A general approach to the exact and near-exact distributions of the main likelihood ratio test statistics used in the complex multivariate Normal setting

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Abstract

In this paper the authors show how, in parallel with what happens in the real case, it is possible to establish a common structure for the exact distribution of the main likelihood ratio test (l.r.t.) statistics used in the complex multivariate Normal setting. This leads to simple expressions for the exact distribution of some of these statistics and to very well-performing approximations for the distribution of the other statistics. Easy to implement near-exact distributions are developed for the l.r.t. statistics to test sphericity and the equality of covariance matrices. Numerical studies show how these near-exact distributions outperform by far any other available approximations.

Keywords: independence, equality of mean vectors, nullity of expected mean matrix, sphericity, equality of covariance matrices, mixtures, Generalized Integer Gamma distribution, Generalized Near-Integer Gamma distribution

1. Introduction

In this paper we show that in the complex multivariate normal setting, just as in the real case (see [20, 7]), the main likelihood ratio test (l.r.t.) statistics, which are: i) the l.r.t. statistic to test independence of sets of variables, ii) the l.r.t. statistic to test equality of mean vectors, iii) the l.r.t. statistic to test the nullity of an expected value matrix, iv) the l.r.t. statistic to test sphericity, and v) the l.r.t. statistic to test equality of covariance matrices, all have a common structure for their exact distribution, which for some $s \in \mathbb{N}$ and $u \in \mathbb{N}_0$, may be stated as

$$\Lambda \overset{st}{\sim} \left(\prod_{j=1}^{p-1} e^{-Z_j}\right) \times \left(\prod_{k=1}^{u} Y_k\right), \tag{1}$$

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where Λ represents the l.r.t. statistic, '$\stackrel{st}{\sim}$' is to be read as 'is stochastically equivalent to', or 'is distributed as', and where,

$$Z_j \sim \Gamma\left(r_j, \frac{n-1-j}{n}\right), \quad \text{with} \quad r_j \in \mathbb{N}, \quad \text{and} \quad Y_k \sim Beta(a_k, b_k) \quad j = 1, \ldots, p-1; \; k = 1, \ldots, u \tag{2}$$

are independent r.v.'s (random variables). In (2), n is the sample size and p represents the number of variables involved in the tests in i), iv) and v) and the sum of the number of variables plus the number of vectors, minus 1, for the test in ii) or the sum of the number of variables involved plus the number of rows in the matrix, for the test in iii). Also, in (2), we have $u = 0$ for the l.r.t. statistics in i), ii) and iii).

Indeed we will be able to show that the similarities among the distributions of the l.r.t. statistics in i)-v) above are even more impressive in the complex case than in the real case.

From the exact expression (1) for the distribution of these statistics we may then obtain

- very simple expressions for the exact p.d.f. and c.d.f. of the statistics in i), ii) and iii)
- very well-fitting manageable near-exact distributions for the statistics in iv) and v).

We will use the definition of the complex multivariate Normal distribution in [34, 11][3, sec. 4.2][1, prob. 2.64]. We will thus say that the random vector $\underline{X}$ ($p\times 1$) has a complex multivariate Normal distribution, with expected value $\underline{\mu}$ and Hermitian variance-covariance matrix Σ if the p.d.f. of $\underline{X}$ is

$$f_{\underline{X}}(\underline{x}) = \pi^{-p}|\Sigma|^{-1}\, e^{-\overline{(\underline{x}-\underline{\mu})}'\Sigma^{-1}(\underline{x}-\underline{\mu})}\,,$$

where $\overline{(\underline{x}-\underline{\mu})}$ denotes the complex conjugate of $(\underline{x}-\underline{\mu})$ and the prime denotes the transpose. In this case we will write

$$\underline{X} \sim CN_p(\underline{\mu}, \Sigma)\,. \tag{3}$$

The complex multivariate Normal distribution has applications in a wide range of areas, from crystallography [26] to spectral analysis in time series [16, 4, 3, 4, 29, 11, 33, 19] and to studies on the performance of radar receivers [9].

In Section 2 we derive the exact distributions for the l.r.t. statistics in i), ii) and iii) and show that they have simple closed form expressions, without involving any infinite sums. In Sections 3 and 4 we obtain respectively the exact distribution of the l.r.t. statistics in iv) and v) in the form in (1), specifying the value of u and the parameters in the distributions of the r.v.'s Y_k. In Section 5 we derive near-exact distributions for these statistics and in Section 6 we carry out some numerical studies which show the very good performance of the near-exact distributions developed.

2. The exact distribution of the l.r.t. statistics to test independence, the equality of mean vectors and the nullity of an expected value matrix

In these three cases the exact distribution of the l.r.t. statistics has already been obtained in the form (1), with $u = 0$, in [2]. However, we will take the opportunity to obtain these distributions using a slightly different approach, which has much in common with the procedures used to obtain the exact distributions in the next section for the l.r.t. statistics in iv) and v) above. Moreover the details of this development will highlight the fact that the distribution which will be obtained for the l.r.t. statistic to test the nullity of an expected value matrix in subsection 2.3 is not the same as that obtained in [18] in which reference, as we will see, there is a small mistake.

The difference from the real case is that in the complex case we are able to obtain the exact distribution for the negative logarithm of the l.r.t. statistics to test independence, the equality of mean vectors or the nullity of an expected value matrix as GIG distributions [5][20, App. B], for any number of variables or mean vectors involved. In this way we are able to obtain very simple expressions for the exact p.d.f. and c.d.f. of the logarithm of the l.r.t. statistics as well as for the l.r.t. statistics themselves. The simplicity of the expressions obtained is quite striking when compared with the expressions obtained by other authors [10, 15, 18, 19, 27, 30], and although bearing some resemblance with the expressions in [31], they have in contrast to these, the advantage of involving only finite sums. Even when compared with the quite close representations in [13, 21, 14], they have the advantage of being more general and with coefficients which have much simpler expressions. This is especially the case with respect to the representation in [14], while in [13] the author did not obtain explicit expressions for such coefficients.

2.1. The l.r.t. statistic to test independence among sets of variables

Let us suppose that the random vector $\underline{X}$ in (3) is split into m subvectors $\underline{X}_k$ $(k = 1, \dots, m)$. This will induce the following partitioning of Σ

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \cdots & \Sigma_{1k} & \cdots & \Sigma_{1m} \\ \vdots & \ddots & \vdots & & \vdots \\ \Sigma_{k1} & \cdots & \Sigma_{kk} & \cdots & \Sigma_{km} \\ \vdots & & \vdots & \ddots & \vdots \\ \Sigma_{m1} & \cdots & \Sigma_{mk} & \cdots & \Sigma_{mm} \end{bmatrix}$$

where $\Sigma_{kk} = Var(\underline{X}_k)$ $(k = 1, \dots, m)$ and $\Sigma_{ij} = Cov(\underline{X}_i, \underline{X}_j)$ $(i, j \in \{1, \dots, m\})$. We wish to test the hypothesis of mutual independence of the random subvectors $\underline{X}_k$,

$$H_0 : \Sigma = diag(\Sigma_{11}, \dots, \Sigma_{kk}, \dots, \Sigma_{mm}) \equiv \Sigma_{ij} = 0, \text{ for all } i \neq j\,. \tag{4}$$

Let us further suppose that, for each k, $\underline{X}_k$ has p_k variables, with

$$p = \sum_{k=1}^{m} p_k\,.$$

Then, for a sample of size n, the l.r.t. statistic to test H_0 in (4), which may be derived in much the same manner as it is in the real case, is

$$\Lambda_1 = \left(\frac{|A|}{\prod_{k=1}^{m} |A_{kk}|} \right)^n, \tag{5}$$

where A is the maximum likelihood estimator (m.l.e.) of Σ and A_{kk} its k-th diagonal block $(k = 1, \dots, m)$, with

$$A = \left(\overline{X - \frac{1}{n} E_{nn} X} \right)' \left(X - \frac{1}{n} E_{nn} X \right) \tag{6}$$

where, once again the bar denotes the complex conjugate, X is the $n{\times}p$ sample matrix and E_{np} is an $n{\times}p$ unitary matrix (see [11] and [1, problem 3.11] for references concerning the maximum likelihood estimators of Σ in the complex case).

Following a similar procedure to that used in the real case we may show that:

i) we may write the statistic Λ_1 in (5) as

$$\Lambda_1 = \prod_{k=1}^{m-1} \Lambda_{1k(k+1,\ldots,m)}$$

where $\Lambda_{1k(k+1,\ldots,m)}$ is the l.r.t. statistic to test the null hypothesis of independence between the k-th set and the set formed by joining the sets $k+1$ through m, and

ii) under H_0 in (4) the $m-1$ statistics $\Lambda_{1k(k+1,\ldots,m)}$ are independent.

In addition, in much the same manner as in the real case, we may show that each statistic $\Lambda_{1k(k+1,\ldots,m)}$ has, in the complex case, the same distribution as

$$\prod_{j=1}^{p_k} (Y_j)^n$$

where Y_j are p_k independent r.v.'s with

$$Y_j \sim Beta\,(n-q_k-j, q_k)\ ,$$

where $q_k = p_{k+1} + \cdots + p_m$.

This way we have

$$E\left(\Lambda_1^h\right) = \prod_{k=1}^{m-1}\prod_{j=1}^{p_k} \frac{\Gamma\,(n-j)}{\Gamma(n-q_k-j)}\,\frac{\Gamma(n-q_k-j+nh)}{\Gamma(n-j+nh)}$$

and thus, for

$$W_1 = -\log\,\Lambda_1\,, \tag{7}$$

we have

$$\begin{aligned} \Phi_{W_1}(t) &= E\left(e^{\mathrm{i}tW_1}\right) = E\left(\Lambda_1^{-\mathrm{i}t}\right) \\ &= \prod_{k=1}^{m-1}\prod_{j=1}^{p_k} \frac{\Gamma\,(n-j)}{\Gamma(n-q_k-j)}\,\frac{\Gamma(n-q_k-j-\mathrm{i}tn)}{\Gamma(n-j-\mathrm{i}tn)}\,. \end{aligned} \tag{8}$$

We should remark that the above expression for $E\left(\Lambda_1^h\right)$ matches the expressions in [19, 10].

