{k} \lambda_j\right) |\xi| \leq \\ &\leq \left(\max_{\substack{\xi \in E \\ \mu \in \mathcal{M}_\xi}} |\mu|\right) \sum_{j=1}^{k} \lambda_j + \left(1 - \sum_{j=1}^{k} \lambda_j\right) \max_{\xi \in E} |\xi|, \end{aligned}$$

one gets

$$1 - \sum_{j=1}^{k} \lambda_j \geq \frac{\frac{1}{2} - \max\limits_{\substack{\xi \in E \\ \mu \in \mathcal{M}_\xi}} |\mu|}{\max\limits_{\xi \in E} |\xi| - \max\limits_{\substack{\xi \in E \\ \mu \in \mathcal{M}_\xi}} |\mu|} > 0$$

as wished.

We argue that repeating the same reasoning with the matrix $\eta + \eta_1$ and so on, after i iterations of this procedure, we obtain a sequence of rank one matrices $\eta_1, ..., \eta_i$ satisfying $[\eta + \eta_1 + \cdots + \eta_{j-1} - \eta_j, \eta + \eta_1 + \cdots + \eta_{j-1} + \eta_j] \subseteq \operatorname{int} K$, $j = 1, \cdots, i$ and $\eta + \eta_1 + ... + \eta_i \in B_\delta(E)$ where $i \leq L(\delta, E, K)$ is independent of η.

Indeed, without loss of generality, assume that $|\eta + \eta_1| \geq |\eta - \eta_1|$. Then it follows that

$$|\eta + \eta_1| \geq C$$

since, by (4.4),

$$2 |\eta + \eta_1|^2 \geq |\eta + \eta_1|^2 + |\eta - \eta_1|^2 = 2 |\eta|^2 + 2 |\eta_1|^2 \geq 2C^2.$$

If $\eta + \eta_1 \notin B_\delta(E)$ we obtain, as before, η_2 such that $|\eta_2| \geq C$ and $[\eta + \eta_1 - \eta_2, \eta + \eta_1 + \eta_2] \subset \operatorname{int} K = K$. Again, assuming that $|\eta + \eta_1 + \eta_2| \geq |\eta + \eta_1 - \eta_2|$, one has

$$2 |\eta + \eta_1 + \eta_2|^2 \geq |\eta + \eta_1 + \eta_2|^2 + |\eta + \eta_1 - \eta_2|^2 = 2 |\eta + \eta_1|^2 + 2 |\eta_2|^2 \geq 4C^2.$$

After i iterations of this procedure one gets $\eta + \eta_1 + ... + \eta_i \in \operatorname{int} K = K$ with

$$|\eta + \eta_1 + ... + \eta_i| \geq \sqrt{i}C.$$

Thus, $|\eta + \eta_1 + ... + \eta_i| \to +\infty$, as $i \to +\infty$, contradicting the fact that K is bounded. Therefore, for some i bounded by a constant $L = L(\delta, E, K)$, we must have $\eta + \eta_1 + ... + \eta_i \in K \cap B_\delta(E)$.

This concludes the proof of condition (H) of Corollary 3.9 and thus the proof. □

4.2 Isotropic differential inclusion

In this section we discuss the differential inclusion problem

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega, \end{cases} \tag{4.5}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ and E is a compact subset of $\mathbb{R}^{n\times n}$ which is isotropic, that is to say, invariant under orthogonal transformations.

We observe that a result due to Dacorogna and Marcellini [9, Theorem 7.28] provides a sufficient condition for existence of solutions to this problem. Indeed, denoting by $\lambda_1(\xi) \le \lambda_2(\xi) \le \cdots \le \lambda_n(\xi)$ the singular values of $\xi \in \mathbb{R}^{n\times n}$, if there exists $\eta \in E$ with $\lambda_i(\eta) = \gamma_i > 0$ and $\varphi \in C^1_{piec}(\overline{\Omega}; \mathbb{R}^n)$ is such that

$$D\varphi \in E \cup \left\{ \xi \in \mathbb{R}^{n\times n} : \prod_{i=\tau}^{n} \lambda_i(\xi) < \prod_{i=\tau}^{n} \gamma_i, \ \tau = 1, ..., n \right\}$$

then (4.5) has $W^{1,\infty}(\Omega; \mathbb{R}^n)$ solutions.

In the 2 dimensional case ($n = 2$), a less restrictive condition can be obtained, although it is more difficult to check in concrete examples. This was studied by Croce [6] (see also [5]), using the Baire categories method that we discussed in Section 3, and by Barroso, Croce and Ribeiro [1] using the convex integration method due to Müller and Šverák [17, 18]. With both methods, the result obtained was the following.

Theorem 4.5 *Let* $E := \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_E\}$, *where* $\Lambda_E \subset \{(x, y) \in \mathbb{R}^2 : 0 < x \le y\}$ *is a compact set and let*

$$K = \left\{ \xi \in \mathbb{R}^{2\times 2} : \ f_\theta(\lambda_1(\xi), \lambda_2(\xi)) < \max_{(a,b)\in\Lambda_E} f_\theta(a, b), \forall \ \theta \in [0, \max_{(a,b)\in\Lambda_E} b] \right\},$$

where $f_\theta(x, y) := xy + \theta(y - x)$. *Then, if* $\Omega \subset \mathbb{R}^2$ *is a bounded open set and if* $\varphi \in C^1_{piec}(\overline{\Omega}, \mathbb{R}^2)$ *is such that* $D\varphi \in E \cup K$ *a.e. in* Ω, *there exists a map* $u \in \varphi + W^{1,\infty}_0(\Omega, \mathbb{R}^2)$ *such that* $Du \in E$ *a.e. in* Ω.

