On some auto-induced regime switching double threshold glued diffusions

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As an heartily and humble token of immense admiration and everlasting gratitude, this work is dedicated by the first author to Jean-Pierre Kahane, his friend, PhD adviser and patient mathematical mentor for more than 20 years, from whom he first learned, at Orsay lectures, the surprising and beautiful level sets behavior of fractional (and usual) Brownian processes.

Abstract

Regime switching processes are usually defined with an external random source driving the regime changes. In this paper we define and study a regime switching diffusion considering two thresholds, and regime switching occurring, by a change in the diffusion drift and volatility, whenever the trajectory touches the upper threshold after having crossed, or touched, the lower threshold or touches the lower threshold after having crossed, or touched, the upper threshold. We develop an estimation procedure for the thresholds and for the regime parameters of the diffusions. We show that a generalized Black-Scholes model with the regime switching diffusion as the law of the risky asset is arbitrage free and complete under an additional hypothesis on the diffusion coefficients of the two regimes diffusions.

1 Introduction

Let us describe some motivation for the study presented in this paper, with a tentative modeling of a simple phenomena. Consider a closed room containing a heating device equipped with a thermostat; whenever the temperature drops below some given threshold c the heating device starts delivering heat until the temperature gets over the threshold. Due to some thermal inertia of the heating device, the temperature will continue to

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augment and, after some time, due to deficient insulation of the walls of the room, the temperature will start to drop. Suppose that the temperature is monitored by a very precise thermometer recording, not only the up and down trends, corresponding to the functioning periods of the heating device but also, the local fluctuations of the temperature due to, for instance, the convection air currents and the irregular heat radiation patterns of the room walls. A natural mathematical model of the phenomena just described would be a stochastic process with linear up and down trends, for instance μt and $-\nu t$, with $\mu, \nu \in]0, +\infty[$, and some additive Gaussian noise given by scaled Brownian processes $\sigma_1 B_t$ and $\sigma_2 B_t$, with $\sigma_1, \sigma_2 > 0$, that is, in such a way that $X_t = \mu t + \sigma_1 B_t$ if the temperature verifies $X_t < c$ and $X_t = -\nu t + \sigma_2 B_t$ if $X_t > c$. Let us use this model to describe the phenomena. With some initial temperature $X_0 = c_0 < c$ the temperature process will be given by $X_t = \mu t + \sigma_1 B_t$ until the temperature reaches the threshold c at the stopping time given by

$$\tau_1 := \inf\{t > 0 : \mu t + \sigma_1 B_t = c\}.$$

Now, with a temperature stochastic process model with continuous trajectories we would want to have that,

$$\mu\tau_1 + \sigma_1 B_{\tau_1} = X_{\tau_1} = c = -\nu\tau_1 + \sigma_2 B_{\tau_1}$$

and for $t > \tau$, we should have $X_t = -\nu t + \sigma_2 B_t$ until the following hitting time of the threshold c given by the stopping time

$$\tau_2 := \inf\{t > \tau_1 : -\nu t + \sigma_2 B_t = c\}.$$

Unfortunately, we will have $\tau_1 = \tau_2$ almost surely. In order to prove this statement, let us observe that by changing $\mathbb{P}$, the initial probability, into some adequate new probability $\mathbb{Q}$ given by Girsanov theorem (see [Øks03, p. 162, 164]), the process $-\nu t + \sigma_2 B_t$ under $\mathbb{Q}$ has the same law as a Brownian process. Finally, by a classical result on the level sets of the Brownian process (see [Str94, p. 247] or [RY99, p. 109]), we may say that the level sets $\{t \in \mathbb{R}^+ : -\nu t + \sigma_2 B_t = c\}$, are perfect sets, that is, closed sets consisting only of limit points of the set itself. In particular, for almost all the trajectories $\omega_0 \in \Omega$, with $t_0 := \tau_1(\omega_0)$, the set $\{t \geq t_0 : -\nu t + \sigma_2 B_t(\omega_0) = c\}$ is closed and composed entirely of limit points of itself; being so, $\tau_2(\omega_0) = t_0$ and consequently $\tau_2 = \tau_1$ almost surely. One way to overcome this modeling difficulty is to consider that the regime switch occurs at the hitting time of the threshold with some significant delay added to this hitting time. This model was explored in a previous work (see [ME]). In this paper we develop, for a general class of diffusions the model of a regime switching with a double threshold, following the previous work [Mot07] where the sudy was carried for the non scaled Brownian process with a linear drift. A model of two interlaced price-liquidity regime switching diffusions, with two thresholds each, was introduced and studied in [REMV09], with a regime switching happening for one of the diffusions whenever the other hit one of its own thresholds.