Theorem 1. *The exact distribution of W_1 in* (7) *is a GIG distribution of depth $p-1$ (see Appendix A) with p.d.f.*

$$f_{W_1}(w) = f^{GIG}\left(w\,\middle|\,r_j;\frac{n-1-j}{n};j=1,\ldots,p-1\right)$$

and c.d.f.

$$F_{W_1}(w) = F^{GIG}\left(w\,\middle|\,r_j;\frac{n-1-j}{n};j=1,\ldots,p-1\right),$$

where

$$r_j = \begin{cases} h_j & j=1 \\ h_j + r_{j-1} & j=2,\dots,p-1 \end{cases} \tag{9}$$

with

$$h_j = (\textit{number of } p_k \textit{ greater or equal to } j) - 1\,, \quad j=1,\dots,p-1\,. \tag{10}$$

Proof. From (8), using the relation

$$\frac{\Gamma(z+a)}{\Gamma(z)} = \prod_{\ell=0}^{a-1}(z+\ell)\,, \quad \text{(for any complex } z \text{ and positive integer } a\text{)}, \tag{11}$$

we may write

$$\begin{aligned} \Phi_{W_1}(t) &= \prod_{k=1}^{m-1}\prod_{j=1}^{p_k}\prod_{\ell=0}^{q_k-1}(n-q_k-j+\ell)\,(n-q_k-j+\ell-\mathrm{i}tn)^{-1} \\ &= \prod_{j=1}^{p-1}(n-1-j)^{r_j}\,(n-1-j-\mathrm{i}tn)^{-r_j} \\ &= \prod_{j=1}^{p-1}\left(\frac{n-1-j}{n}\right)^{r_j}\left(\frac{n-1-j}{n}-\mathrm{i}t\right)^{-r_j}\,, \end{aligned} \tag{12}$$

with r_j given by (9) and (10) in the body of the Theorem. From (12) we may see that the distribution of W_1 is indeed a GIG distribution of depth $p-1$, with shape parameters r_j and rate parameters $\frac{n-1-j}{n}$ $(j=1,\dots,p-1)$. □

Then the following Corollary gives the exact p.d.f. and c.d.f. of $\Lambda_1 = e^{-W_1}$.

Corollary 1. *The exact p.d.f. and c.d.f. of the statistic* $\Lambda_1 = e^{-W_1}$ *in* (5) *are*

$$f_{\Lambda_1}(\ell) = f^{GIG}\left(-\log\,\ell\,\middle|\,r_j;\frac{n-1-j}{n};j=1,\dots,p-1\right)\frac{1}{\ell}$$

and

$$F_{\Lambda_1}(\ell) = 1 - F^{GIG}\left(-\log\,\ell\,\middle|\,r_j;\frac{n-1-j}{n};j=1,\dots,p-1\right),$$

for r_j *given by* (9) *and* (10).

The exact distribution of Λ_1 is thus of the form (1), with $u=0$.

2.2. The l.r.t. statistic to test the equality of several mean vectors

Let us suppose that

$$\underline{X}_j \sim CN_p(\underline{\mu}_j, \Sigma_j)\,, \quad j = 1, \ldots, q$$

and that, assuming $\Sigma_1 = \cdots = \Sigma_q (= \Sigma)$, we want to test the null hypothesis

$$H_0 : \underline{\mu}_1 = \cdots = \underline{\mu}_q\,, \tag{13}$$

based on q independent samples, the j-th of which is from $\underline{X}_j$, with size n_j. Let us also suppose that the j-th sample is stored in the $n_j{\times}p$ matrix X_j.

Then, once again, using a procedure similar to the one used in the real case, it is not hard to determine that the l.r.t. statistic used to test H_0 in (13) is given by

$$\Lambda_2 = \left(\frac{|A|}{|A+B|}\right)^n \tag{14}$$

where $n = \sum_{j=1}^{q} n_j$,

$$A = \sum_{j=1}^{q} \left(\overline{X_j - E_{n_j1}\underline{\widetilde{X}}'_j}\right)' \left(X_j - E_{n_j1}\underline{\widetilde{X}}'_j\right)$$

and

$$B = \sum_{j=1}^{q} n_j \left(\underline{\widetilde{X}}_j - \underline{\widetilde{X}}\right) \left(\overline{\underline{\widetilde{X}}_j - \underline{\widetilde{X}}}\right)'$$

where once again the bar denotes the complex conjugate and

$$\underline{\widetilde{X}}_j = \frac{1}{n_j} X'_j E_{n_j1}$$

is the vector of sample means from the j-th sample, and

$$\underline{\widetilde{X}} = \frac{1}{n} \sum_{j=1}^{q} n_j \underline{\widetilde{X}}_j\,.$$

The $p{\times}p$ matrix A has what is called a complex Wishart distribution [11, 12] with $n - q$ degrees of freedom and parameter matrix Σ. We will denote this fact by

$$A \sim CW_p(n - q, \Sigma)\,.$$

Under H_0 in (13),

$$B \sim CW_p(q - 1, \Sigma)\,.$$

But then, given the independence, for Normal r.v.'s, of the m.l.e.'s of the mean and variance, the matrices A and B are independent and thus

$$A + B \sim CW_p(n - 1, \Sigma)\,.$$

It then follows that (see [12]),

$$2^p\,|A| \sim |\Sigma| \prod_{j=1}^{p} W_j \qquad \text{and} \qquad 2^p\,|A+B| \sim |\Sigma| \prod_{j=1}^{p} Z_j$$

where W_j ($j = 1, \ldots, p$) and Z_j ($j = 1, \ldots, p$) are two independent sets of p independent r.v.'s, with

$$W_j \sim \chi^2_{2(n-q-j+1)} \quad \text{and} \quad Z_j = W_j + W_j^* \sim \chi^2_{2(n-1-j+1)}\,, \qquad j = 1, \ldots, p\,,$$

where each

$$W_j^* \sim \chi^2_{2(q-1)}\,, \qquad j = 1, \ldots, p\,,$$

is independent of W_j ($j = 1, \ldots, p$).

Thus, under H_0 in (13),

$$\Lambda_2 \sim \prod_{j=1}^{p} (Y_j)^n \qquad \text{where} \qquad \begin{array}{l} Y_j \sim Beta\,(n-q-j+1, q-1)\;, \quad j = 1, \ldots, p \\ \text{are } p \text{ independent r.v.'s.} \end{array}$$

But then,

$$E\left(\Lambda_2^h\right) = \prod_{j=0}^{p-1} \frac{\Gamma(n-1-j)}{\Gamma(n-q-j)} \frac{\Gamma(n-q-j+nh)}{\Gamma(n-1-j+nh)} \qquad \left(h > \tfrac{p+q}{n} - 1\right)$$

so that the c.f. of

$$W_2 = -\log\,\Lambda_2 \tag{15}$$

may be written as

$$\Phi_{W_2}(t) = E\left(e^{\mathrm{i}tW_2}\right) = E\left(\Lambda_2^{-\mathrm{i}t}\right) = \prod_{j=0}^{p-1} \frac{\Gamma(n-1-j)}{\Gamma(n-q-j)} \frac{\Gamma(n-q-j-\mathrm{i}tn)}{\Gamma(n-1-j-\mathrm{i}tn)}\,. \tag{16}$$

We have thus the following Theorem.

Theorem 2. *The exact distribution of W_2 in* (15) *is a GIG distribution of depth $p+q-2$ (see Appendix A) with p.d.f.*

$$f_{W_2}(w) = f^{GIG}\left(w \,\middle|\, r_j; \frac{n-1-j}{n}; j = 1, \ldots, p+q-2\right)$$

and c.d.f.

$$F_{W_2}(w) = F^{GIG}\left(w \,\middle|\, r_j; \frac{n-1-j}{n}; j = 1, \ldots, p+q-2\right),$$

where

$$r_j = \begin{cases} h_j & j = 1 \\ h_j + r_{j-1} & j = 2, \ldots, p+q-2 \end{cases} \tag{17}$$

with, for $j = 1, \ldots, p+q-2$,

$$h_j = (\textit{number of elements in } \{p, q-1\} \textit{ greater or equal to } j) - 1\,. \tag{18}$$

Proof. From (16), using the relation in (11), we may write

$$\begin{aligned}\Phi_{W_2}(t) &= \prod_{j=0}^{p-1}\prod_{\ell=0}^{q-2}(n-q-j+\ell)\,(n-q-j+\ell-\mathrm{i}tn)^{-1}\\ &= \prod_{j=1}^{p+q-2}(n-1-j)^{r_j}\,(n-1-j-\mathrm{i}tn)^{-r_j}\\ &= \prod_{j=1}^{p+q-2}\left(\frac{n-1-j}{n}\right)^{r_j}\left(\frac{n-1-j}{n}-\mathrm{i}t\right)^{-r_j}\,,\end{aligned} \tag{19}$$

with r_j given by (17) and (18) in the body of the Theorem. From (19) we may see that the distribution of W_2 is indeed a GIG distribution of depth $p+q-2$, with shape parameters r_j and rate parameters $\frac{n-1-j}{n}$ $(j=1,\ldots,p+q-2)$. □

Then the following Corollary gives the exact p.d.f. and c.d.f. of $\Lambda_2 = e^{-W_2}$.

Corollary 2. *The exact p.d.f. and c.d.f. of the statistic* $\Lambda_2 = e^{-W_2}$ *in* (14) *are*

$$f_{\Lambda_2}(\ell) = f^{GIG}\left(-\log\,\ell\,\Big|\,r_j;\frac{n-1-j}{n};j=1,\ldots,p+q-2\right)\frac{1}{\ell}$$

and

$$F_{\Lambda_2}(\ell) = 1-F^{GIG}\left(-\log\,\ell\,\Big|\,r_j;\frac{n-1-j}{n};j=1,\ldots,p+q-2\right),$$

for r_j *given by* (17) *and* (18).

Once again, the distribution of Λ_2 is of the form (1) with $u=0$.

2.3. The l.r.t. statistic to test nullity of an expected value matrix

We will address in this subsection the test developed in section 3.2 of [18]. Let Z $(p{\times}n)$ be a matrix with a complex multivariate Normal distribution with expected value $\boldsymbol{\mu}M$, where $\boldsymbol{\mu}$ is a $p{\times}q$ complex matrix and M is $q{\times}n$ of rank $q\,(\leq n)$, and variance $I_n\otimes\Sigma$, that is, with $var(vec(Z)) = I_n\otimes\Sigma$. We will denote this fact by

$$Z_{p\times n} \;\sim\; CN_{p\times n}(\boldsymbol{\mu}M, I_n\otimes\Sigma)\,. \tag{20}$$

Let us then suppose that we want to test the hypothesis

$$H_0:\;\boldsymbol{\mu}_{(p\times q)} = 0_{(p\times q)}\,. \tag{21}$$

Then, according to [18], the l..r.t. statistic to test H_0 is

$$\Lambda_3 = \left(\frac{|\Psi|}{\left|\Psi+\frac{1}{n}\beta(M\overline{M}')\overline{\beta}'\right|}\right)^n \tag{22}$$

where

$$\Psi = \frac{1}{n}Z\left(I_n-\overline{M}'\left(M\overline{M}'\right)^{-1}M\right)\overline{Z}' = \frac{1}{n}\left[Z\overline{Z}'-\beta\left(M\overline{M}'\right)\overline{\beta}'\right]$$

and

$$\beta = Z\overline{M}'\left(M\overline{M}'\right)^{-1}$$

are respectively the m.l.e.'s of Σ and μ, and as such independent.

