The exact and near-exact distributions of the likelihood ratio test statistic for testing circular symmetry

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In this paper the exact distribution of the logarithm of the likelihood ratio test (l.r.t.) statistic for testing circular symmetry is obtained, for an odd number of variables *p*, in the form of a Generalized Integer Gamma distribution and, for an even *p*, in the form of an infinite mixture of Generalized Near-Integer Gamma distributions. For the case of an even *p*, two kinds of near-exact distributions are developed for the likelihood ratio test statistic which correspond, for the logarithm of the l.r.t. statistic, to a Generalized Near-Integer Gamma distribution or finite mixtures of these distributions. Numerical studies are conducted in order to assess the quality of these new approximations. Tables of exact quantiles, for odd *p*, and near-exact quantiles, for even *p* are presented.

Keywords: near-exact distributions, asymptotic distributions, circular symmetry, Generalized Integer Gamma distribution, Generalized Near-Integer Gamma distribution, mixtures.

AMS Subject Classification: 62H10, 62E20, 62H05, 62H15, 62E15, 62F03

1. Introduction

Testing if a covariance matrix has a circular or circulant symmetric structure is an interesting problem in different areas of research, concerning different kinds of practical applications. For example, in [7] the author makes a review of works involving circulant matrices and gives examples of several practical problems involving these kind of structures in different fields of interest such as “biological sciences, psychometry, quality control, signal detection” as well as in spatial statistics and engineering.

We will show that when the number of variables is odd, the exact distribution of the negative logarithm of the likelihood ratio test statistic (l.r.t.) corresponds to a Generalized Integer Gamma (GIG) distribution (see Coelho [4] and Appendix C) and when the number of variables is even to an infinite mixture of Generalized Near-Integer Gamma (GNIG) distributions (see Coelho [5] and Appendix C). For the case of an even number of variables two kinds of near-exact distributions are developed for the logarithm of the l.r.t. statistic and for the l.r.t. statistic. Numerical studies are conducted in order to analyze the performance of the near-exact distributions developed. These studies show that both kinds of near-exact distributions have very good asymptotic properties and that at the same time are much more precise than the asymptotic approximation given in [10].

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We consider a sample of size N from a p-variate normal population, $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (see [1]) and we are interested in testing the null hypothesis

$$H_0: \ \Sigma = \begin{pmatrix} b_0 & b_1 & \dots & b_{p-1} \\ b_{p-1} & b_0 & \dots & b_{p-2} \\ \vdots & \vdots & & \vdots \\ b_1 & b_2 & \dots & b_0 \end{pmatrix} \quad \text{vs} \quad H_1 : \Sigma > 0 \tag{1}$$

where $b_j = b_{p-j}$ for $j = 1, \ldots, \lfloor p/2 \rfloor$.

In [10] the authors derive the l.r.t. statistic, Λ,

$$\Lambda = 2^{N(p-m-1)} \left(\frac{|\mathbf{V}|}{\prod_{j=1}^{p} v_j} \right)^{N/2} \tag{2}$$

where $\mathbf{V}$ is given in (A1), v_j are given by (A2) and (A3) in Appendix A and

$$m = \left\lfloor \frac{p}{2} \right\rfloor . \tag{3}$$

The expression for the h-th null moment of the l.r.t. statistic, Λ, is given by (see [9, 10])

$$E\left(\Lambda^h\right) = \frac{\Gamma\left(\frac{N-1}{2}\right)^m \Gamma\left(\frac{N}{2}\right)^{p-m-1}}{\prod_{j=1}^{p-1} \Gamma\left(\frac{N-j-1}{2}\right)} \frac{\prod_{j=1}^{p-1} \Gamma\left(\frac{N-j-1}{2} + \frac{Nh}{2}\right)}{\Gamma\left(\frac{N-1}{2} + \frac{Nh}{2}\right)^m \Gamma\left(\frac{N}{2} + \frac{Nh}{2}\right)^{p-m-1}} . \tag{4}$$

From the expression of the h-th null moment of Λ we may easily derive the characteristic function (c.f.) of the random variable (r.v.) $W = -\frac{2}{N} \log \Lambda$, in the following way

$$\begin{aligned} \Phi_W(t) &= E\left[\mathrm{e}^{\mathrm{i}t(-\frac{2}{N}\log\Lambda)}\right] = E\left[\Lambda^{\left(-\frac{2}{N}\mathrm{i}t\right)}\right] \\ &= \frac{\Gamma\left(\frac{N-1}{2}\right)^m \Gamma\left(\frac{N}{2}\right)^{p-m-1}}{\prod_{j=1}^{p-1} \Gamma\left(\frac{N-j-1}{2}\right)} \frac{\prod_{j=1}^{p-1} \Gamma\left(\frac{N-j-1}{2} - \mathrm{i}t\right)}{\Gamma\left(\frac{N-1}{2} - \mathrm{i}t\right)^m \Gamma\left(\frac{N}{2} - \mathrm{i}t\right)^{p-m-1}} . \end{aligned} \tag{5}$$

We will use a factorization the c.f. of W in (5) to derive the exact distribution of both W and $\Lambda^{2/N}$ for any value of p as well as near-exact distributions for even p.

2. The exact distribution of $\Lambda^{2/N}$

The next results show that the exact distribution of the logarithm of $\Lambda^{2/N}$, when the number of variables is odd, corresponds to a GIG distribution and when the number of variables is even corresponds to an infinite mixture of GNIG distributions. In [9], using a different approach based on the inverse Mellin transform and the residue theorem, the authors derived the exact distribution of $\Lambda^{2/N}$ in series form, which for odd p even yields a finite representation. However, with our approach we will be able to obtain simpler explicit expressions for the exact p.d.f. (probability density function) and c.d.f. (cumulative distribution function) of $\Lambda^{2/N}$, which for odd p allow for an easy direct computation of exact quantiles and p-values and which for even p allow for an easy development of very accurate and manageable near-exact distributions. These near-exact distributions allow for an easy computation of near-exact quantiles, which, given their high closeness to the exact ones, may be used instead of these ones.

2.1. *The case of an odd number of variables*

Let us start with case where p is odd.

Theorem 2.1 : *When p is odd, the c.f. of the r.v. $W = -\frac{2}{N}\log\Lambda$ may be written as*

$$\Phi_W(t) = \prod_{j=2}^{p} \theta_j^{r_j}\left(\theta_j - \mathrm{i}t\right)^{-r_j} \tag{6}$$

with

$$r_j = 1 + \left\lfloor \frac{p-j}{2} \right\rfloor \quad \textit{and} \quad \theta_j = \frac{N}{2} - \frac{j}{2}\,. \tag{7}$$

Proof : See Appendix B □

The c.f. of W in (6) corresponds to the c.f. of the sum of $p-1$ independent Gamma r.v.'s with integer shape parameters r_j and rate parameters θ_j, all different, that is, corresponds to a GIG distribution with depth $p-1$, with those shape and rate parameters. The p.d.f. and c.d.f. of $\Lambda^{2/N}$ are thus obtained by simple transformation and they are given in the next Corollary.

Corollary 2.2: *When p is odd, using the notation in Appendix C, the exact p.d.f. of $\Lambda^{2/N}$ is*

$$f^{GIG}\left(-\log z \mid r_2,\ldots,r_p\,;\;\theta_2,\ldots,\theta_p\,;\;p-1\right)\frac{1}{z} \tag{8}$$

and the c.d.f. is given by

$$1 - F^{GIG}\left(-\log z \mid r_2,\ldots,r_p\,;\;\theta_2,\ldots,\theta_p\,;\;p-1\right), \tag{9}$$

with r_j and θ_j given in (7) *and where* $0 < z < 1$ *represents the running value of the statistic* $\Lambda^{2/N} = e^{-W}$.

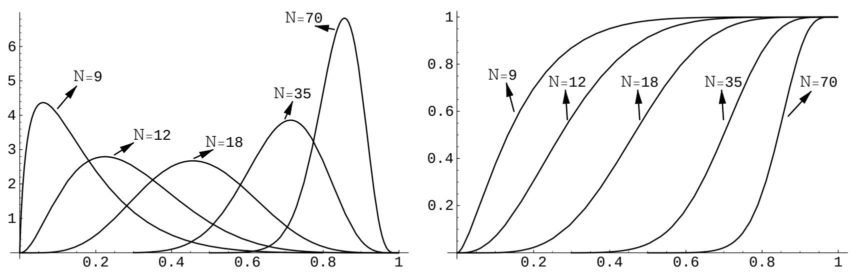
Proof : We only need to consider the relation

$$F_{\Lambda^{2/N}}(z) = 1 - F_W(-\log z)$$

where $F_{\Lambda^{2/N}}(\cdot)$ is the cumulative distribution function of $\Lambda^{2/N}$ and $F_W(\cdot)$ is the cumulative distribution function of $W = -\frac{2}{N}\log\Lambda$. □

In Figure 1 we present some examples of plots for p.d.f.'s and c.d.f.'s of $\Lambda^{2/N}$ for $p = 5$ and for different values of N.

Figure 1. Plots for the exact p.d.f.'s and c.d.f.'s of $\Lambda^{2/N}$ for p=5



These results confirm the results obtained in [2] and enable us to easily compute exact quantiles for virtually any combination of values of p and N. In Tables D1–D6 of Appendix D we present the exact quantiles of $\Lambda^{2/N}$ for different values of $p \geq 8$ and N, which complement the quantiles given in [9].

2.2. *The case of an even number of variables*

Let us now consider the case where the number of variables is even.

Theorem 2.3 : *When p is even the c.f. of $W = -\frac{2}{N}\log\Lambda$ may be written as*

$$\Phi_W(t) = \underbrace{\prod_{j=2}^{p} \theta_j^{r_j^*}\left(\theta_j - \mathrm{i}t\right)^{-r_j^*}}_{\Phi_{W_1}(t)} \times \underbrace{\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N}{2}-1-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-1\right)\Gamma\left(\frac{N-1}{2}-\mathrm{i}t\right)}}_{\Phi_{W_2}(t)} \tag{10}$$

where

$$r_j^* = \begin{cases} \frac{p-2}{2}, & j=2 \\ r_j, & j=3,\ldots,p \end{cases} \tag{11}$$

with θ_j and r_j given by (7).

Proof : See Appendix B. □

In expression (10) we have that $\Phi_{W_1}(t)$ is the c.f. of a GIG distribution with deph $p-1$ and $\Phi_{W_2}(t)$ is the c.f. of a single r.v. with a Logbeta distribution with parameters $\frac{N}{2}-1$ and $\frac{1}{2}$.

Using the results in [12] we will show how is it possible to write the exact distribution of $\Lambda^{2/N}$ as an infinite mixture of GNIG distributions. From the two first expressions in section 5 of [12] and expressions (11) and (14) in the same paper, we may write

$$\frac{\Gamma(a-it)}{\Gamma(a+b-it)} = \sum_{k=0}^{\infty} p_k(b)(a-it)^{-b-k} \tag{12}$$

where $p_k(b)$ is a polynomial of degree k in b with

$$p_k(b) = \frac{1}{k}\sum_{m=0}^{k-1}\left(\frac{\Gamma(1-b-m)}{\Gamma(-b-k)(k-m+1)!} + (-1)^{k+m}b^{k-m+1}\right)p_m(b) \tag{13}$$

where

$$p_0(b) = 1. \tag{14}$$

Then $\Phi_{W_2}(t)$ in (10) may be written as

$$\Phi_{W_2}(t) = \frac{\Gamma\left(\frac{N}{2}-1+\frac{1}{2}\right)\Gamma\left(\frac{N}{2}-1-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-1\right)\Gamma\left(\frac{N}{2}-1+\frac{1}{2}-\mathrm{i}t\right)} \tag{15}$$

$$= \sum_{k=0}^{\infty} p_k^* \left(\frac{N}{2}-1\right)^{\frac{1}{2}+k}\left(\frac{N}{2}-1-it\right)^{-\left(\frac{1}{2}+k\right)} \tag{16}$$

with

$$p_k^* = \frac{\Gamma\left(\frac{N}{2}-1+\frac{1}{2}\right)}{\Gamma\left(\frac{N}{2}-1\right)} \frac{p_k\left(\frac{1}{2}\right)}{\left(\frac{N}{2}-1\right)^{\frac{1}{2}+k}}\,, \tag{17}$$

what shows that the c.f. $\Phi_{W_2}(t)$ may be written as the c.f. of an infinite mixture of Gamma distributions with shape parameters $\frac{1}{2}+k$, all with the same rate parameter $\frac{N}{2}-1$, with weights p_k^* with p_k given by (13) and (14). Then, the c.f. of W in (10) may be written as

$$\begin{aligned}\Phi_W(t) &= \underbrace{\prod_{j=2}^{p} \theta_j^{r_j^*}\left(\theta_j - \mathrm{i}t\right)^{-r_j^*}}_{\Phi_{W_1}(t)} \underbrace{\sum_{k=0}^{\infty} p_k^* \left(\frac{N}{2}-1\right)^{\frac{1}{2}+k} \left(\frac{N}{2}-1-\mathrm{i}t\right)^{-(\frac{1}{2}+k)}}_{\Phi_{W_2}(t)} \\ &= \sum_{k=0}^{\infty} p_k^* \left\{ \prod_{j=2}^{p} \theta_j^{r_j^*}\left(\theta_j - \mathrm{i}t\right)^{-r_j^*} \left(\frac{N}{2}-1\right)^{\frac{1}{2}+k} \left(\frac{N}{2}-1-\mathrm{i}t\right)^{-(\frac{1}{2}+k)} \right\},\end{aligned}$$

which is the c.f. of an infinite mixture of Generalized Near-Integer Gamma distributions with depth $p-1$. We thus have the following Corollary.