The difficulty of defining a regime switching model for diffusions with only one threshold has its roots in the level set behavior of a Brownian type process. The continuous martingale zero set behavior is detailed in [RY99] by means of support of the local time of the martingale. An ingenious detailed study of the Hausdorff dimension of level sets of Hölder mappings, of Gaussian Fourier series and of fractional Brownian motions can be found in [Kah85].

Regime switching diffusion with the regime changes driven by a jump Markov process taking values in a finite set have been extensively studied in the last two decades and even more intensively in recent times. Among the many recent works, dealing with important problems for these processes we single [KZY07] where the stability is studied and references to important previous work are signaled. The statistical problem addressed in this paper, namely the estimation of thresholds, is treated for the external Markov driven regime switching processes in the discrete time framework in [Cha93] and [CT98]; estimation of threshold models in continuous time is developed in [FP98]. Finally, an application of a regime switching model driven by a external Markov source to interest rate theory is given in [HP00].

2 The double threshold process definition

We will now define a stochastic process on some interval $[0, T]$, by gluing together excursions of diffusions defined by successive isolated random times corresponding to the successive alternative hitting of the thresholds. Consider two thresholds $m < M$ defined in the phase space of a real valued diffusion $(X_t^\theta)_{t\geq 0}$ with $\theta \in \mathbf{\Theta} = \{\theta_1, \theta_2\}$ a parameter which will differentiate the regimes. In all the following, we will suppose that $x_0 < m$. Additionally, suppose that we have $(X_t^\theta)_{t\in[0,T]} \equiv (X_t^{\theta_1})_{t\in[0,T]}$ this second process being the diffusion given by the stochastic differential equation:

$$\begin{cases} dX_t^{\theta_1} = \mu(t, X_t^{\theta_1}, \theta_1)dt + \sigma(t, X_t^{\theta_1}, \theta_1)dB_t & t \in [0, T] \\ X_0^{\theta_1} = x_0 \, , \end{cases}$$

with $(B_t)_{t\geq 0}$ a Brownian process defined on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Consider now that there exists a first hitting time of the upper threshold M given by:

$$\tau_1 = \inf \left\{ t > 0 \ : X_t^{\theta_1} = M \right\} \wedge T \, .$$

Let us suppose that $\tau_1 < T$ with positive probability; otherwise, if $\tau_1 = T$ almost surely there will no need to proceed. Given that $(X_t^{\theta_1})_{t\geq 0}$ has continuous trajectories we are sure that by the time τ_1, the process has already crossed the lower threshold m and so, in our construction, a regime switching occurs. For that, redefine the process $(X_t^\theta)_{t\geq 0}$ to be given for $t \in [0, \tau_1]$ as before by $X_t^\theta = X_t^{\theta_1}$ and anew, for $t \geq \tau_1$, by the solution of the stochastic differential equation (2.1) given by the following proposition.

Proposition 1. *The process defined by*

$$\begin{cases} dX_t^{\theta_2} = \mu(t, X_t^{\theta_2}, \theta_2)dt + \sigma(t, X_t^{\theta_2}, \theta_2)dB_t & t \in [\tau_1, T] \\ X_{\tau_1}^{\theta_2} = M = X_{\tau_1}^{\theta_1} \, , \end{cases} \tag{2.1}$$

exists and is uniquely determined if the integrability, the Lipschitz control and the sub-linear growth usual conditions are verified by the diffusion coefficients of the following stochastic differential equation

$$\begin{cases} dX_t = \mu(t, X_t, \theta_2)dt + \sigma(t, X_t, \theta_2)dB_t & t \in [t_0, T] \\ X_{t_0} = Z, \end{cases} \tag{2.2}$$

with some initial condition $Z \in L^2$ for $t_0 \in [0, T]$, ensuring that a solution exists and is uniquely determined.

