A geometrical view of the Nehari manifold

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Abstract

We study the Nehari manifold $\mathcal{N}$ associated to the boundary value problem

$$-\Delta u = f(u)\,, \quad u \in H_0^1(\Omega)\,,$$

where Ω is a bounded regular domain in $\mathbb{R}^n$. Using elementary tools from Differential Geometry, we provide a local description of $\mathcal{N}$ as an hypersurface of the Sobolev space $H_0^1(\Omega)$. We prove that, at any point $u \in \mathcal{N}$, there exists an exterior tangent sphere whose curvature is the limit of the increasing sequence of principal curvatures of $\mathcal{N}$. Also, the H_1-norm of $u \in \mathcal{N}$ depends on the number of principal negative curvatures. Finally, we study basic properties of an angle decreasing flow on the Nehari manifold associated to homogeneous non–linearities.

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1 Introduction

The variational method introduced by Nehari in [9]–[10] was a significant outcome of his research on the non–oscillating nature of solutions to certain classes of second order equations. For instance, concerning the linear problem

$$y'' + p(x)y = 0\,, \quad y(a) = y'(b) = 0\,,$$

where p is a continuous positive function, [Theorem 1, [8]] sets the equivalence between the existence of a positive solution in $[a, +\infty[$ and the fact that the lowest eigenvalue

$$\lambda := \min \frac{\int_a^b y'^2\, dx}{\int_a^b p y^2\, dx}$$

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satisfies $\lambda > 1$ for all $b > a$. In [7], a solution to the non–linear equation

$$y'' + p(x)y^{2n+1} = 0\,, \quad y(a) = y(b) = 0$$

with a prescribed number m of intermediate zeros $a < a_1 < \ldots < a_m < b$ is obtained by minimizing the functional

$$\tilde{J}(u; a_1, \ldots, a_m) := \sum_{\nu=1}^{m+1} [\tilde{J}_\nu]^{\frac{1}{n}}\,,$$

where $u \in C_0^{0,1}[a, b]$ satisfies $u(a_1) = \ldots = u(a_m) = 0$ and

$$\tilde{J}_\nu(w) = \frac{\left(\int_{a_\nu}^{a_\nu+1} w'^2\,dx\right)^{n+1}}{\int_{a_\nu}^{a_\nu+1} pw^{2n+2}\,dx}\,.$$

is the Rayleigh coefficient on $C_0^{0,1}([a_\nu, a_{\nu+1}])$. Similar ideas were later exploited in [14] and [15] although these works don't mention Nehari's early contribution. In fact, as it was defined in [9], the "normalization condition" (known *a posteriori* as the Nehari constrain)

$$\int_a^b y'^2\,dx = \int_a^b y^2 F(y^2, x)\,dx \qquad (u \neq 0)\,,$$

was the basis of a more comprehensive method allowing the proof of the existence of solutions to a second order non-linear equation of type

$$y'' + yF(y^2, x) = 0\,,$$

where the non-homogeneous linear term prevented the method of minimizing a Rayleigh coefficient.

In the past few decades, the Nehari method has been extensively used on the study of existence of ground–state, nodal, multi-spike or multi-bump solutions, in what can be considered as a natural enlargement of Nehari's concerns about oscillatory aspects of second order non-linear differential equations (see for instance [4],[5] and [12]). For the interested reader on an abstract treatment of the Nehari method (or on further references about the subject) we recommend the survey [13]. Our purpose to bring out a clearer picture of a variational framework known since 1960 was, in some sense, stimulated by the study of [2].

In section 1 we obtain classical estimates of the energy of a function satisfying the Nehari constrain and recall basic facts about the Nehari manifold. In section 2 we use the notion of curvature to provide a local description of the Nehari manifold $\mathcal{N}$. Some regularity assumptions will be required both on the nonlinear term of the Nehari constrain as well as on the function $u \in \mathcal{N}$. In the last section, we propose an alternative flow on the Nehari manifold (assuming an homogeneous nonlinearity) whose stable stationnary points are, under appropriate conditions, solutions of the second order equation

$$-\Delta u = f(u)\,, \qquad u \in H_0^1(\Omega)\,.$$

This work is a personal tribute to Nehari's pioneering works [9]–[10] fifty years after their publication. I thank Luis Sanchez and Pedro Girão for their interest and support.

2 Preliminary results

Along this article we consider the space $H_0^1(\Omega)$, where Ω is a bounded and regular domain of $\mathbb{R}^N$. We assume $H_0^1(\Omega)$ is endowed with the norm

$$\|u\|^2 = \langle u, u\rangle := \int_\Omega |\nabla u|^2(x)\, dx\,.$$

As usual, we denote $2^* = \frac{2N}{N-2}$ and $2^* = +\infty$ if $N = 2$, so that the embedding

$$H_0^1(\Omega) \subset L^q(\Omega)$$

is compact for $1 \le q < 2^*$. We introduce the classical Euler-Lagrange functional

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2(x)\, dx - \int_\Omega F(u)(x)\, dx\,, \tag{2.1}$$

defined over $H_0^1(\Omega)$ where $F(u) = \int_0^u f(s)\, ds$. Critical points of J in $H_0^1(\Omega)$ are classical solutions of the elliptic equation

$$-\Delta u = f(u) \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega\,, \tag{2.2}$$

provided well known assumptions on the non-linear term f are verified (see, for instance [11]). In our case, we require

(f1) $f \in C^2(\mathbb{R}, \mathbb{R})$.

(f2) $f(u)u \le \beta f'(u)u^2$ where $0 < \beta < 1$.

(f3) There exist positive constants $\xi_1 \le \xi_2$ such that

$$\xi_1 |u|^{p-2} \le f'(u) \le \xi_2 |u|^{p-2}\,,$$

where $2 < p < 2^*$.

Note that condition (f2) implies that $f(0) = 0$ as well as

$$\zeta F(u) \leq f(u)u\,, \tag{2.3}$$

for some $\zeta > 2$, which is the classical Ambrosetti-Rabinowitz condition. Further, we will require

(f3') There exist positive constants $\xi_1 \leq \xi_2$ such that

$$\xi_1|u|^{p-2} \leq f''(u)u \leq \xi_2|u|^{p-2}\,.$$

Condition (f3') implies (f3) (adapting, if necessary, the constants ξ_1 and ξ_2). We define a sequence (e_n) in $H_0^1(\Omega)$ in the following way. Let e_1 be such that

$$\|e_1\|^2 = \min\left\{\|u\|^2 \,:\, \int_\Omega F(u)(x)\,dx = 1\right\}\,,$$

and for $n > 1$

$$\|e_n\|^2 = \min\left\{\|u\|^2 \,:\, \int_\Omega F(u)(x)\,dx = 1\,, \quad u \in (\text{ span}\{e_1, ..., e_{n-1}\})^\perp\right\}\,. \tag{2.4}$$

We have the following fact whose proof we postpone to the Appendix.

Lemma 1 *The sequence (e_n) is an orthogonal basis of $H_0^1(\Omega)$. Also $(\|e_n\|)$ is non-decreasing and*

$$\lim_{n\to\infty} \|e_n\| = \infty.$$

Remark 1 *Each e_n satisfies the relation*

$$-\Delta e_n = \lambda_{nn} f(e_n) + \sum_{i=1}^{n-1} \lambda_{ni}(-\Delta e_i) \tag{2.5}$$

for some Lagrange multipliers λ_{ni}. In particular, $e_n \in C^{3,\alpha}(\Omega) \cap C_0(\overline{\Omega})$. Multiplying (2.5) by e_n, and integrating by parts we conclude

$$\lambda_{nn} = \frac{\|e_n\|^2}{\int_\Omega f(e_n)e_n\,(x)\,dx} > 0\,.$$

A similar argument yields, for all $m > n$,

$$0 = \int_\Omega \nabla e_n \nabla e_m(x)\,dx = \lambda_{nn} \int_\Omega f(e_n)e_m(x)\,dx. \tag{2.6}$$

Then (2.6) implies

$$\textit{for all } m > n \qquad \langle \nabla J(e_n), e_m\rangle = 0\,.$$

The Nehari manifold is defined as

$$\mathcal{N} := \{u \in H_0^1(\Omega) : u \neq 0 \quad \text{and} \quad \langle \nabla J(u), u \rangle = 0\}. \tag{2.7}$$

Condition $\langle \nabla J(u), u \rangle = 0$ writes

$$\int_\Omega |\nabla u|^2(x)\, dx - \int_\Omega f(u)u(x)\, dx = 0\,. \tag{2.8}$$

In the next Proposition we obtain estimates on a function $u \in \mathcal{N}$ based on the dimension of a space where the second derivative of J at u is negative definite.