We notice that it turns out that $K = \operatorname{int} \operatorname{Rco} E = \operatorname{int} \operatorname{Rco}_f E$. To achieve the previous theorem two fundamental results due to Cardaliaguet and Tahraoui [2] were used. On one hand, they characterized the polyconvex hull of any set E as in the theorem; based on their description, Croce [6] then showed that

$$\operatorname{Pco} E = \left\{ \xi \in \mathbb{R}^{2\times 2} : \ f_\theta(\lambda_1(\xi), \lambda_2(\xi)) \le \max_{(a,b)\in\Lambda_E} f_\theta(a, b), \forall \ \theta \in [0, \max_{(a,b)\in\Lambda_E} b] \right\}.$$

On the other hand, Cardaliaguet and Tahraoui [2] showed that, in dimension 2, compact isotropic sets which are rank one convex are also polyconvex (see also [4]). Since the hull $\operatorname{Rco}_f E$ of a compact isotropic set E is also compact and isotropic (cf. Theorem 4.2) and since it is also rank one convex, one immediately obtains that $\operatorname{Rco}_f E = \operatorname{Pco} E$ and thus a characterization for $\operatorname{Rco}_f E$. This was fundamental to study the differential inclusion by means of the convex integration method. Indeed, to prove the required in-approximation it is necessary to know the hull $\operatorname{Rco}_f E$. In the prior work of Croce [6], where the Baire categories method was used via the approximation property, the appropriate hull to consider was $\operatorname{Rco} E$. Contrary to the case of $\operatorname{Rco}_f E$, a characterization of $\operatorname{Rco} E$ does not follow immediately from Cardaliaguet and Tahraoui's results since, in general, $\operatorname{Rco} E$ may not be compact. For this reason, in [6], $\operatorname{Rco} E$ had to be computed (and the conclusion was that it coincides with $\operatorname{Rco}_f E$). However, thanks to the theory presented in Section 3, the results of Cardaliaguet and Tahraoui are sufficient to obtain Theorem 4.5 using the Baire categories method. We proceed with a brief sketch of this proof which is essentially the one given in [6], but with no need of computing $\operatorname{Rco} E$.

Proof of Theorem 4.5. We recall that by the results in [2] and [6],

$$\operatorname{Rco}_f E = \left\{ \xi \in \mathbb{R}^{2\times 2} : \ f_\theta(\lambda_1(\xi), \lambda_2(\xi)) \leq \max_{(a,b)\in\Lambda_E} f_\theta(a,b), \ \forall\ \theta \in [0, \max_{(a,b)\in\Lambda_E} b] \right\}$$

and

$$\operatorname{int}\operatorname{Rco}_f E = \left\{ \xi \in \mathbb{R}^{2\times 2} : f_\theta(\lambda_1(\xi), \lambda_2(\xi)) < \max_{(a,b)\in\Lambda_E} f_\theta(a,b), \forall \theta \in [0, \max_{(a,b)\in\Lambda_E} b] \right\}.$$

Thus, by Corollary 3.7 and Theorem 3.2, it is enough to construct compact sets E_δ such that $\operatorname{Rco}_f E_\delta \subset \operatorname{int}\operatorname{Rco}_f E$ and satisfying conditions (i) and (ii) of Theorem 3.5 with $K_\delta = \operatorname{Rco}_f E_\delta$. In fact, this is the case if we consider

$$E_\delta = \bigcup_{(a,b)\in\Lambda_E} \left\{ \xi \in \mathbb{R}^{2\times 2} : \ (\lambda_1(\xi), \lambda_2(\xi)) = (a-\delta, b-\delta) \right\},$$

for $0 \leq \delta \leq \min_{(a,b)\in\Lambda_E} \dfrac{a}{2}$. We refer to [6] for the details of the proof. □

4.3 Differential inclusions for some $\mathcal{SO}(n)$ invariant sets

We consider in this section the differential inclusion problem

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega, \end{cases} \tag{4.6}$$

in the case E has the form

$$E = \left\{ \xi \in \mathbb{R}^{n\times n} : (\lambda_1(\xi), \cdots, \lambda_n(\xi)) \in \Lambda_E, \ \det \xi > 0 \right\},$$

with $\Lambda_E \subseteq \{(x_1, \cdots, x_n) \in \mathbb{R}^n : 0 < x_1 \leq \cdots \leq x_n\}$. As already observed, these are not isotropic sets, but just a class of $\mathcal{SO}(n)$ invariant sets.

For $n = 2$, Cardaliaguet and Tahraoui [3] defined the set $R(\Lambda_E)$ for any compact set $\Lambda_E \subset \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2\}$ as the smallest compact subset of $\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2\}$ containing Λ_E such that

$$\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in R(\Lambda_E),\ \det \xi \geq 0\}$$

is rank one convex. This hull can be used to describe $\mathrm{Rco}_f\, E$ (see Lemma 4.8). The representation of this envelop is quite complicated and leads to some difficulty in dealing with it in order to show existence of solutions to problem (4.6). In Theorem 4.11 we consider a particular set E composed by matrices with two possible singular values and, using Cardaliaguet and Tahraoui's results, we give a sufficient condition for existence, relating the gradient of the boundary data and the hull $\mathrm{Rco}_f\, E$.

For $n > 2$, a representation of $\mathrm{Rco}_f\, E$ is not available. Of course, if one wants to ensure existence of solutions to (4.6), one may not need to know the entire hull. Moreover, we notice that in the applications it is more convenient to have simpler conditions to check than those describing the hull $\mathrm{Rco}_f\, E$ obtained by Cardaliaguet and Tahraoui [3] for the 2 dimensional case. In this sense, in Theorems 4.12 and 4.14 we will establish sufficient conditions for existence of solutions to problem (4.6) for certain sets E in dimension 2 and 3. Analogous results could be obtained in higher dimensions however, due to the heavy notation already present in the 3 dimensional case, we have only considered these two settings.

4.3.1 Set of singular values consisting of two points

In this section we are going to consider the case where

$$E = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_E,\ \det \xi > 0\}$$

with $\Lambda_E = \{(a_1, a_2), (b_1, b_2)\}$ and $0 < a_1 < b_1 < a_2 < b_2$. We start by studying the set $\mathrm{Rco}_f\, E$. To this effect we will use the following characterization of $R(\Lambda_E)$ obtained in [3, Proposition 8.6, Theorem 7.1 and Definition 1.1].

Theorem 4.6 *Let Λ be a compact subset of $\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2\}$ such that $R(\Lambda)$ is connected. Then*

$$R(\Lambda) = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2, x_1 \geq \alpha, \sigma_3(x_1) \leq x_2 \leq \inf\{\sigma_1(x_1), \sigma_2(x_1)\}\}$$

where $\alpha = \inf_{(x_1,x_2)\in\Lambda} x_1$,

$$\sigma_1(x_1) = \inf_{(\theta,\gamma)\in\Sigma_1} f^1_{\theta,\gamma}(x_1), \quad f^1_{\theta,\gamma}(x_1) = \begin{cases} \theta + \frac{\gamma-\theta^2}{\theta-x_1}, & x_1 < \theta, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$\sigma_2(x_1) = \inf_{(\theta,\gamma)\in\Sigma_2} f^2_{\theta,\gamma}(x_1), \quad f^2_{\theta,\gamma}(x_1) = \theta + \frac{\gamma - \theta^2}{\theta + x_1},$$