But then, since $\left(I_n - \overline{M}'\left(M\overline{M}'\right)^{-1} M\right)$ is the projector on the null space of the columns of M, and since it is idempotent with

$$rank\left(I_n - \overline{M}'\left(M\overline{M}'\right)^{-1} M\right) = tr\left(I_n - \overline{M}'\left(M\overline{M}'\right)^{-1} M\right) = n - q\,,$$

and given the distribution of Z in (20), we have

$$\Psi = \frac{1}{n} Z\left(I_n - \overline{M}'\left(M\overline{M}'\right)^{-1} M\right)\overline{Z}' \sim CW_p\left(n - q, \frac{1}{n}\Sigma\right).$$

From (20) we may easily see that

$$\beta = Z\overline{M}'\left(M\overline{M}'\right)^{-1} \sim CN_{p\times q}\left(\boldsymbol{\mu}, \left(M\overline{M}'\right)^{-1} \otimes \Sigma\right),$$

so that

$$\beta\left(M\overline{M}'\right)^{1/2} \sim CN_{p\times q}\left(\boldsymbol{\mu}\left(M\overline{M}'\right)^{1/2}, I_q \otimes \Sigma\right),$$

where, under H_0 in (21), $\boldsymbol{\mu}\left(M\overline{M}'\right)^{1/2} = 0$. Consequently under H_0 in (21),

$$\beta\left(M\overline{M}'\right)\overline{\beta}' \sim CW_p(q, \Sigma)\,,$$

independent of Ψ, so that, under H_0 in (21),

$$\Psi + \frac{1}{n}\beta(M\overline{M}')\overline{\beta}' = \frac{1}{n} Z\overline{Z}' \sim CW_p\left(n, \frac{1}{n}\Sigma\right).$$

Thus, following similar steps to those in subsection 3.2, we may verify that

$$\Lambda_3 \sim \prod_{j=1}^{p}\left(Y_j\right)^n \quad \text{where} \quad \begin{array}{l} Y_j \sim Beta\,(n - q - j + 1, q) \quad (j = 1, \ldots, p) \\ \text{are } p \text{ independent r.v.'s} \end{array} \tag{23}$$

for $n > q + p - 1$, so that

$$E\left(\Lambda_3^h\right) = \prod_{j=1}^{p} \frac{\Gamma(n + 1 - j)}{\Gamma(n + 1 - q - j)} \frac{\Gamma(n + 1 - q - j + nh)}{\Gamma(n + 1 - j + nh)} \qquad \left(h > \tfrac{q+p-1}{n} - 1\right).$$

Thus, for

$$W_3 = -\log\,\Lambda_3\,, \tag{24}$$

we have

$$\begin{aligned} \Phi_{W_3}(t) &= E\left(e^{itW_3}\right) = E\left(\Lambda_3^{-it}\right) \\ &= \prod_{j=1}^{p} \frac{\Gamma(n + 1 - j)}{\Gamma(n + 1 - q - j)} \frac{\Gamma(n + 1 - q - j - nit)}{\Gamma(n + 1 - j - nit)} \end{aligned} \tag{25}$$

$$= \prod_{j=1}^{p}\prod_{\ell=0}^{q-1}(n+1-q-j+\ell)\,(n+1-q-j+\ell-n\mathrm{i}t)^{-1}$$

$$= \prod_{j=1}^{p+q-1}(n-j)^{r_j}\,(n-j-n\mathrm{i}t)^{-r_j}$$

$$= \prod_{j=1}^{p+q-1}\left(\frac{n-j}{n}\right)^{r_j}\left(\frac{n-j}{n}-\mathrm{i}t\right)^{-r_j} \tag{26}$$

for r_j given by (17) and (18) in Theorem 2, with q replaced by $q+1$.

When we compare (23) with (5.2.2) and (5.2.3) and (25) with (5.3.1) and (5.3.3) in [18] we see that there is a small mistake in Khatri's paper, in that q has to be subtracted from the first argument of the Beta r.v.'s in (5.2.2) and (5.2.3) and also from the arguments of all Gamma functions in (5.3.1) of [18].

From (26) we have the following Theorem and Corollary.

Theorem 3. *The exact distribution of W_3 in (24) is a GIG distribution of depth $p+q-1$ (see Appendix A) with p.d.f.*

$$f_{W_3}(w) = f^{GIG}\left(w \,\middle|\, r_j; \frac{n-j}{n}; j=1,\ldots,p+q-1\right)$$

and c.d.f.

$$F_{W_3}(w) = F^{GIG}\left(w \,\middle|\, r_j; \frac{n-j}{n}; j=1,\ldots,p+q-1\right),$$

where r_j ($j=1,\ldots,p$) are given by (17) and (18) in Theorem 2.

Corollary 3. *The exact p.d.f. and c.d.f. of the statistic $\Lambda_3 = e^{-W_3}$ in (22) are*

$$f_{\Lambda_3}(\ell) = f^{GIG}\left(-\log\,\ell \,\middle|\, r_j; \frac{n-j}{n}; j=1,\ldots,p+q-1\right)\frac{1}{\ell}$$

and

$$F_{\Lambda_3}(\ell) = 1 - F^{GIG}\left(-\log\,\ell \,\middle|\, r_j; \frac{n-j}{n}; j=1,\ldots,p+q-1\right),$$

for r_j given by (17) and (18).

3. The exact distribution of the l.r.t. statistic to test sphericity of the covariance matrix

Let us suppose that

$$\underline{X} \sim CN_p(\underline{\mu}, \Sigma),$$

and that we want to test the hypothesis

$$H_0 : \Sigma = \sigma^2 I_p \quad \text{(for some unspecified } \sigma^2 > 0). \tag{27}$$

Then, using similar procedures to the ones used in the real case, for a sample of size n, we may show that the l.r.t. statistic to test H_0 in (27), is,

$$\Lambda_4 = \left(\frac{|A|}{\left(tr\, \frac{1}{p} A \right)^p} \right)^n , \tag{28}$$

where A is the m.l.e. of Σ (see (6) and the note after this expression for references on the m.l.e. of Σ), with

$$\Lambda_4 \sim \prod_{j=1}^{p-1} \left(Y_j \right)^n ,$$

where $Y_j \sim Beta\left(n-j-1, \frac{j}{p}+j \right)$ are $p-1$ independent r.v.'s.

But then, for $h > \frac{p}{n} - 1$,

$$E\left(\Lambda_4^h\right) = \prod_{j=1}^{p-1} \frac{\Gamma\left(n-1+\frac{j}{p}\right)}{\Gamma(n-j-1)} \frac{\Gamma(n-j-1+nh)}{\Gamma\left(n-1+\frac{j}{p}+nh\right)} , \tag{29}$$

which, on using the Gamma multiplication formula,

$$\Gamma(nz) = (2\pi)^{\frac{1}{2}(1-n)}\, n^{nz-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n} \right) , \tag{30}$$

yields an expression which in fact agrees with expressions (4.2) in [28] and (2.10) in [19], taking into account that the first of these expressions is designed to yield the non-null moments and that the statistic used in both papers above is indeed the n-th power of the l.r.t. statistic. In the first of these expressions one has to use $k = 0$, $\Sigma = I_p$ and the definition of the complex multivariate gamma function in [17].

Then, if we take $W_4 = -\log\, \Lambda_4$, since the expression in (29) is well-defined for h in a neighborhood of zero, we may write,

$$\begin{aligned}
\Phi_{W_4}(t) &= E\left(e^{\mathrm{i}tW_4}\right) = E\left(\Lambda_4^{-\mathrm{i}t}\right) \\
&= \prod_{j=1}^{p-1} \frac{\Gamma\left(n-1+\frac{j}{p}\right)}{\Gamma(n-j-1)} \frac{\Gamma(n-j-1-n\mathrm{i}t)}{\Gamma\left(n-1+\frac{j}{p}-n\mathrm{i}t\right)} \\
&= \prod_{j=1}^{p-1} \frac{\Gamma\,(n-1)}{\Gamma(n-j-1)} \frac{\Gamma(n-j-1-n\mathrm{i}t)}{\Gamma\,(n-1-n\mathrm{i}t)} \frac{\Gamma\left(n-1+\frac{j}{p}\right)}{\Gamma(n-1)} \frac{\Gamma(n-1-n\mathrm{i}t)}{\Gamma\left(n-1+\frac{j}{p}-n\mathrm{i}t\right)} \\
&= \prod_{j=1}^{p-1} \left\{ \prod_{\ell=0}^{j-1} (n-j-1+\ell)\,(n-j-1+\ell-n\mathrm{i}t)^{-1} \right\} \frac{\Gamma\left(n-1+\frac{j}{p}\right)}{\Gamma(n-1)} \frac{\Gamma(n-1-n\mathrm{i}t)}{\Gamma\left(n-1+\frac{j}{p}-n\mathrm{i}t\right)} \\
&= \left\{ \prod_{j=1}^{p-1} (n-j-1)^{p-j}\,(n-j-1-n\mathrm{i}t)^{-(p-j)} \right\} \left\{ \prod_{j=1}^{p-1} \frac{\Gamma\left(n-1+\frac{j}{p}\right)}{\Gamma(n-1)} \frac{\Gamma(n-1-n\mathrm{i}t)}{\Gamma\left(n-1+\frac{j}{p}-n\mathrm{i}t\right)} \right\} \\
&= \underbrace{\left\{ \prod_{j=1}^{p-1} \left(\frac{n-j-1}{n} \right)^{p-j} \left(\frac{n-j-1}{n} - \mathrm{i}t \right)^{-(p-j)} \right\}}_{\Phi_{1,W_4}(t)} \underbrace{\left\{ \prod_{j=1}^{p-1} \frac{\Gamma\left(n-1+\frac{j}{p}\right)}{\Gamma(n-1)} \frac{\Gamma(n-1-n\mathrm{i}t)}{\Gamma\left(n-1+\frac{j}{p}-n\mathrm{i}t\right)} \right\}}_{\Phi_{2,W_4}(t)}
\end{aligned} \tag{31}$$

which shows that the exact distribution of W_4 is the same as the distribution of the sum of a GIG distributed r.v. with depth $p-1$, with shape parameters $r_j = p - j$ and rate parameters $\frac{n-j-1}{n}$ $(j = 1, \ldots, p-1)$, with $p-1$ independent $Logbeta\left(n-1, \frac{j}{p}\right)$ r.v.'s $(j = 1, \ldots, p-1)$. This shows that the distribution of Λ_4 is thus of the form (1), with $u = p - 1$. The possibility of expressing the exact distribution of Λ_4 in this form will enable us to develop a very well-fitting near-exact distribution for Λ_4 in the Section 5.1.

4. The exact distribution of the l.r.t. statistic to test equality of several covariance matrices

In this section we will only address the case of equal sample sizes. The case of unequal sample sizes, given its compexity, will only be addressed in the next section.