Proposition 1 *Assume $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies (f2)–(f3). Let $u \in \mathcal{N}$ and V_j be a j-dimensional subspace of $H_0^1(\Omega)$ such that*

$$D^2 J_{vv}(u) \leq 0 \quad \text{for all } v \in V_j\,. \tag{2.9}$$

Then

$$J(u) \geq \max\{C_1 \|e_j\|^{\frac{2p}{p-2}}\,,\, C_2\},$$

where e_j was defined in (2.4) and C_1, C_2 are positive constants independent of u.

Proof. By (2.8), our assumptions on f and Sobolev's Embedding Theorem we have, for some constant c_p,

$$\|u\|^2 \leq \frac{\xi_2}{p-1} \int_\Omega |u|^p(x)\, dx \leq \frac{\xi_2 c_p}{p-1} \|u\|^p \tag{2.10}$$

Then, for $C = \left(\frac{p-1}{\xi_2 c_p}\right)^{\frac{1}{p-2}}$, we conclude

$$\|u\| \geq C\,. \tag{2.11}$$

By (2.1), (2.3) and (2.8),

$$J(u) \geq \left(\frac{1}{2} - \frac{1}{\zeta}\right) \|u\|^2\,. \tag{2.12}$$

The previous estimates prove that $J(u) \geq C_2$ with $C_2 = (1/2 - 1/\zeta)C^2$.

Let

$$S = \{v \in V_j : \|v\| = 1\}\,.$$

We have $\gamma(S) = j$ where γ is the the genus of a closed symmetric set (see [11]). Let

$$E_j = (\mathrm{span}\{e_1, ..., e_{j-1}\})^\perp\,.$$

Since $\gamma(S) >$ codimension E_j, we conclude by [Proposition 7.8, [11]] that

$$S \cap E_j \neq \emptyset\,.$$

We may therefore choose $v \in V_j \cap E_j$ and, multiplying if necessary by an appropriate constant, assume $\int_\Omega F(v)(x)\,dx = 1$. We have

$$D^2 J_{vv}(u) = \int_\Omega |\nabla v|^2(x)\,dx - \int_\Omega f'(u)v^2(x)\,dx \leq 0\,. \tag{2.13}$$

By (2.13), Holder inequality and (f3),

$$\int_\Omega |\nabla v|^2(x) \leq \left(\int_\Omega |f'(u)|^{\frac{p}{p-2}}(x)\,dx\right)^{\frac{p-2}{p}} \left(\int_\Omega |v|^p(x)\,dx\right)^{\frac{2}{p}} \leq$$
$$C\left(\int_\Omega |u|^p(x)\,dx\right)^{\frac{p-2}{p}} \left(\int_\Omega F(v)(x)\,dx\right)^{\frac{2}{p}} = C\left(\int_\Omega |u|^p(x)\,dx\right)^{\frac{p-2}{p}}\,. \tag{2.14}$$

where $C = \xi_2^{\frac{p-2}{p}} \left(\frac{(p-1)p}{\xi_1}\right)^{\frac{2}{p}}$. By the definition of (e_n) and our assumptions on v we have,

$$\int_\Omega |\nabla v|^2(x)\,dx \geq \int_\Omega |\nabla e_j|^2(x)\,dx. \tag{2.15}$$

We conclude, by (2.12), (2.14) and (2.15)

$$J(u) \geq \left(\frac{1}{2} - \frac{1}{\zeta}\right) \int_\Omega |\nabla u|^2(x)\,dx = \left(\frac{1}{2} - \frac{1}{\zeta}\right) \int_\Omega f(u)u(x)\,dx \geq \tag{2.16}$$
$$\left(\frac{1}{2} - \frac{1}{\zeta}\right) \frac{\xi_1}{p-1} \int_\Omega |u|^p(x)\,dx \geq C_1 \|e_j\|^{\frac{2p}{p-2}}$$

where $C_1 = C^{-p/(p-2)} \left(\frac{1}{2} - \frac{1}{\zeta}\right) \frac{\xi_1}{p-1}$. ■

Remark 2 *We conclude from Proposition 1 and Lemma 1 that if (u_j) is a sequence in $\mathcal{N}$ such that, for each u_j, there exists a j-dimensional space V_j verifying (2.9) then*

$$\|u_j\| \to \infty\,.$$

Given $u \in \mathcal{N}$ the tangent space $\mathfrak{T}_u$ to $\mathcal{N}$ at u consists on the functions $v \in H_0^1(\Omega)$ such that

$$2\int_\Omega \nabla u \nabla v\,(x)\,dx - \int_\Omega f'(u)uv(x)\,dx - \int_\Omega f(u)v(x)\,dx = 0\,. \tag{2.17}$$

The next proposition sets some well–known facts.

Proposition 2 *Assume f satisfies (f1)–(f3). There exists $C' > 0$ such that*

$$u \in \mathcal{N} \Rightarrow \|u\| \geq C' \,. \tag{2.18}$$

Moreover, $\mathcal{N}$ is locally diffeomorphic to

$$S := \{u \in H_0^1(\Omega), \|u\| = 1\} \,.$$

Given $u \in \mathcal{N}$,

$$\nabla J(u) = 0 \quad \Leftrightarrow \quad \Pi_u(\nabla J(u)) = 0 \,, \tag{2.19}$$

where Π_u is the orthogonal projection on $\mathfrak{T}_u$.

Proof. Condition (2.18) was already proved in Proposition 1. Given $u \in H_0^1(\Omega)\backslash\{0\}$, consider the function

$$g(t) := \langle \nabla J(tu), tu \rangle = t^2 \int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega t f(tu) u\,(x) \, dx \,.$$

By (f2)–(f3), we have $g(t) > 0$ if $0 < t < \epsilon$ for ϵ sufficiently small. Also

$$\lim_{t \to +\infty} g(t) = -\infty \,.$$

Therefore there exists $t_0 > 0$ such that $g(t_0) = 0$. By (2.8) and (f2),

$$g'(t_0) = 2t_0 \int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega f(t_0 u)u + f'(t_0 u)u^2 \, dx < 0 \,.$$

Consequently, $t_0 > 0$ is uniquely determined. Also, by the Implicit Function Theorem,

$$t_0(u) \in C^2(H_0^1(\Omega)\backslash\{0\}), \mathbb{R}\backslash\{0\}) \,.$$

Consider the C^2–application

$$P_{\mathcal{N}} : H_0^1(\Omega)\backslash\{0\} \mapsto \mathcal{N} \qquad u \to t_0(u)u \,.$$

Clearly, the restriction

$$P_{\mathcal{N}}|_S \mapsto \mathcal{N}$$

is a local diffeomorphism.