$$\sigma_3(x_1) = \sup_{(\theta,\gamma)\in\Sigma_3} f^3_{\theta,\gamma}(x_1), \quad f^3_{\theta,\gamma}(x_1) = \begin{cases} -\theta + \frac{\gamma-\theta^2}{x_1-\theta}, & x_1 > \theta, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$\begin{aligned}
\Sigma_0 &= \{(\theta,\gamma) \in \mathbb{R}^2 : \ \theta \geq 0, \ \gamma \geq \theta^2\}, \\
\Sigma_1 &= \{(\theta,\gamma) \in \Sigma_0 : \ x_2 \leq f^1_{\theta,\gamma}(x_1), \ \forall\, (x_1,x_2) \in \Lambda\}, \\
\Sigma_2 &= \{(\theta,\gamma) \in \Sigma_0 : \ x_2 \leq f^2_{\theta,\gamma}(x_1), \ \forall\, (x_1,x_2) \in \Lambda\}, \\
\Sigma_3 &= \{(\theta,\gamma) \in \Sigma_0 : \ x_2 \geq f^3_{\theta,\gamma}(x_1), \ \forall\, (x_1,x_2) \in \Lambda\}.
\end{aligned}$$

Moreover, there exists a convex function $h : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ *such that*

$$\begin{aligned}
&\left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in R(\Lambda), \ \det \xi \geq 0\right\} \\
&= \left\{\xi \in \mathbb{R}^{2\times 2} : h(\xi, \det \xi) \leq 0, \ \det \xi \geq 0\right\}.
\end{aligned}$$

The following sufficient condition for $R(\Lambda)$ to be connected was also proven in [3, Proposition 8.4].

Proposition 4.7 *Let* Λ *be a compact subset of* $\{(x_1,x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2\}$ *and assume that* Λ *satisfies the following property: if there exist* C_1 *and* C_2*, compact subsets of* Λ*, such that* $C_1 \cap C_2 = \emptyset$*,* $C_1 \cup C_2 = \Lambda$ *and* $\sup_{(x_1,x_2)\in C_1} x_2 < \inf_{(x_1,x_2)\in C_2} x_1$*, then either* $C_1 = \emptyset$ *or* $C_2 = \emptyset$*. Then* $R(\Lambda)$ *is connected.*

If $E = \left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_E, \ \det \xi \geq 0\right\}$, where Λ_E is a compact subset of $\{(x_1,x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2\}$, then $R(\Lambda_E)$ describes the hull $\mathrm{Rco}_f\, E$, as we prove in the following lemma.

Lemma 4.8 *Let* $E = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_E, \ \det \xi \geq 0\}$ *where* Λ_E *is a compact subset of* $\{(x_1,x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2\}$ *such that* $R(\Lambda_E)$ *is connected. Then*

$$\mathrm{Rco}_f\, E = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in R(\Lambda_E), \ \det \xi \geq 0\}.$$

Proof. We set

$$\mathcal{E} = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in R(\Lambda_E), \ \det \xi \geq 0\}.$$

Since $R(\Lambda_E)$ is connected, by Theorem 4.6 there exists a convex function $h : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that $\mathcal{E} = \{\xi \in \mathbb{R}^{2\times 2} : h(\xi, \det \xi) \leq 0, \ \det \xi \geq 0\}$. Note that $h(\xi, \det \xi) \leq 0$ for every $\xi \in E$, since $E \subset \mathcal{E}$. According to Definition 2.5 of $\mathrm{Pco}_f\, E$, one has $\mathrm{Pco}_f\, E \subseteq \mathcal{E}$. This implies that $\mathrm{Rco}_f\, E \subseteq \mathcal{E}$.

On the other hand, $\mathrm{Rco}_f\, E$ is a compact and rank one convex set. By Theorem 4.2,

$$\mathrm{Rco}_f\, E = \{\xi \in \mathbb{R}^{2\times 2} : \ (\lambda_1(\xi), \lambda_2(\xi)) \in \tilde{\Lambda}, \ \det \xi \geq 0\}$$

for some compact subset $\tilde{\Lambda}$ of $\{(x_1,x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2\}$. By definition of $R(\Lambda_E)$, $\tilde{\Lambda} \supseteq R(\Lambda_E)$ and thus $\mathrm{Rco}_f\, E \supseteq \mathcal{E}$. □

Using the two previous results we can show the following formula for $\mathrm{Rco}_f\, E$, for the set E considered in this section.

Proposition 4.9 *Let* $E = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_E,\ \det \xi > 0\}$, *where* $\Lambda_E = \{(a_1, a_2), (b_1, b_2)\}$, $0 < a_1 < b_1 < a_2 < b_2$, *and define*

$$\underline{\theta} = \frac{-a_1 a_2 + b_1 b_2}{b_1 + b_2 - a_1 - a_2}.$$

Then

$$\mathrm{Rco}_f\, E = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in R(\Lambda_E),\ \det \xi > 0\} \tag{4.7}$$

where $R(\Lambda_E)$ *is the set of points* $(x_1, x_2) \in \mathbb{R}^2$ *such that*

$$\begin{cases} a_1 \le x_1 \le x_2 \le b_2, \\ x_2 \ge \dfrac{a_1 a_2}{x_1}, \\ x_2 \le \dfrac{-a_1 a_2 + \underline{\theta}(a_1 + a_2) - \underline{\theta} x_1}{-x_1 + \underline{\theta}}, \quad \textit{if } a_1 \le x_1 \le b_1 \\ x_2 \le \dfrac{b_1 b_2}{x_1}. \end{cases} \tag{4.8}$$

Moreover,

$$\mathrm{int}\,\mathrm{Rco}_f\, E = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \mathrm{rel\,int} R(\Lambda_E),\ \det \xi > 0\},$$

where $\mathrm{rel\,int} R(\Lambda_E)$ *is the relative interior of* $R(\Lambda_E)$ *with respect to the set* $\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 \le x_2\}$ *and is the set of points* $(x_1, x_2) \in \mathbb{R}^2$ *such that*

$$\begin{cases} a_1 < x_1 \le x_2 < b_2, \\ x_2 > \dfrac{a_1 a_2}{x_1}, \\ x_2 < \dfrac{-a_1 a_2 + \underline{\theta}(a_1 + a_2) - \underline{\theta} x_1}{-x_1 + \underline{\theta}}, \quad \textit{if } a_1 < x_1 < b_1 \\ x_2 < \dfrac{b_1 b_2}{x_1}. \end{cases}$$