Hence, let us suppose that

$$X_k \sim CN_p(\underline{\mu}_k, \Sigma_k), \quad k = 1, \ldots, q,$$

and that, based on q independent samples, each of size n, we want to test the hypothesis

$$H_0 : \Sigma_1 = \cdots = \Sigma_q .$$

Then, the l.r.t. statistic is

$$\Lambda_5 = \left(q^{pq} \frac{\prod_{k=1}^{q} |A_k|}{|A|^q} \right)^n , \tag{32}$$

where A_k is the m.l.e. of Σ_k $(k = 1, \ldots, q)$, and $A = A_1 + \cdots + A_q$ (see (6) and the note after this expression for references on the m.l.e.'s of Σ_k), with

$$\Lambda_5 \sim \prod_{\substack{j=1 \\ \text{(except for } j=k=1)}}^{p} \prod_{k=1}^{q} \left(Y_{jk}\right)^n$$

where $Y_{jk} \sim Beta\left(n-j, j-1+\frac{k-j}{q}\right)$. But then, either from this fact or from (2.14) in [19], taking $d = 1$, $N_i = n$ and making the corresponding changes resulting from replacing in (2.13) all the n_i by n, and then using (30), we may write

$$\begin{aligned} E\left(\Lambda_5^h\right) &= q^{pqnh} \prod_{j=1}^{p} \left\{ \frac{\Gamma((n-1)q+1-j)}{\Gamma((n-1)q+1-j+nqh)} \prod_{k=1}^{q} \frac{\Gamma(n-j+nh)}{\Gamma(n-j)} \right\} \\ &= \prod_{j=1}^{p} \prod_{k=1}^{q} \frac{\Gamma\left(n-1+\frac{k-j}{q}\right)}{\Gamma\left(n-1+\frac{k-j}{q}+nh\right)} \frac{\Gamma(n-j+nh)}{\Gamma(n-j)} . \end{aligned}$$

Then, if we take $W_5 = -\log \Lambda_5$, we have

$$\begin{aligned} \Phi_{W_5}(t) &= E\left(e^{-\mathrm{i}tW_5}\right) = E\left(\Lambda_5^{-\mathrm{i}t}\right) \\ &= \prod_{j=1}^{p} \prod_{k=1}^{q} \frac{\Gamma\left(n-1+\frac{k-j}{q}\right)}{\Gamma\left(n-1+\frac{k-j}{q}-n\mathrm{i}t\right)} \frac{\Gamma\left(n-j-n\mathrm{i}t\right)}{\Gamma\left(n-j\right)} \end{aligned}$$

$$
\begin{aligned}
&= \prod_{j=1}^{p}\prod_{k=1}^{q} \frac{\Gamma\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor\right)}{\Gamma(n-j)} \frac{\Gamma(n-j-n\mathrm{i}t)}{\Gamma\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor-n\mathrm{i}t\right)} \\
&\qquad \frac{\Gamma\left(n-1+\frac{k-j}{q}\right)}{\Gamma\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor\right)} \frac{\Gamma\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor-n\mathrm{i}t\right)}{\Gamma\left(n-1+\frac{k-j}{q}-n\mathrm{i}t\right)} \\
&= \prod_{j=1}^{p}\prod_{k=1}^{q} \left\{ \prod_{\ell=0}^{j-2+\left\lfloor\frac{k-j}{q}\right\rfloor} (n-j+\ell)(n-j+\ell-n\mathrm{i}t)^{-1} \right\} \\
&\qquad \frac{\Gamma\left(n-1+\frac{k-j}{q}\right)}{\Gamma\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor\right)} \frac{\Gamma\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor-n\mathrm{i}t\right)}{\Gamma\left(n-1+\frac{k-j}{q}-n\mathrm{i}t\right)} \\
&= \prod_{j=1}^{p}\prod_{k=1}^{q} \left\{ \prod_{\ell=0}^{j-2+\left\lfloor\frac{k-j}{q}\right\rfloor} \frac{n-j+\ell}{n}\left(\frac{n-j+\ell}{n}-\mathrm{i}t\right)^{-1} \right\} \\
&\qquad \frac{\Gamma\left(n-1+\frac{k-j}{q}\right)}{\Gamma\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor\right)} \frac{\Gamma\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor-n\mathrm{i}t\right)}{\Gamma\left(n-1+\frac{k-j}{q}-n\mathrm{i}t\right)} \\
&= \underbrace{\left\{ \prod_{j=1}^{p-1} \left(\frac{n-1-j}{n}\right)^{r_j} \left(\frac{n-1-j}{n}-\mathrm{i}t\right)^{-r_j} \right\}}_{\Phi_{1,W_5}(t)} \\
&\qquad \underbrace{\left\{ \prod_{j=1}^{p}\prod_{k=1}^{q} \frac{\Gamma\left(n-1+\frac{k-j}{q}\right)}{\Gamma\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor\right)} \frac{\Gamma\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor-n\mathrm{i}t\right)}{\Gamma\left(n-1+\frac{k-j}{q}-n\mathrm{i}t\right)} \right\}}_{\Phi_{2,W_5}(t)}
\end{aligned} \tag{33}
$$

for

$$
r_j = \begin{cases} q(q-1)\left(j-\frac{1}{2}\right) & j=1,\ldots,\left\lceil\frac{p-1}{q}\right\rceil-1 \\ \frac{1}{2}\left(p-p^2+2jpq+q-3jq-q^2(j-1)^2\right) & j=\left\lceil\frac{p-1}{q}\right\rceil \\ q(p-j) & j=\left\lceil\frac{p-1}{q}\right\rceil+1,\ldots,p-1\,, \end{cases} \tag{34}
$$

which shows that the exact distribution of Λ_5 is of the form (1), with $u = pq$. As a consequence, as it happens with the statistic in the previous section, very well-fitting near-exact distributions can be developed for Λ_5. See Section 5.2.

5. Near-exact distributions

Given the complexity of the exact distributions of the statistics in sections 3 and 4, or rather, of the expressions that might be obtained for their exact p.d.f.'s and c.d.f.'s and the concomitant issues related with their manageability, the development of near-exact distributions for such statistics arises as an sensible goal.

Let then, as a general notation W stand for the negative logarithm of the l.r.t. statistics in sections 3 and 4. The near-exact distributions that we will develop in this section will assume the form of mixtures of GNIG (Generalized Near-Integer Gamma) distributions (see Appendix A), which for the negative logarithm of the l.r.t. statistics in sections 3 and 4 will have p.d.f.'s and c.d.f.'s respectively of the form

$$f_W(w) = \sum_{\ell=0}^{m^*} \pi_\ell \, f^{GNIG}\left(w \,|\, r_1, \ldots, r_{p-1}, r+\ell; \frac{n-2}{n}, \ldots, \frac{n-p}{n}, \lambda; p^* \right) \tag{35}$$

and

$$F_W(w) = \sum_{\ell=0}^{m^*} \pi_\ell \, F^{GNIG}\left(w \,|\, r_1, \ldots, r_{p-1}, r+\ell; \frac{n-2}{n}, \ldots, \frac{n-p}{n}, \lambda; p^* \right), \tag{36}$$

for some $m^* \in \mathbb{N}$, where $r_1, \ldots, r_{p-1} \in \mathbb{N}$ and, for the statistic in Section 3,

$$r = \frac{p-1}{2}\,, \quad \lambda = \frac{n-1}{n}\,, \quad r_j = p-j \;\; (j=1,\ldots,p-1)\,, \quad \text{and} \quad p^* = p \tag{37}$$

while for the statistic in section 4,

$$r = p\,\frac{q-1}{2}\,, \quad \text{and} \quad p^* = p+q$$

where λ^* is the rate parameter in

$$\Phi^*_{W_5}(t) = \Phi_{1,W_5}(t)\left(p_1 (\lambda^*)^{s_1} (\lambda^* - \mathrm{i}t)^{-s_1} + (1-p_1)\,(\lambda^*)^{s_2}\,(\lambda^* - \mathrm{i}t)^{-s_2} \right), \tag{38}$$

which is determined together with s_1, s_2 and p_1 in such a way that the first 4 derivatives of $\Phi^*_{W_5}(t)$ and $\Phi_{W_5}(t)$ in (33) at $t=0$ are the same. For further details please see Section 5.2.

These near-exact distributions are built by leaving $\Phi_{1,W_4}(t)$ and $\Phi_{1,W_5}(t)$, respectively in (31) and (33) unchanged and then asymptotically approximating the Logbeta distributions whose c.f.'s are represented in $\Phi_{2,W_4}(t)$ and $\Phi_{2,W_5}(t)$ in the same expressions by infinite mixtures of Gamma distributions. Indeed any *Logbeta*(a,b) distribution may, for non-integer b, be asymptotically replaced by an infinite mixture of $\Gamma(b+\ell, a)$ $(\ell = 0,1,\ldots)$ distributions, since based on expressions (12) and (14) from [32], which for $z = a - \mathrm{i}t$, $\alpha = 0$ and non-integer $\beta = b$ may be written as

$$\frac{\Gamma(a-\mathrm{i}t)}{\Gamma(a+b-\mathrm{i}t)} \approx \sum_{\ell=0}^{\infty} p_\ell(b)\,(a-\mathrm{i}t)^{-(b+k)} \qquad (\text{as } a \to \infty)$$

with

$$p_\ell(b) = \frac{1}{k} \sum_{m=0}^{k-1} \left(\frac{\Gamma(1-b-m)}{\Gamma(-b-\ell)(\ell-m+1)!} + (-1)^{\ell+m} b^{\ell-m+1} \right) p_m(b)\,, \quad \ell = 1,2,\ldots,$$

and $p_0(b) = 1$, we may then write the c.f. of $Y = -\log X$ where $X \sim Beta(a,b)$ as

$$\Phi_Y(t) = \frac{\Gamma(a+b)}{\Gamma(a)} \, \frac{\Gamma(a-\mathrm{i}t)}{\Gamma(a+b-\mathrm{i}t)} \approx \sum_{\ell=0}^{\infty} \underbrace{\frac{\Gamma(a+b)}{\Gamma(a)} \, \frac{p_\ell(b)}{a^{b+\ell}}}_{p^*_\ell(a,b)} \; a^{b+\ell} (a-\mathrm{i}t)^{-(b+\ell)}\,. \tag{39}$$

It happens then that the right hand side of (39) is the c.f. of an infinite mixture, with weights $p^*_\ell(a,b)$ of $\Gamma(b+\ell, a)$ $(\ell = 0,1,\ldots)$ distributions.

Then we truncate the infinite sum in (39) and instead of the weights $p^*_\ell(a,b)$ we use the weights π_ℓ ($\ell = 0,\dots,m^*$), determined in such a way that

$$\left.\frac{\partial^h}{\partial t^h}\Phi_{W_k}(t)\right|_{t=0} = \left.\frac{\partial^h}{\partial t^h}\left\{\Phi_{1,W_k}(t)\sum_{\ell=0}^{m^*}\pi_\ell a^{b+\ell}(a-\mathrm{i}t)^{-(b+\ell)}\right\}\right|_{t=0}, \quad k=4,5\,, \quad h=1,\dots,m^*\,,$$

with $\pi_{m^*} = 1-\sum_{\ell=0}^{m^*-1}\pi_\ell$. This way these near-exact distributions have, by construction, the first m^* moments equal to the first m^* exact moments.