We now turn to (2.19). The first implication is trivial. Consider the constraint $\phi(u) := \langle \nabla J(u), u \rangle = 0$. By (f2), for any $u \in \mathcal{N}$,

$$\langle \nabla \phi(u), u \rangle = \int_\Omega |\nabla u|^2(x) \, dx - \int_\Omega f'(u)u^2(x) \, dx = \int_\Omega f(u)u - f'(u)u^2(x) \, dx < 0 \,,$$

i.e., $u \notin \mathfrak{T}_u$. Then, $\Pi_u(\nabla J(u)) = 0$ and $\langle \nabla J(u), u \rangle = 0$ imply $\nabla J(u) = 0$. ∎

3 Local geometry of the Nehari manifold

We use basic notions of Differential Geometry to describe the Nehari manifold as an hypersurface of $H_0^1(\Omega)$ (see for instance, [1] and [6]). In the sequence, we will assume that assumptions (f1), (f2) and (f3') are verified. By the Riesz representation of a linear functional in $H_0^1(\Omega)$ and (2.17), the tangent space can also be characterized as

$$\mathfrak{T}_u := \{v \in H_0^1(\Omega) : \quad \langle N(u), v\rangle = 0\,\},$$

with $N(u) = 2u + \Delta^{-1}(h(u))$ and

$$h(u) = f'(u)u + f(u)\,. \tag{3.1}$$

Prescribe

$$n(u) = \frac{N(u)}{\|N(u)\|}\,,$$

as unitary normal to $\mathfrak{T}_u$. By (f2),

$$\langle n(u), u\rangle < 0 \tag{3.2}$$

for all $u \in \mathcal{N}$. Our assumptions on f imply that the map $u \to n(u)$ is of class C^1 in $H_0^1(\Omega)\backslash\{0\}$. Given $u \in \mathcal{N}$, we formally define a Weingarten map

$$L_u : \mathfrak{T}_u \mapsto \mathfrak{T}_u \qquad L_u(v) = Dn(u)[v]\,.$$

In fact, given $u \in \mathcal{N}$, $v \in \mathfrak{T}_u$ and a regular path γ such that

$$\gamma : \quad]-1,1[\mapsto \mathcal{N}\,, \quad \gamma(0) = u\,, \quad \gamma'(0) = v,$$

we have

$$\langle n(\gamma(t)), n(\gamma(t))\rangle = 1 \qquad \forall t \in]-1,1[\,.$$

In particular

$$\langle Dn(\gamma(0))[\gamma'(0)], n(\gamma(0))\rangle = 0\,,$$

i.e.

$$Dn(u)[v] \in \mathfrak{T}_u$$

for all $v \in \mathfrak{T}_u$. We also recall the classical formula

$$Dn(u)[v] = -D\Pi_u(v, n(u))\,. \tag{3.3}$$

Computing,

$$Dn(u)[v] = \frac{1}{\|N(u)\|}\left(2v + \Delta^{-1}(h'(u)v) - n(u)\left\langle 2v + \Delta^{-1}(h'(u)v)\,,\, n(u)\right\rangle\right)\,. \tag{3.4}$$

If we assume $u \in W_0^{1,\infty} \subset H_0^1(\Omega)$ the operator

$$L_u(v) := Dn(u)[v] = \frac{1}{\|N(u)\|}(2I + T_u)$$

where

$$T_u(v) = \Delta^{-1}(h'(u)v) - n(u)\left\langle 2v + \Delta^{-1}(h'(u)v)\,,\, n(u)\right\rangle$$

is well-defined for all $v \in H_0^1(\Omega)$. Moreover the operator

$$T_u : \mathfrak{T}_u \mapsto \mathfrak{T}_u$$

is self-adjoint and compact (note that the term $\langle 2v + \Delta^{-1}(h'(u)v)\,,\, n(u)\rangle$ maps into $\mathbb{R}$). We may therefore provide an orthogonal basis for $\mathfrak{T}_u$ of eigenvectors of T_u. To an eigenvector v of T_u with associated eigenvalue λ corresponds the same eigenvector v of L_u with associated eigenvalue

$$k = \frac{2 + \lambda}{\|N(u)\|}\,. \tag{3.5}$$

Remark 3 *Of course, the assumption that $u \in W_0^{1,\infty}(\Omega)$ may be weakened. For instance, if Ω is a bounded regular subset of $\mathbb{R}^2$, as $H_0^1(\Omega) \subset L^q(\Omega)$ for any $q \in [1, +\infty[$ with compact embedding, the principal curvatures are defined for all $u \in H_0^1(\Omega) \cap \mathcal{N}$. However, the class of functions in $W_0^{1,\infty}(\Omega)$ is of special interest regarding its invariance property for a significant class of energy decreasing flows associated to Euler-Lagrange functionals.*

We have the following property of the non-zero eigenvalues of the compact operator T_u.

Lemma 2 *Given $u \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$, the distinct non-zero eigenvalues of T_u form an increasing sequence $(\lambda_n(u))$ converging to zero.*

Proof. As usual, we determine the sequence of the non-zero eigenvalues and corresponding eigenvectors of T_u by means of a recurrent sequence of minimization problems:

$$\lambda_n := \inf\left\{\langle T_u(v), v\rangle : v \in \mathfrak{T}_u\,, \|v\| = 1\,, v \in (\mathrm{span}\{v_1, ..., v_{n-1}\})^{\perp}\right\}$$

and a corresponding eigenvector v_n is a function where the infimum is attained. Necessarily, (λ_n) is an increasing sequence. In case $\lambda_{n+1} = \lambda_n$ the eigenvalue λ_n has multiplicity greater than 1. Since $\langle n(u), v\rangle = 0$, we have

$$\langle T_n(v), v\rangle = \langle \Delta^{-1}(h'(u)v), v\rangle = -\int_\Omega h'(u)v^2(x)\,dx\,,$$

and conclude $\lambda_n \leq 0$ for all $n \in \mathbb{N}$.

Assume, for some n that $\lambda_n = 0$ and $\lambda_{n-1} < 0$. Then for any $k \geq n$, we have $\lambda_k = 0$ and the corresponding eigenfunction v_k satisfies

$$\int h'(u) v_k^2(x)\, dx = 0\,.$$

Then, by (f3),

$$v_k \equiv 0 \qquad \text{in} \quad \text{supp}(u) \qquad \forall k \geq n\,.$$

As any w such that

$$\text{support}(w) \subset \text{support}(u)$$

is orthogonal to v_k with $k \geq n$, w necessarily belongs to $\text{span}\{v_1, ..., v_{n-1}\}$. This would imply, for any bounded regular domain ω such that $\omega \subset \text{supp}(u)$,

$$\left(H_0^1(\omega) \cap \mathfrak{T}_u\right) \subset \text{span}\{v_1, ..., v_{n-1}\}$$

which is absurd since the first subspace is infinite dimensional. ■

If

$$\int_\Omega h'(u) v^2(x)\, dx \; > 0\,, \quad \forall v \in \mathfrak{T}_u \backslash \{0\}\,,$$

the sequence (v_i) of eigenvectors associated to the sequence of non-zero eigenvalues (λ_i) provides an Hilbert basis of $\mathfrak{T}_u$. This is the case if $u(x) \neq 0$ a.e. in Ω. In general, we may write

$$\mathfrak{T}_u = \text{Ker}(T_u) \oplus R(T_u)\,,$$

where $R(T_u)$ is the closure of the subspace generated by the family $\{v_i\}$.

In view of (3.5), we will refer an eigenvalue k_i of L_u as a (signed) principal curvature of $\mathcal{N}$ at u if the corresponding eigenvalue λ_i of T_u satisfies $\lambda_i < 0$. The sequence (k_i) is increasing and converges to $2/\|N(u)\|$. We denote by $\mathfrak{K}_u$ the set of all eigenvalues of L_u. We have

$$\mathfrak{K}_u \subseteq \{k_i\}_{i \in \mathbb{N}} \cup \{2/\|N(u)\|\}\,, \tag{3.6}$$

with equality of sets in the degenerate case $\text{Ker}(T_u) \neq \{0\}$. In particular, at any point $u \in \mathcal{N}$, the principal curvatures are positive, except at most for a finite number.