Proof. Notice that, since $a_1 > 0$ and $\lambda_1(\xi)\lambda_2(\xi) = |\det \xi|$, our set E satisfies the hypotheses of Lemma 4.8 since, by Proposition 4.7, the set $R(\Lambda_E)$ is connected. Thus (4.7) holds and to establish (4.8) we will write the inequalities $\sigma_3(x_1) \le x_2 \le \inf\{\sigma_1(x_1), \sigma_2(x_1)\}$, given by Theorem 4.6, for $x_1 \ge a_1$, in a more explicit way. Let us start by studying $x_2 \ge \sigma_3(x_1)$. It is easy to see that

$$\Sigma_3 = \left\{(\theta, \gamma) \in \mathbb{R}^2 : 0 \le \theta < a_1,\ \theta^2 \le \gamma \le a_1 a_2 - \theta(a_2 - a_1)\right\},$$

as $0 < a_1 < b_1 < a_2 < b_2$. Therefore $x_2 \ge \sigma_3(x_1)$ is equivalent to

$$x_1 x_2 - \theta(x_2 - x_1) \ge a_1 a_2 - \theta(a_2 - a_1),\ \ \theta \in [0, a_1)$$

since $x_1 > \theta$, that is,

$$x_2 \ge \sup_{\theta \in [0, a_1)} \frac{a_1 a_2 - \theta(a_2 - a_1) - \theta x_1}{x_1 - \theta} = \frac{a_1 a_2}{x_1}. \tag{4.9}$$

In the same way, to study $x_2 \le \sigma_2(x_1)$, we remark that

$$\Sigma_2 = \big\{(\theta, \gamma) \in \mathbb{R}^2 : \theta \ge 0,\ \gamma \ge \max\{\theta^2, b_1 b_2 + \theta(b_2 - b_1)\}\big\}\,.$$

Therefore $x_2 \le \sigma_2(x_1)$ is equivalent to

$$x_1 x_2 + \theta(x_2 - x_1) \le \max\{\theta^2, b_1 b_2 + \theta(b_2 - b_1)\}\,, \quad \theta \ge 0\,,$$

that is,

$$x_2 \le \min\left\{\frac{b_1 b_2}{x_1}, b_2\right\}. \tag{4.10}$$

We now analize the inequality $x_2 \le \sigma_1(x_1)$. It is easy to see that

$$\Sigma_1 = \{(\theta, \gamma) \in \mathbb{R}^2 : \theta \ge 0,\ \gamma \ge \max\{\theta^2, -a_1 a_2 + \theta(a_1 + a_2), -b_1 b_2 + \theta(b_1 + b_2)\}\}\,.$$

Therefore $x_2 \le \sigma_1(x_1)$ is equivalent to

$$x_2 \le \frac{\gamma - \theta x_1}{\theta - x_1}, \quad \forall\, (\theta, \gamma) \in \Sigma_1 : \theta > x_1\,,$$

that is,

$$-x_1 x_2 + \theta(x_1 + x_2) \le \max\{\theta^2, -a_1 a_2 + \theta(a_1 + a_2), -b_1 b_2 + \theta(b_1 + b_2)\}, \quad \forall\, \theta > x_1\,.$$

By (4.10), $x_2 \le b_2$ and so

$$-x_1 x_2 + \theta(x_1 + x_2) \le \max\{\theta^2, -a_1 a_2 + \theta(a_1 + a_2), -b_1 b_2 + \theta(b_1 + b_2)\},\ \theta \in (x_1, b_2].$$

Since $\theta > x_1 \ge a_1$ and $0 < a_1 < b_1 < a_2 < b_2$, one has

$$x_2 \le \inf_{\theta \in (x_1, b_2]} \frac{\max\{-a_1 a_2 + \theta(a_1 + a_2), -b_1 b_2 + \theta(b_1 + b_2)\} - \theta x_1}{-x_1 + \theta}\,. \tag{4.11}$$

We observe that

$$b_1 < \underline{\theta} < a_2 \tag{4.12}$$

and

$$\max\{-a_1 a_2 + \theta(a_1 + a_2), -b_1 b_2 + \theta(b_1 + b_2)\} = \begin{cases} -a_1 a_2 + \theta(a_1 + a_2), & a_1 \le \theta \le \underline{\theta} \\ -b_1 b_2 + \theta(b_1 + b_2), & \underline{\theta} \le \theta \le b_2\,. \end{cases}$$

To study (4.11) we distinguish the cases $x_1 \ge \underline{\theta}$ and $x_1 < \underline{\theta}$. In the first case, (4.11) gives

$$x_2 \le \inf_{\theta \in (x_1, b_2]} \frac{-b_1 b_2 + \theta(b_1 + b_2) - \theta x_1}{-x_1 + \theta}\,.$$

Notice that the sign of the derivative of the function

$$g(\theta) := \frac{-b_1 b_2 + \theta(b_1 + b_2) - \theta x_1}{-x_1 + \theta}$$

does not depend on θ. Therefore we have

$$x_2 \leq \min\Big\{g(b_2), \lim_{\theta\to x_1^+} g(\theta)\Big\} = g(b_2) = b_2\,, \text{ if } x_1 \geq \underline{\theta}\,. \tag{4.13}$$

In the second case, (4.11) yields

$$x_2 \leq \min\Big\{\inf_{\theta\in(x_1,\underline{\theta}]} \frac{-a_1a_2+\theta(a_1+a_2)-\theta x_1}{-x_1+\theta}, \min_{\theta\in[\underline{\theta},b_2]} \frac{-b_1b_2+\theta(b_1+b_2)-\theta x_1}{-x_1+\theta}\Big\}.$$

As above, the sign of the derivatives of $g(\theta)$ and

$$f(\theta) := \frac{-a_1a_2+\theta(a_1+a_2)-\theta x_1}{-x_1+\theta}$$

does not depend on θ, so we obtain

$$x_2 \leq \min\Big\{f(\underline{\theta}), \lim_{\theta\to x_1^+} f(\theta), g(\underline{\theta}), g(b_2)\Big\} = \min\big\{f(\underline{\theta}), b_2\big\},$$

that is,

$$x_2 \leq \min\left\{\frac{-a_1a_2+\underline{\theta}(a_1+a_2)-\underline{\theta}x_1}{-x_1+\underline{\theta}}, b_2\right\}, \text{ if } x_1 < \underline{\theta}\,.$$

Therefore if $x_1 < \underline{\theta}$

$$x_2 \leq \begin{cases} \dfrac{-a_1a_2+\underline{\theta}(a_1+a_2)-\underline{\theta}x_1}{-x_1+\underline{\theta}}, & x_1 \leq b_1 \\ b_2, & b_1 < x_1 < \underline{\theta}\,. \end{cases} \tag{4.14}$$