Further details on the construction of these near-exact distributions are given in the subsections ahead.

5.1. Near-exact distribution for the l.r.t. statistic in section 3

Using (39) we will as a first step replace $\Phi_{2,W_4}(t)$ in (31), which is the c.f. of the sum of $p-1$ independent *Logbeta*$\left(n-1,\frac{j}{p}\right)$ r.v.'s ($j=1,\dots,p-1$), multiplied by n, by the c.f. of the sum of $p-1$ independent infinite mixtures of $\Gamma\left(\frac{j}{p}+\ell, n-1\right)$ distributions ($\ell=0,1,\dots$), multiplied by n, which is the c.f. of the sum of $p-1$ independent infinite mixtures of $\Gamma\left(\frac{j}{p}+\ell,\frac{n-1}{n}\right)$ distributions ($\ell=0,1,\dots$), and which in turn, and given that the rate parameters of the Gamma distributions are not functions of either j or ℓ, is an infinite mixture of $\Gamma\left(\left(\sum_{j=1}^{p-1}\frac{j}{p}\right)+\ell,\frac{n-1}{n}\right)$ distributions, where $\sum_{j=1}^{p-1}\frac{j}{p}=\frac{p-1}{2}$. We may note that in this way we obtain a representation of the exact distribution which bears some resemblance with the representations in [23, 24, 31], however with coefficients which are much simpler to compute, and having the advantage of allowing for the easy development of very accurate near-exact approximations, as we will see next. Moreover, our representation does not have any parameters that are not well-defined, in contrast to the representation in [31].

In this way, in order to obtain a near-exact distribution for W_4 we will replace $\Phi_{2,W_4}(t)$ in (31) by the c.f. of a finite mixture of $\Gamma\left(\frac{p-1}{2}+\ell,\frac{n-1}{n}\right)$ distributions, for $\ell=0,\dots,m^*$,

$$\Phi^*_{2,W_4}(t) = \sum_{\ell=0}^{m^*}\pi_\ell\left(\frac{n-1}{n}\right)^{\frac{p-1}{2}+\ell}\left(\frac{n-1}{n}-\mathrm{i}t\right)^{-\left(\frac{p-1}{2}+\ell\right)} \tag{40}$$

where the weights π_ℓ are determined in such a way that the first m^* derivatives of $\Phi^*_{2,W_4}(t)$ and $\Phi_{2,W_4}(t)$ with respect to t, at $t=0$, are equal, that is, $\pi_0,\dots,\pi_{m^*-1}$ are determined as the numerical solution of the system of m^* equations

$$\left.\frac{\partial^h}{\partial t^h}\,\Phi^*_{2,W_4}(t)\right|_{t=0} = \left.\frac{\partial^h}{\partial t^h}\,\Phi_{2,W_4}(t)\right|_{t=0}, \quad h=1,\dots,m^* \tag{41}$$

and with $\pi_{m^*}=1-\sum_{\ell=0}^{m^*-1}\pi_\ell$, and we will then use

$$\Phi^*_{W_4}(t) = \Phi_{1,W_4}(t)\,\Phi^*_{2,W_4}(t)\,,$$

with $\Phi_{1,W_4}(t)$ given by (31) and $\Phi^*_{2,W_4}(t)$ given by (40) above, as a near-exact c.f. for W_4, to which correspond the p.d.f. and the c.d.f. in (35) and (36), with r, λ and p^* given by (37).

We have thus the following Theorem and Corollary.

Theorem 4. *Let $W_4=-\log\Lambda_4$, where Λ_4 is the l.r.t. statistic in* (28). *Then, for some $m^*\in\mathbb{N}$, distributions with p.d.f.*

$$f(w) = \sum_{\ell=0}^{m^*}\pi_\ell\, f^{GNIG}\left(w\,|\,r_1,\dots,r_{p-1},\frac{p-1}{2}+\ell;\frac{n-2}{n},\dots,\frac{n-p}{n},\frac{n-1}{n};p\right) \quad (w>0)$$

and c.d.f.

$$F(w)=\sum_{\ell=0}^{m^*}\pi_\ell\, F^{GNIG}\left(w\,|\,r_1,\ldots,r_{p-1},\frac{p-1}{2}+\ell;\frac{n-2}{n},\ldots,\frac{n-p}{n},\frac{n-1}{n};p\right)\quad (w>0)$$

where $r_j=p-j$ $(j=1,\ldots,p-1)$ *and* $\pi_0,\ldots,\pi_{m^*-1}$ *are determined from* (41) *above, with* $\Phi^*_{2,W_4}(t)$ *given by* (40) *and* $\pi_{m^*}=1-\sum_{\ell=0}^{m^*-1}\pi_\ell$*, are near-exact distributions for* W_4.

From the above Theorem we may then easily obtain the following Corollary.

Corollary 4. *Let* Λ_4 *be the l.r.t. statistic in* (28)*. Then distributions with p.d.f.*

$$f(z)=\sum_{\ell=0}^{m^*}\pi_\ell\, f^{GNIG}\left(\log z\,|\,r_1,\ldots,r_{p-1},\frac{p-1}{2}+\ell;\frac{n-2}{n},\ldots,\frac{n-p}{n},\frac{n-1}{n};p\right)\frac{1}{z}\quad (0<z<1)$$

and c.d.f.

$$F(z)=\sum_{\ell=0}^{m^*}\pi_\ell\left(1-F^{GNIG}\left(\log z\,|\,r_1,\ldots,r_{p-1},\frac{p-1}{2}+\ell;\frac{n-2}{n},\ldots,\frac{n-p}{n},\frac{n-1}{n};p\right)\right)\quad (0<z<1)$$

where $0<z<1$ *represents the running value for* Λ_4 *and* r_j $(j=1,\ldots,p-1)$ *and* π_ℓ $(\ell=0,\ldots,m^*)$ *are defined as in the previous Theorem, are near-exact distributions for* Λ_4.

We should note that although for generality the components of the mixtures in the previous Theorem and Corollary were denoted as GNIG distributions, since for odd p, $r=\frac{p-1}{2}$ is indeed an integer, in this case the components of the mixtures are GIG distributions.

These near-exact distributions are asymptotic not only for increasing sample sizes but also for increasing number of variables involved, as it is shown by the numerical studies carried out in the next Section. These studies also show the extreme closeness of these near-exact distributions to the corresponding exact distributions, which is, in general, even more striking than for the real case.

These near-exact distributions are very flexible, their parameters are very simple to determine and their implementation is very easy with the help of adequate software.

5.2. Near-exact distribution for the l.r.t. statistic in section 4

5.2.1. The case of equal sample sizes

For the statistic Λ_5 in Section 4, in the case where all q samples have size n, using (39) we may asymptotically replace $\Phi_{2,W_5}(t)$ in (33), which is the c.f. of the sum of $pq-\min(p,q)$ independent *Logbeta*$\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor,\frac{k-j}{q}-\left\lfloor\frac{k-j}{q}\right\rfloor\right)$ r.v.'s $(j=1,\ldots,p;k=1,\ldots,q;j\neq k)$, multiplied by n, by the c.f. of the sum of $pq-\min(p,q)$ independent infinite mixtures of $\Gamma\left(\frac{k-j}{q}-\left\lfloor\frac{k-j}{q}\right\rfloor+\ell,\frac{1}{n}\left(n-1+\left\lfloor\frac{k-j}{q}\right\rfloor\right)\right)$ distributions $(\ell=0,1,\ldots)$. If it were the case that the rate parameters in these Gamma distributions were functions of neither j nor k, as happened in the previous subsection, then this sum of mixtures would yield a simple mixture of Gamma distributions. However, since now the rate parameters of the Gamma distributions in the sum of mixtures are functions of both j and k, it renders it difficult to add the different mixtures of Gamma distributions. For this reason we decide to use as an asymptotic replacement for $\Phi_{2,W_5}(t)$ in (33) the c.f. of a finite mixture of m^*+1 c.f.'s of $\Gamma\left(\sum_{j=1}^{p}\sum_{k=1}^{q}\frac{k-j}{q}-\left\lfloor\frac{k-j}{q}\right\rfloor+\ell,\lambda^*\right)$ distributions $(\ell=0,\ldots,m^*)$, which is

$$\Phi^*_{2,W_5}(t)=\sum_{\ell=0}^{m^*}\pi_\ell(\lambda^*)^{r+\ell}(\lambda^*-\mathrm{i}t)^{-(r+\ell)}\,,\tag{42}$$

with

$$r=\sum_{j=1}^{p}\sum_{k=1}^{q}\frac{k-j}{q}-\left\lfloor\frac{k-j}{q}\right\rfloor=p\,\frac{q-1}{2}\,, \tag{43}$$

and where λ^* is the common rate parameter of a mixture of two Gamma distributions whose first four moments match the first four moments of the sum of independent Logbeta r.v.'s whose c.f. is $\Phi_{2,W_5}(t)$ in (33), that is, where λ^* is the rate parameter in (38), which is determined together with s_1, s_2 and p_1 in such a way that, for $\Phi^*_{W_5}(t)$ in (38),

$$\left.\frac{\partial^h}{\partial t^h}\,\Phi^*_{W_5}(t)\right|_{t=0}=\left.\frac{\partial^h}{\partial t^h}\,\Phi_{W_5}(t)\right|_{t=0}, \quad h=1,\dots,4\,, \tag{44}$$

and where the weights π_ℓ ($\ell=0,\dots,m^*-1$), are then determined in such a way that

$$\left.\frac{\partial^h}{\partial t^h}\,\Phi^*_{2,W_5}(t)\right|_{t=0}=\left.\frac{\partial^h}{\partial t^h}\,\Phi_{2,W_5}(t)\right|_{t=0}, \quad h=1,\dots,m^*\,, \tag{45}$$

with $\pi_{m^*}=1-\sum_{\ell=0}^{m^*-1}\pi_\ell$.

This way, we will then use

$$\Phi^*_{W_5}(t)=\Phi_{1,W_5}(t)\,\Phi^*_{2,W_5}(t)\,,$$

with $\Phi_{1,W_5}(t)$ given by (33) and $\Phi^*_{2,W_5}(t)$ given by (42) above, as a near-exact c.f. for W_5, and as such we may thus enunciate the results summarized in the following Theorem and Corollary.