Let P be a plane containing the inward normal $n(u)$ and a direction $v(u)$ associated to a positive curvature. Using the reference frame of center u and vectors $v(u)$ and $n(u)$, if $w \in P \cap \mathcal{N} \backslash \{u\}$ is sufficiently close to u, then

$$w = x\, v(u) + y\, n(u) \quad \text{with} \quad (x, y) \in \mathbb{R}^2\,, \quad y < 0\,.$$

Remark 4 *We may describe the above mentionned property saying that, at any point $u \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$, there exists an "exterior" tangent sphere to the Nehari manifold, with center*

$$C(u) = u - \frac{\|N(u)\|}{2} \cdot n(u) = -\frac{1}{2}\Delta^{-1}(h(u)),$$

and radius $\|N(u)\|/2$, whose curvature is approximated by the sequence of principal curvatures of the Nehari manifold.

We have the following estimates on the curvatures of the Nehari manifold.

Lemma 3 *There exists $C > 0$ such that, for every $u \in W_0^{1,\infty}(\Omega) \cap \mathcal{N}$ and $i \in \mathbb{N}$*

$$-\frac{C(2 + \|u\|^{2(p-2)/p})}{\|u\|} \leq k_i(u) \leq \frac{C}{\|u\|} \tag{3.7}$$

Proof. As

$$\langle N(u), u\rangle = 2\|u\|^2 - \int f(u)u\,dx - \int f'(u)u^2\,dx$$

by (2.8) and (f2)

$$|\langle N(u), u\rangle| \geq \frac{1-\beta}{\beta} \cdot \|u\|^2,$$

and, by Schwarz inequality,

$$\|N(u)\| \geq \frac{1-\beta}{\beta}\,\|u\|. \tag{3.8}$$

In view of (3.5), we conclude from Lemma 2 and (3.8) the right hand-side of (3.7). In order to prove the complete estimate it suffices to set the inequality to k_1. Assume $\|v\| = 1$. Necessarily

$$\lambda_1 \geq \lambda := \min_{\|v\|=1} -\int_\Omega h'(u)v^2(x)\,dx.$$

By (f3') and (3.1),

$$h'(u) \leq C_1\,|u|^{p-2}$$

for $C_1 = \xi_2/(p-1)$. Then, by Holder inequality, (2.8) and Sobolev Imbedding Theorem, for some constant $C_2 > 0$

$$\begin{aligned}\int_\Omega h'(u)v^2(x)\,dx &\leq C_1\left(\int_\Omega |u|^p(x)\,dx\right)^{\frac{p-2}{p}}\left(\int_\Omega |v|^p(x)\,dx\right)^{\frac{2}{p}}\\ &\leq C_2\left(\int_\Omega f(u)u(x)\,dx\right)^{2(p-2)/p} = C_2\|u\|^{2(p-2)/p},\end{aligned} \tag{3.9}$$

thereby proving inequality (3.7). ∎

Remark 5 *Note that, if $p \leq 4$, the curvatures are uniformly bounded below on the Nehari manifold by a negative constant. In particular, there exists $\overline{K} > 0$ such that, for all $u \in \mathcal{N}$,*

$$|k_i(u)| \leq \overline{K} \qquad \forall i \in \mathbb{N}\,.$$

Analogously to Proposition 1, we obtain lower bounds on the the energy of $u \in \mathcal{N}$ based on the number of negative principal curvatures of the Weingarten map L_u.

Proposition 3 *Assume (f1)-(f2)-(f3'). Let $u \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$ be such that*

$$k_i(u) \leq 0\,, \qquad i = 1, ..., j\,.$$

Then, there exist positive constants C_1 and C_2 independent of u such that

$$J(u) \geq \max\{C_1 \|e_j\|^{\frac{2p}{p-2}}\,,\, C_2\}\,,$$

where e_j was defined in (2.4).

Proof. The proof is similar to the proof of Proposition 1 so we omit the details. Consider the subspaces

$$V_j := \text{span}\{v_1, ...v_{\bar{j}}\} \quad \text{and} \quad E_j = (\text{span}\{e_1, ..., e_{j-1}\})^{\perp}$$

where the v_i's are eigenvectors associated to $k_1, ..., k_j$ (necessarily, $\bar{j} \geq j$). For any $v \in V_j$,

$$\langle Dn_u(v), v\rangle = \frac{1}{\|N(u)\|}\langle 2v + T_u(v), v\rangle = \frac{1}{\|N(u)\|}\left(2\|v\|^2 - \int_\Omega h'(u)v^2(x)\,dx\right) \leq 0\,, \tag{3.10}$$

or

$$\|v\|^2 - \frac{1}{2}\int_\Omega h'(u)v^2(x)\,dx \;\leq 0\,.$$

As in Lemma 1, we may choose $v \in V_j \cap E_j$ such that $\int_\Omega H(v)(x)\,dx = 1$ for $H(v) = \int_0^v h(s)\,ds$. Recalling that, by (3.1), $h'(u) = 2f'(u) + f''(u)u$, we may estimate as in (2.14)–(2.16) and conclude the proof. ■

Remark 6 *We may assert the existence of points on the Nehari manifold with an arbitrarily large number of negative principal curvatures. In fact, let us consider a multi-bump function*

$$u := \sum_{k=1}^{n} v_k$$

where, for $i \neq j$,

$$support\,(v_i) \cap support\,(v_j) = \emptyset$$

and

$$v_k \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$$

for all $k = 1, ..., n$. Since

$$2\|v_i\|^2 - \int_\Omega h'(u)v_i^2(x)\,dx = 2\|v_i\|^2 - \int_\Omega h'(u)v_i^2(x)\,dx$$

and the set of functions $\{v_i\}_{i=1,...,n}$ is orthogonal, we conclude that

$$k_1 < ... < k_{n-1} < 0\,,$$

where k_i is the sequence of eigenvalues of L_u.

4 An angle-decreasing flow.

In the next section, we assume

$$f(u) = \begin{cases} c_1|u|^{p-2}u & \text{if } u \leq 0 \\ c_2|u|^{p-2}u\,, & \text{if } u > 0\,, \end{cases} \tag{4.1}$$

where $c_1, c_2 > 0$. In case where the non-linearity f is as in (4.1), then

$$J(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 \quad \forall u \in \mathcal{N}\,.$$

In particular, critical points of the distance functional $u \mapsto \|u\|$ constrained to $\mathcal{N}$ are solutions of (2.2).

We introduce an auxiliary functional on the Nehari manifold:

$$\theta_u \equiv \theta(u) =: \left\langle n(u), \frac{u}{\|u\|} \right\rangle\,.$$

The functional θ is the restriction to $\mathcal{N}$ of a functional of class $C^1(H_0^1(\Omega)\backslash\{0\}, \mathbb{R})$ that we will denote by θ. Note that, by (3.2) and Schwarz inequality

$$\theta(\mathcal{N}) \subset [-1, 0[\,.$$

Also, $\arccos(\theta_u)$ corresponds to the angle between the vectors u and $n(u)$.