In conclusion, from (4.9), (4.10), (4.13) and (4.14) we get

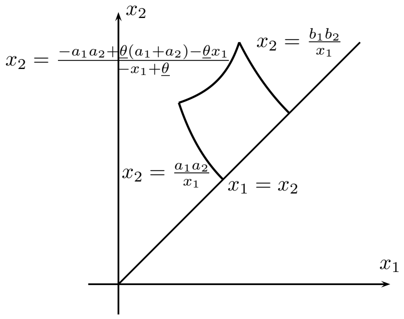
$$\frac{a_1a_2}{x_1} \leq x_2 \leq \min\left\{b_2, \frac{b_1b_2}{x_1}\right\}, \text{ if } x_1 > b_1$$

and

$$\frac{a_1a_2}{x_1} \leq x_2 \leq \min\left\{b_2, \frac{b_1b_2}{x_1}, \frac{-a_1a_2+\underline{\theta}(a_1+a_2)-\underline{\theta}x_1}{-x_1+\underline{\theta}}\right\}, \text{ if } x_1 \leq b_1.$$

By (4.12) and the fact that $x_1 \to \dfrac{-a_1a_2+\underline{\theta}(a_1+a_2)-\underline{\theta}x_1}{-x_1+\underline{\theta}}$ passes through (a_1,a_2) and (b_1,b_2), we get (4.8).
The formula of $\operatorname{int}\operatorname{Rco}_f E$ is easy to obtain from the above representation. □

Remark 4.10 *In the following figure we provide a representation of the set* $R(\Lambda_E)$.



We are now in position to prove an existence result for problem (4.6).

Theorem 4.11 *Let* $E = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_E,\ \det\xi > 0\}$ *where* $\Lambda_E = \{(a_1, a_2), (b_1, b_2)\}$ *and* $0 < a_1 < b_1 < a_2 < b_2$*. Let* $\Omega \subset \mathbb{R}^2$ *be a bounded open set and let* $\varphi \in C^1_{piec}(\overline{\Omega}, \mathbb{R}^2)$ *be such that* $D\varphi \in E \cup \operatorname{int}\operatorname{Rco}_f E$ *a.e. in* Ω*. Then there exists a map* $u \in \varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ *such that* $Du(x) \in E$ *for a.e.* x *in* Ω*.*

Proof. We will prove the result using Theorem 3.2 and Corollary 3.7. Let $\delta > 0$ be sufficiently small such that $a_1 + \delta < b_1 < a_2 < b_2 - \delta$. We define

$$E_\delta = \{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_{E_\delta},\ \det\xi > 0\},$$

with $\Lambda_{E_\delta} = \{(a_1 + \delta, a_2), (b_1, b_2 - \delta)\}$. Observe that E_δ is a compact set. By Proposition 4.9, $\operatorname{Rco}_f E_\delta$ is given by the matrices $\xi \in \mathbb{R}^{2\times 2}$ with positive determinant such that $(x_1, x_2) = (\lambda_1(\xi), \lambda_2(\xi))$ satisfies

$$a_1 + \delta \le x_1 \le x_2 \le b_2 - \delta \tag{4.15}$$

$$x_2 \ge \frac{(a_1 + \delta)a_2}{x_1} \tag{4.16}$$

$$x_2 \le \frac{-(a_1 + \delta)a_2 + \underline{\theta}_\delta(a_1 + \delta + a_2) - \underline{\theta}_\delta x_1}{-x_1 + \underline{\theta}_\delta}, \quad \text{if } a_1 + \delta \le x_1 \le b_1 \tag{4.17}$$

$$x_2 \le \frac{b_1(b_2 - \delta)}{x_1} \tag{4.18}$$

where

$$\underline{\theta}_\delta = \frac{-(a_1 + \delta)a_2 + b_1(b_2 - \delta)}{b_1 + b_2 - a_1 - a_2 - 2\delta}.$$

We are going to verify the hypotheses of Corollary 3.7. We start by proving that $\operatorname{Rco}_f E_\delta \subset \operatorname{int}\operatorname{Rco}_f E$. Let $\xi \in \operatorname{Rco}_f E_\delta$ and denote $(\lambda_1(\xi), \lambda_2(\xi))$ by (x_1, x_2). Since (x_1, x_2) satisfies inequalities (4.15), (4.16) and (4.18), it is clear that $a_1 < x_1 \le x_2 < b_2$, $x_2 > \dfrac{a_1a_2}{x_1}$ and $x_2 < \dfrac{b_1b_2}{x_1}$. It remains to show that

$$x_2 < \frac{-a_1a_2 + \underline{\theta}(a_1 + a_2) - \underline{\theta}x_1}{-x_1 + \underline{\theta}}, \quad \text{if } a_1 < x_1 < b_1 \tag{4.19}$$

where $\underline{\theta} = \dfrac{-a_1a_2 + b_1b_2}{b_1 + b_2 - a_1 - a_2}$. Since $x_1 \geq a_1 + \delta$, it suffices to show that if $x_1 \in [a_1 + \delta, b_1)$, then

$$\frac{-(a_1+\delta)a_2 + \underline{\theta}_\delta(a_1+\delta+a_2) - \underline{\theta}_\delta x_1}{-x_1 + \underline{\theta}_\delta} < \frac{-a_1a_2 + \underline{\theta}(a_1+a_2) - \underline{\theta}x_1}{-x_1 + \underline{\theta}}.$$

As $b_1 < \underline{\theta} < a_2$ and $b_1 < \underline{\theta}_\delta < a_2$, the above inequality is equivalent to

$$(\underline{\theta}_\delta - \underline{\theta})(x_1 - a_1)(x_1 - a_2) < \delta(-x_1 + \underline{\theta})(a_2 - \underline{\theta}_\delta)\,, \tag{4.20}$$

for $x_1 \in [a_1 + \delta, b_1)$. This inequality holds whenever $\underline{\theta}_\delta - \underline{\theta} > 0$, since the left hand side of (4.20) is negative and the right hand side is positive. To show it also holds in the case $\underline{\theta}_\delta - \underline{\theta} < 0$, we notice that the graph of $x_1 \to (\underline{\theta}_\delta - \underline{\theta})(x_1 - a_1)(x_1 - a_2)$ is a concave parabola passing through $(a_1, 0), (a_2, 0)$, whereas the graph of $x_1 \to \delta(-x_1 + \underline{\theta})(a_2 - \underline{\theta}_\delta)$ is a straight line with negative slope passing through $(\underline{\theta}, 0)$. Therefore it is sufficient to prove (4.20) for $x_1 = b_1$, that is,

$$\left[\frac{-(a_1+\delta)a_2 + b_1(b_2-\delta)}{b_1+b_2-a_1-a_2-2\delta} - \frac{-a_1a_2+b_1b_2}{b_1+b_2-a_1-a_2}\right](b_1-a_1)(b_1-a_2) <$$

$$< \delta\left[-b_1 + \frac{-a_1a_2+b_1b_2}{b_1+b_2-a_1-a_2}\right]\left[a_2 - \frac{-(a_1+\delta)a_2 + b_1(b_2-\delta)}{b_1+b_2-a_1-a_2-2\delta}\right].$$

It is not difficult to see that the above inequality holds if and only if $\delta < b_1 - a_1$ which is satisfied by the hypotheses on δ.