Theorem 5. *Let $W_5=-\log\,\Lambda_5$, where Λ_5 is the l.r.t. statistic in* (32). *Then, for some $m^*\in\mathbb{N}$, distributions with p.d.f.*

$$f(w)=\sum_{\ell=0}^{m^*}\pi_\ell\,f^{GNIG}\left(w\,|\,r_1,\dots,r_{p-1},r+\ell;\frac{n-2}{n},\dots,\frac{n-p}{n},\lambda^*;p\right)\quad(w>0) \tag{46}$$

and c.d.f.

$$F(w)=\sum_{\ell=0}^{m^*}\pi_\ell\,F^{GNIG}\left(w\,|\,r_1,\dots,r_{p-1},r+\ell;\frac{n-2}{n},\dots,\frac{n-p}{n},\lambda^*;p\right)\quad(w>0) \tag{47}$$

where r_j ($j=1,\dots,p-1$) are given by (34)*, r given by* (43)*, λ^* by solving* (44) *in order to λ^*, s_1, s_2 and p_1, and $\pi_0,\dots,\pi_{m^*-1}$ are determined from* (45) *above, with $\Phi^*_{2,W_5}(t)$ given by* (42) *and $\pi_{m^*}=1-\sum_{\ell=0}^{m^*-1}\pi_\ell$, are near-exact distributions for W_5.*

From the above Theorem we may then easily obtain the following Corollary, by simple transformation of the r.v.'s involved, taking into account that $\Lambda_5=e^{-W_5}$.

Corollary 5. *Let Λ_5 be the l.r.t. statistic in* (32). *Then distributions with p.d.f.*

$$f(z)=\sum_{\ell=0}^{m^*}\pi_\ell\,f^{GNIG}\left(\log\,z\,|\,r_1,\dots,r_{p-1},r+\ell;\frac{n-2}{n},\dots,\frac{n-p}{n},\lambda^*;p\right)\frac{1}{z}\quad(0<z<1) \tag{48}$$

and c.d.f.

$$F(z) = \sum_{\ell=0}^{m^*} \pi_\ell \left(1 - F^{GNIG}\left(\log z \,|\, r_1, \ldots, r_{p-1}, r+\ell; \frac{n-2}{n}, \ldots, \frac{n-p}{n}, \lambda^*; p\right)\right) \quad (0 < z < 1) \tag{49}$$

where $0 < z < 1$ *represents the running value for* Λ_5 *and* r_j $(j = 1, \ldots, p-1)$, r, λ^* *and* π_ℓ $(\ell = 0, \ldots, m^*)$ *are defined as in the previous Theorem, are near-exact distributions for* Λ_5.

These near-exact distributions are asymptotic not only for increasing sample sizes but also for increasing number of variables and increasing number of matrices involved, as it is shown by the numerical studies carried out in the next section. As it happened with the near-exact distributions developed in the previous subsection, these studies also show the extreme closeness of these near-exact distributions to the corresponding exact distributions, in general, once again, even more accentuated than for the real case.

Once again, also these near-exact distributions are very flexible, their parameters very simple to determine and their implementation rendered very easy with the help of adequate software.

5.2.2. The case of unequal sample sizes

When the samples have different sizes, with the k-th sample having size n_k, the l.r.t. statistic Λ_5 is now

$$\Lambda_5 = \frac{N^{Np}}{\prod_{k=1}^{q} n_k^{n_k p}} \frac{\prod_{k=1}^{q} |A_k|^{n_k}}{|A|^N},$$

where

$$N = \sum_{k=1}^{q} n_k,$$

and A_k $(k = 1, \ldots, q)$ and A are the same as in (32).

Then, from (2.14) in [19], taking $d = 1$ and $N_i = n_i$ $(i = 1, \ldots, q)$, the h-th moment of Λ_5 is in this case, for $h > \max_{1 \le k \le q} \frac{p-n_k}{n_k}$,

$$E\left(\Lambda_5^h\right) = \frac{N^{Nph}}{\prod_{k=1}^{q} n_k^{n_k ph}} \prod_{j=1}^{p} \left\{ \frac{\Gamma(N-q+1-j)}{\Gamma(N-q+1-j+Nh)} \prod_{k=1}^{q} \frac{\Gamma(n_k - j + n_k h)}{\Gamma(n_k - j)} \right\},$$

and the c.f. of $W_5 = -\log \Lambda_5$ is thus

$$\Phi_{W_5}(t) = E\left(\Lambda_5^{-\mathrm{i}t}\right) = \frac{N^{-Np\mathrm{i}t}}{\prod_{k=1}^{q} n_k^{-n_k p\mathrm{i}t}} \prod_{j=1}^{p} \left\{ \frac{\Gamma(N-q+1-j)}{\Gamma(N-q+1-j-N\mathrm{i}t)} \prod_{k=1}^{q} \frac{\Gamma(n_k - j - n_k \mathrm{i}t)}{\Gamma(n_k - j)} \right\}.$$

It happens that the exact distribution of Λ_5 in (32) in this case of unequal sample sizes is quite complicated and it is not even possible to give it a structure as the one in (1).

However, using a procedure similar to the one used in [8], we may write

$$\Phi_{W_5}(t) = \Phi_{1,W_5}(t) \frac{\Phi_{W_5}(t)}{\Phi_{1,W_5}(t)},$$

where $\Phi_{1,W_5}(t)$ is given by (33), now with $n = N/q$, and, in order to build a near-exact approximation for W_5 we will leave $\Phi_{1,W_5}(t)$ unchanged and replace $\frac{\Phi_{W_5}(t)}{\Phi_{1,W_5}(t)}$ by $\Phi^*_{2,W_5}(t)$, given by (42), once again with λ^* defined in a similar manner to the one used in the previous subsection and with r either equal to s_1 in (33) or given by (43). As we will see in the next section, while the first choice for r will yield near-exact distributions that are asymptotic for increasing sample sizes as well as for increasing values of p and q, that is, respectively the number of variables and matrices involved, the second choice may give slightly better approximations for small values of p and even better approximations for large sample sizes. However, these latter near-exact distributions will no longer be asymptotic for increasing values of p or q, but only for increasing sample sizes.

As such, in terms of near-exact distribution, for this case of unequal sample sizes, similar results to the ones in Theorem 5 and Corollary 5 still hold, with the due changes in the parameters, namely with all the occurrences of n in expressions (46)-(49) changed to N/q.

6. Numerical studies

In order to assess the proximity of the near-exact distributions developed in the previous section to the exact distribution and their performance in different situations, we will use the measure

$$\Delta = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\Phi_W(t) - \Phi^*_W(t)}{t}\right| dt\,,$$

with

$$\max w \left|F_W(w) - F^*_W(w)\right| \leq \Delta\,,$$

where W represents generally either W_4 or W_5, $\Phi_W(t)$ represents the exact c.f. of W and $\Phi^*_W(t)$ its approximate c.f. usually a near-exact c.f., but occasionally an asymptotic or other c.f., $F_W(\cdot)$ the exact c.d.f. of W, corresponding to $\Phi_W(t)$, and $F^*_W(\cdot)$ the c.d.f. corresponding to $\Phi^*_W(t)$. Some further details on this measure and its relation with the Berry-Esseen bound are analyzed in [8]. We should note that, quite clearly, a given value of this measure for any given approximation to the exact distribution of either $W_4 = -\log \Lambda_4$ or $W_5 = -\log \Lambda_5$, for any given combination of parameters, will have exactly the same value as the corresponding measure concerning the corresponding approximation to the distribution of respectively either Λ_4 or Λ_5. In other words, computing the value of Δ for a given approximation e.g. for W_4 will be the same as computing its value for the corresponding approximation for Λ_4.

6.1. Numerical studies on the approximations for Λ_4

In Table 1 we have the values of the measure Δ for the statistic Λ_4, for values of p, the number of variables involved, ranging from 3 to 50 and sample sizes n which exceed p by 2, 12, 50 and 100. In order to compare the performance of the near-exact distributions with other available approximations in the literature, we have also used the Box type asymptotic distribution in [25] and the Pearson type I distribution used in [19]. The Pearson type I distribution was fit, by matching the first four exact moments of $\Lambda_4^{1/nb}$. In each case we have used for b the positive integer value which would give a better fit, this way obtaining indeed much better approximations than the ones obtained in [19]. These values of b are specified in Table 1, inside square brackets, right after the value of Δ for the Pearson type I approximation. We may note that this Pearson type I approximation, with the slight change we introduced, has a much better performance than truncations, even with a rather large number of terms, of any of the expansions in [24].

Table 1 – Values of the measure Δ for the near-exact and asymptotic distributions for the l.r.t. statistic to test sphericity

		Near-exact distribution # of exact moments matched				Box	Pearson type I
p	n	4	6	10	15		
3	5	1.09×10^{-8}	2.74×10^{-11}	3.57×10^{-13}	9.28×10^{-16}	5.75×10^{-2}	2.17×10^{-3} [1]
	15	6.27×10^{-11}	1.21×10^{-13}	1.39×10^{-18}	4.27×10^{-23}	9.12×10^{-3}	2.78×10^{-5} [1]
	53	2.21×10^{-11}	3.42×10^{-14}	1.19×10^{-18}	7.73×10^{-23}	2.88×10^{-3}	6.85×10^{-7} [1]
	103	1.08×10^{-12}	6.43×10^{-16}	1.16×10^{-21}	2.71×10^{-27}	1.52×10^{-3}	9.58×10^{-8} [1]
6	8	9.23×10^{-14}	2.15×10^{-17}	7.93×10^{-24}	1.31×10^{-30}	3.43×10^{-1}	7.60×10^{-4} [2]
	18	8.88×10^{-14}	3.39×10^{-17}	7.24×10^{-24}	1.37×10^{-30}	1.14×10^{-1}	3.37×10^{-6} [3]
	56	1.71×10^{-14}	4.07×10^{-18}	7.35×10^{-25}	6.12×10^{-32}	3.20×10^{-2}	9.33×10^{-8} [3]
	106	2.02×10^{-15}	2.32×10^{-19}	1.00×10^{-26}	7.68×10^{-35}	1.64×10^{-2}	1.33×10^{-8} [3]
10	12	3.54×10^{-16}	2.18×10^{-20}	2.61×10^{-28}	2.39×10^{-37}	8.07×10^{-1}	3.56×10^{-4} [3]
	22	2.19×10^{-15}	7.50×10^{-20}	2.26×10^{-27}	1.64×10^{-35}	3.52×10^{-1}	2.17×10^{-6} [5]
	60	1.21×10^{-15}	3.02×10^{-20}	7.29×10^{-28}	2.04×10^{-36}	1.16×10^{-1}	7.88×10^{-8} [5]
	110	9.24×10^{-17}	7.55×10^{-22}	3.06×10^{-30}	2.87×10^{-40}	$6,17\times10^{-2}$	1.19×10^{-8} [5]
20	22	3.47×10^{-18}	6.90×10^{-24}	2.37×10^{-33}	2.60×10^{-45}	2.19×10^{0}	——
	32	6.80×10^{-16}	3.46×10^{-22}	3.56×10^{-30}	1.18×10^{-39}	1.19×10^{0}	2.51×10^{-6} [8]
	70	5.49×10^{-16}	7.14×10^{-23}	1.73×10^{-30}	2.02×10^{-40}	4.90×10^{-1}	3.42×10^{-8} [11]
	120	1.48×10^{-16}	4.59×10^{-24}	8.87×10^{-32}	1.01×10^{-42}	3.04×10^{-1}	6.07×10^{-9} [11]
50	52	6.91×10^{-18}	3.35×10^{-25}	8.87×10^{-35}	8.46×10^{-47}	6.48×10^{0}	——
	62	2.57×10^{-16}	6.21×10^{-23}	1.88×10^{-32}	4.66×10^{-43}	4.85×10^{0}	1.27×10^{-6} [16]
	100	1.56×10^{-16}	3.48×10^{-23}	3.92×10^{-33}	8.04×10^{-46}	2.48×10^{0}	2.91×10^{-8} [24]
	150	8.77×10^{-18}	6.80×10^{-25}	5.13×10^{-36}	3.71×10^{-48}	1.57×10^{0}	3.97×10^{-9} [27]

The results in Table 1 show that although the Pearson type I distribution, with the improvement we introduced of finding the integer value of b which gives the best fit to the exact distribution, that is, the smallest value of Δ, has a very good performance, with an asymptotic behavior not only for increasing sample sizes but also for increasing values of p, it still is no match even for the near-exact distribution which matches 4 moments. Besides, finding the integer value of b which gives the best fit for the Pearson type I distribution is not always that easy of a task, and it was not possible to fit any such distribution for the cases where p was larger and the sample size only exceeds p by 2. The Box type asymptotic distribution has the poorest behavior of all the approximations. Its performance being worse for larger values of p, with values of Δ above 1, which shows that in these cases the asymptotic distribution yielded by the approximation is not a true distribution. Moreover, it produces values of Δ quite close to 1 for a number of other cases. This asymptotic approximation has, however, as a virtue its simplicity.