Corollary 2.4: *When p is even, using the notation in Appendix C, the exact p.d.f. of* $\Lambda^{2/N}$ *is given by*

$$\sum_{k=0}^{\infty} p_k^* f^{GNIG}\left(-\log z \,|\, r_3^*,\ldots,r_p^*, r_2^*+\frac{1}{2}+k\,;\; \theta_3,\ldots,\theta_p, \frac{N}{2}-1\,;\; p-1\right)\frac{1}{z} \tag{18}$$

and the c.d.f. by

$$1-\sum_{k=0}^{\infty} p_k^* F^{GNIG}\left(-\log z \,|\, r_3^*,\ldots,r_p^*, r_2^*+\frac{1}{2}+k\,;\; \theta_3,\ldots,\theta_p, \frac{N}{2}-1\,;\; p-1\right) \tag{19}$$

with r_j^* *and* θ_j *given respectively by* (11) *and* (7), *and* p_k^* *given by* (17) *and where* $0<z<1$ *represents the running value of the statistic* $\Lambda^{2/N} = e^{-W}$.

Proof: The proof is similar to the one of Corollary 2.2. □

3. Near-exact distributions for *W* when the number of variables is even

The exact c.d.f. of $\Lambda^{2/N}$, when p is even, is given in expression (19) in the form of an infinite mixture of GNIG distributions, what makes its use a bit hard in practical terms. For this reason we will now derive two kinds of near-exact distributions for W and for $\Lambda^{2/N}$, which will be asymptotic both for n and p and which, by construction, will equate a given number of the first exact moments.

3.1. *Near-exact distributions - Type I*

The c.f.'s of these near-exact distributions will have the following form

$$\underbrace{\Phi_{W_1}(t)}_{\text{GIG dist.}} \times \Phi_{M_G}(t)\,, \tag{20}$$

where $\Phi_{W_1}(t)$ is given in (10), while $\Phi_{M_G}(t)$, for this kind of near-exact distributions, may be either the c.f. of a single Gamma distribution or of a mixture of two or three Gamma distributions, depending on the number of exact moments we want to match. The c.f. $\Phi_{M_G}(t)$ will have, accordingly, the same 2, 4 or 6 first derivatives (with respect to t at

$t = 0$) as $\Phi_{W_2}(t)$ in (10), that is

$$\frac{d^j}{dt^j}\,\Phi_{M_G}(t)\bigg|_{t=0} = \frac{d^j}{dt^j}\,\Phi_{W_2}(t)\bigg|_{t=0}, \quad j = 1, \ldots, h \tag{21}$$

for $h = 2, 4$ or 6, according to the case of $\Phi_{M_G}(t)$ being the c.f. of a single Gamma distribution, or the c.f. of a mixture of 2 or 3 Gamma distributions with the same rate parameter, that is,

$$\Phi_{M_G}(t) = \sum_{k=1}^{h/2} \omega_k\, \lambda^{s_k}\, (\lambda - \mathrm{i}t)^{-s_k}\,, \tag{22}$$

with weights $\omega_k > 0$ ($k = 1, \ldots, h/2$) and $\sum_{k=1}^{h/2} \omega_k = 1$.

Since as shown in (16), $\Phi_{W_2}(t)$ may be seen as the c.f. of an infinite mixture of Gamma distributions, all with the same rate parameter, as already remarked in [8], the replacement of a Logbeta random variable by a single Gamma distribution or a mixture of two or three Gamma distributions, all with the same rate parameter, matching the first 2, 4 or 6 exact moments is a much adequate decision.

This allows us to write the near-exact c.f. of the negative logarithm of the l.r.t. statistic in the form in (20) with $\Phi_{M_G}(t)$ given by (22), being thus the near-exact distributions obtained in this way, either a GNIG distribution, or a mixture of two or three GNIG distributions, which have very manageable expressions, allowing this way for an easy computation of very accurate near-exact quantiles.

Theorem 3.1 : *If we replace $\Phi_{W_2}(t)$ in* (10) *by $\Phi_{M_G}(t)$ given by* (22)*, we obtain, by simple transformation, near-exact distributions for $\Lambda^{2/N}$ with p.d.f.'s given by*

$$\sum_{\nu=1}^{h/2} \omega_\nu\, f^{\,GNIG}\Big(-\log z \;\mid\; r_2^*, \ldots, r_p^*, s_\nu; \theta_2, \ldots, \theta_p, \lambda\,;\; p\Big)\frac{1}{z} \tag{23}$$

and c.d.f.'s

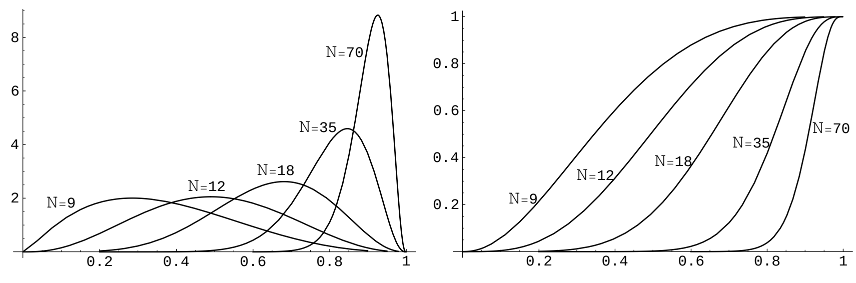
$$1 - \sum_{\nu=1}^{h/2} \omega_\nu\, F^{\,GNIG}\Big(-\log z \;\mid\; r_2^*, \ldots, r_p^*, s_\nu\,;\; \theta_2, \ldots, \theta_p, \lambda\,;\; p\Big) \tag{24}$$

where $0 < z < 1$ represents the running value of the statistic $\Lambda^{2/N} = e^{-W}$, with r_j^* given in (11) *and θ_j in* (7)*. The values of ω_ν, s_ν and λ are obtained from the numerical solution of the system of equations in* (21)*, with $\omega_{h/2} = 1 - \sum_{k=1}^{h/2-1} \omega_k$, for $h = 2, 4$ or 6 (according to the case of $\Phi_{M_G}(t)$ being the c.f. of a single Gamma distribution, or the c.f. of a mixture of 2 or 3 Gamma distributions with the same rate parameter).*

Proof : See Appendix B. □

In Figure 2 we present some examples of plots for c.d.f.'s and p.d.f.'s of the type I near-exact distributions for $\Lambda^{2/N}$ when $h = 6$, $p = 4$ and for different values of N.

Figure 2. Plots for the near-exact p.d.f.'s and c.d.f.'s of $\Lambda^{2/N}$ for $h = 6$ and p=4



3.2. *Near-exact distributions - Type II*

The c.f.'s of these near-exact distributions will have the form

$$\underbrace{\Phi_{W_1}(t)}_{\text{GIG dist.}} \times \Phi^*_{M_G}(t)\,. \tag{25}$$

In this case, $\Phi^*_{M_G}(t)$ will have a similar structure to the representation of $\Phi_{W_2}(t)$ in (16), and as such it will be the c.f. of a mixture of $m^* + 1$ Gamma distributions, all with rate parameters $N/2 - 1$ and with shape parameters $1/2 + k$ with $k = 0, \ldots, m^*$. That is, we will take

$$\Phi^*_{M_G}(t) = \sum_{k=0}^{m^*} \omega^*_k \left(\frac{N}{2} - 1\right)^{\frac{1}{2}+k} \left(\frac{N}{2} - 1 - \mathrm{i}t\right)^{-\frac{1}{2}-k}, \tag{26}$$

with

$$\left.\frac{d^j}{dt^j}\,\Phi^*_{M_G}(t)\right|_{t=0} = \left.\frac{d^j}{dt^j}\,\Phi_{W_2}(t)\right|_{t=0}, \quad j = 1, \ldots, m^*\,, \tag{27}$$

being thus the weights ω^*_k ($k = 0, \ldots, m^* - 1$) in (26) determined in such a way that $\Phi^*_{M_G}(t)$ has the same first m^* derivatives (with respect to t at $t = 0$) as $\Phi_{W_2}(t)$, with $\sum_{k=0}^{m^*} \omega^*_k = 1$ and where the expression for $\Phi_{W_2}(t)$ to be used is the one in (15).

We should note that, since the system of equation in (26) has in every case a unique solution and as such is very easy to determine, using this approach we may consider in $\Phi^*_{M_G}(t)$ a mixture of Gamma distributions with as many terms as we wish, that is, we may obtain near-exact distributions which equate as many of the exact moments as we wish.

The near-exact c.f.'s for the negative logarithm of the l.r.t. statistic will have the form in (25) with $\Phi^*_{M_G}(t)$ given by (26), being thus the near-exact distributions obtained in this way, mixtures of $m^* + 1$ GNIG distributions. The type II near-exact distributions for $\Lambda^{2/N}$ are given in the following Theorem.

Theorem 3.2 : *If we replace* $\Phi_{W_2}(t)$ *given in* (15) *by* $\Phi^*_{M_G}(t)$ *in* (26) *we obtain near-exact distributions for* $\Lambda^{2/N}$ *with c.d.f.'s given by*

$$1 - \sum_{k=0}^{m^*} \omega^*_k F^{GNIG}\left(-\log z \mid r^*_3, \ldots, r^*_p, r^*_2 + \frac{1}{2} + k\,;\ \theta_3, \ldots, \theta_p, \frac{N}{2} - 1\,;\ p - 1\right) \tag{28}$$

and p.d.f.'s

$$\sum_{k=0}^{m^*} \omega^*_k f^{GNIG}\left(-\log z \mid r^*_3, \ldots, r^*_p, r^*_2 + \frac{1}{2} + k\,;\ \theta_3, \ldots, \theta_p, \frac{N}{2} - 1\,;\ p - 1\right)\frac{1}{z} \tag{29}$$

where $0 < z < 1$ *represents the running value of the statistic* $\Lambda^{2/N} = e^{-W}$, *with* r^*_j *given in* (11) *and* θ_j *in* (7) *and the values of* ω^*_k *are obtained from the numerical solution of the system of equations in* (27), *that is*

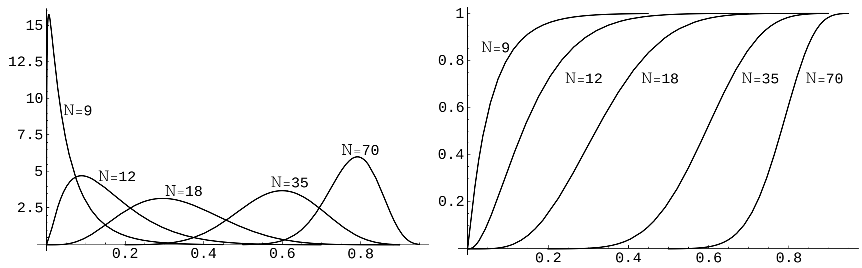
$$\left.\frac{d^j}{dt^j}\,\Phi^*_{M_G}(t)\right|_{t=0} = \left.\frac{d^j}{dt^j}\,\Phi_{W_2}(t)\right|_{t=0}, \quad j = 1, \ldots, m^* \quad \text{with} \quad \omega^*_{m^*} = 1 - \sum_{k=0}^{m^*-1} \omega^*_k\,.$$

Proof : Similar to the proof of Theorem 3.1. □

In Figure 3 we present some examples of plots for c.d.f.'s and p.d.f.'s of the type II near-exact distributions for $\Lambda^{2/N}$ when $p = 6$ and $m^* = 4$, for different values of N.

In Tables E1–E7 of Appendix E we present near-exact quantiles of $\Lambda^{2/N}$ for different values of p and N. In these Tables we used the type II near-exact distribution which,

Figure 3. Plots for the near-exact p.d.f.'s and c.d.f.'s of $\Lambda^{2/N}$ for p=6 and $m^* = 4$



matching the least number of exact moments, also matches at least 10 significant decimal digits of the corresponding exact quantile. We have chosen to use the type II near-exact distributions because they are more manageable, in the sense that we can match as many exact moments as we wish, and because, as we will see in the next Section, these kind of near-exact distributions reveals a higher level of precision than the type I near-exact distributions, when the same number of exact moments is matched.

4. Numerical studies

In order to evaluate the quality of the near-exact approximations developed in this work, for the case of an even number of variables, we will use a measure of proximity between c.f.'s which is also a measure of proximity between c.d.f.'s. This measure is,

$$\Delta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\Phi_W(t)-\Phi_{ne}(t)}{t}\right| \mathrm{d}t\,, \tag{30}$$

where $\Phi_W(t)$ represents the exact c.f. of the negative logarithm of the l.r.t. statistic and $\Phi_{ne}(t)$ represents an approximate c.f. for the same statistic. Taking S for the support of W, we have,

$$\max_{w\in S}|F_W(w)-F_{ne}(w)| \;\leq\; \Delta\,, \tag{31}$$

where $F_W(\cdot)$ represents the exact c.d.f. of W and $F_{ne}(\cdot)$ represents the c.d.f. corresponding to $\Phi_{ne}(t)$.

The relation in (31) may be derived directly from inversion formulas and was already used in [6, 8] to study the accuracy of near-exact approximations.