Assuming $u \in W_0^{1,\infty}(\Omega)$, we use our previous decomposition of the tangent space $\mathfrak{T}_u$ to calculate

$$\Pi_u(\nabla\theta_u)\,.$$

For any $v \in \mathfrak{T}_u$,

$$\langle \nabla\theta_u, v\rangle = D\theta_u(v) = \left\langle Dn(u)[v], \frac{u}{\|u\|}\right\rangle - \langle n(u), u\rangle \frac{\langle u, v\rangle}{\|u\|^3} \tag{4.2}$$

Choosing v an eigenvector with corresponding eigenvalue k, as $\langle n, v\rangle = 0$ we obtain by (3.4),

$$\langle \nabla\theta_u, v\rangle = \left(k - \frac{\theta_u}{\|u\|}\right)\left\langle v, \frac{u}{\|u\|}\right\rangle . \tag{4.3}$$

We may write, in the non-degenerate case $\mathrm{Ker}(T_u) = \{0\}$,

$$\Pi_u(\nabla\theta_u) = \frac{1}{\|u\|}\sum_{i=1}^{\infty}\left(k_i - \frac{\theta_u}{\|u\|}\right)\langle v_i, u\rangle \cdot v_i . \tag{4.4}$$

More generally, denoting by Π_u^0 the projection on $\mathrm{Ker}(T_u) \subset \mathfrak{T}_u$,

$$\Pi_u(\nabla\theta_u) = \frac{1}{\|u\|}\sum_{i=1}^{\infty}\left(k_i - \frac{\theta_u}{\|u\|}\right)\langle v_i, u\rangle \cdot v_i + \frac{2-\theta_u}{\|u\|^2}\cdot \Pi_u^0(u) \tag{4.5}$$

Remark 7 *Using (4.4)–(4.5) and Lemma 3, a simple estimate shows that, for some $C > 0$,*

$$\|\Pi_u(\nabla\theta_u)\| \le C\|u\|^{(p-4)/p} \le C\|u\| , \qquad \forall u \in \mathcal{N} .$$

In case $\nabla J(u) = 0$ then $\nabla\theta_u = 0$ but the inverse is not true. However, in case $\theta_u/\|u\| \notin \mathfrak{K}_u$,

$$\nabla J(u) = 0 \quad \Leftrightarrow \quad \nabla\theta_u = 0 .$$

Note that eventually unstable stationary points of the H^1–distance decreasing flow on the Nehari–manifold are minimizers of the angle functional. We have the following

Proposition 4 *Let*

$$\Phi : W_0^{1,\infty}(\Omega) \cap \mathcal{N} \mapsto W_0^{1,\infty}(\Omega) , \quad \Phi(u) = \Pi_u(\nabla\theta_u) .$$

Given $u_0 \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$, the initial value problem

$$\eta(0, u_0) = u_0 , \qquad \frac{d\eta}{dt}(t) = -\Phi(\eta(t, u_0)) . \tag{4.6}$$

has a unique solution

$$\eta : \mathcal{N} \times [0, \tau_0[\mapsto \mathcal{N} ,$$

for some $\tau_0 > 0$. In case Ω is a bounded regular domain of $\mathbb{R}^2$ then $\tau_0 = +\infty$ for all $u_0 \in \mathcal{N}$. Moreover, for $0 < t_1 < t_2$

$$\theta(\eta(t_1, u_0)) \ge \theta(\eta(t_2, u_0)) . \tag{4.7}$$

The proof of Proposition 4 will follow from the next lemmas.

Lemma 4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. Define*

$$\Psi : W_0^{1,\infty}(\Omega) \mapsto W_0^{1,\infty}(\Omega) , \quad u \to \Delta^{-1}(f(u)) .$$

Then Ψ is locally Lipschitz continuous.

Proof. Trivially, $W_0^{1,\infty}(\Omega) \subset C_0(\Omega)$ with continuous injection. By standard regularity theory (see [3], theorems 8.33-8.34) we have $\Delta^{-1}(f(u)) \in C_0^{1,\alpha}\left(\overline{\Omega}\right)$ so Ψ is well-defined.

Let $B_\epsilon(u)$ be the ball of radius ϵ and center u in $C_0(\Omega)$. By our assumptions on f, for any $v \in B_\epsilon(u)$, we have

$$|f(u) - f(v)| \leq K_\epsilon |u - v| ,$$

for some $K_\epsilon > 0$. We conclude that the functional

$$\psi : W_0^{1,\infty}(\Omega) \mapsto C\left(\overline{\Omega}\right) , \qquad u \to f(u)$$

is locally Lipschitz continuous. Since $\Delta^{-1} : C\left(\overline{\Omega}\right) \mapsto C_0^{1,\alpha}(\Omega)$ is Lipschitz continuous, we conclude that $\Psi = \Delta^{-1} \circ \psi$ is locally Lipschitz continuous. The proof is complete. ■

Remark 8 *With similar arguments, we may prove that, for locally Lipschitz functions $f, g : \mathbb{R} \mapsto \mathbb{R}$,*

$$u \mapsto \Delta^{-1}[\Delta^{-1}(f(u))g(u)]$$

is locally Lipschitz continuous in $W_0^{1,\infty}(\Omega)$.

Lemma 5 *Let*

$$\Phi : W_0^{1,\infty}(\Omega) \cap \mathcal{N} \mapsto W_0^{1,\infty}(\Omega) , \qquad \Phi(u) = \Pi_u(\nabla\theta_u) .$$

For any $u_1 \in W_0^{1,\infty}(\Omega) \cap \mathcal{N}$, there exists a $W^{1,\infty}$-ball B_1 centered at u_1 and a Lipschitz continuous function

$$F : B_1 \cap W_0^{1,\infty}(\Omega) \mapsto W_0^{1,\infty}(\Omega)$$

such that

$$F(u) = \Phi(u) , \qquad \forall u \in B_1 \cap W_0^{1,\infty}(\Omega) \cap \mathcal{N} .$$

Proof. Let B_1 be a $W^{1,\infty}$-ball centered at u_1 such that $\|u\|$ and $\|N(u)\|$ are uniformly bounded below by a positive constant in B_1. We consider the following extensions $N, n : B_1 \mapsto W_0^{1,\infty}(\Omega)$ and $\theta : B_1 \mapsto \mathbb{R}$,

$$N(u) = 2u + \Delta^{-1}(h(u)) \, , n(u) = \frac{N(u)}{\|N(u)\|} \, , \theta(u) = \left\langle n(u), \frac{u}{\|u\|} \right\rangle \, .$$

In the homogeneous case, $h(u) = f(u) + f'(u)u = pf(u)$. Then,

$$\theta(u) = \frac{1}{\|N(u)\| \cdot \|u\|} \left(2\|u\|^2 - p \int_\Omega f(u)u \, dx \right) \, .$$

Define

$$J_1(u) = \|N(u)\|^2 = 4\|u\|^2 + 4p\langle u, \Delta^{-1}(f(u))\rangle + p^2\|\Delta^{-1}(f(u))\|^2 \, ,$$

and

$$J_2(u) = \left(2\|u\| - \frac{p}{\|u\|} \int_\Omega f(u)u \, dx \right) \, ,$$

so that

$$\theta(u) = \frac{J_2(u)}{\sqrt{J_1(u)}} \, . \tag{4.8}$$

By Lemma 4 $J_1 : W_0^{1,\infty}(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous. Moreover

$$\begin{aligned} &\langle \nabla J_1(u), v\rangle = \\ &8\langle u, v\rangle + 4p\langle v, \Delta^{-1}(f(u))\rangle + \\ &4p\langle u, \Delta^{-1}(f'(u)v)\rangle + 2p^2\langle \Delta^{-1}(f'(u)v), \Delta^{-1}(f(u))\rangle \, . \end{aligned} \tag{4.9}$$

As

$$\langle u, \Delta^{-1}(f'(u)v)\rangle = -\int_\Omega f'(u)uv \, dx = (p-1)\langle v, \Delta^{-1}(f(u))\rangle \, ,$$

and

$$\langle \Delta^{-1}(f'(u)v), \Delta^{-1}(f(u))\rangle = (p-1)\langle v, \Delta^{-1}[\Delta^{-1}(f(u))f'(u)]\rangle \, ,$$

we conclude that

$$\nabla J_1(u) = 8u + 4p^2\Delta^{-1}(f(u)) + 2p^2(p-1)\Delta^{-1}[\Delta^{-1}(f(u))f'(u)] \, . \tag{4.10}$$