The near-exact distributions always show a very good performance, with very low values of Δ. They exhibit a very good performance even for the smaller sample sizes, always showing an asymptotic behavior both for increasing sample sizes as well as for increasing values of p, and, of course, with the near-exact distribution which match a larger number of exact moments showing an increasingly better performance.

Since with the software and computers nowadays available it is not hard to implement such distributions and since the computation of their parameters is really simple, they seem to be the right choice both in terms of accuracy as well as in terms of convenience.

6.2. Numerical studies on the approximations for Λ_5

Concerning the case of equal sample sizes, we have in Table 2 the values of the measure Δ for the statistic Λ_5, for different values of p, q and n, respectively the number of variables involved, the number of covariance matrices involved and the common sample size of the q independent samples. In order to compare the performance of the near-exact distributions with other available approximations in the literature, we have also used the mixture of two Beta distributions in [10] and the Pearson type I distribution used by Krishnaiah et al. [19]. These two distributions were fit by matching respectively the first two and four exact moments of $\Lambda_5^{1/nb}$, for the choice of $b \in \mathbb{N}$ which gives the lowest value of Δ. These values of b are specified in Table 2, inside square brackets, right after the value of Δ for each of these approximations. The mixture of two Beta distributions we used is a mixture of two Beta distributions with the same first parameter and a second parameter given by expressions (4.7) and (4.9) in [10]. Then the first weight in the mixture and the first parameter in the Beta distributions are determined by equating the two first exact moments of $\Lambda_5^{1/nb}$ and the two first moments of the mixture. Although the mixture of two Beta distributions in [10] is indeed not implemented exactly in this way, we decided to implement it the way we did since not only the definition of the parameter s in (4.9) and (4.10) in [10] yields a conflicting definition for A_2, as well as in defining the parameters the way we did we think the approximation will indeed benefit from a much better performance.

From the results in Table 2 we may see how the Pearson type I approximation always outperforms the mixture of two Beta distributions, mainly for the larger sample sizes. However, the Pearson type I distribution is always itself largely outperformed by the near-exact distribution which matches only 4 moments. The near-exact distributions once again show not only a marked asymptotic behavior for increasing sample sizes but also for increasing values of both p and q, as well as a consistent extremely good performance for very small sample sizes, avoiding the cumbersomeness of the determination of the best value for the parameter b in the Pearson type I and in the mixture of Betas approximations.

Table 2 – Values of the measure Δ for the near-exact and asymptotic distributions for the l.r.t. statistic to test the equality of q covariance matrices

			Near-exact distribution # of exact moments matched				Mixture of Beta dists.	Pearson type I
p	q	n	4	6	10	15		
5	3	7	7.26×10^{-9}	3.05×10^{-11}	1.07×10^{-15}	7.40×10^{-21}	9.63×10^{-5} [6]	1.36×10^{-5} [6]
		50	4.57×10^{-13}	1.23×10^{-17}	1.12×10^{-25}	1.55×10^{-34}	7.67×10^{-6} [4]	4.78×10^{-9} [7]
		100	4.03×10^{-15}	8.06×10^{-20}	4.38×10^{-29}	1.95×10^{-39}	2.68×10^{-7} [8]	5.54×10^{-10} [7]
	10	7	2.15×10^{-10}	1.13×10^{-13}	4.19×10^{-20}	1.20×10^{-27}	1.29×10^{-4} [26]	3.24×10^{-6} [22]
		50	1.67×10^{-14}	2.41×10^{-19}	5.06×10^{-25}	6.29×10^{-39}	6.12×10^{-8} [30]	1.56×10^{-9} [28]
		100	5.02×10^{-16}	1.84×10^{-21}	3.52×10^{-32}	7.86×10^{-44}	2.18×10^{-8} [30]	1.50×10^{-10} [29]
	15	7	1.53×10^{-10}	7.06×10^{-14}	2.29×10^{-20}	4.14×10^{-28}	6.80×10^{-6} [37]	1.60×10^{-6} [34]
		50	1.33×10^{-14}	1.63×10^{-19}	3.60×10^{-29}	1.03×10^{-40}	5.16×10^{-8} [46]	7.25×10^{-10} [44]
		100	4.05×10^{-16}	1.26×10^{-21}	1.77×10^{-32}	1.16×10^{-45}	6.09×10^{-9} [45]	8.85×10^{-11} [44]
10	3	12	1.50×10^{-10}	9.18×10^{-14}	6.08×10^{-20}	5.39×10^{-27}	6.63×10^{-4} [11]	2.22×10^{-5} [9]
		50	3.13×10^{-14}	1.66×10^{-18}	4.69×10^{-26}	1.67×10^{-35}	4.93×10^{-6} [11]	1.32×10^{-8} [13]
		100	1.65×10^{-15}	5.46×10^{-20}	4.26×10^{-29}	3.38×10^{-40}	2.98×10^{-9} [14]	1.41×10^{-9} [13]
	10	12	8.52×10^{-14}	2.94×10^{-18}	3.29×10^{-26}	2.75×10^{-36}	2.96×10^{-4} [46]	3.23×10^{-6} [34]
		50	8.63×10^{-16}	7.48×10^{-21}	4.20×10^{-31}	3.96×10^{-44}	1.90×10^{-7} [61]	2.17×10^{-9} [54]
		100	4.07×10^{-17}	8.32×10^{-23}	3.05×10^{-34}	3.61×10^{-46}	1.55×10^{-8} [58]	2.01×10^{-10} [56]
	15	12	5.10×10^{-13}	2.25×10^{-17}	5.19×10^{-26}	1.33×10^{-36}	5.33×10^{-5} [67]	1.69×10^{-6} [52]
		50	1.03×10^{-15}	5.51×10^{-21}	1.73×10^{-31}	2.25×10^{-44}	2.97×10^{-7} [88]	1.08×10^{-9} [84]
		100	2.47×10^{-17}	3.68×10^{-23}	8.45×10^{-35}	5.78×10^{-48}	1.07×10^{-8} [89]	1.09×10^{-10} [86]
15	3	17	6.72×10^{-11}	3.03×10^{-14}	1.11×10^{-20}	2.50×10^{-28}	4.02×10^{-4} [16]	1.67×10^{-5} [12]
		50	6.52×10^{-13}	5.25×10^{-17}	5.90×10^{-25}	2.30×10^{-34}	3.07×10^{-6} [23]	1.90×10^{-8} [19]
		100	1.15×10^{-14}	2.35×10^{-19}	1.67×10^{-28}	3.89×10^{-39}	4.69×10^{-7} [23]	1.26×10^{-9} [20]
	10	17	9.66×10^{-14}	2.56×10^{-18}	2.54×10^{-27}	2.57×10^{-38}	3.99×10^{-4} [65]	2.88×10^{-6} [43]
		50	1.51×10^{-15}	1.03×10^{-20}	6.47×10^{-31}	2.57×10^{-43}	1.29×10^{-6} [83]	2.78×10^{-9} [79]
		100	3.45×10^{-17}	6.79×10^{-23}	3.46×10^{-34}	1.29×10^{-47}	8.30×10^{-8} [86]	2.27×10^{-10} [83]
	15	17	6.17×10^{-15}	2.48×10^{-20}	6.27×10^{-30}	1.43×10^{-41}	6.35×10^{-5} [95]	1.76×10^{-6} [63]
		50	5.09×10^{-17}	5.13×10^{-22}	1.13×10^{-32}	4.75×10^{-46}	1.27×10^{-7} [117]	1.64×10^{-9} [119]
		100	6.36×10^{-18}	9.02×10^{-24}	1.21×10^{-35}	1.85×10^{-50}	6.96×10^{-7} [118]	1.25×10^{-10} [127]

For these reasons we would strongly recommend the use of the near-exact distributions, particularly because the present availability of fast computers and adequate software facilitates their computational implementation.

Concerning the case of different sample sizes, we were not able to fit the Pearson type I approximation to any case. Actually the authors in [19] have only used this approximation for the case of equal sample sizes. However, it was possible to fit the mixture of two Beta distributions to powers of $\Lambda_5^{1/b}$. But, it happens that although surprisingly enough the numerical solutions of the systems of equations in order to determine the first weight and the common first parameter in the Beta distributions seem to be more stable in this different sample sizes case, when trying to find the best integer value of b we found what seemed to be many local minima. Yet, since when using either the Pearson type I or the mixture of two Beta distributions approximations we equate the moments of $\Lambda_5^{1/b}$ or $\Lambda_5^{1/nb}$ to the moments of these approximations, we cannot find the value for b and the parameters in these approximations in one only single step but we have rather to use a two step iterative process, rendering the whole process of determining the best value of b a very frustrating and time consuming task. These facts, together with the facts that only for $p = 5$ and smaller sample sizes it was possible to find a better value for the measure Δ for the mixture of two Beta distributions than for the near-exact distribution that matches four exact moments, and that the near-exact distributions continue to have a very good performance also for these different sample size cases, for all sample sizes and for all values of p and q, we think that it is indeed much preferable to use the near-exact approximations, which seem to be the only approximations with a very good and consistent performance.