The other conditions of Corollary 3.7 are easy to check. Indeed, any $\eta \in E_\delta$ can be written as $R\,\mathrm{diag}(a_1+\delta, a_2)\,S$ or $R\,\mathrm{diag}(b_1, b_2-\delta)\,S$, for some $R, S \in \mathcal{SO}(2)$. In both cases $\mathrm{dist}(\eta; E) \leq \delta$. This proves that for every $\varepsilon > 0$ $\mathrm{dist}(\eta; E) \leq \varepsilon$ for every $\eta \in E_\delta$, with $\delta \leq \varepsilon$.

To prove the last condition of Corollary 3.7, let $\eta \in \mathrm{int}\,\mathrm{Rco}_f\,E$. Since $a_1 + \delta \to a_1$, $b_2 - \delta \to b_2$ and $\underline{\theta}_\delta \to \underline{\theta}$, as $\delta \to 0$, we have that $(\lambda_1(\eta), \lambda_2(\eta))$ satisfies (4.16), (4.17) and (4.18) for sufficiently small δ, that is, $\eta \in \mathrm{Rco}_f E_\delta$ for sufficiently small δ. □

4.3.2 Set of singular values containing a line segment

In this section we establish sufficient conditions for existence of solutions to problem (4.6) when $n = 2$ and $n = 3$. Our results rely on the hypothesis that the set of singular values of the matrices in E contains a line segment. We start by considering the 2 dimensional case.

Theorem 4.12 *Let* $E = \left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_E,\ \det\xi > 0\right\}$, *where* $\Lambda_E \subseteq \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 \leq x_2\}$, *and assume that*

$$\Gamma := \{(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2)) :\ t \in [0,1]\} \subseteq \Lambda_E,$$

with $a_1 < b_1$, $a_2 < b_2$, *and either* $a_1 < a_2$ *or* $b_1 < b_2$. *Let*

$$K := \left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_K,\ \det\xi > 0\right\},$$

where

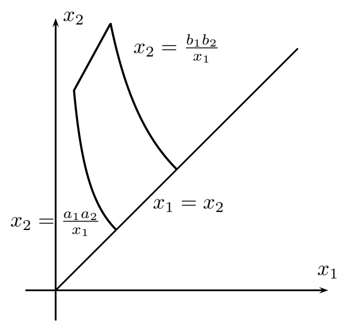
$$\Lambda_K := \bigcup_{(\alpha_1,\alpha_2)\in \mathrm{rel\,int}\,\Gamma} \{(x_1,x_2)\in\mathbb{R}^2 : \ x_1x_2=\alpha_1\alpha_2,\ 0<x_1\le x_2<\alpha_2\},$$

and $\mathrm{rel\,int}\,\Gamma$ *is the relative interior of* Γ *with respect to the line joining* (a_1,a_2) *and* (b_1,b_2)*, that is,*

$$\mathrm{rel\,int}\,\Gamma := \{(a_1+t(b_1-a_1), a_2+t(b_2-a_2)) : \ t\in(0,1)\}.$$

Let $\Omega\subset\mathbb{R}^2$ *be a bounded open set and let* $\varphi\in C^1_{piec}(\overline{\Omega},\mathbb{R}^2)$ *be such that* $D\varphi(x)\in E\cup K$ *for a.e.* x *in* Ω*. Then there exists a map* $u\in\varphi+W^{1,\infty}_0(\Omega,\mathbb{R}^2)$ *such that* $Du(x)\in E$ *for a.e.* x *in* Ω*.*

Remark 4.13 *Saying that* $D\varphi(x)\in K$ *means that* $\det D\varphi(x)>0$ *and the singular values of* $D\varphi(x)$ *lie in the region given by the figure below.*



Moreover, analogous results can be achieved for other cases of the extreme points of the segment, (a_1,a_2) *and* (b_1,b_2)*, but each case must be treated separately.*

Proof. We start by noticing that, by the regularity of the boundary condition φ, one can assume that E is compact. Let $\widetilde{E}$ be the subset of E whose singular values lie in the segment joining (a_1,a_2) and (b_1,b_2):

$$\widetilde{E} := \left\{\xi\in\mathbb{R}^{2\times2} : (\lambda_1(\xi),\lambda_2(\xi))\in\Lambda_{\widetilde{E}},\ \det\xi>0\right\},\ \text{where } \Lambda_{\widetilde{E}} := \Gamma.$$

Using Theorem 3.5, we will show that K has the relaxation property with respect to $\widetilde{E}$ and thus with respect to E, since $\widetilde{E}\subseteq E$. The result will then follow as an application of Theorem 3.2. Notice first that $\widetilde{E}$ is a bounded set since $\Lambda_{\widetilde{E}}$ is compact and λ_2 is a norm.

We will prove that the set Λ_K is open in $\{(x_1,x_2)\in\mathbb{R}^2 : \ x_1\le x_2\}$. Let $y=(y_1,y_2)\in\Lambda_K$ and assume, by contradiction, that, for each $\delta>0$, there exists $x=(x_1,x_2)\in B_\delta(y)$ with $x_1\le x_2$ and $x\notin\Lambda_K$. Therefore, it is possible to construct a sequence $x^n=(x_1^n,x_2^n)$ converging to y with $x_1^n\le x_2^n$ and $x^n\notin\Lambda_K$. Observe that, due to the hypotheses on a_1,a_2,b_1,b_2, the function

$\psi(t) := (a_1 + t(b_1 - a_1))(a_2 + t(b_2 - a_2))$ is strictly increasing in $[0, 1]$ and thus, a continuous bijection between $[a_1a_2, b_1b_2]$ and $\Lambda_{\widetilde{E}}$ is defined. By definition of Λ_K, there exists $(\alpha_1, \alpha_2) \in \operatorname{rel\,int}\Gamma$ such that $y_1y_2 = \alpha_1\alpha_2 \in (a_1a_2, b_1b_2)$ with $y_2 < \alpha_2$. By continuity of the product, $\lim_{n\to+\infty} x_1^n x_2^n = \alpha_1\alpha_2$. Hence, for sufficiently large $n \in \mathbb{N}$, $x_1^n x_2^n \in (a_1a_2, b_1b_2)$ and thus, the existence of the bijection referred to above implies that $x_1^n x_2^n = \beta_1^n \beta_2^n$, for some $(\beta_1^n, \beta_2^n) \in \operatorname{rel\,int}\Gamma$ with $\lim_{n\to+\infty} \beta_1^n\beta_2^n = \alpha_1\alpha_2$. In particular, again by the continuity of the bijection between $[a_1a_2, b_1b_2]$ and $\Lambda_{\widetilde{E}}$, $\lim_{n\to+\infty}(\beta_1^n, \beta_2^n) = (\alpha_1, \alpha_2)$. Since, by hypothesis, $x^n \notin \Lambda_K$, then $x_2^n \geq \beta_2^n$ and passing to the limit, as $n \to +\infty$, we get $y_2 \geq \alpha_2$, which is a contradiction. So we conclude that Λ_K is open.