Then, by Remark 8, we conclude that $\nabla J_1 : W_0^{1,\infty}(\Omega) \to W_0^{1,\infty}(\Omega)$ is locally Lipschitz continuous. Similarly, we may prove that $J_2 : W_0^{1,\infty}(\Omega) \to \mathbb{R}$ and $\nabla J_2 : W_0^{1,\infty}(\Omega) \to W_0^{1,\infty}(\Omega)$ are locally Lipschitz continuous. We conclude from (4.8) that $\nabla\theta : W_0^{1,\infty}(\Omega) \cap B_1 \to W_0^{1,\infty}(\Omega)$ is Lipschitz continuous. Finally, writing

$$F(u) := \nabla\theta(u) - \langle \nabla\theta(u), n(u)\rangle n(u)$$

we conclude that

$$F : W_0^{1,\infty}(\Omega) \cap B_1 \mapsto W_0^{1,\infty}(\Omega)\,,$$

is Lipschitz continuous and

$$F(u) = \Pi_u(\nabla\theta_u)\,, \quad \forall u \in \mathcal{N}\,.$$

■

Proof of Proposition 4:

Assuming $[0, \tau_0[$ is the maximal domain of $\eta(t, u_0)$ in $W_0^{1,\infty}(\Omega)$, one easily verifies that $\eta(t, u_0) \in \mathcal{N}$ for all $t \in [0, \tau_0[$. Consider the case where Ω is a bounded regular domain of $\mathbb{R}^2$. Suppose in view of a contradiction that $\tau_0 < \infty$. Then, by Remark 7 and classical Gronwall estimates, as $t \to \tau_0$ necessarily $\eta(t, u_0) \to w \in \mathcal{N}$ in H^1-norm. Consider the H^1-ball $B_R(w)$ centered at w and radius $R = \|w\|/2$. Noting that $B_R(w)$ is bounded in $L^q(\Omega)$ for arbitrarily large q, by standard regularity theory (see section 8.11–[3]), we have, for all $u \in B_R(w)$,

$$\|\Delta^{-1}(f(u))\|_{W^{1,\infty}(\Omega)} \le \|\Delta^{-1}(f(u))\|_{C^{1,\alpha}(\Omega)} \le C$$

and

$$\|\Delta^{-1}[\Delta^{-1}(f(u))f'(u)]\|_{W^{1,\infty}(\Omega)} \le \|\Delta^{-1}[\Delta^{-1}(f(u))f'(u)]\|_{C^{1,\alpha}(\Omega)} \le C$$

for some $C > 0$. Also, $\|u\|$ and $\|N(u)\|$ are uniformly bounded below in $B_R(w)$ by a positive constant. Adapting the arguments in Lemma 5 we may consider $F : B_R(w) \cap W_0^{1,\infty}(\Omega) \to W_0^{1,\infty}(\Omega)$ such that

$$\Pi_\eta(\nabla\theta_\eta) = F(\eta)\,, \quad \forall \eta \in B_R(w) \cap W_0^{1,\infty}(\Omega) \cap \mathcal{N}$$

and, for some $K_B > 0$,

$$|F(w_1) - F(w_2)|_{W^{1,\infty}} \le K_B |w_1 - w_2|_{W^{1,\infty}}\,, \ \ \forall\, w_1, w_2 \in B_R(w) \cap W_0^{1,\infty}(\Omega)\,.$$

Then there exists a constant ϵ such that, for any $w' \in B_{R/2}(w) \cap W_0^{1,\infty}(\Omega)$, the maximal domain of definition in $W_0^{1,\infty}(\Omega)$ of $\eta(w', t)$ contains $[0, \epsilon[$. This implies that the maximal domain of $\eta(t, u_0)$ contains $[0, \tau_0 + \epsilon[$, contradicting our assumption on τ_0. Finally, since

$$\frac{d}{dt}\,\theta(\eta(t)) = \langle \nabla\theta(\eta), -\Pi_\eta(\nabla\theta(\eta))\rangle \le 0,$$

we conclude the monotone property (4.7). ■

As $\eta([0,\tau_0[) \subset \mathcal{N} \cap W_0^{1,\infty}(\Omega)$, for any $u \in \eta([0,\tau_0[)$ we may provide an orthonormal basis of $\mathfrak{T}_u$ consisting of eigenvectors of L_u. Let us study how the norm of the projection $\Pi_\eta(\eta)$ and of the normal component $\langle \eta, n\rangle \cdot n$ evolve along the flow defined in (4.6). For simplicity of notation, we assume $\mathrm{Ker}(T_u) = \{0\}$ although minor changes provide the more general case.

$$\frac{d}{dt}\left(\frac{1}{2}\|\Pi_{\eta(t)}\eta(t)\|^2\right) = \\ \Big\langle D\Pi_{\eta(t)}(\eta'(t),\eta(t)), \Pi_{\eta(t)}(\eta(t))\Big\rangle + \Big\langle \Pi_{\eta(t)}(\eta'(t)), \Pi_{\eta(t)}(\eta(t))\Big\rangle \tag{4.11}$$

Denoting $\eta(t) = u$ and $n(u) = n$, we have, by (4.4),

$$\Big\langle \Pi_{\eta(t)}(\eta'(t)), \Pi_{\eta(t)}(\eta(t))\Big\rangle = \langle -\Pi_u(\nabla\theta_u), u\rangle = \frac{1}{\|u\|}\sum_{i=1}^{\infty}\left(-k_i + \frac{\theta_u}{\|u\|}\right)\langle v_i, u\rangle^2 . \tag{4.12}$$

Also

$$\Big\langle D\Pi_{\eta(t)}(\eta'(t),\eta(t)), \Pi_{\eta(t)}(\eta(t))\Big\rangle = \langle D\Pi_u(-\Pi_u(\nabla\theta_u), u), \Pi_u(u)\rangle . \tag{4.13}$$

We decompose

$$D\Pi_u(-\Pi_u(\nabla\theta_u), u) = D\Pi_u(-\Pi_u(\nabla\theta_u), \Pi_u(u) + \langle u, n\rangle n) = \\ D\Pi_u(-\Pi_u(\nabla\theta_u), \Pi_u(u)) + \langle u, n\rangle D\Pi_u(-\Pi_u(\nabla\theta_u), n)$$

and observe that

$$D\Pi_u(-\Pi_u(\nabla\theta_u), \Pi_u(u)) \in \mathfrak{T}_u^{\perp}$$

(since it is the second fundamental form of $\mathcal{N}$ at u). Then, by (3.3), we may re-write (4.13)

$$\langle D\Pi_u(-\Pi_u(\nabla\theta_u), u), \Pi_u(u)\rangle = \langle u, n\rangle\langle Dn(u)[\Pi_u(\nabla\theta_u)], u\rangle = \\ \sum_{i=1}^{\infty}\theta_u k_i\left(k_i - \frac{\theta_u}{\|u\|}\right)\langle u, v_i\rangle^2 . \tag{4.14}$$

Combining (4.11), (4.12) and (4.14) we obtain, for $u = \eta(t)$,

$$\frac{d}{dt}\left(\frac{1}{2}\|\Pi_\eta\eta\|^2\right) = \sum_{i=1}^{\infty}\left(k_i(\eta) - \frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i(\eta) - \frac{1}{\|\eta\|}\right)\langle \eta, v_i(\eta)\rangle^2 . \tag{4.15}$$

Let us turn to the study of the normal component $\langle \eta, n\rangle n$. Differentiating in t, assuming $\eta(t) = u$, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\langle \eta(t), n(\eta(t))\rangle^2\right) = \langle u, n\rangle\left(\langle u, Dn_u(-\Pi_u(\nabla\theta_u))\rangle + \langle -\Pi_u(\nabla\theta_u), n\rangle\right) .$$

Noting that $\langle -\Pi_u(\nabla\theta_u), n\rangle = 0$, we may write, for $u = \eta(t)$,

$$\frac{d}{dt}\left(\frac{1}{2}\langle \eta(t), n(\eta(t))\rangle^2\right) = \sum_{i=1}^{\infty} \theta_\eta k_i(\eta)\left(-k_i(\eta) + \frac{\theta_\eta}{\|\eta\|}\right)\langle \eta, v_i(\eta)\rangle^2 . \tag{4.16}$$

We have the following monotone property of the angle decreasing flow:

Proposition 5 *Let $u \in \mathcal{N} \cap W_0^{1,\infty}(\Omega)$. Consider the solution $\eta(t,u)$ of (4.6) and denote*

$$\eta^\top = \Pi_\eta \eta \quad \textit{and} \quad \eta^\perp = \eta - \eta^\top .$$

If

$$\mathfrak{K}_u \cap \left](\theta_u\|u\|)^{-1}, \theta_u\|u\|^{-1}\right[\; = \; \emptyset ,$$

then $\frac{d}{dt}\|\eta^\top\|_{\eta=u} \le 0$. In case

$$\mathfrak{K}_u \cap]\theta_u/\|u\|, 0[\; = \; \emptyset$$

then $\frac{d}{dt}\|\eta^\perp\|_{\eta=u} \ge 0$.