Proof. We observe that equation (2.1) is to be interpreted, for all $t \in [0, T]$, as

$$X_t^{\theta_2} = X_{\tau_1}^{\theta_1} \mathbb{I}_{[\tau_1,T]}(t) + \int_0^t \mu(t, X_u^{\theta_2}, \theta_2)\mathbb{I}_{[\tau_1,T]}(u)du + \int_0^t \sigma(t, X_u^{\theta_2}, \theta_2)\mathbb{I}_{[\tau_1,T]}(u)dB_u \tag{2.3}$$

Now consider some standard existence theorem (see [Øks03, p. 289] or [LS01, p. 134]). Then as $X_{\tau_1}^{\theta_1} = M$ it is clear that the initial condition of (2.1) is in L^2. Moreover, as

$$\left|\mu(t, X_u^{\theta_2}, \theta_2)\mathbb{I}_{[\tau_1,T]}(u)\right| \leq \left|\mu(t, X_u^{\theta_2}, \theta_2)\right|$$

and

$$\left|\sigma(t, X_u^{\theta_2}, \theta_2)\mathbb{I}_{[\tau_1,T]}(u)\right| \leq \left|\sigma(t, X_u^{\theta_2}, \theta_2)\right|$$

it is clear that the integrability, the Lipschitz control and the sub-linear growth conditions verified by the diffusion coefficients of (2.2) are still verified by the diffusion coefficients of equation (2.1) in the equivalent formulation (2.3) and so the result is proved. $\square$

Consider now the next regime switching stopping time given by the hitting time of the lower threshold

$$\tau_2 = \inf\left\{t > \tau_1 \, : X_t^{\theta_2} = m\right\} \wedge T \, .$$

Again, let us suppose that $\tau_2 < T$ with positive probability. And again, by time τ_2 the process is for sure below M and a new regime switching is trigered. We may now redefine the process $(X_t^\theta)_{t\geq 0}$ for $t \geq \tau_2$ by $X_t^\theta = X_t^{\theta_1}$ given by the solution of

$$\begin{cases} dX_t^{\theta_1} = \mu(t, X_t^{\theta_1}, \theta_1)dt + \sigma(t, X_t^{\theta_1}, \theta_1)dB_t & t \geq \tau_2 \\ X_{\tau_2}^{\theta_1} = m \, , \end{cases}$$

We may pursue by induction defining $(\tau_n)_{n\geq 1}$ a strictly increasing sequence of isolated stopping times triggering the regime switchings. We will now prove that these stopping times are isolated for almost all $\omega \in \Omega$. Consider some $n \geq 1$; for any $\omega \in \Omega_0$, a set of full probability, the diffusion trajectory $X_t^\theta(\omega)$, defined by the correspondent stochastic differential equation for $t \geq \tau_n$, is uniformly continuous in $[\tau_n(\omega), T]$. So, there exists $\eta = \eta(\omega, n)$ such that if $|r - s| < \eta$, with $r, s \geq \tau_n(\omega)$, we have that $\left|X_r^\theta(\omega) - X_s^\theta(\omega)\right| < M - m$. It clear then that as $\left|X_{\tau_n(\omega)}^\theta(\omega) - X_{\tau_{n+1}(\omega)}^\theta(\omega)\right| = |M - m|$, we surely have $|\tau_{n+1}(\omega) - \tau_n(\omega)| \geq \eta$.

Finally, we can summarize the construction above in the following result.