We will compute the values of the measure Δ between the exact distribution of $W = -\frac{2}{N}\log\Lambda$ and the two kinds of near-exact distributions proposed, as well as between the exact distribution and the asymptotic approximation obtained in [10] using a Box style procedure (see [3]). In these calculations we will use the exact c.f. in (5) and the near-exact c.f.'s in (20) and (22), corresponding to the type I near-exact distributions given in Theorem 3.1, for $h = 2, 4$ or 6, and the near-exact c.f.'s in (25) and (26), corresponding to the type II near-exact distributions in Theorem 3.2, for $m^* = 2, 4, 6$ or 12. We will denote respectively by "*GNIG*", "*M*2*GNIG*" and "*M*3*GNIG*" the type I near-exact distributions corresponding to $h = 2, 4$ and 6 and we will denote by "*M***K***GNIG**", the type II near-exact distributions corresponding to the mixture of **K** GNIG distributions, with $\mathbf{K} = 3, 5, 7$ or 13, by taking respectively $m^* = 2, 4, 6$ or 12 in (25) and (26). We will also denote, in our Tables, the asymptotic approximation obtained in [10] by "*Box*".

From Tables 1 and 2 we may observe that the values of Δ for both kinds of near-exact distributions are, in every case considered, substantially smaller than the ones for the asymptotic approximation presented in [10]. Moreover, the near-exact approximations reveal good asymptotic properties not only for increasing values of N but also for increasing values of p. We may also see that the type I near-exact distributions have larger values of

Table 1. Values of Δ for the approximating distributions for $W = -\frac{2}{N}\log\Lambda$

		Measure Δ							
p	N	*GNIG*	*M2GNIG*	*M3GNIG*	*M**3**GNIG**	*M**5**GNIG**	*M**7**GNIG**	*M**13**GNIG**	*Box*
8	10	1.4×10^{-6}	2.6×10^{-9}	1.3×10^{-11}	1.2×10^{-7}	1.0×10^{-10}	1.9×10^{-13}	4.5×10^{-18}	8.5×10^{-2}
10	12	3.4×10^{-7}	3.2×10^{-10}	4.5×10^{-13}	2.5×10^{-8}	9.8×10^{-12}	3.1×10^{-15}	3.2×10^{-21}	1.4×10^{-1}
12	14	1.1×10^{-7}	5.9×10^{-11}	2.7×10^{-14}	7.2×10^{-9}	1.4×10^{-12}	5.0×10^{-16}	8.7×10^{-23}	2.1×10^{-1}
14	16	4.4×10^{-8}	1.4×10^{-11}	2.0×10^{-15}	2.5×10^{-9}	2.5×10^{-13}	6.6×10^{-17}	1.6×10^{-24}	2.7×10^{-1}
16	18	2.0×10^{-8}	4.1×10^{-12}	1.7×10^{-15}	1.0×10^{-9}	5.8×10^{-14}	9.9×10^{-18}	4.0×10^{-26}	3.5×10^{-1}
18	20	9.9×10^{-9}	1.4×10^{-12}	1.3×10^{-17}	4.6×10^{-10}	1.6×10^{-14}	1.8×10^{-18}	1.3×10^{-27}	4.3×10^{-1}
20	22	5.3×10^{-9}	5.3×10^{-13}	7.6×10^{-17}	2.3×10^{-10}	5.0×10^{-15}	3.8×10^{-19}	5.9×10^{-29}	5.1×10^{-1}
22	24	3.1×10^{-9}	2.2×10^{-13}	4.4×10^{-17}	1.2×10^{-10}	1.7×10^{-15}	9.1×10^{-20}	3.3×10^{-30}	5.9×10^{-1}
24	26	1.9×10^{-9}	1.0×10^{-13}	2.5×10^{-18}	6.7×10^{-11}	6.7×10^{-16}	2.5×10^{-20}	2.2×10^{-31}	6.7×10^{-1}
30	32	5.2×10^{-10}	1.3×10^{-14}	2.3×10^{-19}	1.5×10^{-11}	5.7×10^{-17}	8.5×10^{-22}	1.4×10^{-34}	9.0×10^{-1}
50	52	2.9×10^{-11}	1.4×10^{-16}	6.5×10^{-22}	5.3×10^{-13}	2.2×10^{-19}	3.6×10^{-25}	1.4×10^{-40}	1.5×10^{0}
100	102	6.6×10^{-13}	3.2×10^{-19}	1.9×10^{-25}	6.0×10^{-15}	1.2×10^{-22}	1.1×10^{-29}	9.2×10^{-49}	2.9×10^{0}

Table 2. Values of Δ for the approximating distributions for $W = -\frac{2}{N}\log\Lambda$

		Measure Δ							
p	N	*GNIG*	*M2GNIG*	*M3GNIG*	*M**3**GNIG**	*M**5**GNIG**	*M**7**GNIG**	*M**13**GNIG**	*Box*
8	10	1.4×10^{-6}	2.6×10^{-9}	1.3×10^{-11}	1.2×10^{-7}	1.0×10^{-10}	1.9×10^{-13}	4.5×10^{-18}	8.5×10^{-2}
8	15	1.1×10^{-6}	2.5×10^{-9}	3.1×10^{-12}	6.1×10^{-8}	4.4×10^{-11}	6.7×10^{-14}	2.3×10^{-19}	7.1×10^{-3}
8	20	6.6×10^{-7}	1.4×10^{-9}	4.1×10^{-13}	2.9×10^{-8}	1.3×10^{-11}	1.8×10^{-14}	1.1×10^{-20}	1.8×10^{-3}
8	30	3.1×10^{-7}	5.0×10^{-10}	4.9×10^{-13}	9.0×10^{-9}	2.0×10^{-12}	1.5×10^{-15}	2.2×10^{-23}	3.6×10^{-4}
8	50	1.1×10^{-7}	1.2×10^{-10}	1.2×10^{-13}	2.0×10^{-9}	1.6×10^{-13}	4.8×10^{-17}	5.1×10^{-26}	5.7×10^{-5}
8	100	2.8×10^{-8}	1.6×10^{-11}	9.8×10^{-15}	2.5×10^{-10}	5.0×10^{-15}	3.8×10^{-19}	1.2×10^{-29}	5.8×10^{-6}

Δ when compared with the values of the type II near-exact distributions that match the same number of exact moments, that is, when we compare the values of Δ for the *GNIG*, *M2GNIG* and *M3GNIG* distributions with the values of Δ for the *M**3**GNIG**, *M**5**GNIG** and *M**7**GNIG** distributions. We should however note that when the two kinds of near-exact distributions equate the same number of exact moments, the type I near-exact distributions have a much simpler structure, with less terms in the mixture. The system of equations (27) used to evaluate the parameters of the type II near-exact distributions is much simpler to solve than the one in (21) and it has always a unique solution which makes possible to match as many exact moments as we wish.

In Tables 3–6 we present values of the tail probability error, that is, the absolute value of the difference between the approximating probability and exact probability (α), we also present the approximating quantiles - with all the decimal places, except the last one, equal to the exact ones - for the near-exact distributions developed in this paper and for the approximation based on Box (1949) method given in [10], for $p = 8, 12, 16, 20$ and $N = p + 1, 50, 100$ and also $N = 200$, only for $p = 16$ and $p = 20$. In order to evaluate the approximating probabilities in Tables 3–6 we have determined the exact quantiles, for the cases considered, from the numerical inversion of the c.f. of $-\frac{2}{N}\log\Lambda$ by using the Gil-Pelaez inversion formulas (see [11]) and the bisection method, which is not a practical method to use especially if we want to work with a considerable precision, owing to the very long computation times involved. The approximating probability is then the value given by the approximating c.d.f.'s at the exact quantile. Therefore in Tables 3–6 we have, for each choice of p and N, in the first line the difference between the approximating probability and exact probability ($\alpha = 0.01$ or 0.05), and in the second line, for the same choice of N, the approximating quantile. From these Tables we may draw, from the tail probability error and from the near-exact quantiles, similar conclusions that we have reached by observing Tables 1 and 2, namely concerning the asymptotic characteristics in terms of sample size and number of variables. Once again, we have good asymptotic properties for both kinds of the near-exact distributions and we may also see that the type I near-exact distributions present a slightly higher tail error probability when compared we the type II near-exact distributions that equate the same number of exact moments.

Table 3. Values of the tail probability error = |approx. probability - α| and values for the approximating quantiles (app. quantile) of $\Lambda^{2/N}$ with the number of decimal places equal to the exact ones for $p = 8$

			$\alpha = 0.01$		
N		*M*3*GNIG**	*M*5*GNIG**	*M*7*GNIG**	*M*13*GNIG**
9	tail error	4.2×10^{-9}	6.8×10^{-13}	1.2×10^{-15}	1.8×10^{-21}
	app. quantile	4.686555×10^{-8}	$4.6865592237 \times 10^{-8}$	$4.686559223097 \times 10^{-8}$	$4.686559223098218506 \times 10^{-8}$
50	tail error	4.9×10^{-10}	3.6×10^{-14}	9.9×10^{-18}	4.3×10^{-23}
	app. quantile	$3.19134871 \times 10^{-1}$	$3.191348725282 \times 10^{-1}$	$3.191348725283870 \times 10^{-1}$	$3.191348725283871055798 \times 10^{-1}$
100	tail error	6.2×10^{-11}	1.2×10^{-15}	8.5×10^{-20}	4.9×10^{-31}
	app. quantile	$5.797822350 \times 10^{-1}$	$5.79782235171871 \times 10^{-1}$	$5.797822351718742427 \times 10^{-1}$	$5.797822351718742429860847541 1 \times 10^{-1}$
			$\alpha = 0.05$		
9	tail error	1.4×10^{-8}	2.3×10^{-12}	3.4×10^{-15}	4.0×10^{-20}
	app. quantile	1.2515632×10^{-6}	$1.2515639611 \times 10^{-6}$	$1.25156396122 \times 10^{-6}$	$1.251563961230040475 \times 10^{-6}$
50	tail error	3.3×10^{-11}	6.9×10^{-16}	1.3×10^{-18}	1.7×10^{-24}
	app. quantile	$3.737120292 \times 10^{-1}$	$3.737120294907 \times 10^{-1}$	$3.7371202949080183 \times 10^{-1}$	$3.737120294908018405200 33 \times 10^{-1}$
100	tail error	2.8×10^{-10}	2.6×10^{-14}	6.8×10^{-21}	3.6×10^{-30}
	app. quantile	$6.2511360441 \times 10^{-1}$	$6.25113604433524 \times 10^{-1}$	$6.2511360443352521341 \times 10^{-1}$	$6.2511360443352521340841979061 4 \times 10^{-1}$

			$\alpha = 0.01$		
N		*GNIG*	*M2GNIG*	*M3GNIG*	*Box*
9	tail error	4.1×10^{-8}	1.3×10^{-11}	1.1×10^{-14}	9.9×10^{-3}
	app. quantile	4.68652×10^{-8}	$4.68655921 \times 10^{-8}$	$4.68655922308 \times 10^{-8}$	————
50	tail error	2.4×10^{-8}	1.4×10^{-11}	4.3×10^{-15}	9.1×10^{-7}
	app. quantile	3.1913480×10^{-1}	$3.1913487256 \times 10^{-1}$	$3.1913487252837 \times 10^{-1}$	3.1916×10^{-1}
100	tail error	6.1×10^{-9}	1.9×10^{-12}	5.1×10^{-16}	1.0×10^{-5}
	app. quantile	5.7978221×10^{-1}	$5.79782235176 \times 10^{-1}$	$5.79782235171872 \times 10^{-1}$	5.79784×10^{-1}
			$\alpha = 0.05$		
9	tail error	1.4×10^{-7}	2.7×10^{-11}	1.2×10^{-13}	4.8×10^{-2}
	app. quantile	1.25155×10^{-6}	$1.251563962 \times 10^{-6}$	$1.25156396122 \times 10^{-6}$	————
50	tail error	6.0×10^{-9}	4.3×10^{-12}	2.4×10^{-15}	2.8×10^{-5}
	app. quantile	3.7371200×10^{-1}	$3.737120295 \times 10^{-1}$	$3.737120294907 \times 10^{-1}$	3.7373×10^{-1}
100	tail error	2.4×10^{-8}	-3.3×10^{-11}	2.8×10^{-14}	2.7×10^{-6}
	app. quantile	$6.25113600 \times 10^{-1}$	$6.25113604436 \times 10^{-1}$	$6.25113604433523 \times 10^{-1}$	6.25115×10^{-1}

Table 4. Values of the tail probability error = |approx. probability - α| and values for the approximating quantiles (app. quantile) of $\Lambda^{2/N}$ with the number of decimal places equal to the exact ones for $p = 12$

		$\alpha = 0.01$			
N		$M\mathbf{3}GNIG^{*}$	$M\mathbf{5}GNIG^{*}$	$M\mathbf{7}GNIG^{*}$	$M\mathbf{13}GNIG^{*}$
13	tail error	2.6×10^{-10}	1.1×10^{-14}	4.8×10^{-19}	6.0×10^{-27}
	app. quantile	$5.2608253 \times 10^{-10}$	$5.26082563311 \times 10^{-10}$	$5.2608256331063615 \times 10^{-10}$	$5.26082563310636103177993 \times 10^{-10}$
50	tail error	1.4×10^{-10}	3.5×10^{-15}	2.8×10^{-19}	4.0×10^{-29}
	app. quantile	$1.00633144 \times 10^{-1}$	$1.00633145162002 \times 10^{-1}$	$1.006331451620065703 \times 10^{-1}$	$1.006331451620065707536966603 \times 10^{-1}$
100	tail error	1.9×10^{-11}	1.4×10^{-16}	3.4×10^{-21}	2.0×10^{-32}
	app. quantile	$3.4067949402 \times 10^{-1}$	$3.40679494063601 \times 10^{-1}$	$3.40679494063602016781 \times 10^{-1}$	$3.406794940636020167880527881306 \times 10^{-1}$
		$\alpha = 0.05$			
13	tail error	8.6×10^{-10}	2.7×10^{-14}	5.1×10^{-18}	9.9×10^{-26}
	app. quantile	1.43203×10^{-8}	$1.43204002485 \times 10^{-8}$	$1.4320400248600756 \times 10^{-8}$	$1.43204002486007531381658 \times 10^{-8}$
50	tail error	1.4×10^{-10}	7.1×10^{-15}	9.8×10^{-19}	9.7×10^{-30}
	app. quantile	$1.26083992 \times 10^{-1}$	$1.2608399304088 \times 10^{-1}$	$1.260839930408908825 \times 10^{-1}$	$1.2608399304089088293035045 \times 10^{-1}$
100	tail error	1.7×10^{-11}	2.4×10^{-16}	8.5×10^{-21}	1.3×10^{-32}
	app. quantile	$3.7858770382 \times 10^{-1}$	$3.785877038373587 \times 10^{-1}$	$3.7858770383735886710 \times 10^{-1}$	$3.78587703837358867110747571647458 \times 10^{-1}$