Proof. In the non-degenerate case, the proof follows from (4.16), (4.15), recalling that $\theta_\eta < 0$ for all $\eta \in \mathcal{N}$. In the general case, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|\Pi_\eta\eta\|^2\right) = \sum_{i=1}^{\infty}\left(k_i(\eta) - \frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i(\eta) - \frac{1}{\|\eta\|}\right)\langle \eta, v_i(\eta)\rangle^2 + K_1\|\Pi_\eta^0(\eta)\|^2 . \tag{4.17}$$

where $K_1 = (2/\|\eta\| - \theta_\eta\|\eta\|^{-1})(2\theta_\eta/\|N(\eta)\| - \|\eta\|^{-1}) < 0$ and

$$\frac{d}{dt}\left(\frac{1}{2}\langle \eta(t), n(\eta(t))\rangle^2\right) = \sum_{i=1}^{\infty} \theta_\eta k_i(\eta)\left(-k_i(\eta) + \frac{\theta_\eta}{\|\eta\|}\right)\langle \eta, v_i(\eta)\rangle^2 + K_2\|\Pi_\eta^0(\eta)\|^2 . \tag{4.18}$$

where $K_2 = 2\theta_\eta/\|N(\eta)\|\,(-2/\|N(\eta)\| + \theta_\eta/\|\eta\|) > 0$ and the proof follows from similar estimates. ∎

An Example. We will now study an example of convergence of the angle decreasing flow $\eta(t,u_0)$ to a critical point of the distance functional on $\mathcal{N}$ –i.e. a solution of (2.2). We assume Ω is a bounded regular domain of $\mathbb{R}^2$ so that, by Proposition 4, $\eta(.,u_0)$ is defined in $[0,+\infty[$. Moreover we assume that, for all $t > 0$, $\mathrm{Ker}(T_{\eta(t,u_0)}) = \{0\}$. This last hypotheses may be removed provided minor changes are added to the following assumptions. We shall denote $\theta(u_0) = c$ (recall $-1 \le c < 0$) and suppose the following:

(N1) There exists a positive sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty}\alpha_n < \infty$ and

$$u \in \eta([0,+\infty[, u_0) \Rightarrow \qquad \langle u, v_n(u)\rangle^2 \leq \alpha_n \langle u, v_1(u)\rangle^2 \,.$$

(As usual, $(v_n(u))$ is the basis of $\mathfrak{T}_u$ composed by the eigenvectors of the Weingarten map L_u.)

(N2) There exists $K, \rho > 0$ and $\bar{n}$ such that, for all $u \in \eta([0,+\infty[, u_0)$,

$$-K \leq k_{\bar{n}}(u) \leq -\left(\rho + \frac{1}{|c|}\right)\frac{1}{\|u\|}\,.$$

(N3) For (α_n), $\bar{n}$ and ρ defined in (N1)-(N2), for some $C_1 > 0$,

$$-|c|\rho^2 + \left(\frac{1}{2c} - \frac{c}{2}\right)^2 \sum_{i=\bar{n}+1}^{\infty} \alpha_i \leq -C_1 \cdot \sum_{n=1}^{\infty} \alpha_n \tag{4.19}$$

$$|c|\rho^2 - \frac{1}{4}\sum_{i=\bar{n}+1}^{\infty}\alpha_i \geq C_1 \tag{4.20}$$

Intuitively, assumptions (N1)–(N3) impose that, all along the flow, $\Pi_\eta(\eta)(t)$ mainly concentrates on directions of the tangent space associated to certain negative eigenvalues of L_u. We have the following convergence property.

Suppose conditions (N1)–(N3) are verified. Then, as $t \to \infty$, $\eta(t, u_0)$ converges in H^1-norm to a critical point u^* of J. Moreover, $\|\eta^\top(t, u_0)\|$ is a decreasing function of t whereas $\|\eta^\perp(t, u_0)\|$ is an increasing function of t.

The proof of the convergence will follow after a number of steps. For simplicity, denote $\eta(t) := \eta(t, u_0)$. We have $\theta_\eta \leq c$ so that, by (N1)

$$|\langle \eta(t), v_n(\eta(t))\rangle| \leq \alpha_n |\langle \eta(t), v_1(\eta(t))\rangle| \qquad \forall n \in \mathbb{N}\,, \forall t \in [0,+\infty[\,.$$

Step 1: Increasing of $\|\eta^\perp(t)\|$

We prove that the norm of the normal component is an increasing function of t. As usual, we shall denote $k_i := k_i(\eta)$ and $v_i := v_i(\eta)$. We have, by (4.16),

$$\frac{d}{dt}\frac{1}{2}\|\eta^\perp\|^2 = \sum_{i=1}^{\bar{n}} \theta_\eta k_i \left(-k_i + \frac{\theta_\eta}{\|\eta\|}\right)\langle \eta, v_i\rangle^2 + \sum_{i=\bar{n}+1}^{\infty} \theta_\eta k_i \left(-k_i + \frac{\theta_\eta}{\|\eta\|}\right)\langle \eta, v_i\rangle^2\,. \tag{4.21}$$

Note that, by (N1),

$$\sum_{i=\bar{n}+1}^{\infty} \theta_\eta k_i \left(-k_i + \frac{\theta_\eta}{\|\eta\|}\right) \langle \eta, v_i \rangle^2 \geq \sum_{i=\bar{n}+1}^{\infty} \frac{\theta_\eta^3}{4\|\eta\|^2} \langle \eta, v_i \rangle^2$$
$$\geq -\frac{1}{4\|\eta\|^2} \langle \eta, v_1 \rangle^2 \sum_{i=\bar{n}+1}^{\infty} \alpha_i \,. \tag{4.22}$$

By (N2), (4.20), (4.21) and (4.22)

$$\frac{d}{dt}\frac{1}{2}\|\eta^\perp\|^2 \geq \left(\frac{|c|\rho^2}{\|\eta\|^2} - \frac{1}{4\|\eta\|^2} \sum_{i=\bar{n}+1}^{\infty} \alpha_i\right) \langle u, v_1 \rangle^2 \geq \frac{C_1}{\|\eta\|^2} \langle u, v_1 \rangle^2 \geq 0\,. \tag{4.23}$$

Step 2: Decreasing of $\|\eta^\top(t)\|$

By (4.15), (N1)–(N3) we write

$$\frac{d}{dt}\frac{1}{2}\|\eta^\top(t,u_1)\|^2 =$$
$$\sum_{i=1}^{\bar{n}} \left(k_i - \frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i - \frac{1}{\|\eta\|}\right) \langle \eta, v_i \rangle^2 + \sum_{i=\bar{n}+1}^{\infty} \left(k_i - \frac{\theta_\eta}{\|\eta\|}\right)\left(\theta_\eta k_i - \frac{1}{\|\eta\|}\right) \langle \eta, v_i \rangle^2$$
$$\leq -\frac{|c|\rho^2}{\|\eta\|^2} \langle \eta, v_1 \rangle^2 + \frac{\gamma}{\|\eta\|^2} \langle \eta, v_1 \rangle^2 \cdot \sum_{i=\bar{n}+1}^{\infty} \alpha_i \,,$$

where

$$\gamma = \left(\frac{c}{2} - \frac{1}{2c}\right)^2 .$$

Therefore, by (4.19),

$$\frac{d}{dt}\frac{1}{2}\|\eta^\top(t)\|^2 \leq -\frac{C_1 \sum_{n=1}^{\infty} \alpha_n}{\|\eta(t)\|^2} \langle \eta(t), v_1 \rangle^2 \,.$$