Theorem 2.1. *Consider a parameter set* $\mathbf{\Theta} = \{\theta_1, \theta_2\}$*with two regime parameters. Let* $\mu(t, x, \theta)$ *and* $\sigma(t, x, \ \theta)$ *be two real valued functions defined on* $\mathbb{R} \times \mathbf{\Theta}$ *such that for any random variable* $Z \in L^2$, $\theta \in \mathbf{\Theta}$ *and* $t_0 \in [0, T]$ *the integrability, the Lipschitz control and the sub-linear growth usual conditions are verified by the diffusion coefficients of the following stochastic differential equation*

$$\begin{cases} dX_t = \mu(t, X_t, \theta)dt + \sigma(t, X_t, \theta)dB_t \quad t \in [t_0, T] \\ X_{t_0} = Z \ , \end{cases}$$

ensuring existence and unicity of the solution. Define, for $n \geq 0$,

$$\widehat{n} := \frac{1 - (-1)^n}{2} + 1 \ .$$

Then, there exist a strictly increasing sequence of stopping times $(\tau_n)_{n \geq 0}$*, isolated almost surely, such that* $\tau_0 \equiv 0$ *and* $\tau_0 \leq \tau_n < \tau_{n+1} \leq T$ *for* $n \geq 1$*, and such that the stochastic differential equation defined with* $X_0^\theta = x_0 < m$ *for* $t \in [0, T]$*, by*

$$\begin{aligned} X_t^\theta = & \left(\sum_{n=0}^{+\infty} X_{\tau_n}^{\theta_{\widehat{n}}} \mathbb{1}_{[\tau_n, \tau_{n+1}[}(t) \right) + \int_0^t \left(\sum_{n=0}^{+\infty} \mu \left(t, X_u^{\theta_{\widehat{n}}}, \theta_{\widehat{n}} \right) \mathbb{1}_{[\tau_n, \tau_{n+1}[}(u) \right) du + \\ & + \int_0^t \left(\sum_{n=0}^{+\infty} \sigma \left(t, X_u^{\theta_{\widehat{n}}}, \theta_{\widehat{n}} \right) \mathbb{1}_{[\tau_n, \tau_{n+1}[}(u) \right) dB_u \end{aligned} \tag{2.4}$$

has an unique almost surely continuous solution which is a regime switching process $(X_t^\theta)_{t \in [0,T]}$ *given by:*

$$X_t^\theta = \sum_{n=0}^{+\infty} X_{n,t}^{\theta_{\widehat{n}}} \mathbb{1}_{[\tau_n, \tau_{n+1}[}(t) \tag{2.5}$$

with the excursion process $(X_{n,t}^{\theta_{\widehat{n}}})_{t \in [\tau_n, \tau_{n+1}[}$ *given by the solution of the stochastic differential equation*

$$\begin{cases} dX_{n,t}^{\theta_{\widehat{n}}} = \mu(t, X_{n,t}^{\theta_{\widehat{n}}}, \theta_{\widehat{n}})dt + \sigma(t, X_{n,t}^{\theta_{\widehat{n}}}, \theta_{\widehat{n}})dB_t \quad \tau_{n+1} \leq t < \tau_{n+2} \\ X_{n,\tau_{n+1}}^{\theta_{\widehat{n}}} = m\mathbb{1}_{\{\widehat{n}=1\}} + M\mathbb{1}_{\{\widehat{n}=2\}} \ . \end{cases}$$

For this solution process, we have that:

- *If* $X_t^\theta < m$*, then* X_t^θ *has drift* $\mu\left(t, X_t^{\theta_1}, \theta_1\right)$ *and volatility* $\sigma\left(t, X_t^{\theta_1}, \theta_1\right)$*;*
- *If* $X_t^\theta > M$*, then* X_t^θ *has drift* $\mu\left(t, X_t^{\theta_2}, \theta_2\right)$ *and volatility* $\sigma\left(t, X_t^{\theta_2}, \theta_2\right)$.

Remark 1. Given $\mathcal{T} = (\tau_n)_{n \geq 0}$, the excursion processes $(X_{n,t}^{\theta_{\widehat{n}}})_{t \in [\tau_n, \tau_{n+1}[}$, for $n \geq 0$, are independent by construction.