		$\alpha = 0.01$			
N		$GNIG$	$M2GNIG$	$M3GNIG$	Box
13	tail error	3.5×10^{-9}	3.9×10^{-13}	2.9×10^{-17}	1.0×10^{-2}
	app. quantile	5.260821×10^{-10}	$5.260825632 \times 10^{-10}$	$5.26082563310633 \times 10^{-10}$	———
50	tail error	6.9×10^{-9}	1.2×10^{-12}	1.4×10^{-16}	6.9×10^{-5}
	app. quantile	1.0063313×10^{-1}	$1.00633145163 \times 10^{-1}$	$1.006331451620067 \times 10^{-1}$	1.007×10^{-1}
100	tail error	1.9×10^{-9}	1.9×10^{-13}	5.7×10^{-18}	5.3×10^{-6}
	app. quantile	$3.40679490 \times 10^{-1}$	$3.406794940639 \times 10^{-1}$	$3.4067949406360202 \times 10^{-1}$	3.4069×10^{-1}
		$\alpha = 0.05$			
13	tail error	1.2×10^{-8}	6.4×10^{-13}	3.8×10^{-16}	5.0×10^{-2}
	app. quantile	1.43203×10^{-8}	$1.43204002489 \times 10^{-8}$	$1.43204002486005 \times 10^{-8}$	———
50	tail error	8.5×10^{-9}	4.8×10^{-12}	1.7×10^{-15}	2.0×10^{-4}
	app. quantile	1.2608398×10^{-1}	$1.26083993042 \times 10^{-1}$	$1.260839930408902 \times 10^{-1}$	1.261×10^{-1}
100	tail error	2.2×10^{-9}	7.0×10^{-13}	1.8×10^{-16}	1.6×10^{-5}
	app. quantile	$3.78587702 \times 10^{-1}$	$3.785877038377 \times 10^{-1}$	$3.785877038373587 \times 10^{-1}$	3.7859×10^{-1}

Table 5. Values of the tail probability error = |approx. probability - α| and values for the approximating quantiles (app. quantile) of $\Lambda^{2/N}$ with the number of decimal places equal to the exact ones for $p = 16$

		$\alpha = 0.01$			
N		$M\mathbf{3}GNIG^*$	$M\mathbf{5}GNIG^*$	$M\mathbf{7}GNIG^*$	$M\mathbf{13}GNIG^*$
17	tail error	3.8×10^{-11}	5.5×10^{-16}	1.6×10^{-20}	6.0×10^{-30}
	app. quantile	$6.95434542 \times 10^{-12}$	$6.954345480219 \times 10^{-12}$	$6.954345480218604 \times 10^{-12}$	$6.95434548021860398307916420 \times 10^{-12}$
50	tail error	5.4×10^{-11}	5.5×10^{-16}	3.1×10^{-21}	7.4×10^{-31}
	app. quantile	$2.011798021 \times 10^{-2}$	$2.01179802370857 \times 10^{-2}$	$2.0117980237085932393 \times 10^{-2}$	$2.011798023708593239405362518 64 \times 10^{-2}$
100	tail error	7.9×10^{-12}	2.5×10^{-17}	1.4×10^{-22}	9.6×10^{-35}
	app. quantile	$1.6593579463 \times 10^{-1}$	$1.6593579464440622 \times 10^{-1}$	$1.65935794644406256 7176 \times 10^{-1}$	$1.6593579464440625671778937723 09464 \times 10^{-1}$
200	tail error	1.0×10^{-12}	8.8×10^{-19}	1.7×10^{-24}	1.4×10^{-39}
	app. quantile	$4.2107594790 \times 10^{-1}$	$4.21075947900821562 \times 10^{-1}$	$4.2107594790082156376 8826 \times 10^{-1}$	$4.2107594790082156376882890264 9968732831 \times 10^{-1}$
		$\alpha = 0.05$			
17	tail error	1.3×10^{-10}	1.1×10^{-15}	1.5×10^{-19}	9.1×10^{-29}
	app. quantile	$1.91996584 \times 10^{-10}$	$1.9199658564262 \times 10^{-10}$	$1.91996585642633442 \times 10^{-10}$	$1.91996585642633440867788017 \times 10^{-10}$
50	tail error	6.1×10^{-11}	1.7×10^{-15}	1.3×10^{-19}	2.9×10^{-30}
	app. quantile	$2.710964811 \times 10^{-2}$	$2.71096481224043 \times 10^{-2}$	$2.71096481224045128 \times 10^{-2}$	$2.71096481224045129147816245 91 \times 10^{-2}$
100	tail error	8.5×10^{-12}	6.6×10^{-17}	1.5×10^{-21}	1.0×10^{-33}
	app. quantile	$1.90221991370 \times 10^{-1}$	$1.90221991373322 \times 10^{-1}$	$1.90221991373323022005 \times 10^{-1}$	$1.9022199137332302200647076052 6404 \times 10^{-1}$
100	tail error	1.1×10^{-12}	2.2×10^{-18}	1.3×10^{-23}	1.5×10^{-37}
	app. quantile	$4.49680394561 \times 10^{-1}$	$4.49680394562087063 \times 10^{-1}$	$4.4968039456208706416 2554 \times 10^{-1}$	$4.4968039456208706416255938810 92039055 \times 10^{-1}$

		$\alpha = 0.01$			
N		$GNIG$	$M2GNIG$	$M3GNIG$	Box
17	tail error	6.7×10^{-10}	3.3×10^{-14}	2.8×10^{-19}	9.9×10^{-3}
	app. quantile	6.954344×10^{-12}	$6.9543454801 \times 10^{-12}$	$6.9543454802186035 \times 10^{-12}$	———
50	tail error	2.7×10^{-9}	1.7×10^{-13}	3.7×10^{-17}	2.9×10^{-4}
	app. quantile	2.011797×10^{-2}	$2.01179802371 \times 10^{-2}$	$2.011798023708594 \times 10^{-2}$	2.02×10^{-2}
100	tail error	7.8×10^{-10}	3.4×10^{-14}	3.5×10^{-18}	2.0×10^{-5}
	app. quantile	$1.65935793 \times 10^{-1}$	$1.6593579464444 \times 10^{-1}$	$1.6593579464440626 \times 10^{-1}$	1.6596×10^{-1}
200	tail error	2.1×10^{-10}	5.0×10^{-15}	———	1.7×10^{-6}
	app. quantile	$4.210759475 \times 10^{-1}$	$4.21075947900829 \times 10^{-1}$	———	4.21078×10^{-1}
		$\alpha = 0.05$			
17	tail error	2.3×10^{-9}	5.0×10^{-14}	3.3×10^{-18}	5.0×10^{-2}
	app. quantile	$1.9199656 \times 10^{-10}$	$1.91996585643 \times 10^{-10}$	$1.9199658564263341 \times 10^{-10}$	———
50	tail error	3.6×10^{-9}	1.0×10^{-12}	1.8×10^{-16}	8.6×10^{-4}
	app. quantile	2.7109647×10^{-2}	$2.71096481225 \times 10^{-2}$	$2.71096481224044 \times 10^{-2}$	2.72×10^{-2}
100	tail error	1.0×10^{-9}	1.7×10^{-13}	2.2×10^{-17}	6.0×10^{-5}
	app. quantile	$1.902219910 \times 10^{-1}$	$1.9022199137338 \times 10^{-1}$	$1.9022199137332301 \times 10^{-1}$	1.9024×10^{-1}
200	tail error	2.6×10^{-10}	2.3×10^{-14}	———	5.5×10^{-6}
	app. quantile	$4.496803944 \times 10^{-1}$	$4.4968039456209 \times 10^{-1}$	———	4.49682×10^{-1}

Table 6. Values of the tail probability error = |approx. probability - α| and values for the approximating quantiles (app. quantile) of $\Lambda^{2/N}$ with the number of decimal places equal to the exact ones for $p = 20$

$\alpha = 0.01$					
N		*M***3***GNIG**	*M***5***GNIG**	*M***7***GNIG**	*M***13***GNIG**
21	tail error	8.7×10^{-12}	5.1×10^{-17}	6.9×10^{-22}	1.2×10^{-32}
	app. quantile	$9.97410882 \times 10^{-14}$	$9.9741088398785 \times 10^{-14}$	$9.974108839878466639 \times 10^{-14}$	$9.9741088398784666377420968307 \times 10^{-14}$
50	tail error	2.5×10^{-11}	1.1×10^{-16}	2.5×10^{-21}	5.0×10^{-32}
	app. quantile	$2.403097861 \times 10^{-3}$	$2.403097863214333 \times 10^{-3}$	$2.4030978632143391742 \times 10^{-3}$	$2.403097863214339174133353847685 \times 10^{-3}$
100	tail error	3.9×10^{-12}	6.2×10^{-18}	1.2×10^{-23}	2.6×10^{-35}
	app. quantile	$6.6351689067 \times 10^{-2}$	$6.635168906994063 \times 10^{-2}$	$6.635168906994064141959 \times 10^{-2}$	$6.635168906994064141958287517445508 \times 10^{-2}$
200	tail error	5.3×10^{-13}	2.3×10^{-19}	2.2×10^{-26}	4.4×10^{-39}
	app. quantile	$2.73763300041 \times 10^{-2}$	$2.737633000425506026 \times 10^{-2}$	$2.7376330004255060294031387 \times 10^{-2}$	$2.7376330004255060294031384614313292215 \times 10^{-2}$
$\alpha = 0.05$					
21	tail error	2.9×10^{-11}	9.8×10^{-17}	6.7×10^{-21}	1.7×10^{-31}
	app. quantile	$2.785206144 \times 10^{-12}$	$2.7852061477834 \times 10^{-12}$	$2.785206147783500651 \times 10^{-12}$	$2.78520614778350065035096629827 \times 10^{-12}$
50	tail error	3.1×10^{-11}	4.9×10^{-16}	3.0×10^{-22}	1.0×10^{-31}
	app. quantile	$3.506020572 \times 10^{-3}$	$3.506020573451140 \times 10^{-3}$	$3.5060205734511497991 \times 10^{-3}$	$3.506020573451149799511776093562 \times 10^{-3}$
100	tail error	4.6×10^{-12}	2.2×10^{-17}	2.2×10^{-20}	7.2×10^{-35}
	app. quantile	$7.8543932312 \times 10^{-2}$	$7.854393231348867 \times 10^{-2}$	$7.85439323134886829940 \times 10^{-2}$	$7.854393231348868299414686849429664 \times 10^{-2}$
200	tail error	6.1×10^{-13}	7.7×10^{-19}	2.8×10^{-24}	1.4×10^{-38}
	app. quantile	$2.966993838063 \times 10^{-2}$	$2.966993838065558834 \times 10^{-2}$	$2.96699383806555883659100 \times 10^{-2}$	$2.9669938380655588365910167550902836147 \times 10^{-2}$

$\alpha = 0.01$					
p	*N*	*GNIG*	*M2GNIG*	*M3GNIG*	*Box*
21	tail error	1.9×10^{-10}	4.8×10^{-15}	1.0×10^{-19}	1.0×10^{-2}
	app. quantile	$9.9741084 \times 10^{-14}$	$9.97410883986 \times 10^{-14}$	$9.9741088398784664 \times 10^{-14}$	———
50	tail error	1.2×10^{-9}	3.5×10^{-14}	8.3×10^{-18}	9.4×10^{-4}
	app. quantile	$2.40309780 \times 10^{-3}$	$2.403097863216 \times 10^{-3}$	$2.4030978632143395 \times 10^{-3}$	2.45×10^{-3}
100	tail error	3.9×10^{-10}	8.4×10^{-15}	1.0×10^{-18}	5.9×10^{-5}
	app. quantile	6.6351688×10^{-2}	$6.6351689069945 \times 10^{-2}$	$6.6351689069940642 \times 10^{-2}$	6.638×10^{-2}
200	tail error	1.1×10^{-10}	1.3×10^{-15}	———	4.7×10^{-6}
	app. quantile	$2.73763300 \times 10^{-2}$	$2.73763300042552 \times 10^{-2}$	———	2.73769×10^{-2}
$\alpha = 0.05$					
21	tail error	6.4×10^{-10}	7.4×10^{-15}	1.2×10^{-18}	5.0×10^{-2}
	app. quantile	$2.7852060 \times 10^{-12}$	$2.785206147784 \times 10^{-12}$	$2.7852061477835005 \times 10^{-12}$	———
50	tail error	1.7×10^{-9}	2.8×10^{-13}	2.7×10^{-17}	2.8×10^{-3}
	app. quantile	$3.50602054 \times 10^{-3}$	$3.50602057349 \times 10^{-3}$	$3.5060205734511493 \times 10^{-3}$	3.55×10^{-3}
100	tail error	5.2×10^{-10}	5.3×10^{-14}	4.1×10^{-18}	1.8×10^{-4}
	app. quantile	$7.85439322 \times 10^{-2}$	$7.854393231349 \times 10^{-2}$	$7.85439323134886822 \times 10^{-2}$	7.857×10^{-2}
200	tail error	1.4×10^{-10}	7.7×10^{-15}	———	1.5×10^{-5}
	app. quantile	$2.966993837 \times 10^{-2}$	$2.96699383806558 \times 10^{-2}$	———	2.9670×10^{-2}

An analogous conclusion may be drawn when we consider the number of exact decimal places that both kinds of near-exact quantiles bear. Although the performance of all the near-exact distributions is excellent for the very small sample sizes, we may note that the in order to be able to observe their asymptotic character we have sometimes to consider quite large values of N, namely in cases where p is larger and namely concerning the type I near-exact distributions. The asymptotic quantiles, given by the approximation based on Box method, only start to match a few exact decimal places when larger values of N are considered. We should also remark the very good results given by the $M\mathbf{13}GNIG^*$ distribution both for the value of the error tail probability and the number of decimal places equal to the exact quantile.