Table 3 – Values of the measure Δ for the near-exact and asymptotic distributions for the l.r.t. statistic to test the equality of q covariance matrices, for different sample sizes n_k

			Near-exact distribution # of exact moments matched				Mixture of Beta dists.	
p	q	n_k(*)	4	6	10	15		
5	5	7(3)19	9.21×10^{-5}	9.27×10^{-6}	1.86×10^{-7}	3.28×10^{-9}	2.83×10^{-5}	[68]
	10	7(3)34	4.10×10^{-5}	3.09×10^{-6}	4.59×10^{-8}	7.10×10^{-10}	1.66×10^{-4}	[212]
	15	7(3)49	2.45×10^{-5}	1.56×10^{-6}	1.77×10^{-8}	1.94×10^{-10}	9.93×10^{-5}	[400]
10	5	12(3)24	8.03×10^{-5}	7.26×10^{-6}	1.56×10^{-7}	3.99×10^{-9}	1.02×10^{-4}	[254]
	10	12(3)39	3.41×10^{-5}	2.38×10^{-6}	3.32×10^{-8}	4.64×10^{-10}	1.96×10^{-4}	[650]
	15	12(3)54	2.02×10^{-5}	1.21×10^{-6}	1.21×10^{-8}	1.11×10^{-10}	1.82×10^{-4}	[1150]
15	5	17(3)29	6.60×10^{-5}	5.50×10^{-6}	1.06×10^{-7}	2.32×10^{-9}	1.25×10^{-4}	[527]
	10	17(3)44	2.81×10^{-5}	1.84×10^{-6}	2.23×10^{-8}	2.60×10^{-10}	6.78×10^{-5}	[1233]
	15	17(3)59	1.69×10^{-5}	9.50×10^{-7}	8.24×10^{-9}	6.32×10^{-11}	2.22×10^{-4}	[2245]
5	5	37(3)49	4.85×10^{-9}	2.97×10^{-10}	3.40×10^{-12}	3.96×10^{-14}	8.68×10^{-9}	[573]
	10	37(3)64	3.74×10^{-9}	1.04×10^{-10}	2.74×10^{-13}	5.56×10^{-16}	7.23×10^{-5}	[352]
	15	37(3)79	3.21×10^{-9}	5.78×10^{-11}	6.66×10^{-14}	4.98×10^{-17}	8.38×10^{-6}	[550]
10	5	42(3)54	3.97×10^{-9}	5.03×10^{-11}	3.74×10^{-14}	1.69×10^{-17}	8.83×10^{-8}	[1358]
	10	42(3)69	4.65×10^{-9}	2.41×10^{-11}	3.61×10^{-15}	2.46×10^{-19}	3.03×10^{-7}	[3148]
	15	42(3)84	4.78×10^{-9}	1.57×10^{-11}	9.21×10^{-16}	2.20×10^{-20}	5.36×10^{-7}	[3288]
15	5	47(3)59	4.22×10^{-9}	1.80×10^{-11}	2.05×10^{-15}	1.13×10^{-19}	1.55×10^{-6}	[2023]
	10	47(3)74	6.79×10^{-9}	1.38×10^{-11}	1.96×10^{-16}	1.29×10^{-21}	2.09×10^{-7}	[4050]
	15	47(3)89	7.41×10^{-9}	1.16×10^{-11}	3.51×10^{-17}	1.09×10^{-23}	8.91×10^{-7}	[7764]

(*) the notation 'i(s)f' for sample sizes, indicates the initial value, the step and the 'final' value, so that for example, 7(3)19 stands for values of $n_1, \ldots, n_5$ equal to 7,10,13,16,19.

In Table 3 we may see the values of Δ for the near-exact distributions which use for λ^* and r the values of λ^* and s_1 in (38), obtained by solving the system of equations in (44) as well as the values of Δ for the mixture of two Beta distributions, with the indication, inside square brackets, of the value of b used, which is the integer value which would give the lowest value of Δ. We may see how the near-exact distributions exhibit a very good performance even for the smaller sample sizes and a clear asymptotic behavior not only for increasing sample sizes but also for increasing values of p and q, and how the near-exact distributions which match four exact moments always outperform the mixture of two Beta distributions except for the smaller sample sizes for $p = 5$, with the measures for this mixture exhibiting a bit erratic behavior.

In Table 4 we have the values of Δ for the near-exact distributions which use the same value of λ^* as the ones in Table 3, but use for r the value given by (43). Although these near-exact distributions give better values than the ones considered previously, for the smaller sample sizes, only for $p = 5$, and for all the larger sample sizes, they have lost their asymptotic character for increasing values of p, the number of variables involved. It seems that these near-exact distributions will be adequate for small values of p as long as the sample sizes considered are rather small or for larger values of sample sizes, for any p.

Table 4 – Values of the measure Δ for a second version of near-exact distributions for the l.r.t. statistic to test the equality of q covariance matrices, for different sample sizes n_k

			# of exact moments matched by the near-exact dist.			
p	q	n_k(*)	4	6	10	15
5	5	7(3)19	2.94×10^{-5}	1.75×10^{-6}	7.20×10^{-8}	1.67×10^{-9}
	10	7(3)34	2.23×10^{-5}	5.41×10^{-7}	1.63×10^{-8}	2.66×10^{-10}
	15	7(3)49	1.59×10^{-5}	2.51×10^{-7}	5.65×10^{-9}	6.41×10^{-11}
5	5	37(3)49	1.76×10^{-10}	1.65×10^{-13}	3.27×10^{-19}	6.24×10^{-26}
	10	37(3)64	2.16×10^{-10}	1.40×10^{-13}	1.12×10^{-19}	6.24×10^{-27}
	15	37(3)79	2.34×10^{-10}	1.36×10^{-13}	7.19×10^{-20}	2.04×10^{-27}
10	5	42(3)54	8.32×10^{-10}	1.14×10^{-12}	4.80×10^{-18}	2.49×10^{-24}
	10	42(3)69	1.30×10^{-9}	1.52×10^{-12}	3.24×10^{-18}	5.12×10^{-25}
	15	42(3)84	1.14×10^{-9}	1.38×10^{-12}	2.39×10^{-18}	2.21×10^{-25}
15	5	47(3)59	1.80×10^{-9}	2.96×10^{-12}	1.65×10^{-17}	1.13×10^{-23}
	10	47(3)74	1.67×10^{-9}	3.35×10^{-12}	1.33×10^{-17}	3.58×10^{-24}
	15	47(3)89	2.92×10^{-9}	2.68×10^{-12}	5.50×10^{-19}	1.42×10^{-24}

(*) please see the note after Table 3.

7. Conclusions

Once we work towards and succeed in giving the exact distribution of the l.r.t. statistics under study a form of the type in (1) we may readily not only show that actually there is a common structure for the exact distribution of these l.r.t. statistics which goes beyond the fact that their exact distribution may be represented as the distribution of the product of independent Beta r.v.'s but we also get a much deeper insight into the true structure of such distributions and are then able to obtain quite simple expressions for the exact distributions of the l.r.t. statistics to test the independence of several groups of variables, the equality of several mean vectors and the nullity of an expected value matrix and to develop very well-fitting near-exact approximations for the l.r.t. statistics to test sphericity and the equality of covariance matrices. These near-exact approximations show very good performances even for very small sample sizes and display an asymptotic behavior not only for increasing sample sizes as well as for increasing number of variables and matrices involved, outperforming by far any other available approximations. Although they may seem at first sight somewhat complicated to implement, with the present availability of fast computers and adequate software, their practical implementation is rendered very simple. Moreover, the numerical determination of their parameters is well defined and numerically stable. Benefiting from all these features, the near-exact approximations can be recommended whenever the exact distributions exhibit a very complicated structure, especially if this structure involves infinite series representations.

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Appendix A. The GIG and GNIG distributions

We will say that the r.v. X has a Gamma distribution with rate parameter $\lambda > 0$ and shape parameter $r > 0$, if its p.d.f. may be written as

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda x} x^{r-1}, \quad (x > 0)$$

and we will denote this fact by

$$X \sim \Gamma(r, \lambda).$$

Let

$$X_j \sim \Gamma(r_j, \lambda_j) \qquad j = 1, \dots, p$$

be p independent r.v.'s with Gamma distributions with shape parameters $r_j \in \mathbb{N}$ and rate parameters $\lambda_j > 0$, with $\lambda_j \neq \lambda_{j'}$, for all $j, j' \in \{1, \dots, p\}$. We will say that then the r.v.

$$Y = \sum_{j=1}^{p} X_j$$

has a GIG (Generalized Integer Gamma) distribution of depth p, with shape parameters r_j and rate parameters λ_j, $(j = 1, \dots, p)$, and we will denote this fact by

$$Y \sim GIG(r_j, \lambda_j;\ p).$$

The p.d.f. and c.d.f. of Y are respectively given by (see [5])

$$f^{GIG}(y \,|\, r_1, \dots, r_p; \lambda_1, \dots, \lambda_p; p) = K \sum_{j=1}^{p} P_j(y)\, e^{-\lambda_j y}, \quad (y > 0)$$

and

$$F^{GIG}(y \,|\, r_1, \dots, r_j; \lambda_1, \dots, \lambda_p; p) = 1 - K \sum_{j=1}^{p} P^*_j(y)\, e^{-\lambda_j y}, \quad (y > 0)$$

where K is given by (5) in Coelho (1998), and $P_j(y)$ and $P^*_j(y)$ are given by (7) and (16) in the same reference.

The GNIG (Generalized Near-Integer Gamma) distribution of depth $p + 1$ (see Coe04) is the distribution of the r.v.

$$Z = Y_1 + Y_2$$

where Y_1 and Y_2 are independent, Y_1 having a GIG distribution of depth p and Y_2 with a Gamma distribution with a non-integer shape parameter r and a rate parameter $\lambda \neq \lambda_j$ ($j = 1, \ldots, p$). The p.d.f. of Z is given by

$$f^{GNIG}(z\,|\,r_1, \ldots, r_p, r;\ \lambda_1, \ldots, \lambda_p, \lambda; p+1) =$$

$$K\lambda^r \sum_{j=1}^{p} e^{-\lambda_j z} \sum_{k=1}^{r_j} \left\{ c_{j,k} \frac{\Gamma(k)}{\Gamma(k+r)} z^{k+r-1} {}_1F_1(r, k+r, -(\lambda-\lambda_j)z) \right\}, \qquad (z > 0)$$

and the c.d.f. given by

$$F^{GNIG}(z\,|\,r_1, \ldots, r_p, r;\ \lambda_1, \ldots, \lambda_p, \lambda; p+1) = \frac{\lambda^r z^r}{\Gamma(r+1)} {}_1F_1(r, r+1, -\lambda z)$$

$$-K\lambda^r \sum_{j=1}^{p} e^{-\lambda_j z} \sum_{k=1}^{r_j} c^*_{j,k} \sum_{i=0}^{k-1} \frac{z^{r+i}\lambda_j^i}{\Gamma(r+1+i)} {}_1F_1(r, r+1+i, -(\lambda-\lambda_j)z), \qquad (z > 0)$$

where

$$c^*_{j,k} = \frac{c_{j,k}}{\lambda_j^k} \Gamma(k)$$

with $c_{j,k}$ given by (11) through (13) in Coelho (1998). In the above expressions ${}_1F_1(a, b; z)$ is the Kummer confluent hypergeometric function. This function typically has very good convergence properties and is nowadays easily handled by a number of software packages.

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