Remark 2. Under the hypothesis that $\mu(t,x,\theta) = \mu(x,\theta)$ and $\sigma(t,x,\theta) = \sigma(x,\theta)$ we have that almost surely $\mathcal{T}$ is a finite sequence. In fact, every excursion with a given parameter $\theta \in \mathbf{\Theta}$ is determined by the same diffusion coefficients and by the fact that either they start with the diffusion in M and ending in m or vice-versa. This means that the stopping times in $\mathcal{T}$ are, alternatively, the random time that a certain diffusion takes to go from the value M to the value m or the time that another diffusion takes to go from the value m to the value M. This being said, suppose that $\mathcal{T}$ is infinite. Then, we have that the sequence of random variables $(\tau_{n+1} - \tau_n)_{n\geq 1}$ is a sequence of non negative random variables such that the subsequences $(\tau_{2p+1} - \tau_{2p})_{p\geq 0}$ and $(\tau_{2p+2} - \tau_{2p+1})_{p\geq 0}$ are sequences of identically distributed random variables verifying:

$$\sum_{n=1}^{+\infty} (\tau_{n+1} - \tau_n) \leq T \text{ and so } \sum_{n=1}^{+\infty} \mathbb{E}\left[\tau_{n+1} - \tau_n\right] \leq T \ .$$

But, as $\mathbb{E}\left[\tau_{n+1} - \tau_n\right]$ takes only two real values depending on the parity of n, we must have for all $n \geq 1$ that $\mathbb{E}\left[\tau_{n+1} - \tau_n\right] = 0$ which implies that for all $n \geq 1$, $\tau_{n+1} - \tau_n = 0$ almost surely, having as a consequence that for all $N \geq 2$,

$$\tau_N - \tau_1 = \sum_{n=1}^{N-1} (\tau_{n+1} - \tau_n) = 0 \text{ almost surely,}$$

allowing us to conclude that $\mathcal{T}$ has only one element, namely τ_1.

Remark 3. If $x_0 > M$, similar results to those presented will follow if we suppose that the starting regime is regime 2. The same will happen if $x_0 \in [m, M]$ for any of the two possible choices of the starting regime, either regime 1 or regime 2.

3 On the thresholds estimation

For some trajectory $\omega \in \Omega$, consider $C_n(\omega) = \{X^\theta_{t_1}(\omega), \ldots, X^\theta_{t_{p_n}}(\omega)\}$ the finite set of discrete observations at time $n \in \mathbb{N}$ and suppose that for all $n \geq 1$ we have that $C_n(\omega) \subseteq C_{n+1}(\omega)$. We suppose that we are able to decide to what regime in $\{1, 2\}$ each observation $X^\theta_{t_j}(\omega)$ belongs, thus defining a random variable R^n_j such that $R^n_j(\omega) = 1$ if $X^\theta_{t_j}(\omega)$ is regime 1 and $R^n_j(\omega) = 2$ if $X^\theta_{t_j}(\omega)$ is regime 2. A fundamental remark is that

- If $R^n_j(\omega) = 1$, that is $X^\theta_{t_j}(\omega)$ is regime 1, then $X^\theta_{t_j}(\omega) \leq M$;
- If $R^n_j(\omega) = 2$, that is $X^\theta_{t_j}(\omega)$ is regime 2, then $X^\theta_{t_j}(\omega) \geq m$.

We now define the natural partition of the sets of observations given by:

$$C^M_n(\omega) = \{X^\theta_{t_j}(\omega) : R^n_j(\omega) = 1, j = 1, \ldots, p_n\} \ ,$$

and

$$C^m_n(\omega) = \{X^\theta_{t_j}(\omega) : R^n_j(\omega) = 2, j = 1, \ldots, p_n\} \ .$$

The following result shows that, under the hypothesis of increasing the number of observations with time made above, there exists strongly consistent estimators of the thresholds.