5. Conclusions

We have shown that the exact distribution of the negative logarithm of $\Lambda^{2/N}$, for odd p, is a GIG distribution with depth $p-1$ which is a very manageable distribution that allows the easy evaluation of quantiles and p-values for $W = -\frac{2}{N}\log\Lambda$ and for $\Lambda^{2/N}$. Using this distribution, exact quantiles of $\Lambda^{2/N}$, for larger values of p than those presented in [9], were calculated and are shown in Tables D1-D6 in Appendix D.

When p is even, the exact distribution of $W = -\frac{2}{N}\log\Lambda$ may be represented as an infinite mixture of GNIG distributions. For this case, two kinds of near-exact distributions, with very good asymptotic properties, were developed in the form of finite mixtures of GNIG distributions. Both types of near-exact distributions show very good performances. The type I near exact distributions have a much simpler structure, with a smaller number of terms in the mixture but anyway they present very good results when we consider the measure Δ or when we consider the tail probability error. However the system of equations in (21) used to evaluate the parameters involved in this kind of near-exact distributions is more difficult to solve, specially when we consider high values of p and do not allow us to match more than 6 exact moments. In Tables 5 and 6 it was even not possible to compute the parameters for M3GNIG for the larger sample sizes. The type II near-exact distributions present even better results than the type I near-exact distributions and have the advantage that the system of equations in (27) is much simpler to solve, in every case. However when the number of exact moments matched increases, the large number of GNIG distributions involved can make the mixture a little heavy to deal in practice.

In Tables E1-E7 near-exact quantiles are presented for $\Lambda^{2/N}$. Tables E1-E7 show that in order to have near-exact quantiles with the same 10 significant decimal digits as the exact quantiles we have to use the $M\mathbf{5}GNIG^*$ distribution for values of $N \geq 30$ or for values of $p \geq 12$. When $p \leq 12$, in some cases, we may have to use the $M\mathbf{7}GNIG^*$ distribution. It is expected that, for higher values of N, it will only be necessary to use the $M\mathbf{3}GNIG^*$.

In all cases both kinds of near-exact distributions are asymptotic for the sample size as well as for the number of variables involved and are much more precise than the asymptotic approximation presented in [10].

6. Acknowledgment

This research was financially supported by the Portuguese Foundation for Science and Technology (FCT/MCTES) through CMA/UNL - 'Centro de Matemática e Aplicações' of Universidade Nova de Lisboa (Financiamento Base 2010 ISFL-1-297 from FCT/MCTES/PT).

Appendix A. Notation used in the expression of the l.r.t. statistic

In this Appendix we summarize the main results in [10] and establish the notation related with the l.r.t. statistic used in this paper.

Under H_0 in (1), we have that $\mathbf{\Sigma}$ is circular symmetric, thus its eigenvalues are real and there exists an orthogonal matrix $\mathbf{P}$, such that,

$$\mathbf{\Sigma} = \mathbf{P}\mathbf{D}_\lambda\mathbf{P}'$$

with $\mathbf{D}_\lambda = diag\left(\lambda_1, ..., \lambda_p\right)$. The columns of the matrix $\mathbf{P} = [\mathbf{u}_{jk}]$ are the eigenvectors of $\mathbf{\Sigma}$, corresponding to $\lambda_1, ..., \lambda_p$, and may be given by

$$\mathbf{u}_{jk} = \frac{1}{\sqrt{p}}\left\{cos\left[\frac{2\pi}{p}(j-1)(k-1)\right] + sin\left[\frac{2\pi}{p}(j-1)(k-1)\right]\right\}, \quad j,k = 1, ..., p\,.$$

We may note that the $\mathbf{u}_{jk}$ do not depend on the elements of $\mathbf{\Sigma}$, only the eigenvalues $\lambda_1, \ldots, \lambda_p$ will do (see [10] for details).

Consider a random sample of size $N = n + 1$ from the distribution $N_p\left(\boldsymbol{\mu}, \mathbf{\Sigma}\right)$ and let $\mathbf{X}_{N\times p}$ be the sample matrix. Let $\mathbf{E}_{N1}$ be an $N \times 1$ unitary vector. Let

$$\overline{\mathbf{X}} = \left[\overline{\mathbf{X}}_1 \ldots \overline{\mathbf{X}}_p\right] = \frac{1}{N}\mathbf{X}'\mathbf{E}_{N1}$$

be the vector of sample means, and

$$\mathbf{S} = \left(\mathbf{X} - \mathbf{E}_{N1}\overline{\mathbf{X}}'\right)'\left(\mathbf{X} - \mathbf{E}_{N1}\overline{\mathbf{X}}'\right).$$

Let us take $\mathbf{y} = \frac{1}{\sqrt{N}}\overline{\mathbf{X}}\mathbf{P}$ and $\mathbf{V} = \mathbf{P}'\mathbf{S}\mathbf{P}$. Then, $\mathbf{y}$ and $\mathbf{V} = [v_{ij}]$ are independently distributed with

$$\mathbf{y} \sim N_p\left(\frac{1}{\sqrt{N}}\boldsymbol{\mu}\,\mathbf{P}, \overline{\mathbf{\Sigma}}\right) \qquad \mathbf{V} \sim W_p\left(N-1, \overline{\mathbf{\Sigma}}\right) \tag{A1}$$

where $\overline{\mathbf{\Sigma}} = \mathbf{P}'\mathbf{\Sigma}\mathbf{P}$. If $\mathbf{\Sigma}$ is circular then

$$\overline{\mathbf{\Sigma}} = \mathbf{D}_\lambda = diag\left(\lambda_1, ..., \lambda_p\right), \quad \lambda_j = \lambda_{p-j+2} \; j = 2, ..., p\,.$$

We then define, for even p,

$$v_j = \begin{cases} v_{jj}, & j = 1 \text{ or } j = m+1 \\ v_{jj} + v_{p-j+2,p-j+2}, & j = 2, ..., m, \end{cases} \tag{A2}$$

while for odd p,

$$v_j = \begin{cases} v_{jj}, & j = 1 \\ v_{jj} + v_{p-j+2,p-j+2}, & j = 2, ..., m+1, \end{cases} \tag{A3}$$

with $v_{p-j+2} = v_j$ for $(j = 2, \ldots, p)$, and, for even p,

$$w_j = \begin{cases} y_{jj}^2, & j = 1 \text{ or } j = m+1 \\ y_j^2 + y_{p-j+2}^2, & j = 2, ..., m, \end{cases} \tag{A4}$$

while for odd p,

$$w_j = \begin{cases} y_1^2, & j = 1 \\ y_j^2 + y_{p-j+2}^2, & j = 2, \ldots, m+1. \end{cases} \tag{A5}$$

Appendix B. Proof of theorems 2.1 and 2.3 and 3.1

Proof of Theorem 2.1

Proof: From the expression of the c.f. of W in (5) and considering that for an odd p we have $m = \left\lfloor \frac{p}{2} \right\rfloor = \frac{p-1}{2}$, we may write the c.f. of W as

$$\begin{aligned}
\Phi_W(t) &= \left(\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N}{2} - \mathrm{i}t\right)}{\Gamma\left(\frac{N-1}{2} - \mathrm{i}t\right)\Gamma\left(\frac{N}{2}\right)} \right)^{\frac{p-1}{2}} \times \prod_{j=1}^{p-1} \frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N}{2} - \frac{j+1}{2} - \mathrm{i}t\right)}{\Gamma\left(\frac{N}{2} - \frac{j+1}{2}\right)\Gamma\left(\frac{N}{2} - \mathrm{i}t\right)} \\
&= \left(\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N}{2} - \mathrm{i}t\right)}{\Gamma\left(\frac{N-1}{2} - \mathrm{i}t\right)\Gamma\left(\frac{N}{2}\right)} \right)^{\frac{p-1}{2}} \times \prod_{\substack{j=1 \\ \text{step } 2}}^{p-2} \frac{\Gamma\left(\frac{N}{2} - \frac{j+1}{2} + \frac{j+1}{2}\right)\Gamma\left(\frac{N}{2} - \frac{j+1}{2} - \mathrm{i}t\right)}{\Gamma\left(\frac{N}{2} - \frac{j+1}{2}\right)\Gamma\left(\frac{N}{2} - \frac{j+1}{2} + \frac{j+1}{2} - \mathrm{i}t\right)} \\
&\quad \times \prod_{\substack{j=2 \\ \text{step } 2}}^{p-1} \frac{\Gamma\left(\frac{N}{2} - \frac{j+1}{2} + \frac{j}{2}\right)\Gamma\left(\frac{N}{2} - \frac{j+1}{2} - \mathrm{i}t\right)}{\Gamma\left(\frac{N}{2} - \frac{j+1}{2}\right)\Gamma\left(\frac{N}{2} - \frac{j+1}{2} + \frac{j}{2} - \mathrm{i}t\right)} \prod_{\substack{j=2 \\ \text{step } 2}}^{p-1} \frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N}{2} - \frac{j+1}{2} + \frac{j}{2} - \mathrm{i}t\right)}{\Gamma\left(\frac{N}{2} - \frac{j+1}{2} + \frac{j}{2}\right)\Gamma\left(\frac{N}{2} - \mathrm{i}t\right)} \\
&= \left(\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N}{2} - \mathrm{i}t\right)}{\Gamma\left(\frac{N-1}{2} - \mathrm{i}t\right)\Gamma\left(\frac{N}{2}\right)} \right)^{\frac{p-1}{2}} \times \prod_{\substack{j=1 \\ \text{step } 2}}^{p-2} \prod_{k=0}^{\frac{j+1}{2}-1} \left(\frac{N-j-1}{2} + k \right) \left(\frac{N-j-1}{2} + k - \mathrm{i}t \right)^{-1} \\
&\quad \times \prod_{\substack{j=2 \\ \text{step } 2}}^{p-1} \prod_{k=0}^{\frac{j}{2}-1} \left(\frac{N-j-1}{2} + k \right) \left(\frac{N-j-1}{2} + k - \mathrm{i}t \right)^{-1} \left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N}{2} - \frac{1}{2} - \mathrm{i}t\right)}{\Gamma\left(\frac{N}{2} - \frac{1}{2}\right)\Gamma\left(\frac{N}{2} - \mathrm{i}t\right)} \right)^{\frac{p-1}{2}}
\end{aligned}$$

noticing that

$$\left\lfloor \frac{j+1}{2} \right\rfloor = \begin{cases} \frac{j+1}{2} & \text{if } j \text{ is odd} \\ \frac{j}{2} & \text{if } j \text{ is even} \end{cases} \tag{B1}$$

$$\Phi_W(t) = \prod_{j=1}^{p-1} \prod_{k=0}^{\left\lfloor \frac{j+1}{2} - 1 \right\rfloor} \left(\frac{N-j-1}{2} + k \right) \left(\frac{N-j-1}{2} + k - \mathrm{i}t \right)^{-1} \tag{B2}$$

$$= \prod_{j=2}^{p} \left(\frac{N}{2} - \frac{j}{2} \right)^{1 + \left\lfloor \frac{p-j}{2} \right\rfloor} \left(\frac{N}{2} - \frac{j}{2} - \mathrm{i}t \right)^{-1 - \left\lfloor \frac{p-j}{2} \right\rfloor}. \tag{B3}$$

Since the c.f. in (B2) corresponds to the c.f. of the sum of independent Exponential r.v.'s, counting the number of Exponential distributions with the same rate parameter we obtain the representation in (B3) for the c.f. of W.