Since the singular values are continuous functions and $\lambda_1 \leq \lambda_2$, it follows that K is an open set. It remains thus to show that K has the relaxation property with respect to $\widetilde{E}$, which will be achieved through Theorem 3.5.

Before proceeding, we observe that $K \subset \operatorname{int}\operatorname{Rco}\widetilde{E}$. Indeed, it follows from a result due to Dacorogna and Tanteri [12] (see also [7, Theorem 7.43]) that, for each $(\alpha_1, \alpha_2) \in \operatorname{rel\,int}\Lambda_{\widetilde{\mathrm{E}}}$,

$$\begin{aligned}\operatorname{Rco}\left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) = (\alpha_1, \alpha_2),\ \det\xi > 0\right\} = \\ = \left\{\xi \in \mathbb{R}^{2\times 2} :\ \det\xi = \alpha_1\alpha_2,\ \lambda_2(\xi) \leq \alpha_2\right\}.\end{aligned} \tag{4.21}$$

Therefore, it is clear that $K \subseteq \operatorname{Rco}\widetilde{E}$ and since it is open, the desired inclusion follows. Moreover this inclusion implies that K is bounded since $\widetilde{E}$ is bounded.

We also note that, since $\Lambda_{\widetilde{E}} \subset \{(x_1, x_2) \in \mathbb{R}^2 :\ 0 < x_1 \leq x_2\}$,

$$\Lambda_K = \bigcup_{(\alpha_1,\alpha_2)\in \operatorname{rel\,int}\Gamma} \left\{\left(\alpha_1 + c, \frac{\alpha_1\alpha_2}{\alpha_1 + c}\right) \in \mathbb{R}^2 :\ 0 < c \leq \sqrt{\alpha_1\alpha_2} - \alpha_1\right\}. \tag{4.22}$$

Now we will prove the relaxation property introducing convenient approximating sets $\widetilde{E}_\delta$ and K_δ. For sufficiently small δ, let

$$c(\delta, \alpha_1, \alpha_2) = \min\{\delta, \sqrt{\alpha_1\alpha_2} - \alpha_1\},$$

$$\widetilde{E}_\delta := \left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_{\widetilde{E}_\delta},\ \det\xi > 0\right\},$$

$$K_\delta := \left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in \Lambda_{K_\delta},\ \det\xi > 0\right\},$$

where

$$\begin{aligned}\Lambda_{\widetilde{E}_\delta} := \left(\bigcup_{(\alpha_1,\alpha_2)\in \operatorname{rel\,int}\Lambda_{\widetilde{\mathrm{E}}}} \left\{\left(\alpha_1 + c(\delta, \alpha_1, \alpha_2), \frac{\alpha_1\alpha_2}{\alpha_1 + c(\delta, \alpha_1, \alpha_2)}\right)\right\}\right) \bigcap \\ \bigcap \left\{(x_1, x_2) \in \mathbb{R}^2 :\ 0 < x_1 \leq x_2,\ a_1a_2 + \delta \leq x_1x_2 \leq b_1b_2 - \delta\right\},\end{aligned}$$

$$\Lambda_{K_\delta} := \bigcup_{(\alpha_1,\alpha_2)\in\Lambda_{\widetilde{E}_\delta}} \{(x_1, x_2) \in \mathbb{R}^2 :\ x_1x_2 = \alpha_1\alpha_2,\ 0 < x_1 \leq x_2 \leq \alpha_2\}.$$

We proceed with the proof of conditions (i), (ii) and (iii) of Theorem 3.5.

To prove condition (i), by the matrix decomposition (4.2), it is enough to show that, for any given $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$\operatorname{dist}((x_1, x_2); \Lambda_{\widetilde{E}}) \leq \varepsilon, \ \forall \ (x_1, x_2) \in \Lambda_{\widetilde{E}_\delta}, \ \delta \in (0, \delta_0].$$

The elements of $\Lambda_{\widetilde{E}_\delta}$ are of the form $\left(\alpha_1 + c(\delta, \alpha_1, \alpha_2), \frac{\alpha_1\alpha_2}{\alpha_1 + c(\delta,\alpha_1,\alpha_2)}\right)$ for some $(\alpha_1, \alpha_2) \in \operatorname{rel\,int} \Lambda_{\widetilde{\mathrm{E}}} \subset \Lambda_{\widetilde{E}}$. Thus, choosing $\delta_0 = \frac{\varepsilon}{\sqrt{1+b_2^2/a_1^2}}$, we achieve the desired condition, since

$$\left| \left(\alpha_1 + c(\delta, \alpha_1, \alpha_2), \frac{\alpha_1\alpha_2}{\alpha_1 + c(\delta, \alpha_1, \alpha_2)}\right) - (\alpha_1, \alpha_2) \right| \leq \delta\sqrt{1 + b_2^2/a_1^2},$$

where we have used the fact that any $(\alpha_1, \alpha_2) \in \operatorname{rel\,int} \Lambda_{\widetilde{\mathrm{E}}}$ satisfies $\alpha_1 \geq a_1$ and $\alpha_2 \leq b_2$ (this follows from the hypotheses on a_1, a_2, b_1, b_2).