As $\langle \eta, v_1 \rangle^2 \geq \left(\sum_{n=1}^{\infty} \alpha_n\right)^{-1} \|\eta^\top\|^2$, we conclude

$$\frac{d}{dt}\frac{1}{2}\|\eta^\top(t)\|^2 \leq -\frac{C_1}{\|\eta(t)\|^2} \|\eta^\top(t)\|^2 \,. \tag{4.24}$$

Step 3: Convergence of $\eta(t)$ *to a global minimum of* θ

By the previous steps, we have

$$\frac{d}{dt}\|\eta\|^2 = \frac{d}{dt}\|\eta^\perp\|^2 + \frac{d}{dt}\|\eta^\top\|^2 \leq \frac{d}{dt}\|\eta^\perp\|^2 \,, \tag{4.25}$$

and, by (N2) and (4.21), for some $\overline{K}$ such that $|k_i| \leq \overline{K}$,

$$\frac{d}{dt}\|\eta^\perp\|^2 \leq \overline{K}^2 \|\eta^\top\|^2 \,. \tag{4.26}$$

We conclude from (4.25)–(4.26)

$$\frac{d}{dt}\|\eta\|^2 \leq \overline{K}^2 \|\eta^\top\|^2 \,.$$

or

$$\|\eta\|^2(t) \leq \|\eta\|^2(0) + \overline{K}^2 \int_0^t \|\eta^\top\|^2(s)\,ds \,.$$

Then, by (4.24),

$$\frac{d}{dt}\|\eta^\top(t)\|^2 \leq -\frac{2C_1\|\eta^\top(t)\|^2}{\|\eta(0)\|^2 + \overline{K}^2 \int_0^t \|\eta^\top\|^2(s)\,ds} \,.$$

By Lemma 7 (Appendix), we conclude that

$$\int_0^{+\infty} \|\eta^\top\|(t)\,dt \leq C(C_1, \|\eta(0)\|, \overline{K}) \,.$$

In particular, by (2.18)and (4.4),

$$\|\eta'(t)\| = \|\Pi_\eta(\nabla\theta_\eta)\| \leq \frac{\overline{K}}{\|\eta\|}\|\eta^\top\| \leq M\|\eta^\top\| \tag{4.27}$$

for an adequate constant M independent of η. Then

$$\int_0^{+\infty} \|\eta'(t)\|\,dt \leq C(C_1, \|\eta(0)\|, \overline{K}) \,,$$

and the flow $\eta(t)$ necessarily converges in H^1-norm to u^*. By (N1) and (4.23), using a simple approximation argument, one concludes that $\Pi_{u^*}(u^*) = 0$. Then $\theta(u^*) = -1$, u^* is a critical point of the distance functional on the Nehari Manifold and a solution to (2.2). ■

Remark 9 *Note that, in view of Remark 7 and estimate (4.27), the H^1-convergence of $\eta(t)$ and $\eta^\perp(t)$ are equivalent.*

5 Appendix

5.1 A suitable basis of $H_0^1(\Omega)$.

Let $F \in C(\mathbb{R}, \mathbb{R})$ be such that $F(0) = 0$, $F(u) > 0$ if $u \neq 0$. Moreover, assume

$$\lim_{u \to \pm\infty} F(u) = +\infty \,, \tag{5.1}$$

and

$$\lim_{u \to \pm\infty} \frac{F(u)}{|u|^q} = 0 \,, \tag{5.2}$$

for some $1 \leq q < 2^*$.

We define by recurrence a family of orthogonal vectors. Consider the following minimization problem:

$$\min \left\{ \int_\Omega |\nabla u|^2(x)\, dx \;:\; u \in H_0^1(\Omega), \int_\Omega F(u)(x)\, dx = 1 \right\}. \tag{5.3}$$

By (5.1)–(5.2), a minimizer exists, that we shall denote by e_1. More generally, we define e_n to be a minimizer of the Dirichlet integral $\int_\Omega |\nabla u|^2(x)\, dx$ over the weakly closed set

$$\left\{ u \in H_0^1(\Omega) : \int_\Omega F(u)(x)\, dx = 1 \;\text{ and }\; u \in \langle e_1, ..., e_{n-1} \rangle^\perp \right\} .$$

Lemma 6 *The sequence (e_n) is an orthogonal basis of $H_0^1(\Omega)$. Also $(\|e_n\|)$ is non-decreasing and*

$$\lim_{n \to \infty} \|e_n\| = \infty.$$

Proof.

Trivially, the sequence $(\|e_n\|)$ is non-decreasing. We assert that

$$\lim_{n \to \infty} \|e_n\| = \infty \,.$$

Suppose, in view of a contradiction, the existence of $C > 0$ such that $\|e_n\| \leq C$ for all $n \in \mathbb{N}$. Passing to a weakly convergent subsequence, denoted by (e_{n_j}), we have

$$e_{n_j} \rightharpoonup v \quad \text{ and } \quad \int_\Omega F(v)(x)\, dx = 1 \,. \tag{5.4}$$

Let $n_j \in \mathbb{N}$ be fixed. We have

$$\langle v, e_{n_j} \rangle = \lim_{k \to \infty} \langle e_{n_k}, e_{n_j} \rangle = 0 \,.$$

Now letting $n_j \to \infty$ we conclude $\|v\| = 0$ and contradict (5.4). The assertion is proved.

Let $w \in H_0^1(\Omega)$ be such that

$$\langle w, e_i \rangle = 0 \quad \text{for all } i \in \mathbb{N}\,. \tag{5.5}$$

If $w \neq 0$ assume (without loss of generality)

$$\int_\Omega F(w)(x)\,dx = 1\,.$$

The previous assertion, together with (5.5), imply that there exists $n \in \mathbb{N}$ such that $\|e_{n-1}\| \leq \|w\| < \|e_n\|$. This, contradicts the definition of the function e_n. Then $w = 0$ and the proof is complete. ∎

5.2 A Gronwall type estimate

Lemma 7 *Let $f \in C^1([0, +\infty[, \mathbb{R}^+)$ be such that*

$$f'(t) \leq -\frac{f(t)}{a + b\int_0^t f(u)\,du} \tag{5.6}$$

for some $a, b > 0$. Then

$$\int_0^\infty \sqrt{f}(u)\,du \leq C(a, b, f(0))\,. \tag{5.7}$$

Proof. Integrating equation (5.6),

$$f(t) - f(0) \leq -\frac{1}{b}\left[\ln\left(a + b\int_0^s f(u)du\right)\right]_0^t,$$

or

$$f(t) + \frac{1}{b}\ln\left(a + b\int_0^t f(u)\,du\right) \leq f(0) + \frac{\ln(a)}{b}$$

and, as $f(t) \geq 0$, we conclude, by passing to the limit in t,

$$\ln\left(a + b\int_0^{+\infty} f(u)\,du\right) \leq bf(0) + \ln(a)$$

or

$$\int_0^{+\infty} f(u)\,du \leq C_1 \tag{5.8}$$

where $C_1 = (ae^{bf(0)} - a)/b$. Writing $f(t) = h^2(t)$ with $h(t) > 0$, inequality (5.6) becomes

$$2h(t)h'(t) \leq -\frac{h^2(t)}{a + b\int_0^t f(u)\,du}\,.$$

By (5.6)–(5.8), we conclude

$$h'(t) \leq -\frac{h(t)}{2(a+bC_1)}$$

or

$$h(t) \leq \sqrt{f(0)}e^{-C_2 t},$$

where $C_2 = (2(a+bC_1))^{-1}$. This proves the lemma. ■

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