Theorem 3.1. *Let $\widehat{M}_n(\omega) = \max C_n^M(\omega)$ and $\widehat{m}_n(\omega) = \min C_n^m(\omega)$. Suppose that the observations are regularly distributed in $[0,T]$, in the sense that,*

$$\lim_{n\to+\infty} \max_{1\leq i\leq p_n-1} |t_{i+1} - t_i| = 0 \ . \tag{3.1}$$

Then, if there is at least a regime change from regime 1 to regime 2, the correspondent estimator of the upper threshold M, $\widehat{M}$ defined by $\widehat{M}(\omega) := \lim_{n\to+\infty} \widehat{M}_n(\omega) \in \mathbb{R}$ is strongly consistent and, if there is at least a regime change from regime 2 to regime 1, the estimator of the lower threshold m, $\widehat{m}$ defined by $\widehat{m}(\omega) := \lim_{n\to+\infty} \widehat{m}_n(\omega) \in \mathbb{R}$ is also strongly consistent, that is:

$$\widehat{M} = M \ \textit{a.s. and } \widehat{m} = m \ \textit{a.s.}$$

Proof. As $C_n(\omega) \subseteq C_{n+1}(\omega)$ we have that $\widehat{M}_n \leq \widehat{M}_{n+1}$ and $\widehat{m}_n \geq \widehat{m}_{n+1}$. Then, as a consequence of the fundamental remark above, we have that $\widehat{M}(\omega) = \lim_{n\to+\infty} \widehat{M}_n(\omega) \leq M$ and $\widehat{m}(\omega) = \lim_{n\to+\infty} \widehat{m}_n(\omega) \geq m$. Let us suppose, for instance, that for some $\epsilon > 0$ we have $\widehat{M}(\omega) < M - \epsilon$ and consider the first time of regime switching from regime 1 to regime 2, that is:

$$\tau(\omega) = \inf \left\{ t > 0 : X_t^\theta(\omega) = M \right\} \ ,$$

which is well defined by the hypothesis. By the almost surely uniform continuity of the trajectory $(X_t^\theta(\omega))_{t\in[0,T]}$, there exists $\eta > 0$ such that for $|r - s| < \eta$, with $r, s \in [0,T]$, we have $\left|X_r^\theta(\omega) - X_s^\theta(\omega)\right| < \epsilon$. Now, it is enough to choose some $n_0 \in \mathbb{N}$ for which we have that $\max_{1\leq i\leq p_{n_0}-1} |t_{i+1} - t_i| < \eta$ and to select $i_0 \in \{1, \ldots, p_{n_0}\}$ such that $\tau(\omega) \in [t_{i_0}, t_{i_0+1}]$. As a consequence of these choices, we have that

$$|\tau(\omega) - t_{i_0}| \leq |t_{i_0+1} - t_{i_0}| < \eta$$

and so

$$\left| X_{\tau(\omega)}^\theta(\omega) - X_{t_{i_0}}^\theta(\omega) \right| = \left| M - X_{t_{i_0}}^\theta(\omega) \right| < \epsilon \ ,$$

thus implying that $\left|\widehat{M}_{n_0}(\omega) - M\right| < \epsilon$ and furthermore $\left|\widehat{M}(\omega) - M\right| < \epsilon$, contradicting our initial assumption that $\widehat{M}(\omega) < M - \epsilon$. The proof that the above defined estimator of m is strongly consistent being similar, our proof is complete. □

Remark 4. The result obtained in theorem 3.1 suggests a practical way to estimate the thresholds and the remaining parameters of the regime switching process. In fact, the method is easily described in the following simple steps.

- Consider given some values $\widetilde{m}$ and $\widetilde{M}$ for the thresholds. Using these values, classify your process observations according to this thresholds choice in order to obtain a partition of the set of observations in the set of observation in regime 1 and in regime 2..

- Use some adequate estimators for the parameter θ, in order to obtain an estimator $\widehat{\theta}_1$ for the parameter value θ_1 using the observations in the set corresponding to the regime 1 observations and the same to obtain $\widehat{\theta}_2$ using the observations in the set corresponding to the regime 2 observations.