□

Proof of Theorem 2.3

Proof: When the number of variables, p, is even we have that $m = \left\lfloor \frac{p}{2} \right\rfloor = \frac{p}{2}$, and then we may rewrite the c.f. of W in (5) in the form

$$\Phi_W(t)=\left(\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{N-1}{2}-\mathrm{i}t\right)\Gamma\left(\frac{N}{2}\right)}\right)^{\frac{p}{2}}\times\prod_{j=1}^{p-1}\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N}{2}-\frac{j+1}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-\frac{j+1}{2}\right)\Gamma\left(\frac{N}{2}-\mathrm{i}t\right)},$$

or,

$$\begin{aligned}\Phi_W(t)=&\left(\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{N-1}{2}-\mathrm{i}t\right)\Gamma\left(\frac{N}{2}\right)}\right)^{\frac{p}{2}}\\&\times\left\{\prod_{\substack{j=3\\\text{step }2}}^{p-1}\frac{\Gamma\left(\frac{N}{2}-\frac{j+1}{2}+\frac{j+1}{2}\right)\Gamma\left(\frac{N}{2}-\frac{j+1}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-\frac{j+1}{2}\right)\Gamma\left(\frac{N}{2}-\frac{j+1}{2}+\frac{j+1}{2}-\mathrm{i}t\right)}\right\}\times\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N}{2}-1-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-1\right)\Gamma\left(\frac{N}{2}-\mathrm{i}t\right)}\\&\times\prod_{\substack{j=2\\\text{step }2}}^{p-2}\frac{\Gamma\left(\frac{N}{2}-\frac{j+1}{2}+\frac{j}{2}\right)\Gamma\left(\frac{N}{2}-\frac{j+1}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-\frac{j+1}{2}\right)\Gamma\left(\frac{N}{2}-\frac{j+1}{2}+\frac{j}{2}-\mathrm{i}t\right)}\prod_{\substack{j=2\\\text{step }2}}^{p-2}\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N}{2}-\frac{j+1}{2}+\frac{j}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-\frac{j+1}{2}+\frac{j}{2}\right)\Gamma\left(\frac{N}{2}-\mathrm{i}t\right)}\\=&\left(\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{N-1}{2}-\mathrm{i}t\right)\Gamma\left(\frac{N}{2}\right)}\right)^{\frac{p}{2}}\\&\times\left\{\prod_{\substack{j=3\\\text{step }2}}^{p-1}\prod_{k=0}^{\frac{j+1}{2}-1}\left(\frac{N-j-1}{2}+k\right)\left(\frac{N-j-1}{2}+k-\mathrm{i}t\right)^{-1}\right\}\times\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N}{2}-1-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-1\right)\Gamma\left(\frac{N}{2}-\mathrm{i}t\right)}\\&\times\left\{\prod_{\substack{j=2\\\text{step }2}}^{p-2}\prod_{k=0}^{\frac{j}{2}-1}\left(\frac{N-j-1}{2}+k\right)\left(\frac{N-j-1}{2}+k-\mathrm{i}t\right)^{-1}\right\}\left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N}{2}-\frac{1}{2}-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-\frac{1}{2}\right)\Gamma\left(\frac{N}{2}-\mathrm{i}t\right)}\right)^{\frac{p-2}{2}},\end{aligned}$$

which, after some simplifications and using the equality in (B1), may be written as

$$\begin{aligned}\Phi_W(t)=&\prod_{j=2}^{p-1}\prod_{k=0}^{\left\lfloor\frac{j+1}{2}-1\right\rfloor}\left(\frac{N-j-1}{2}+k\right)\left(\frac{N-j-1}{2}+k-\mathrm{i}t\right)^{-1}\times\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N}{2}-1-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-1\right)\Gamma\left(\frac{N-1}{2}-\mathrm{i}t\right)}\\=&\underbrace{\prod_{j=2}^{p}\theta_j^{r_j^*}\left(\theta_j-\mathrm{i}t\right)^{-r_j^*}}_{\Phi_{W_1}(t)}\times\underbrace{\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N}{2}-1-\mathrm{i}t\right)}{\Gamma\left(\frac{N}{2}-1\right)\Gamma\left(\frac{N-1}{2}-\mathrm{i}t\right)}}_{\Phi_{W_2}(t)}\end{aligned}$$

with r_j^* and θ_j given in (11) and (7).

□

Proof of Theorem 3.1

Proof : In this proof we will consider only the case of $h=6$, since the cases $h=2$ and $h=4$ are derived in a similar way.

If in the c.f. of W given in (10) we replace $\Phi_{W_2}(t)$ by

$$\Phi_{M_G}(t) = \sum_{k=1}^{3} \omega_k \, \lambda^{s_k} \, (\lambda - \mathrm{i}t)^{-s_k} \,,$$

we obtain

$$\Phi_W(t) \approx \Phi_{W_1}(t) \times \underbrace{\sum_{k=1}^{3} \omega_k \, \lambda^{s_k} \, (\lambda - \mathrm{i}t)^{-s_k}}_{\Phi_3^*(t)}$$

$$\approx \sum_{k=1}^{3} \omega_k \underbrace{\underbrace{\Phi_{W_1}(t)}_{\text{GIG distribution}} \times \underbrace{\lambda^{s_k} \, (\lambda - \mathrm{i}t)^{-s_k}}_{\text{Gamma distribution}}}_{\text{GNIG distribution}}$$

that is the c.f. of the mixture of three GNIG distributions of depth p with c.d.f.'s and p.d.f.'s given in (24) and (23). The parameters p_ν, s_ν and λ are defined in such a way that

$$\frac{d^j}{dt^j} \, \Phi_{M_G}(t) \bigg|_{t=0} = \frac{d^j}{dt^j} \, \Phi_{W_2}(t) \bigg|_{t=0} \,, \quad j = 1, \ldots, 6 \,,$$

what gives rise to a near-exact distribution that matches the first six exact moments of W. By simple transformation it's easy to derive the near-exact c.d.f's and p.d.f.'s for $\Lambda^{2/N}$. □

Appendix C. The Gamma, GIG (Generalized Integer Gamma) and GNIG (Generalized Near-Integer Gamma) distributions

We will use this Appendix to establish some notation concerning distributions used in the paper, as well as to give the expressions for the p.d.f.'s (probability density functions) and c.d.f.'s (cumulative distribution functions) of the GIG (Generalized Integer Gamma) and GNIG (Generalized Near-Integer Gamma) distributions.

We will say that the r.v. X has a Gamma distribution with rate parameter $\lambda > 0$ and shape parameter $r > 0$, if its p.d.f. may be written as

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} \, e^{-\lambda x} \, x^{r-1} \,, \quad (x > 0)$$

and we will denote this fact by

$$X \sim \Gamma(r, \lambda) \,.$$

Let

$$X_j \sim \Gamma(r_j, \lambda_j) \qquad j = 1, \ldots, p$$

be p independent r.v.'s with Gamma distributions with shape parameters $r_j \in \boldsymbol{N}$ and rate parameters $\lambda_j > 0$, with $\lambda_j \neq \lambda_{j'}$, for all $j, j' \in \{1, \ldots, p\}$. We will say that then the r.v.

$$Y = \sum_{j=1}^{p} X_j$$

has a GIG (Generalized Integer Gamma) distribution of depth p, with shape parameters r_j and rate parameters λ_j, ($j = 1, \ldots, p$), and we will denote this fact by

$$Y \sim GIG(r_j, \lambda_j; \; p) \,.$$

The p.d.f. and c.d.f. (cumulative distribution function) of Y are respectively given by

(Coelho, 1998)

$$f^{GIG}(y|r_1,\ldots,r_p;\lambda_1,\ldots,\lambda_p;p) = K\sum_{j=1}^{p} P_j(y)\,e^{-\lambda_j y}\,, \quad (y>0) \tag{C1}$$

and

$$F^{GIG}(y|r_1,\ldots,r_j;\lambda_1,\ldots,\lambda_p;p) = 1-K\sum_{j=1}^{p} P_j^*(y)\,e^{-\lambda_j y}\,, \quad (y>0) \tag{C2}$$

where K is given by (5) in Coelho (1998), and $P_j(y)$ and $P_j^*(y)$ are given by (7) and (16) in the same reference.

The GNIG (Generalized Near-Integer Gamma) distribution of depth $p+1$ (Coelho, 2004) is the distribution of the r.v.

$$Z = Y_1 + Y_2$$

where Y_1 and Y_2 are independent, Y_1 having a GIG distribution of depth p and Y_2 with a Gamma distribution with a non-integer shape parameter r and a rate parameter $\lambda \neq \lambda_j$ $(j=1,\ldots,p)$. The p.d.f. (probability density function) of Z is given by

$$f^{GNIG}(z|r_1,\ldots,r_p,r;\ \lambda_1,\ldots,\lambda_p,\lambda;p+1) = K\lambda^r\sum_{j=1}^{p} e^{-\lambda_j z}\sum_{k=1}^{r_j}\left\{c_{j,k}\frac{\Gamma(k)}{\Gamma(k+r)}z^{k+r-1}{}_1F_1(r,k+r,-(\lambda-\lambda_j)z)\right\}, \quad (z>0) \tag{C3}$$

and the c.d.f. (cumulative distribution function) given by

$$F^{GNIG}(z|r_1,\ldots,r_p,r;\ \lambda_1,\ldots,\lambda_p,\lambda;p+1) = \frac{\lambda^r z^r}{\Gamma(r+1)}{}_1F_1(r,r+1,-\lambda z) \tag{C4}$$

where

$$c_{j,k}^* = \frac{c_{j,k}}{\lambda_j^k}\Gamma(k)$$

with $c_{j,k}$ given by (11) through (13) in Coelho (1998). In the above expressions ${}_1F_1(a,b;z)$ is the Kummer confluent hypergeometric function. This function typically has very good convergence properties and is nowadays easily handled by a number of software packages.

Appendix D. Exact quantiles of $\Lambda^{2/N}$ when p is odd

In the next tables we present exact quantiles of $\Lambda^{2/N}$ for odd $p = 9, 11, 13, 15, 17, 19$, obtained using the expression of the c.d.f. in (9).

Table D1. Exact quantiles of $\Lambda^{2/N}$ for $p = 9$

N	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
10	$1.398876592 \times 10^{-8}$	$8.983583921 \times 10^{-7}$	$3.757348403 \times 10^{-7}$	$1.642173830 \times 10^{-6}$
11	$4.523677076 \times 10^{-6}$	$1.256448287 \times 10^{-5}$	$2.833455268 \times 10^{-5}$	$6.759040932 \times 10^{-5}$
12	$5.384478548 \times 10^{-5}$	$1.143595025 \times 10^{-4}$	$2.101798876 \times 10^{-4}$	$4.065237399 \times 10^{-4}$
13	$2.452349709 \times 10^{-4}$	$4.527786581 \times 10^{-4}$	$7.456926288 \times 10^{-4}$	$1.284969659 \times 10^{-3}$
14	$7.098772307 \times 10^{-4}$	$1.198146738 \times 10^{-3}$	$1.837246495 \times 10^{-3}$	$2.933312299 \times 10^{-3}$
15	$1.580255561 \times 10^{-3}$	$2.501934422 \times 10^{-3}$	$3.644318397 \times 10^{-3}$	$5.504905546 \times 10^{-3}$
16	$2.967186930 \times 10^{-3}$	$4.475023996 \times 10^{-3}$	$6.267313456 \times 10^{-3}$	$9.072505840 \times 10^{-3}$
17	$4.949303379 \times 10^{-3}$	$7.182578510 \times 10^{-3}$	$9.749892039 \times 10^{-3}$	$1.364215396 \times 10^{-3}$
18	$7.571685491 \times 10^{-3}$	$1.064840933 \times 10^{-2}$	$1.408954724 \times 10^{-2}$	$1.917173430 \times 10^{-2}$
19	$1.084960208 \times 10^{-2}$	$1.486327662 \times 10^{-2}$	$1.925013312 \times 10^{-2}$	$2.558813429 \times 10^{-2}$
20	$1.477435036 \times 10^{-2}$	$1.979383416 \times 10^{-2}$	$2.517329484 \times 10^{-2}$	$3.280089943 \times 10^{-2}$
25	$4.283110726 \times 10^{-2}$	$5.320933032 \times 10^{-2}$	$6.361069924 \times 10^{-2}$	$7.744381046 \times 10^{-2}$
30	$8.047619864 \times 10^{-2}$	$9.567282836 \times 10^{-2}$	$1.103154209 \times 10^{-1}$	$1.290702177 \times 10^{-1}$
40	$1.649519329 \times 10^{-1}$	$1.866116998 \times 10^{-1}$	$2.065710595 \times 10^{-1}$	$2.310695961 \times 10^{-1}$
50	$2.457238344 \times 10^{-1}$	$2.704796003 \times 10^{-1}$	$2.927344898 \times 10^{-1}$	$3.194135656 \times 10^{-1}$

Table D2. Exact quantiles of $\Lambda^{2/N}$ for $p = 11$

N	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
12	$1.506045150 \times 10^{-9}$	$9.707519060 \times 10^{-9}$	$4.083316288 \times 10^{-8}$	$1.803481600 \times 10^{-7}$
13	$5.613520378 \times 10^{-7}$	$1.583417809 \times 10^{-6}$	$3.630109889 \times 10^{-6}$	$8.858221501 \times 10^{-6}$
14	$7.708755980 \times 10^{-6}$	$1.673825606 \times 10^{-5}$	$3.143329705 \times 10^{-5}$	$6.250541651 \times 10^{-5}$
15	$4.004229292 \times 10^{-5}$	$7.583072675 \times 10^{-5}$	$1.279047671 \times 10^{-4}$	$2.270703363 \times 10^{-4}$
16	$1.303981206 \times 10^{-4}$	$2.260981056 \times 10^{-4}$	$3.554247459 \times 10^{-4}$	$5.850529205 \times 10^{-4}$
17	$3.223832414 \times 10^{-4}$	$5.246747138 \times 10^{-4}$	$7.836863005 \times 10^{-4}$	$1.220511252 \times 10^{-3}$
18	$6.646556562 \times 10^{-4}$	$1.030516079 \times 10^{-3}$	$1.479786649 \times 10^{-3}$	$2.207765052 \times 10^{-3}$
19	$1.205335014 \times 10^{-3}$	$1.797850698 \times 10^{-3}$	$2.501377324 \times 10^{-3}$	$3.605116795 \times 10^{-3}$
20	$1.987751022 \times 10^{-3}$	$2.871998393 \times 10^{-3}$	$3.893084961 \times 10^{-3}$	$5.452728945 \times 10^{-3}$
25	$1.048329613 \times 10^{-2}$	$1.367510805 \times 10^{-2}$	$1.704232270 \times 10^{-2}$	$2.175685008 \times 10^{-2}$
30	$2.708084020 \times 10^{-2}$	$3.338465344 \times 10^{-2}$	$3.970707056 \times 10^{-2}$	$4.813850984 \times 10^{-2}$
40	$7.771521975 \times 10^{-2}$	$9.006563585 \times 10^{-2}$	$1.017821307 \times 10^{-1}$	$1.165920743 \times 10^{-1}$
50	$1.381582783 \times 10^{-1}$	$1.548536737 \times 10^{-1}$	$1.702228683 \times 10^{-1}$	$1.890880425 \times 10^{-1}$