Now we prove condition (ii). Let $\eta \in \operatorname{int} K = K$. It suffices to show that, for sufficiently small δ, $\lambda_1(\eta)\lambda_2(\eta) = \beta_1\beta_2$, for some $(\beta_1, \beta_2) \in \Lambda_{\widetilde{E}_\delta}$ with $\lambda_2(\eta) \leq \beta_2$. By (4.22), $(\lambda_1(\eta), \lambda_2(\eta)) = \left(\alpha_1 + c, \frac{\alpha_1\alpha_2}{\alpha_1+c}\right)$ for some $(\alpha_1, \alpha_2) \in \operatorname{rel\,int} \Lambda_{\widetilde{\mathrm{E}}}$ and $0 < c \leq \sqrt{\alpha_1\alpha_2} - \alpha_1$. The monotonicity of the function ψ introduced above, yields $\alpha_1\alpha_2 \in (a_1a_2, b_1b_2)$. Thus, for small δ, one has $\alpha_1\alpha_2 \in [a_1a_2 + \delta, b_1b_2 - \delta]$. Defining $(\beta_1, \beta_2) = \left(\alpha_1 + c(\delta, \alpha_1, \alpha_2), \frac{\alpha_1\alpha_2}{\alpha_1+c(\delta,\alpha_1,\alpha_2)}\right)$, we observe that $(\beta_1, \beta_2) \in \Lambda_{\widetilde{E}_\delta}$. Finally, $\lambda_2(\eta) < \beta_2$ is equivalent to $c > c(\delta, \alpha_1, \alpha_2)$ and this is true for sufficiently small δ, since $\lim_{\delta\to 0} c(\delta, \alpha_1, \alpha_2) = 0$ and $c > 0$.

It remains to prove condition (iii). We notice that, using (4.21), we conclude that any $\xi \in K_\delta$ belongs to

$$\operatorname{Rco}\left\{\xi \in \mathbb{R}^{2\times 2} : (\lambda_1(\xi), \lambda_2(\xi)) = (\alpha_1, \alpha_2), \ \det \xi > 0\right\},$$

for some $(\alpha_1, \alpha_2) \in \Lambda_{\widetilde{E}_\delta}$. Since $\Lambda_{\widetilde{E}_\delta} \subset \Lambda_K$, this hull is a subset of $K = \operatorname{int} K$ and condition (iii) follows from (2.2) (see also Remark 2.10 - 3)). □

We consider next the 3 dimensional version of the previous result.

Theorem 4.14 *Let* $E = \left\{\xi \in \mathbb{R}^{3\times 3} : (\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi)) \in \Lambda_E, \ \det \xi > 0\right\}$, *where* $\Lambda_E \subseteq \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 \leq x_2 \leq x_3\}$, *and assume that*

$$\Gamma := \{(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2), a_3 + t(b_3 - a_3)) : \ t \in [0, 1]\} \subseteq \Lambda_E,$$

with $a_1 < b_1$, $a_2 < b_2$, $a_3 < b_3$ *and either* $a_1 < a_2 < a_3$ *or* $b_1 < b_2 < b_3$. *Let*

$$K := \left\{\xi \in \mathbb{R}^{3\times 3} : (\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi)) \in \Lambda_K, \ \det \xi > 0\right\},$$

where

$$\Lambda_K := \bigcup_{(\alpha_1,\alpha_2,\alpha_3)\in \operatorname{rel\,int}\Gamma} \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1x_2x_3 = \alpha_1\alpha_2\alpha_3, \right.$$
$$\left. 0 < x_1 \leq x_2 \leq x_3 < \alpha_3, \ x_2x_3 < \alpha_2\alpha_3\right\},$$

and $\operatorname{rel\,int}\Gamma$ *is the relative interior of* Γ *with respect to the line joining* (a_1, a_2, a_3) *and* (b_1, b_2, b_3)*, that is,*

$$\operatorname{rel\,int}\Gamma := \{(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2), a_3 + t(b_3 - a_3)) : \ t \in (0, 1)\}.$$

Let $\Omega \subset \mathbb{R}^3$ *be a bounded open set and let* $\varphi \in C^1_{piec}(\overline{\Omega}, \mathbb{R}^3)$ *be such that* $D\varphi(x) \in E \cup K$ *for a.e.* x *in* Ω*. Then there exists a map* $u \in \varphi + W^{1,\infty}_0(\Omega, \mathbb{R}^3)$ *such that* $Du(x) \in E$ *for a.e.* x *in* Ω*.*

Proof. The proof of this result follows the lines of that of Theorem 4.12. Due to the heavy notation we won't present it here in full detail. The reader can follow the proof of Theorem 4.12 taking into account that in this case the set Λ_K can be written in the form

$$\Lambda_K = \bigcup_{(\alpha_1,\alpha_2,\alpha_3)\in \operatorname{rel\,int}\Gamma} \left\{ \left(\alpha_1 + c_1, \frac{\alpha_1\alpha_2}{\alpha_1 + c_1} + c_2, \frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2 + c_2(\alpha_1 + c_1)} \right) \in \mathbb{R}^3 : \right.$$
$$\left. c_1 > 0, \ c_2 \geq \alpha_1 + c_1 - \frac{\alpha_1\alpha_2}{\alpha_1 + c_1}, \ 0 < c_2 \leq \sqrt{\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1 + c_1}} - \frac{\alpha_1\alpha_2}{\alpha_1 + c_1} \right\}.$$

The approximating sets $\widetilde{E}_\delta$ and K_δ can be defined, for sufficiently small δ, by

$$\widetilde{E}_\delta := \left\{ \xi \in \mathbb{R}^{3\times 3} : (\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi)) \in \Lambda_{\widetilde{E}_\delta}, \ \det \xi > 0 \right\},$$

$$K_\delta := \left\{ \xi \in \mathbb{R}^{3\times 3} : (\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi)) \in \Lambda_{K_\delta}, \ \det \xi > 0 \right\},$$

where $\Lambda_{\widetilde{E}_\delta}$ is the set

$$\left(\bigcup_{(\alpha_1,\alpha_2,\alpha_3)\in \operatorname{rel\,int}\Gamma} \left\{ \left(\alpha_1 + \delta, \frac{\alpha_1\alpha_2}{\alpha_1 + \delta} + c(\delta, \alpha), \frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2 + c(\delta,\alpha)(\alpha_1 + \delta)} \right) \right\} \right)$$
$$\bigcap \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 \leq x_2 \leq x_3, a_1a_2a_3 + \delta \leq x_1x_2x_3 \leq b_1b_2b_3 - \delta \right\}$$

$$c(\delta, \alpha) = \min \left\{ \delta, \sqrt{\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1 + \delta}} - \frac{\alpha_1\alpha_2}{\alpha_1 + \delta} \right\}, \ \alpha = (\alpha_1, \alpha_2, \alpha_3) \text{ and}$$

$$\Lambda_{K_\delta} := \bigcup_{(\alpha_1,\alpha_2,\alpha_3)\in \Lambda_{\widetilde{E}_\delta}} \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1x_2x_3 = \alpha_1\alpha_2\alpha_3, \right.$$
$$\left. 0 < x_1 \leq x_2 \leq x_3 \leq \alpha_3, \ x_2x_3 \leq \alpha_2\alpha_3 \right\}.$$

□

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