- Finally, minimize a loss function of the conditional least squares type:

$$\mathrm{CLS}_{\widetilde{m},\widetilde{M}} := \sum_i \left(X_i - \mathbb{E}_{\widehat{\theta}_1,\widehat{\theta}_2,\widetilde{m},\widetilde{M}} \left[X_i \mid X_1, \ldots X_{i-1} \right] \right)^2$$

  over $\widetilde{m}$ and $\widetilde{M}$.

A method similar to this one was successfully applied in [ME] for the case of a regime switching diffusion with one threshold and delay, the initially given diffusions being geometric brownian motions.

4 An application to continuous time finance

Stock price modeling having in view option pricing is a natural field of application of auto-induced regime switching diffusions. In fact, when some prices are observed over a sufficiently long period to encompass fluctuation cycles, these prices seem to go up with some drift and volatility and then to come down with differents drift and volatility, oscillating around some kind of target price. This behavior suggests the existence of a threshold and two different regimes governing the diffusion above and below the threshold. Moreover, as prices are usually given to the cents if, in the double threshold model, we consider that $M - m$ is strictly less than one cent, for all practical purposes we will get a model that may describe properly the the regime switching prices around one single price threshold. Nevertheless in order for a two narrowly separated threshold model to be properly defined, the regime changes can only occur after a delay; in the absence of a delay for the regime changes to occur, a process with small volatility relatively to the drift could be, most of the time, trapped between the two narrowly separated thresholds. Another approach, perhaps more realistic, is to consider two widely spaced thresholds forcing the price process to oscillate between these two thresholds, as if it was a process loosely contained between two barriers. Our purpose in this section is to show that, under some additional hypothesis, the regime switching process (2.4) of theorem 2.1 is appropriate to define generalized Black-Scholes complete arbitrage free models for option pricing.

It is well known (see [Øks03, p. 288]) that a market model of one risk free asset given by

$$\rho(t) = \exp\left(\int_0^t r(u) du \right) ,$$

and one risky asset X_t driven by a stochastic differential equation

$$\begin{cases} dX_t = \mu(t)X_t dt + \sigma(t)X_t dB_t & t \in [0,T] \\ X_0 = x_0 \, , \end{cases}$$

with r, μ, σ stochastic processes not depending on X, is arbitrage free and complete if the Novikok condition is satisfied, more precisely if

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 dt\right)\right] < +\infty \, .$$

If so, the price at time $t=0$ of an European option with payoff given by a contingent T-claim $f(X_T)$, given non random price X_{t_0}, is unique and it is given by the usual formula:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{f(X_T)}{r(t)}\right] \, ,$$

with $\mathbb{Q}$ the martingale measure. Now suppose that we intend to model the risky asset with a double threshold regime switching model as the one introduced above. We then have the following result.

Theorem 4.1. *With the notations of theorem 2.1 and under the assumptions of this section for ρ, μ and σ, suppose additionally that:*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \max\left(\left(\frac{\mu(t,\theta_1)-r(t)}{\sigma(t,\theta_1)}\right)^2, \left(\frac{\mu(t,\theta_2)-r(t)}{\sigma(t,\theta_2)}\right)^2\right) dt\right)\right] < +\infty \, . \tag{4.1}$$

Then, the market model with the risky asset described by the regime switching process (2.5) is arbitrage free and complete.

Remark 5. The condition in formula (4.1) is obviously satisfied if ρ, μ and σ are constant, that is, if the diffusions in the regimes are as the usual Black-Scholes model of stock prices.