Table D3. Exact quantiles of $\Lambda^{2/N}$ for $p = 13$

N	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
14	$1.693072502 \times 10^{-10}$	$1.094765074 \times 10^{-9}$	$4.627229945 \times 10^{-9}$	$2.061698665 \times 10^{-8}$
15	$7.099274581 \times 10^{-8}$	$2.028889904 \times 10^{-7}$	$4.715929471 \times 10^{-7}$	$1.172601206 \times 10^{-6}$
16	$1.101116106 \times 10^{-6}$	$2.436018347 \times 10^{-6}$	$4.658086703 \times 10^{-6}$	$9.478535756 \times 10^{-6}$
17	$6.412077408 \times 10^{-6}$	$1.240773821 \times 10^{-5}$	$2.135407142 \times 10^{-5}$	$3.886895652 \times 10^{-5}$
18	$2.317613785 \times 10^{-5}$	$4.112166159 \times 10^{-5}$	$6.602309617 \times 10^{-5}$	$1.115201562 \times 10^{-4}$
19	$6.297222882 \times 10^{-5}$	$1.049472052 \times 10^{-4}$	$1.601655637 \times 10^{-4}$	$2.560242995 \times 10^{-4}$
20	$1.414060474 \times 10^{-4}$	$2.245600970 \times 10^{-4}$	$3.294926026 \times 10^{-4}$	$5.045051522 \times 10^{-4}$
25	$1.795346330 \times 10^{-3}$	$2.475104259 \times 10^{-3}$	$3.232765045 \times 10^{-3}$	$4.353519247 \times 10^{-3}$
30	$7.115175534 \times 10^{-3}$	$9.125129767 \times 10^{-3}$	$1.122547762 \times 10^{-2}$	$1.414410336 \times 10^{-2}$
40	$3.141849615 \times 10^{-2}$	$3.734831748 \times 10^{-2}$	$4.313536937 \times 10^{-2}$	$5.066172680 \times 10^{-2}$
50	$6.953886143 \times 10^{-2}$	$7.941995351 \times 10^{-2}$	$8.871971869 \times 10^{-2}$	$1.003974867 \times 10^{-1}$

Table D4. Exact quantiles of $\Lambda^{2/N}$ for $p = 15$

N	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
16	$1.960321100 \times 10^{-11}$	$1.271118733 \times 10^{-10}$	$5.395270073 \times 10^{-10}$	$2.422103789 \times 10^{-9}$
17	$9.091912431 \times 10^{-9}$	$2.628085207 \times 10^{-8}$	$6.181529158 \times 10^{-8}$	$1.561825475 \times 10^{-7}$
18	$1.568221074 \times 10^{-7}$	$3.526069140 \times 10^{-7}$	$6.848044628 \times 10^{-7}$	$1.421187284 \times 10^{-6}$
19	$1.010500048 \times 10^{-6}$	$1.992343855 \times 10^{-6}$	$3.489027517 \times 10^{-6}$	$6.488552735 \times 10^{-6}$
20	$4.011006632 \times 10^{-6}$	$7.261165858 \times 10^{-6}$	$1.187394417 \times 10^{-5}$	$2.050897308 \times 10^{-5}$
25	$1.994343571 \times 10^{-4}$	$2.932378682 \times 10^{-4}$	$4.044427220 \times 10^{-4}$	$5.793661231 \times 10^{-4}$
30	$1.410099205 \times 10^{-3}$	$1.889061326 \times 10^{-3}$	$2.411781895 \times 10^{-3}$	$3.170400613 \times 10^{-3}$
40	$1.076176079 \times 10^{-2}$	$1.314161382 \times 10^{-2}$	$1.553118938 \times 10^{-2}$	$1.872892053 \times 10^{-2}$
50	$3.112821033 \times 10^{-2}$	$3.625322690 \times 10^{-2}$	$4.118291978 \times 10^{-2}$	$4.751011074 \times 10^{-2}$

Table D5. Exact quantiles of $\Lambda^{2/N}$ for $p = 17$

N	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
18	$2.318898698 \times 10^{-12}$	$1.507409986 \times 10^{-11}$	$6.422191675 \times 10^{-11}$	$2.902311699 \times 10^{-10}$
19	$1.174798963 \times 10^{-9}$	$3.430197208 \times 10^{-9}$	$8.152533611 \times 10^{-9}$	$2.088750413 \times 10^{-8}$
20	$2.226856090 \times 10^{-8}$	$5.079008592 \times 10^{-8}$	$9.999293781 \times 10^{-8}$	$2.111105908 \times 10^{-7}$
25	$1.279914651 \times 10^{-5}$	$2.035356356 \times 10^{-5}$	$2.998519629 \times 10^{-5}$	$4.627103570 \times 10^{-5}$
30	$2.014348624 \times 10^{-4}$	$2.833947551 \times 10^{-4}$	$3.771742411 \times 10^{-4}$	$5.198579355 \times 10^{-4}$
40	$3.077113443 \times 10^{-3}$	$3.866719411 \times 10^{-3}$	$4.683030919 \times 10^{-3}$	$5.807826691 \times 10^{-3}$
50	$1.230232622 \times 10^{-2}$	$1.462315349 \times 10^{-2}$	$1.690457497 \times 10^{-2}$	$1.989727167 \times 10^{-2}$

Table D6. Exact quantiles of $\Lambda^{2/N}$ for $p = 19$

N	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
20	$2.788030869 \times 10^{-13}$	$1.816522449 \times 10^{-12}$	$7.765313775 \times 10^{-12}$	$3.530167332 \times 10^{-11}$
25	$3.890145982 \times 10^{-7}$	$6.855615248 \times 10^{-7}$	$1.100282575 \times 10^{-6}$	$1.868630646 \times 10^{-6}$
30	$1.948325702 \times 10^{-5}$	$2.899701949 \times 10^{-5}$	$4.047594613 \times 10^{-5}$	$5.889012736 \times 10^{-5}$
40	$7.215514632 \times 10^{-4}$	$9.349562031 \times 10^{-4}$	$1.162396952 \times 10^{-3}$	$1.485482733 \times 10^{-3}$
50	$4.257494451 \times 10^{-3}$	$5.169928898 \times 10^{-3}$	$6.086838052 \times 10^{-3}$	$7.316438970 \times 10^{-3}$

Appendix E. Near-exact quantiles of $\Lambda^{2/N}$ when p is even

In the following Tables we present the near-exact quantiles of $\Lambda^{2/N}$ for different values of p and N. We present near-exact quantiles that match the 10 significant decimal digits of the exact quantile. In every case we also indicate the number of exact moments, m^*, that the type II near-exact distribution has to match in order to assure such precision.

Table E1. Near-exact quantiles of $\Lambda^{2/N}$ for $p = 8$

N	$\alpha = 0.01$	m^*	$\alpha = 0.025$	m^*	$\alpha = 0.05$	m^*	$\alpha = 0.1$	m^*
9	$4.686559223 \times 10^{-8}$	(6)	$3.002952347 \times 10^{-7}$	(6)	$1.251563961 \times 10^{-6}$	(6)	$5.434185751 \times 10^{-6}$	(6)
10	$1.385848451 \times 10^{-5}$	(4)	$3.812798308 \times 10^{-5}$	(6)	$8.510669506 \times 10^{-5}$	(6)	$2.001387329 \times 10^{-4}$	(6)
11	$1.511192620 \times 10^{-4}$	(6)	$3.166285911 \times 10^{-4}$	(6)	$5.742008279 \times 10^{-4}$	(6)	$1.091535076 \times 10^{-4}$	(6)
12	$6.360677270 \times 10^{-4}$	(6)	$1.156291330 \times 10^{-3}$	(6)	$1.876534755 \times 10^{-3}$	(6)	$3.174470348 \times 10^{-3}$	(6)
13	$1.717780972 \times 10^{-3}$	(6)	$2.852250938 \times 10^{-3}$	(6)	$4.307739809 \times 10^{-3}$	(6)	$6.749907734 \times 10^{-3}$	(6)
14	$3.598285658 \times 10^{-3}$	(6)	$5.603012854 \times 10^{-3}$	(6)	$8.037924450 \times 10^{-3}$	(6)	$1.191780067 \times 10^{-2}$	(4)
15	$6.404653330 \times 10^{-3}$	(6)	$9.500505503 \times 10^{-3}$	(12)	$1.310669908 \times 10^{-2}$	(6)	$1.863006290 \times 10^{-2}$	(4)
16	$1.019035241 \times 10^{-2}$	(6)	$1.454882486 \times 10^{-2}$	(6)	$1.945993019 \times 10^{-2}$	(6)	$2.674882194 \times 10^{-2}$	(4)
17	$1.494952758 \times 10^{-2}$	(6)	$2.069035955 \times 10^{-2}$	(6)	$2.698587267 \times 10^{-2}$	(6)	$3.609184117 \times 10^{-2}$	(4)
18	$2.063418043 \times 10^{-2}$	(6)	$2.782955934 \times 10^{-2}$	(6)	$3.554329118 \times 10^{-2}$	(6)	$4.646297223 \times 10^{-2}$	(4)
19	$2.716959420 \times 10^{-2}$	(6)	$3.585128331 \times 10^{-2}$	(6)	$4.498087478 \times 10^{-2}$	(6)	$5.767056429 \times 10^{-2}$	(4)
20	$3.446646866 \times 10^{-2}$	(6)	$4.463366876 \times 10^{-2}$	(6)	$5.514952296 \times 10^{-2}$	(6)	$6.953746750 \times 10^{-2}$	(6)
25	$7.912126252 \times 10^{-2}$	(6)	$9.603011546 \times 10^{-2}$	(6)	$1.125400659 \times 10^{-1}$	(4)	$1.339261252 \times 10^{-1}$	(4)
30	$1.303872415 \times 10^{-1}$	(4)	$1.522743568 \times 10^{-1}$	(4)	$1.729216926 \times 10^{-1}$	(4)	$1.988062776 \times 10^{-1}$	(4)
40	$2.314877691 \times 10^{-1}$	(4)	$2.587229109 \times 10^{-1}$	(4)	$2.834205299 \times 10^{-1}$	(4)	$3.132436030 \times 10^{-1}$	(4)
50	$3.191348725 \times 10^{-1}$	(4)	$3.480571366 \times 10^{-1}$	(4)	$3.737120295 \times 10^{-1}$	(4)	$4.040491392 \times 10^{-1}$	(4)

Table E2. Near-exact quantiles of $\Lambda^{2/N}$ for $p = 10$

N	$\alpha = 0.01$	m^*	$\alpha = 0.025$	m^*	$\alpha = 0.05$	m^*	$\alpha = 0.1$	m^*
11	$4.819299757 \times 10^{-9}$	(4)	$3.100619760 \times 10^{-8}$	(4)	$1.300507301 \times 10^{-7}$	(4)	$5.713907591 \times 10^{-7}$	(6)
12	$1.673475177 \times 10^{-6}$	(4)	$4.683951125 \times 10^{-6}$	(6)	$1.065022271 \times 10^{-5}$	(4)	$2.569624350 \times 10^{-5}$	(6)
13	$2.138716446 \times 10^{-5}$	(4)	$4.592625114 \times 10^{-5}$	(6)	$8.532056646 \times 10^{-5}$	(6)	$1.673286238 \times 10^{-5}$	(6)
14	$1.039466277 \times 10^{-4}$	(6)	$1.943545971 \times 10^{-4}$	(6)	$3.239176314 \times 10^{-4}$	(6)	$5.665354455 \times 10^{-4}$	(6)
15	$3.188259494 \times 10^{-4}$	(6)	$5.453635959 \times 10^{-4}$	(6)	$8.466601121 \times 10^{-4}$	(6)	$1.372460269 \times 10^{-4}$	(6)
16	$7.471045545 \times 10^{-4}$	(6)	$1.199106333 \times 10^{-3}$	(6)	$1.768520327 \times 10^{-3}$	(6)	$2.712261311 \times 10^{-3}$	(4)
17	$1.468165583 \times 10^{-3}$	(4)	$2.244711148 \times 10^{-3}$	(6)	$3.182880108 \times 10^{-3}$	(6)	$4.676924949 \times 10^{-3}$	(6)
18	$2.550292945 \times 10^{-3}$	(6)	$3.751472626 \times 10^{-3}$	(6)	$5.154766416 \times 10^{-3}$	(6)	$7.318945097 \times 10^{-3}$	(4)
19	$4.045771874 \times 10^{-3}$	(6)	$5.765941504 \times 10^{-3}$	(6)	$7.720715357 \times 10^{-3}$	(6)	$1.065656313 \times 10^{-2}$	(4)
20	$5.989546332 \times 10^{-3}$	(6)	$8.312851569 \times 10^{-3}$	(6)	$1.089223762 \times 10^{-2}$	(4)	$1.468065097 \times 10^{-2}$	(4)
25	$2.263930216 \times 10^{-2}$	(4)	$2.878335629 \times 10^{-2}$	(4)	$3.509460054 \times 10^{-2}$	(4)	$4.369810098 \times 10^{-2}$	(4)
30	$4.908152887 \times 10^{-2}$	(4)	$5.936650772 \times 10^{-2}$	(4)	$6.947040905 \times 10^{-2}$	(4)	$8.266692187 \times 10^{-2}$	(4)
40	$1.171486442 \times 10^{-1}$	(4)	$1.340641655 \times 10^{-1}$	(4)	$1.498738589 \times 10^{-1}$	(4)	$1.695589156 \times 10^{-1}$	(4)
50	$1.890986165 \times 10^{-1}$	(4)	$2.099536064 \times 10^{-1}$	(4)	$2.289191971 \times 10^{-1}$	(4)	$2.519228977 \times 10^{-1}$	(4)

Table E3. Near-exact quantiles of $\Lambda^{2/N}$ for $p = 12$

N	$\alpha = 0.01$	m^*	$\alpha = 0.025$	m^*	$\alpha = 0.05$	m^*	$\alpha = 0.1$	m^*
13	$5.260825633 \times 10^{-10}$	(4)	$3.396305935 \times 10^{-9}$	(4)	$1.300507301 \times 10^{-8}$	(4)	$1.432040025 \times 10^{-8}$	(4)
14	$2.079917339 \times 10^{-7}$	(4)	$5.905254714 \times 10^{-7}$	(4)	$1.363180219 \times 10^{-6}$	(4)	$3.357883304 \times 10^{-6}$	(4)
15	$3.034654946 \times 10^{-6}$	(4)	$6.650960447 \times 10^{-6}$	(4)	$1.260330652 \times 10^{-5}$	(4)	$2.535244403 \times 10^{-5}$	(4)
16	$1.668159295 \times 10^{-5}$	(4)	$3.193231809 \times 10^{-5}$	(4)	$5.440466114 \times 10^{-5}$	(6)	$9.779802411 \times 10^{-5}$	(4)
17	$5.719299189 \times 10^{-5}$	(4)	$1.003103315 \times 10^{-4}$	(4)	$1.593554188 \times 10^{-4}$	(4)	$2.657082239 \times 10^{-4}$	(4)
18	$1.481185171 \times 10^{-4}$	(4)	$2.439193222 \times 10^{-4}$	(4)	$3.682532765 \times 10^{-4}$	(6)	$5.810021427 \times 10^{-4}$	(4)
19	$3.184306605 \times 10^{-4}$	(6)	$4.996149484 \times 10^{-4}$	(4)	$7.251566129 \times 10^{-4}$	(4)	$1.095934006 \times 10^{-3}$	(4)
20	$5.997244051 \times 10^{-4}$	(6)	$9.051976277 \times 10^{-4}$	(4)	$1.272859560 \times 10^{-3}$	(4)	$1.857990869 \times 10^{-3}$	(4)
25	$4.667502215 \times 10^{-3}$	(4)	$6.250514630 \times 10^{-3}$	(4)	$7.965091558 \times 10^{-3}$	(4)	$1.042973956 \times 10^{-2}$	(4)
30	$1.463947251 \times 10^{-2}$	(4)	$1.839165970 \times 10^{-2}$	(4)	$2.223027304 \times 10^{-2}$	(4)	$2.745212649 \times 10^{-2}$	(4)
40	$5.117637964 \times 10^{-2}$	(4)	$6.003720439 \times 10^{-2}$	(4)	$6.855986070 \times 10^{-2}$	(4)	$7.948313616 \times 10^{-2}$	(4)
50	$1.006331452 \times 10^{-1}$	(4)	$1.138167081 \times 10^{-1}$	(4)	$1.260839930 \times 10^{-1}$	(4)	$1.413121632 \times 10^{-1}$	(2)

Table E4. Near-exact quantiles of $\Lambda^{2/N}$ for $p = 14$

N	$\alpha = 0.01$	m^*	$\alpha = 0.025$	m^*	$\alpha = 0.05$	m^*	$\alpha = 0.1$	m^*
15	$5.968007341 \times 10^{-11}$	(4)	$3.864358294 \times 10^{-10}$	(4)	$1.636768946 \times 10^{-9}$	(4)	$7.320277569 \times 10^{-9}$	(4)
16	$2.631877572 \times 10^{-8}$	(4)	$7.564372294 \times 10^{-8}$	(4)	$1.768699381 \times 10^{-7}$	(4)	$4.433213878 \times 10^{-7}$	(4)
17	$4.303895570 \times 10^{-7}$	(4)	$9.598868708 \times 10^{-7}$	(4)	$1.849779949 \times 10^{-6}$	(4)	$3.801312060 \times 10^{-6}$	(4)
18	$2.635434985 \times 10^{-6}$	(4)	$5.147581731 \times 10^{-6}$	(4)	$8.936403861 \times 10^{-6}$	(4)	$1.644163547 \times 10^{-5}$	(4)
19	$9.977587414 \times 10^{-6}$	(6)	$1.788143092 \times 10^{-5}$	(4)	$2.897343157 \times 10^{-5}$	(4)	$4.948756698 \times 10^{-5}$	(4)
20	$2.828395716 \times 10^{-5}$	(4)	$4.762737988 \times 10^{-5}$	(4)	$7.337016162 \times 10^{-5}$	(4)	$1.186146150 \times 10^{-4}$	(4)
25	$6.507492492 \times 10^{-4}$	(4)	$9.248943591 \times 10^{-4}$	(4)	$1.239566302 \times 10^{-3}$	(4)	$1.718864890 \times 10^{-3}$	(4)
30	$3.355262830 \times 10^{-3}$	(4)	$4.393950494 \times 10^{-3}$	(4)	$5.502334209 \times 10^{-3}$	(4)	$7.075150920 \times 10^{-3}$	(4)
40	$1.906963623 \times 10^{-2}$	(4)	$2.296472997 \times 10^{-2}$	(4)	$2.681903407 \times 10^{-2}$	(4)	$3.190185732 \times 10^{-2}$	(4)
50	$4.779459232 \times 10^{-2}$	(4)	$5.510388888 \times 10^{-2}$	(4)	$6.205673620 \times 10^{-2}$	(4)	$7.088131886 \times 10^{-2}$	(4)

Table E5. Near-exact quantiles of $\Lambda^{2/N}$ for $p = 16$

N	$\alpha = 0.01$	m^*	$\alpha = 0.025$	m^*	$\alpha = 0.05$	m^*	$\alpha = 0.1$	m^*
17	$6.954345480 \times 10^{-12}$	(4)	$4.514993295 \times 10^{-11}$	(4)	$1.919965856 \times 10^{-10}$	(4)	$8.647924597 \times 10^{-10}$	(4)
18	$3.370977451 \times 10^{-9}$	(4)	$9.793105080 \times 10^{-9}$	(4)	$2.315444013 \times 10^{-8}$	(4)	$5.891190293 \times 10^{-8}$	(4)
19	$6.094359732 \times 10^{-8}$	(4)	$1.380094214 \times 10^{-7}$	(4)	$2.698618337 \times 10^{-7}$	(4)	$5.648824057 \times 10^{-7}$	(4)
20	$4.108569123 \times 10^{-7}$	(4)	$8.167747384 \times 10^{-7}$	(4)	$1.441312042 \times 10^{-6}$	(4)	$2.705710796 \times 10^{-6}$	(4)
25	$5.592434762 \times 10^{-5}$	(4)	$8.530201441 \times 10^{-5}$	(4)	$1.213442085 \times 10^{-4}$	(4)	$1.800030465 \times 10^{-4}$	(4)
30	$5.680480407 \times 10^{-4}$	(4)	$7.790352764 \times 10^{-4}$	(4)	$1.014598234 \times 10^{-3}$	(4)	$1.364325792 \times 10^{-3}$	(4)
40	$5.978325871 \times 10^{-3}$	(4)	$7.402155558 \times 10^{-3}$	(4)	$8.852253879 \times 10^{-3}$	(4)	$1.082058114 \times 10^{-2}$	(4)
50	$2.011798024 \times 10^{-2}$	(4)	$2.366316754 \times 10^{-2}$	(4)	$2.710964812 \times 10^{-2}$	(4)	$3.158052634 \times 10^{-2}$	(4)

Table E6. Near-exact quantiles of $\Lambda^{2/N}$ for $p = 18$

N	$\alpha = 0.01$	m^*	$\alpha = 0.025$	m^*	$\alpha = 0.05$	m^*	$\alpha = 0.1$	m^*
19	$8.265521794 \times 10^{-13}$	(4)	$5.379152077 \times 10^{-12}$	(4)	$2.295600345 \times 10^{-11}$	(4)	$1.040502518 \times 10^{-10}$	(4)
20	$4.355362880 \times 10^{-10}$	(4)	$1.277429693 \times 10^{-9}$	(4)	$3.050154120 \times 10^{-9}$	(4)	$7.863215324 \times 10^{-9}$	(4)
25	$2.552384884 \times 10^{-6}$	(4)	$4.254251580 \times 10^{-6}$	(4)	$6.518623271 \times 10^{-6}$	(4)	$1.051247048 \times 10^{-5}$	(4)
30	$6.737270305 \times 10^{-5}$	(4)	$9.736170594 \times 10^{-5}$	(4)	$1.325584728 \times 10^{-4}$	(4)	$1.874841858 \times 10^{-4}$	(4)
40	$1.551433278 \times 10^{-3}$	(4)	$1.978725976 \times 10^{-3}$	(4)	$2.427061596 \times 10^{-3}$	(4)	$3.054048472 \times 10^{-3}$	(4)
50	$7.447089156 \times 10^{-3}$	(4)	$8.944356252 \times 10^{-3}$	(4)	$1.043212842 \times 10^{-2}$	(4)	$1.240490010 \times 10^{-2}$	(2)

Table E7. Near-exact quantiles of $\Lambda^{2/N}$ for $p = 20$

N	$\alpha = 0.01$	m^*	$\alpha = 0.025$	m^*	$\alpha = 0.05$	m^*	$\alpha = 0.1$	m^*
21	$9.974108840 \times 10^{-14}$	(4)	$6.505371782 \times 10^{-13}$	(4)	$2.785206148 \times 10^{-12}$	(4)	$1.269572642 \times 10^{-11}$	(4)
25	$4.681479730 \times 10^{-8}$	(4)	$8.828836369 \times 10^{-8}$	(4)	$1.498370532 \times 10^{-7}$	(4)	$2.707054392 \times 10^{-7}$	(4)
30	$5.192701461 \times 10^{-6}$	(4)	$7.976800129 \times 10^{-6}$	(4)	$1.143603125 \times 10^{-5}$	(4)	$1.714984152 \times 10^{-5}$	(4)
40	$3.268365754 \times 10^{-4}$	(4)	$4.303572406 \times 10^{-4}$	(4)	$5.424280625 \times 10^{-4}$	(4)	$7.041416220 \times 10^{-4}$	(4)
50	$2.403097863 \times 10^{-3}$	(4)	$2.950167844 \times 10^{-3}$	(4)	$3.506020573 \times 10^{-3}$	(4)	$4.259716576 \times 10^{-3}$	(4)

References

[1] T.W. Anderson, *An Introduction to Multivariate Statistical Analysis*, 3rd ed. New York: J. Wiley & Sons, 2003.

[2] B.C. Arnold, C.A. Coelho and F.J. Marques, *The distribution of the product of powers of independent Uniform random variables – a simple but useful tool to address and better understand the structure of some distributions*, submitted to Journal of Multivariate Analysis (2010).

[3] G.E.P. Box, *A general distribution theory for a class of likelihood criteria*, Biometrika 36 (1949), pp. 317-346.

[4] C.A. Coelho, *The Generalized Integer Gamma Distribution – A basis for distributions in Multivariate Statistics*, Journal of Multivariate Analysis 64 (1998), pp. 86-102.

[5] C.A. Coelho, *The Generalized Near-Integer Gamma distribution: a basis for 'near-exact' approximations to the distributions of statistics which are the product of an odd number of independent Beta random variables*, Journal of Multivariate Analysis 89 (2004), pp. 191-218.

[6] C.A. Coelho and F.J. Marques, *Near-exact distributions for the independence and sphericity likelihood ratio test statistics*, Journal of Multivariate Analysis 101 (2010), pp. 583-593.

[7] R. Khattree, *Multivariate statistical inference involving circulant matrices: a review*, in: Gupta, A. K., Girko, V. L. (Eds.), *Multidimensional Statistical Analysis and Theory of Random Matrices*, 1996, pp. 101-110, VSP, Netherlands.

[8] F.J. Marques and C.A. Coelho, *Near-exact distributions for the sphericity likelihood ratio test*, Journal of Statistical Planning and Inference 138 (2008), pp. 726-741.

[9] D.K. Nagar, J. Chen and A.K. Gupta, *Distribution and percentage points of the likelihood ratio statistic for testing circular symmetry*, Computational Statistics and Data Analysis, 47 (2004), pp. 79-89.

[10] I. Olkin and S.J. Press, *Testing and estimation for a circular stationary model*, Annals Mathematical Statistics 40 (1969), pp. 1358-1373.

[11] J. Gil-Pelaez, *Note on the inversion theorem*, Biometrika 38 (1951), pp. 481-482.

[12] F.G. Tricomi and A. Erdélyi, *The asymptotic expansion of a ratio of Gamma functions*, Pacific J. Math. 1 (1951), pp. 133-142.