Proof. Let us show that the Novikov condition is satisfied for the diffusion parameters given in formula (2.4) for the regime switching process. As we have that:

$$\begin{aligned} \frac{\left(\sum_{n=0}^{+\infty} \left(\mu\left(t,\theta_{\widehat{n}}\right)-r(t)\right) \mathbb{I}_{[\tau_n,\tau_{n+1}[}(t)\right)^2}{\left(\sum_{n=0}^{+\infty} \sigma\left(t,\theta_{\widehat{n}}\right) \mathbb{I}_{[\tau_n,\tau_{n+1}[}(t)\right)^2} &= \frac{\sum_{n=0}^{+\infty} \left(\mu\left(t,\theta_{\widehat{n}}\right)-r(t)\right)^2 \mathbb{I}_{[\tau_n,\tau_{n+1}[}(t)}{\sum_{n=0}^{+\infty} \sigma\left(t,\theta_{\widehat{n}}\right)^2 \mathbb{I}_{[\tau_n,\tau_{n+1}[}(t)} \leq \\ &\leq \sum_{n=0}^{+\infty} \frac{\left(\mu\left(t,\theta_{\widehat{n}}\right)-r(t)\right)^2}{\sigma\left(t,\theta_{\widehat{n}}\right)^2} \mathbb{I}_{[\tau_n,\tau_{n+1}[}(t) \, , \end{aligned}$$

we have the following upper estimates for the expectation in the Novikov condition formulated for the diffusion parameters of the regime switching process:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \frac{\left(\sum_{n=0}^{+\infty}\left(\mu\left(t,\theta_{\widehat{n}}\right)-r(t)\right)\mathbb{1}_{[\tau_n,\tau_{n+1}[}(t)\right)^2}{\left(\sum_{n=0}^{+\infty}\sigma\left(t,\theta_{\widehat{n}}\right)\mathbb{1}_{[\tau_n,\tau_{n+1}[}(t)\right)^2}dt\right)\right] \leq$$
$$\leq \mathbb{E}\left[\exp\left(\frac{1}{2}\sum_{n=0}^{+\infty}\int_{\tau_n}^{\tau_{n+1}}\left(\frac{\left(\mu\left(t,\theta_{\widehat{n}}\right)-r(t)\right)^2}{\sigma\left(t,\theta_{\widehat{n}}\right)^2}\right)dt\right)\right] \leq$$
$$\leq \mathbb{E}\left[\exp\left(\frac{1}{2}\sum_{n=0}^{+\infty}\int_{\tau_n}^{\tau_{n+1}}\max\left(\left(\frac{\mu(t,\theta_1)-r(t)}{\sigma(t,\theta_1)}\right)^2,\left(\frac{\mu(t,\theta_2)-r(t)}{\sigma(t,\theta_2)}\right)^2\right)dt\right)\right] \leq$$
$$\leq \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \max\left(\left(\frac{\mu(t,\theta_1)-r(t)}{\sigma(t,\theta_1)}\right)^2,\left(\frac{\mu(t,\theta_2)-r(t)}{\sigma(t,\theta_2)}\right)^2\right)dt\right)\right] < +\infty\ .$$

So, the Novikov condition is satisfied for the regime switching diffusion coefficients and the theorem is proved. □

Remark 6. If we only suppose that for all $\theta = \theta_1, \theta_2$ we have

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T\left(\frac{\mu(t,\theta)-r(t)}{\sigma(t,\theta)}\right)^2 dt\right)\right] < +\infty\ .$$

does it follows that the market model is still s arbitrage free and complete, as in theorem 4.1? This would show that the procedure for building a regime switching double threshold diffusion model for a risky asset preserves the arbitrage free and completeness properties of the initially given building diffusions, with no need of other conditions than the usually conditions for the initially given diffusions .

5 Conclusion

In this work we have introduced a procedure to build, on a closed and bounded time interval, a regime switching double threshold diffusion from two initially given diffusions. The infinitesimal parameters of the built diffusion are given by means of the infinitesimal parameters of the initially given diffusions together with a sequence of the successive hitting times of the upper and lower thresholds. These stopping times, are shown to be well defined and to be almost surely isolated. We develop a procedure to estimate the thresholds together with the regime parameters of the diffusions. Finally, we show that if the models with the initially given diffusions are arbitrage free and complete, by force of verification of the Novikov condition for the Sharpe ratios determined by the infinitesimal parameters of these initially given diffusions then, under and additional Novikov condition for the maximum of the Sharpe ratios of the initially given diffusions, the regime switching double threshold model is also arbitrage free and complete.

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